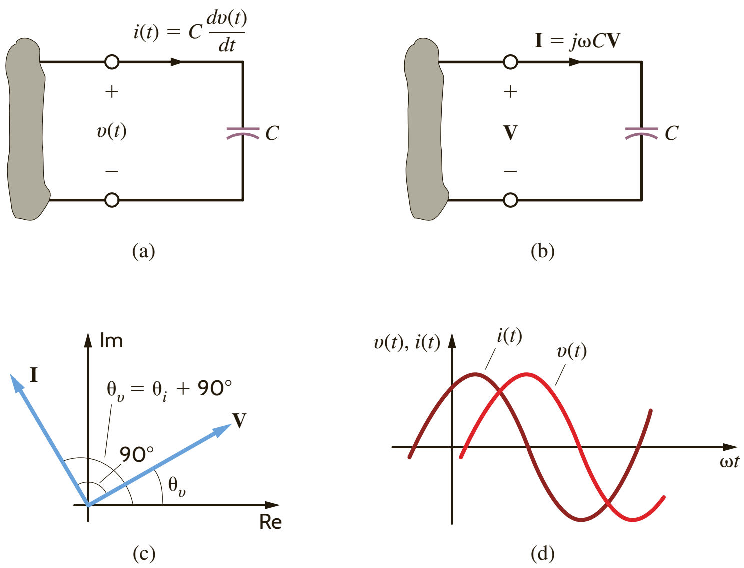


Figure 8.8 Voltage–current relationships for a capacitor.

EXAMPLE 8.8

The voltage $v(t) = 100 \cos(314t + 15°)$ V is applied to a 100-μF capacitor as shown in Fig. 8.8a. Find the current.

SOLUTION

The resultant phasor current is

$$\begin{aligned}\mathbf{I} &= j\omega C(100\underline{/15°}) \\ &= (314)(100 \times 10^{-6}\underline{/90°})(100\underline{/15°}) \\ &= 3.14\ \underline{/105°}\ \text{A}\end{aligned}$$

Therefore, the current written as a time function is

$$i(t) = 3.14 \cos(314t + 105°)\ \text{A}$$

HINT

Applying $\mathbf{I} = j\omega C\mathbf{V}$

LEARNING ASSESSMENT

E8.7 The current in a 150-μF capacitor is $\mathbf{I} = 3.6\underline{/-145°}$ A. If the frequency of the current is 60 Hz, determine the voltage across the capacitor.

ANSWER: $v_C(t) = 63.66 \cos(377t - 235°)$ V.

8.5 Impedance and Admittance

We have examined each of the circuit elements in the frequency domain on an individual basis. We now wish to treat these passive circuit elements in a more general fashion. We define the two-terminal input *impedance* **Z**, also referred to as the driving point impedance, in exactly the same manner in which we defined resistance earlier. Later we will examine another type of impedance, called transfer impedance.

Impedance is defined as the ratio of the phasor voltage **V** to the phasor current **I**:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \tag{8.31}$$

at the two terminals of the element related to one another by the passive sign convention, as illustrated in Fig. 8.9. Since **V** and **I** are complex, the impedance **Z** is complex and

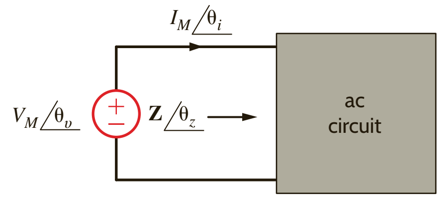


Figure 8.9
General impedance relationship.

$$\mathbf{Z} = \frac{V_M \angle \theta_v}{I_M \angle \theta_i} = \frac{V_M}{I_M} \angle \theta_v - \theta_i = Z \angle \theta_z \qquad \textbf{8.32}$$

Since $\mathbf{Z}$ is the ratio of $\mathbf{V}$ to $\mathbf{I}$, the units of $\mathbf{Z}$ are ohms. Thus, impedance in an ac circuit is analogous to resistance in a dc circuit. In rectangular form, impedance is expressed as

$$\mathbf{Z}(\omega) = R(\omega) + jX(\omega) \qquad \textbf{8.33}$$

where $R(\omega)$ is the real, or resistive, component and $X(\omega)$ is the imaginary, or reactive, component. In general, we simply refer to R as the resistance and X as the reactance. It is important to note that R and X are real functions of ω and therefore $\mathbf{Z}(\omega)$ is frequency dependent. Equation (8.33) clearly indicates that $\mathbf{Z}$ is a complex number; however, it is not a phasor, since phasors denote sinusoidal functions.

