3.1 Nodal Analysis

In a nodal analysis, the variables in the circuit are selected to be the node voltages. The node voltages are defined with respect to a common point in the circuit. One node is selected as the reference node, and all other node voltages are defined with respect to that node. Quite often, this node is the one to which the largest number of branches are connected. It is commonly called *ground* because it is said to be at ground-zero potential, and it sometimes represents the chassis or ground line in a practical circuit.

We will select our variables as being positive with respect to the reference node. If one or more of the node voltages are actually negative with respect to the reference node, the analysis will indicate it.

In order to understand the value of knowing all the node voltages in a network, we consider once again the network in Fig. 2.32, which is redrawn in **Fig. 3.1**. The voltages, V_S, V_a, V_b, and V_c, are all measured with respect to the bottom node, which is selected as the reference and labeled with the ground symbol ⏚. Therefore, the voltage at node 1 is $V_S = 12$ V with respect to the reference node 5, the voltage at node 2 is $V_a = 3$ V with respect to the reference node 5, and so on. Now note carefully that once these node voltages are known, we can immediately calculate any branch current or the power supplied or absorbed by any element, since we know the voltage across every element in the network. For example, the voltage V_1 across the leftmost 9-kΩ resistor is the difference in potential between the two ends of the resistor; that is,

$$\begin{aligned} V_1 &= V_S - V_a \\ &= 12 - 3 \\ &= 9 \text{ V} \end{aligned}$$

This equation is really nothing more than an application of KVL around the leftmost loop; that is,

$$-V_S + V_1 + V_a = 0$$

In a similar manner, we find that

$$V_3 = V_a - V_b$$

and

$$V_5 = V_b - V_c$$

Then the currents in the resistors are

$$\begin{aligned} I_1 &= \frac{V_1}{9\text{k}} = \frac{V_S - V_a}{9\text{k}} \\ I_3 &= \frac{V_3}{3\text{k}} = \frac{V_a - V_b}{3\text{k}} \\ I_5 &= \frac{V_5}{9\text{k}} = \frac{V_b - V_c}{9\text{k}} \end{aligned}$$

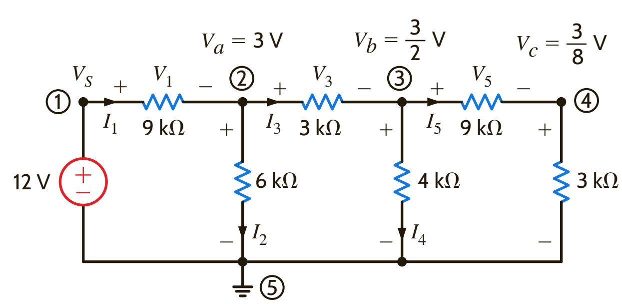
In addition,

$$\begin{aligned} I_2 &= \frac{V_a - 0}{6\text{k}} \\ I_4 &= \frac{V_b - 0}{4\text{k}} \end{aligned}$$

since the reference node 5 is at zero potential.

Figure 3.1

Circuit with known node voltages.



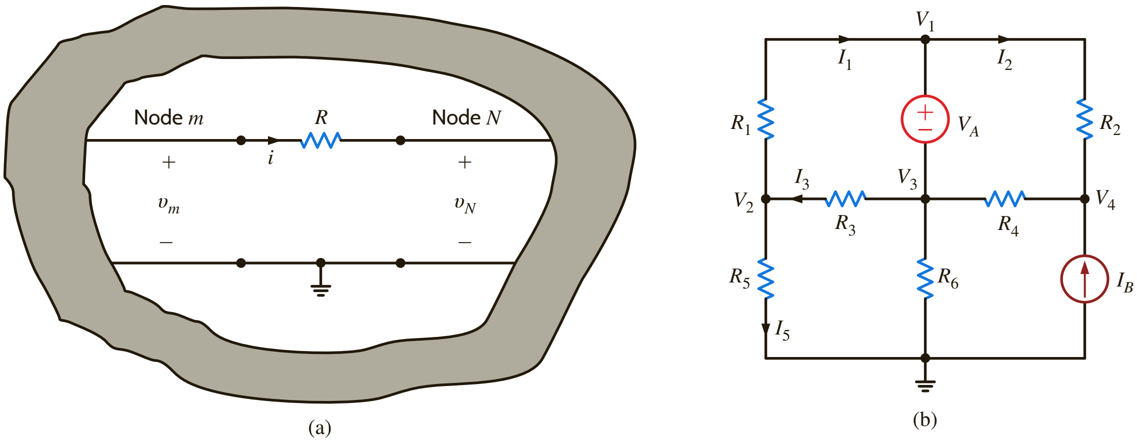


Figure 3.2

Circuit used to illustrate Ohm's law in a multiple-node network.

