HW Problems for Assignment 5 - Part I
Due 6:00 PM Tuesday, November 29, 2016

Note : this is the first half of HW 5. It covers material from Lecture 10 on Tuesday, November 15, 2016. The second half of HW 5 will be posted on Tuesday, November 22, 2016, and covers material from Lecture 11 on Tuesday, November 22, 2016. Both halves are to be submitted by 6 PM on Tuesday, November 29, 2016. There are 50 points possible for the questions covering Lecture 10, and 50 points possible for the questions covering Lecture 11.

1. (15 Points) The Distribution of S_N in a Discrete Setting. If we assume a finite number of losses and loss severities, sometimes it is possible to manually identify the distribution of S_N, and hence the associated risk measures. Here, we will do just that. Assume that in the next year there are either $N = 0, 1$ or 2 operational risk losses. The likelihoods are

$$N = \begin{cases} 0 & \text{p} = .5 \\ 1 & \text{p} = .3 \\ 2 & \text{p} = .2 \end{cases}.$$

Note that when $N = 0$ we have no losses (i.e. $S_N = 0$) by definition. For the loss severities X we also assume three values with probabilities

$$X = \begin{cases} 1000 & \text{p} = .4 \\ 10000 & \text{p} = .5 \\ 100000 & \text{p} = .1 \end{cases}.$$

(a) For $S_N = \sum_{k=1}^{N} X_k$ where the $\{X_k\}$ are i.i.d. copies of X, which are also independent of N, explicitly give the distribution of S_N.
(b) In the setting of part (a), compute $\text{VaR}_{.95}(S_N)$ and $\text{VaR}_{.99}(S_N)$.

2. (15 Points) Approximating a Compound Poisson Random Variable. In this exercise you will reproduce (most of) the plot on slide 23 of the lecture. Specially, you will approximate the tail of the cumulative distribution function (c.d.f.) a compound Poisson random variable. Here, the random variable is $S_N = \sum_{k=1}^{N} X_k$ where $N \sim \text{Poi}(\lambda)$ where the $\{X_k\}$ are i.i.d. copies (also independent of N) of X which is X is log-normally distributed with parameters μ, σ^2. In other words $\log(X) \sim N(\mu, \sigma^2)$. You will estimate the tail of the c.d.f. two ways: using the normal approximation and using the translated Gamma approximation. For the normal approximation we assume that

$$S_N \sim E[S_N] + \sqrt{\text{Var}[S_N]}Z; \qquad Z \sim N(0, 1).$$

To compute the mean and variance, use the formulas in class, along with the distributional assumptions on N and X. For the translated Gamma approximation we assume

$$S_N \sim k + Y; \qquad Y \sim \text{Gamma}\,(\alpha, \beta)\,.$$

Here, k, α, β are chosen to match the mean, variance, and skewness of S_N. To compute these values see page 519 of the Quantitative Risk Management book. For parameter values use

$$\lambda = 100; \qquad \mu = .10; \qquad \sigma = .4.$$

For your plot, produce a log-log plot of $1 - F_{S_N}(x)$ versus x with the two above approximations to F_{S_N}. For comparison purposes, sample $M = 100,000$ copies of S_N and plot the tail of the empirical c.d.f. as well. For the range of x, choose the low value to be $\text{VaR}_{.95}$ associated to the empirical distribution. For the high value, take $\text{VaR}_{.99999}$ of the empirical distribution. Which approximation works better: Normal or Gamma?

3 (20 Points) LVaR via Simulation for Random Spreads. As mentioned in class, the Value at Risk adjustment LVaR in both the constant and random spread setting is some-what ad-hoc in that we take our regular VaR and add the liquidity cost to obtain LVaR. This does not really reflect the liquidity risk associated to holding the position because it does not specify what might happen *in the future* if we liquidate our position. In this exercise we will estimate LVaR_α via simulation in the random spread case using a theoretically more justified approach, and compare it to the simple approximation.