Equations (8.32) and (8.33) indicate that

$$Z \angle \theta_z = R + jX \qquad \textbf{8.34}$$

Therefore,

$$\begin{aligned} Z &= \sqrt{R^2 + X^2} \\ \theta_z &= \tan^{-1}\frac{X}{R} \end{aligned} \qquad \textbf{8.35}$$

where

$$\begin{aligned} R &= Z\cos\theta_z \\ X &= Z\sin\theta_z \end{aligned}$$

For the individual passive elements the impedance is as shown in Table 8.2. However, just as it was advantageous to know how to determine the equivalent resistance in dc circuits, we want to learn how to determine the equivalent impedance in ac circuits.

KCL and KVL are both valid in the frequency domain. We can use this fact, as was done in Chapter 2 for resistors, to show that impedances can be combined using the same rules that we established for resistor combinations. That is, if $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \cdots, \mathbf{Z}_n$ are connected in series, the equivalent impedance $\mathbf{Z}_s$ is

$$\mathbf{Z}_s = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_n \qquad \textbf{8.36}$$

and if $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_n$ are connected in parallel, the equivalent impedance is given by

$$\frac{1}{\mathbf{Z}_p} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_n} \qquad \textbf{8.37}$$

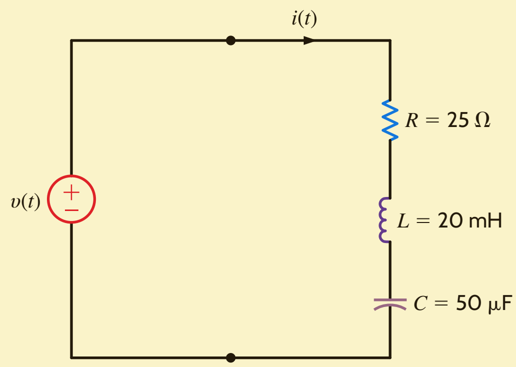
TABLE 8.2 Passive element impedance

PASSIVE ELEMENT	IMPEDANCE
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L = jX_L,\ X_L = \omega L$
C	$\mathbf{Z} = \frac{1}{j\omega C} = -\frac{j}{\omega C} = -jX_C,\ X_C = \frac{1}{\omega C}$

EXAMPLE 8.9

Determine the equivalent impedance of the network shown in Fig. 8.10 if the frequency is $f = 60$ Hz. Then compute the current $i(t)$ if the voltage source is $v(t) = 50\cos(\omega t + 30°)$ V. Finally, calculate the equivalent impedance if the frequency is $f = 400$ Hz.

Figure 8.10
Series ac circuit.



SOLUTION

The impedances of the individual elements at 60 Hz are

$$\begin{aligned}\mathbf{Z}_R &= 25\ \Omega\\ \mathbf{Z}_L &= j\omega L = j(2\pi \times 60)(20 \times 10^{-3}) = j7.54\ \Omega\\ \mathbf{Z}_C &= \frac{-j}{\omega C} = \frac{-j}{(2\pi \times 60)(50 \times 10^{-6})} = -j53.05\ \Omega\end{aligned}$$

Since the elements are in series,

$$\begin{aligned}\mathbf{Z} &= \mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C\\ &= 25 - j45.51\ \Omega\end{aligned}$$

The current in the circuit is given by

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\angle 30°}{25 - j45.51} = \frac{50\angle 30°}{51.93\angle -61.22°} = 0.96\angle 91.22°\ \text{A}$$

or in the time domain, $i(t) = 0.96\cos(377t + 91.22°)$ A.