Thus, as a general rule, if we know the node voltages in this circuit, we can calculate the current through any resistive element using Ohm's law; that is

$$I = \frac{(V_m - V_n)}{R} \tag{3.1}$$

as illustrated in **Fig. 3.2a**. Note carefully that both voltages V_m and V_n are both measured with respect to the same point, i.e., ground. In a nodal analysis, this concept is central to the manner in which we write the equations necessary to determine all the node voltages. Consider for example the network in **Fig. 3.2b**, where the voltages labeled V_1, V_2, V_3, and V_4 represent the voltages at those nodes with respect to the ground node. Then we can write the following KVL equations:

$$-V_2 + I_1R_1 + V_1 = 0$$
$$-V_1 + I_2R_2 + V_4 = 0$$
$$-V_3 + I_3R_3 + V_2 = 0$$

which yields equations for the currents as follows:

$$I_1 = \frac{(V_2 - V_1)}{R_1}$$
$$I_2 = \frac{(V_1 - V_4)}{R_2}$$
$$I_3 = \frac{(V_3 - V_2)}{R_3}$$

In addition, the KVL equation

$$-V_1 + V_A + V_3 = 0$$

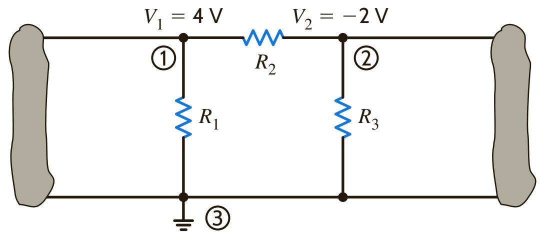
indicates that $V_1 - V_3 = V_A$. Finally, Ohm's law yields $I_5 = V_2/R_5$.

Now that we have demonstrated the value of knowing all the node voltages in a network, let us determine the manner in which to calculate them. In a nodal analysis, we employ KCL equations in such a way that the variables contained in these equations are the unknown node voltages of the network. As we have indicated, one of the nodes in an N-node circuit is selected as the reference node, and the voltages at all the remaining $N - 1$ nonreference nodes are measured with respect to this reference node. Using network topology, it can be shown that exactly $N - 1$ linearly independent KCL equations are required to determine the $N - 1$ unknown node voltages. Therefore, theoretically once one of the nodes in an N-node circuit has been selected as the reference node, our task is reduced to identifying the remaining $N - 1$ nonreference nodes and writing one KCL equation at each of them.

In a multiple-node circuit, this process results in a set of $N - 1$ linearly independent simultaneous equations in which the variables are the $N - 1$ unknown node voltages. To

Figure 3.3

An illustration of node voltages.



help solidify this idea, consider once again Example 2.5. Note that in this circuit, only four (i.e., any four) of the five KCL equations, one of which is written for each node in this five-node network, are linearly independent. Furthermore, many of the branch currents in this example (those not contained in a source) can be written in terms of the node voltages as illustrated in **Fig. 3.2a** and expressed in Eq. (3.1). It is in this manner, as we will illustrate in the sections that follow, that the KCL equations contain the unknown node voltages.

It is instructive to treat nodal analysis by examining several different types of circuits and illustrating the salient features of each. We begin with the simplest case. However, as a prelude to our discussion of the details of nodal analysis, experience indicates that it is worthwhile to digress for a moment to ensure that the concept of node voltage is clearly understood.

At the outset it is important to specify a reference. For example, to state that the voltage at node A is 12 V means nothing unless we provide the reference point; that is, the voltage at node A is 12 V with respect to what? The circuit in **Fig. 3.3** illustrates a portion of a network containing three nodes, one of which is the reference node.

The voltage $V_1 = 4$ V is the voltage at node 1 with respect to the reference node 3. Similarly, the voltage $V_2 = -2$ V is the voltage at node 2 with respect to node 3. In addition, however, the voltage at node 1 with respect to node 2 is $+6$ V, and the voltage at node 2 with respect to node 1 is -6 V. Furthermore, since the current will flow from the node of higher potential to the node of lower potential, the current in R_1 is from top to bottom, the current in R_2 is from left to right, and the current in R_3 is from bottom to top.