Assume at t we already own $\Delta_t > 0$ shares of the asset S and do not plan on liquidating our position immediately. As such the theoretical value of our portfolio is

$$V_t = \Delta_t S_t.$$

Next, assume we hold Δ_t shares over the interval $[t, t+\Delta]$ but then liquidate our portfolio at time $t+\Delta$. The value of our portfolio at $t+\Delta$ is thus

$$V_{t+\Delta} = \Delta_t S^b_{t+\Delta}.$$

The losses are thus

$$L_{t+\Delta} = -\Delta_t \left(S^b_{t+\Delta} - S_t \right).$$

To estimate the risk of our losses we make the following assumptions:

(i) The theoretical price satisfies $S_{t+\Delta} = S_t e^{X_{t+\Delta}}$ where given $\mathcal{F}_t$ we know that $X_{t+\Delta} \sim N\left(\mu_{t+\Delta}, \sigma^2_{t+\Delta}\right)$.

(ii) At time $t+\Delta$ define the relative bid-ask spread $s_{t+\Delta}$ through the equation

$$\frac{1}{2} s_{t+\Delta} = \frac{S_{t+\Delta} - S^b_{t+\Delta}}{S_{t+\Delta}}.$$

Then, given $\mathcal{F}_t$, we assume $s_{t+\Delta} \sim N\left(\mu_{s,t+\Delta}, \sigma^2_{s,t+\Delta}\right)$ and is independent of $X_{t+\Delta}$.

In this setting complete the following tasks:

(a) Explicitly write the losses $L_{t+\Delta}$ in terms of quantities known at time t and the random variables $X_{t+\Delta}, s_{t+\Delta}$.

(b) Use the approximation

$$\mathrm{LVaR}_\alpha^{ind} = \mathrm{VaR}_\alpha + \mathrm{LC},$$

where VaR_α is the Value at Risk associated to holding Δ_t shares in a theoretical asset with price process S and LC is the industry suggested liquidity cost $\mathrm{LC} = \frac{1}{2}\Delta_t S_t \left(\mu_{s,t+\Delta} + k\sigma_{s,t+\Delta}\right)$. Here, compute the exact value of VaR_α in a similar manner to what we did in class, to obtain an exact value for $\mathrm{LVaR}_\alpha^{ind}$.

(c) Estimate the risk-adjusted $\mathrm{LVaR}_\alpha^{sim}$ via simulation. In other words, for $m = 1, ..., M$ sample $X_{t+\Delta}^m, s_{t+\Delta}^m$, compute the losses $\ell_m = L_{t+\Delta}^m$ and then output $\mathrm{LVaR}_\alpha^{sim} = \ell_{(\lceil M\alpha \rceil)}$.

For parameter values use $\Delta_t = 100$, $S_t = 59$, $\mu_{t+\Delta} = 0$, $\sigma_{t+\Delta} = .4/\sqrt{250}$, $\mu_{s,t+\Delta} = .2\%$, $\sigma_{s,t+\Delta} = .08\%$, $k = 3$, $M = 100,000$ samples, and a confidence $\alpha = .99$. For the simulation in part (c) output the following quantities

(1) The confidence α.

(2) The estimate of $\mathrm{LVaR}_\alpha^{sim}$ found via simulation.

(3) The theoretical VaR_α from part (b) - here you will have to compute this in the program.

(4) The estimated liquidity cost $\mathrm{LC}^{sim} = \mathrm{LVaR}_\alpha^{sim} - \mathrm{VaR}_\alpha$.

(5) The estimated percentage increase in the risk measure: $100\left(\frac{\mathrm{LVaR}_\alpha^{sim}}{\mathrm{VaR}_\alpha} - 1\right)$.

(6) The industry approximate $\mathrm{LVaR}_\alpha^{ind}$.

(7) The industry liquidity cost LC.

(8) The industry percentage increase in the risk measure: $100\frac{\mathrm{LC}}{\mathrm{VaR}_\alpha}$.

How do the risk measures and liquidity costs compare?