If the frequency is 400 Hz, the impedance of each element is

$$\begin{aligned}\mathbf{Z}_R &= 25\ \Omega\\ \mathbf{Z}_L &= j\omega L = j50.27\ \Omega\\ \mathbf{Z}_C &= \frac{-j}{\omega C} = -j7.96\ \Omega\end{aligned}$$

The total impedance is then

$$\mathbf{Z} = 25 + j42.31 = 49.14\angle 59.42°\ \Omega$$

At the frequency $f = 60$ Hz, the reactance of the circuit is capacitive; that is, if the impedance is written as $R + jX$, $X < 0$. However, at $f = 400$ Hz the reactance is inductive since $X > 0$.

PROBLEM-SOLVING STRATEGY

BASIC AC ANALYSIS

STEP 1. Express $v(t)$ as a phasor and determine the impedance of each passive element.

STEP 2. Combine impedances and solve for the phasor $\mathbf{I}$.

STEP 3. Convert the phasor $\mathbf{I}$ to $i(t)$.

LEARNING ASSESSMENT

E8.8 Find the current $i(t)$ in the network in Fig. E8.8.

ANSWER:
$i(t) = 3.88 \cos(377t - 39.2°)$ A.

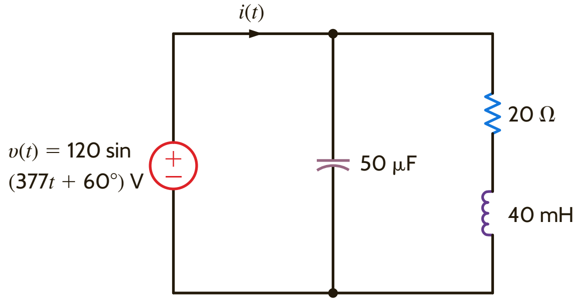


Figure E8.8

Another quantity that is very useful in the analysis of ac circuits is the two-terminal input *admittance*, which is the reciprocal of impedance; that is,

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \qquad \textbf{8.38}$$

The units of **Y** are siemens, and this quantity is analogous to conductance in resistive dc circuits. Since **Z** is a complex number, **Y** is also a complex number.

$$\mathbf{Y} = Y_M \angle \theta_y \qquad \textbf{8.39}$$

which is written in rectangular form as

$$\mathbf{Y} = G + jB \qquad \textbf{8.40}$$

where G and B are called *conductance* and *susceptance*, respectively. Because of the relationship between **Y** and **Z**, we can express the components of one quantity as a function of the components of the other. From the expression

$$G + jB = \frac{1}{R + jX} \qquad \textbf{8.41}$$

HINT
Technique for taking the reciprocal:

$$\frac{1}{R + jX} = \frac{R - jX}{(R + jX)(R - jX)} = \frac{R - jX}{R^2 + X^2}$$

we can show that

$$G = \frac{R}{R^2 + X^2}, \qquad B = \frac{-X}{R^2 + X^2} \qquad \textbf{8.42}$$

and in a similar manner, we can show that

$$R = \frac{G}{G^2 + B^2}, \qquad X = \frac{-B}{G^2 + B^2} \qquad \textbf{8.43}$$

It is very important to note that, in general, R and G are *not* reciprocals of one another. The same is true for X and B. The purely resistive case is an exception. In the purely reactive case, the quantities are negative reciprocals of one another.

The admittance of the individual passive elements are

$$\mathbf{Y}_R = \frac{1}{R} = G$$

$$\mathbf{Y}_L = \frac{1}{j\omega L} = \frac{-j}{\omega L} \qquad \textbf{8.44}$$

$$\mathbf{Y}_C = j\omega C$$

Once again, since KCL and KVL are valid in the frequency domain, we can show, using the same approach outlined in Chapter 2 for conductance in resistive circuits, that the rules for combining admittances are the same as those for combining conductances; that is, if $\mathbf{Y}_1$, $\mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_n$ are connected in parallel, the equivalent admittance is

$$\mathbf{Y}_p = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n \qquad 8.45$$

and if $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ are connected in series, the equivalent admittance is

$$\frac{1}{\mathbf{Y}_S} = \frac{1}{\mathbf{Y}_1} + \frac{1}{\mathbf{Y}_2} + \cdots + \frac{1}{\mathbf{Y}_n} \qquad 8.46$$

EXAMPLE 8.10

Calculate the equivalent admittance $\mathbf{Y}_p$ for the network in Fig. 8.11 and use it to determine the current $\mathbf{I}$ if $\mathbf{V}_S = 60\angle 45^\circ$ V.