These concepts have important ramifications in our daily lives. If a man were hanging in midair with one hand on one line and one hand on another and the dc line voltage of each line was exactly the same, the voltage across his heart would be zero and he would be safe. If, however, he let go of one line and let his feet touch the ground, the dc line voltage would then exist from his hand to his foot with his heart in the middle. He would probably be dead the instant his foot hit the ground.

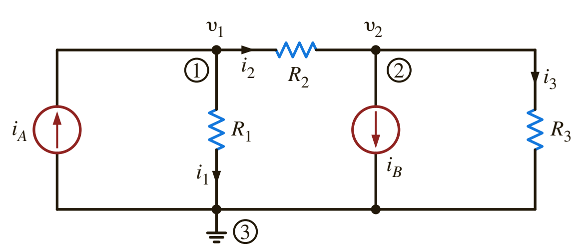
In the town where we live, a young man tried to retrieve his parakeet that had escaped its cage and was outside sitting on a power line. He stood on a metal ladder and with a metal pole reached for the parakeet; when the metal pole touched the power line, the man was killed instantly. Electric power is vital to our standard of living, but it is also very dangerous. The material in this book *does not* qualify you to handle it safely. Therefore, always be extremely careful around electric circuits.

Now as we begin our discussion of nodal analysis, our approach will be to begin with simple cases and proceed in a systematic manner to those that are more challenging. Numerous examples will be the vehicle used to demonstrate each facet of this approach. Finally, at the end of this section, we will outline a strategy for attacking any circuit using nodal analysis.

CIRCUITS CONTAINING ONLY INDEPENDENT CURRENT SOURCES Consider the network shown in **Fig. 3.4**. Note that this network contains three nodes, and thus we know that exactly $N - 1 = 3 - 1 = 2$ linearly independent KCL equations will be required

Figure 3.4

A three-node circuit.



to determine the $N - 1 = 2$ unknown node voltages. First, we select the bottom node as the reference node, and then the voltage at the two remaining nodes labeled v_1 and v_2 will be measured with respect to this node.

The branch currents are assumed to flow in the directions indicated in the figures. If one or more of the branch currents are actually flowing in a direction opposite to that assumed, the analysis will simply produce a branch current that is negative.

Applying KCL at node 1 yields

$$-i_A + i_1 + i_2 = 0$$

Using Ohm's law ($i = Gv$) and noting that the reference node is at zero potential, we obtain

$$-i_A + G_1(v_1 - 0) + G_2(v_1 - v_2) = 0$$

HINT
Employing the passive sign convention.

or

$$(G_1 + G_2)v_1 - G_2v_2 = i_A$$

KCL at node 2 yields

$$-i_2 + i_B + i_3 = 0$$

or

$$-G_2(v_1 - v_2) + i_B + G_3(v_2 - 0) = 0$$

which can be expressed as

$$-G_2v_1 + (G_2 + G_3)v_2 = -i_B$$

Therefore, the two equations for the two unknown node voltages v_1 and v_2 are

$$\begin{aligned} (G_1 + G_2)v_1 - G_2v_2 &= i_A \\ -G_2v_1 + (G_2 + G_3)v_2 &= -i_B \end{aligned} \qquad \mathbf{3.2}$$

Note that the analysis has produced two simultaneous equations in the unknowns v_1 and v_2. They can be solved using any convenient technique, and modern calculators and personal computers are very efficient tools for this application.

In what follows, we will demonstrate three techniques for solving linearly independent simultaneous equations: Gaussian elimination, matrix analysis, and the MATLAB mathematical software package. A brief refresher that illustrates the use of both Gaussian elimination and matrix analysis in the solution of these equations is provided in the Problem-Solving Companion for this text. Use of the MATLAB software is straightforward, and we will demonstrate its use as we encounter the application.

The KCL equations at nodes 1 and 2 produced two linearly independent simultaneous equations:

$$\begin{aligned} -i_A + i_1 + i_2 &= 0 \\ -i_2 + i_B + i_3 &= 0 \end{aligned}$$

The KCL equation for the third node (reference) is

$$+i_A - i_1 - i_B - i_3 = 0$$

Note that if we add the first two equations, we obtain the third. Furthermore, any two of the equations can be used to derive the remaining equation. Therefore, in this $N = 3$ node circuit, only $N - 1 = 2$ of the equations are linearly independent and required to determine the $N - 1 = 2$ unknown node voltages.