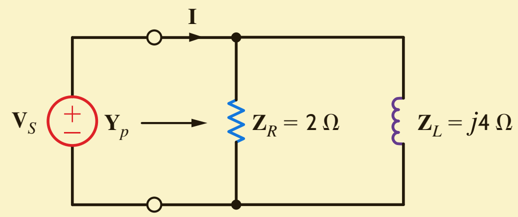


Figure 8.11 Example parallel circuit.

SOLUTION From Fig. 8.11 we note that

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{2}\text{ S}$$
$$\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{-j}{4}\text{ S}$$

Therefore,

$$\mathbf{Y}_p = \frac{1}{2} - j\frac{1}{4}\text{ S}$$

and hence,

$$\mathbf{I} = \mathbf{Y}_p\mathbf{V}_S$$
$$= \left(\frac{1}{2} - j\frac{1}{4}\right)(60\angle 45^\circ)$$
$$= 33.5\angle 18.43^\circ\text{ A}$$

HINT
Admittances add in parallel.

LEARNING ASSESSMENT

E8.9 Find the current $\mathbf{I}$ in the network in Fig. E8.9.

ANSWER:
$\mathbf{I} = 9.01\angle 53.7^\circ$ A.

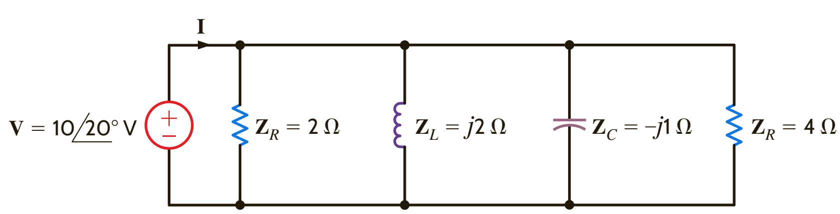


Figure E8.9

As a prelude to our analysis of more general ac circuits, let us examine the techniques for computing the impedance or admittance of circuits in which numerous passive elements are interconnected. The following example illustrates that our technique is analogous to our earlier computations of equivalent resistance.

EXAMPLE 8.11

Consider the network shown in Fig. 8.12a. The impedance of each element is given in the figure. We wish to calculate the equivalent impedance of the network $\mathbf{Z}_{eq}$ at terminals A–B.

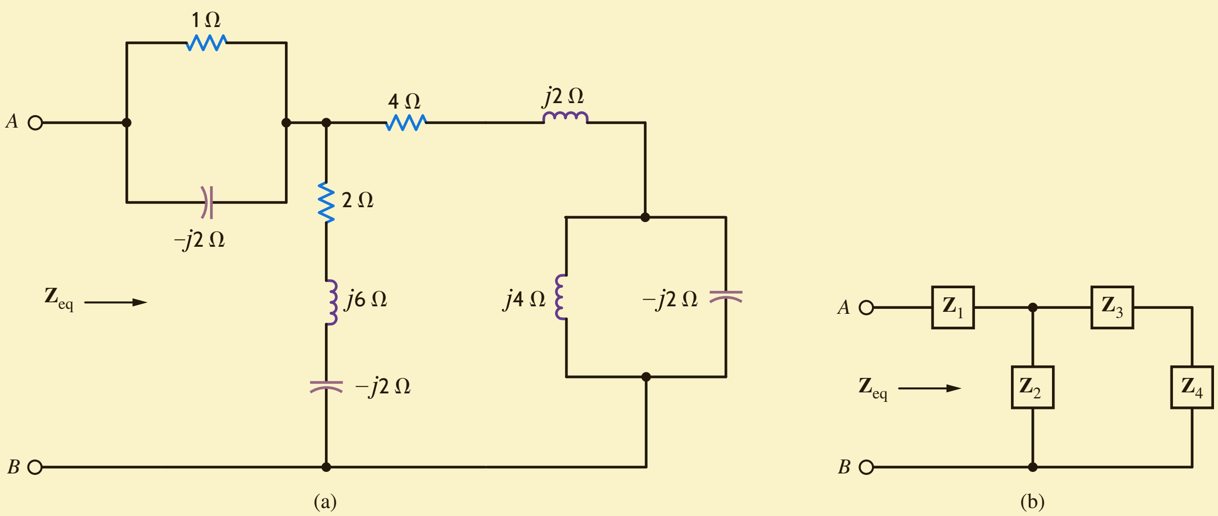


Figure 8.12
Example circuit for determining equivalent impedance in two steps.

SOLUTION

The equivalent impedance $\mathbf{Z}_{eq}$ could be calculated in a variety of ways; we could use only impedances, or only admittances, or a combination of the two. We will use the latter. We begin by noting that the circuit in Fig. 8.12a can be represented by the circuit in Fig. 8.12b.

Note that

$$\begin{aligned}\mathbf{Y}_4 &= \mathbf{Y}_L + \mathbf{Y}_C \\ &= \frac{1}{j4} + \frac{1}{-j2} \\ &= j\frac{1}{4}\ \mathrm{S}\end{aligned}$$

Therefore,

$$\mathbf{Z}_4 = -j4\ \Omega$$

Now

$$\begin{aligned}\mathbf{Z}_{34} &= \mathbf{Z}_3 + \mathbf{Z}_4 \\ &= (4 + j2) + (-j4) \\ &= 4 - j2\ \Omega\end{aligned}$$

and hence,

$$\begin{aligned}\mathbf{Y}_{34} &= \frac{1}{\mathbf{Z}_{34}} \\ &= \frac{1}{4 - j2} \\ &= 0.20 + j0.10\ \mathrm{S}\end{aligned}$$

Since

$$\begin{aligned}\mathbf{Z}_2 &= 2 + j6 - j2\\ &= 2 + j4\ \Omega\end{aligned}$$

then

$$\begin{aligned}\mathbf{Y}_2 &= \frac{1}{2 + j4}\\ &= 0.10 - j0.20\ \text{S}\\ \mathbf{Y}_{234} &= \mathbf{Y}_2 + \mathbf{Y}_{34}\\ &= 0.30 - j0.10\ \text{S}\end{aligned}$$

The reader should carefully note our approach: we are adding impedances in series and adding admittances in parallel.

From $\mathbf{Y}_{234}$ we can compute $\mathbf{Z}_{234}$ as

$$\begin{aligned}\mathbf{Z}_{234} &= \frac{1}{\mathbf{Y}_{234}}\\ &= \frac{1}{0.30 - j0.10}\\ &= 3 + j1\ \Omega\end{aligned}$$

Now

$$\begin{aligned}\mathbf{Y}_1 &= \mathbf{Y}_R + \mathbf{Y}_C\\ &= \frac{1}{1} + \frac{1}{-j2}\\ &= 1 + j\frac{1}{2}\ \text{S}\end{aligned}$$

and then

$$\begin{aligned}\mathbf{Z}_1 &= \frac{1}{1 + j\frac{1}{2}}\\ &= 0.8 - j0.4\ \Omega\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{Z}_{\text{eq}} &= \mathbf{Z}_1 + \mathbf{Z}_{234}\\ &= 0.8 - j0.4 + 3 + j1\\ &= 3.8 + j0.6\ \Omega\end{aligned}$$

PROBLEM-SOLVING STRATEGY

COMBINING IMPEDANCES AND ADMITTANCES

STEP 1. Add the admittances of elements in parallel.

STEP 2. Add the impedances of elements in series.

STEP 3. Convert back and forth between admittance and impedance in order to combine neighboring elements.