Note that a nodal analysis employs KCL in conjunction with Ohm's law. Once the direction of the branch currents has been *assumed*, then Ohm's law, as illustrated by Fig. 3.2 and expressed by Eq. (3.1), is used to express the branch currents in terms of the unknown node voltages. We can assume the currents to be in any direction. However, once we assume a particular direction, we must be very careful to write the currents correctly in terms of the node voltages using Ohm's law.

EXAMPLE 3.1

Suppose that the network in Fig. 3.4 has the following parameters: $I_A = 1$ mA, $R_1 = 12$ kΩ, $R_2 = 6$ kΩ, $I_B = 4$ mA, and $R_3 = 6$ kΩ. Let us determine all node voltages and branch currents.

SOLUTION

For purposes of illustration we will solve this problem using Gaussian elimination, matrix analysis, and MATLAB. Using the parameter values, Eq. (3.2) becomes

$$V_1\left[\frac{1}{12\text{k}} + \frac{1}{6\text{k}}\right] - V_2\left[\frac{1}{6\text{k}}\right] = 1 \times 10^{-3}$$

$$-V_1\left[\frac{1}{6\text{k}}\right] + V_2\left[\frac{1}{6\text{k}} + \frac{1}{6\text{k}}\right] = -4 \times 10^{-3}$$

where we employ capital letters because the voltages are constant. The equations can be written as

$$\frac{V_1}{4\text{k}} - \frac{V_2}{6\text{k}} = 1 \times 10^{-3}$$

$$-\frac{V_1}{6\text{k}} + \frac{V_2}{3\text{k}} = -4 \times 10^{-3}$$

Using Gaussian elimination, we solve the first equation for V_1 in terms of V_2:

$$V_1 = V_2\left(\frac{2}{3}\right) + 4$$

This value is then substituted into the second equation to yield

$$\frac{-1}{6\text{k}}\left(\frac{2}{3}V_2 + 4\right) + \frac{V_2}{3\text{k}} = -4 \times 10^{-3}$$

or

$$V_2 = -15 \text{ V}$$

This value for V_2 is now substituted back into the equation for V_1 in terms of V_2, which yields

$$\begin{aligned} V_1 &= \frac{2}{3}V_2 + 4 \\ &= -6 \text{ V} \end{aligned}$$

The circuit equations can also be solved using matrix analysis. The general form of the matrix equation is

$$\mathbf{GV} = \mathbf{I}$$

where in this case

$$\mathbf{G} = \begin{bmatrix} \frac{1}{4\text{k}} & -\frac{1}{6\text{k}} \\ -\frac{1}{6\text{k}} & \frac{1}{3\text{k}} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \text{ and } \mathbf{I} = \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix}$$

The solution to the matrix equation is

$$\mathbf{V} = \mathbf{G}^{-1}\mathbf{I}$$

and therefore,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4\text{k}} & \frac{-1}{6\text{k}} \\ \frac{-1}{6\text{k}} & \frac{1}{3\text{k}} \end{bmatrix}^{-1} \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix}$$

To calculate the inverse of **G**, we need the adjoint and the determinant. The adjoint is

$$\text{Adj } \mathbf{G} = \begin{bmatrix} \frac{1}{3\text{k}} & \frac{1}{6\text{k}} \\ \frac{1}{6\text{k}} & \frac{1}{4\text{k}} \end{bmatrix}$$

and the determinant is

$$\begin{aligned} |\mathbf{G}| &= \left(\frac{1}{3\text{k}}\right)\left(\frac{1}{4\text{k}}\right) - \left(\frac{-1}{6\text{k}}\right)\left(\frac{-1}{6\text{k}}\right) \\ &= \frac{1}{18\text{k}^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= 18\text{k}^2 \begin{bmatrix} \frac{1}{3\text{k}} & \frac{1}{6\text{k}} \\ \frac{1}{6\text{k}} & \frac{1}{4\text{k}} \end{bmatrix} \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix} \\ &= 18\text{k}^2 \begin{bmatrix} \frac{1}{3\text{k}^2} - \frac{4}{6\text{k}^2} \\ \frac{1}{6\text{k}^2} - \frac{1}{\text{k}^2} \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ -15 \end{bmatrix} \end{aligned}$$

In the MATLAB solution, we simplify the form of the equations by multiplying both equations by 12k, yielding

$$\begin{aligned} 3V_1 - 2V_2 &= 12 \\ -2V_1 + 4V_2 &= -48 \end{aligned}$$

In matrix form, the equation is

$$\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -48 \end{bmatrix}$$

Then the data entries and solution using MATLAB are as follows:

```
>> G = [3 -2; -2 4]
   G =
         3   -2
        -2    4
>> I = [12; -48]
   I =
         12
        -48
>> V = inv(G)*I
   V =
         -6.0000
        -15.0000
```

Knowing the node voltages, we can determine all the currents using Ohm's law:

$$I_1 = \frac{V_1}{R_1} = \frac{-6}{12\text{k}} = -\frac{1}{2} \text{ mA}$$

$$I_2 = \frac{V_1 - V_2}{6\text{k}} = \frac{-6 - (-15)}{6\text{k}} = \frac{3}{2} \text{ mA}$$

and

$$I_3 = \frac{V_2}{6k} = \frac{-15}{6k} = -\frac{5}{2}\text{ mA}$$

Fig. 3.5 illustrates the results of all the calculations. Note that KCL is satisfied at every node.

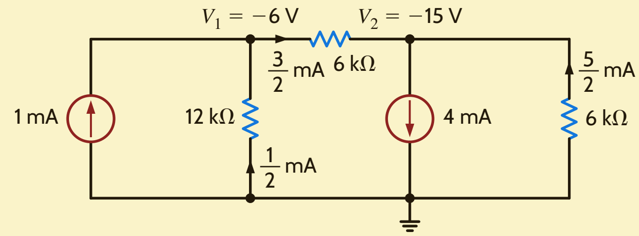


Figure 3.5
Circuit used in Example 3.1.

Let us now examine the circuit in Fig. 3.6. The current directions are assumed as shown in the figure.

We note that this network has four nodes. The node at the bottom of the circuit is selected as the reference node and labeled with the ground symbol. Since $N = 4$, $N - 1 = 3$ linearly independent KCL equations will be required to determine the three unknown nonreference node voltages labeled v_1, v_2, and v_3.

At node 1, KCL yields

$$i_1 - i_A + i_2 - i_3 = 0$$

or

$$\frac{v_1}{R_1} - i_A + \frac{v_1 - v_2}{R_2} - \frac{v_3 - v_1}{R_3} = 0$$

$$v_1\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - v_2\frac{1}{R_2} - v_3\frac{1}{R_3} = i_A$$

At node 2, KCL yields

$$-i_2 + i_4 - i_5 = 0$$

or

$$-\frac{v_1 - v_2}{R_2} + \frac{v_2}{R_4} - \frac{v_3 - v_2}{R_5} = 0$$

$$-v_1\frac{1}{R_2} + v_2\left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5}\right) - v_3\frac{1}{R_5} = 0$$

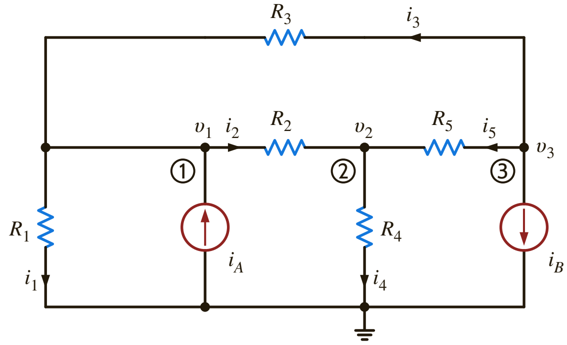


Figure 3.6
A four-node circuit.

At node 3, the equation is

$$i_3 + i_5 + i_B = 0$$

or

$$\frac{v_3 - v_1}{R_3} + \frac{v_3 - v_2}{R_5} + i_B = 0$$

$$-v_1 \frac{1}{R_3} - v_2 \frac{1}{R_5} + v_3 \left(\frac{1}{R_3} + \frac{1}{R_5}\right) = -i_B$$

Grouping the node equations together, we obtain

$$\begin{aligned} v_1 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - v_2 \frac{1}{R_2} - v_3 \frac{1}{R_3} &= i_A \\ -v_1 \frac{1}{R_2} + v_2 \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5}\right) - v_3 \frac{1}{R_5} &= 0 \\ -v_1 \frac{1}{R_3} - v_2 \frac{1}{R_5} + v_3 \left(\frac{1}{R_3} + \frac{1}{R_5}\right) &= -i_B \end{aligned} \tag{3.3}$$

Note that our analysis has produced three simultaneous equations in the three unknown node voltages v_1, v_2, and v_3. The equations can also be written in matrix form as

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} & -\frac{1}{R_5} \\ -\frac{1}{R_3} & -\frac{1}{R_5} & \frac{1}{R_3} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} i_A \\ 0 \\ -i_B \end{bmatrix} \tag{3.4}$$

At this point it is important that we note the symmetrical form of the equations that describe the two previous networks. Eqs. (3.2) and (3.3) exhibit the same type of symmetrical form. The **G** matrix for each network is a symmetrical matrix. This symmetry is not accidental. The node equations for networks containing only resistors and independent current sources can always be written in this symmetrical form. We can take advantage of this fact and learn to write the equations by inspection. Note in the first equation of (3.2) that the coefficient of v_1 is the sum of all the conductances connected to node 1 and the coefficient of v_2 is the negative of the conductances connected between node 1 and node 2. The right-hand side of the equation is the sum of the currents entering node 1 through current sources. This equation is KCL at node 1. In the second equation in (3.2), the coefficient of v_2 is the sum of all the conductances connected to node 2, the coefficient of v_1 is the negative of the conductance connected between node 2 and node 1, and the right-hand side of the equation is the sum of the currents entering node 2 through current sources. This equation is KCL at node 2. Similarly, in the first equation in (3.3), the coefficient of v_1 is the sum of the conductances connected to node 1, the coefficient of v_2 is the negative of the conductance connected between node 1 and node 2, the coefficient of v_3 is the negative of the conductance connected between node 1 and node 3, and the right-hand side of the equation is the sum of the currents entering node 1 through current sources. The other two equations in (3.3) are obtained in a similar manner.

In general, if KCL is applied to node j with node voltage v_j, the coefficient of v_j is the sum of all the conductances connected to node j and the coefficients of the other node voltages (e.g., v_{j-1}, v_{j+1}) are the negative of the sum of the conductances connected directly between these nodes and node j. The right-hand side of the equation is equal to the sum of the currents entering the node via current sources. Therefore, the left-hand side of the equation represents the sum of the currents leaving node j and the right-hand side of the equation represents the currents entering node j.

EXAMPLE 3.2

Let us apply what we have just learned to write the equations for the network in Fig. 3.7 by inspection. Then, given the following parameters, we will determine the node voltages using MATLAB: $R_1 = R_2 = 2\text{ k}\Omega$, $R_3 = R_4 = 4\text{ k}\Omega$, $R_5 = 1\text{ k}\Omega$, $i_A = 4$ mA, and $i_B = 2$ mA.

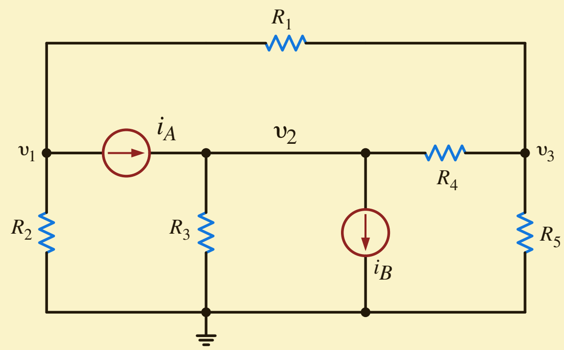


Figure 3.7
Circuit used in Example 3.2.

SOLUTION

The equations are

$$v_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - v_2(0) - v_3\left(\frac{1}{R_1}\right) = -i_A$$

$$-v_1(0) + v_2\left(\frac{1}{R_3} + \frac{1}{R_4}\right) - v_3\left(\frac{1}{R_4}\right) = i_A - i_B$$

$$-v_1\left(\frac{1}{R_1}\right) - v_2\left(\frac{1}{R_4}\right) + v_3\left(\frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}\right) = 0$$

which can also be written directly in matrix form as

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & 0 & -\frac{1}{R_1} \\ 0 & \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_1} & -\frac{1}{R_4} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -i_A \\ i_A - i_B \\ 0 \end{bmatrix}$$

Both the equations and the **G** matrix exhibit the symmetry that will always be present in circuits that contain only resistors and current sources.

If the component values are now used, the matrix equation becomes

$$\begin{bmatrix} \frac{1}{2\text{k}} + \frac{1}{2\text{k}} & 0 & -\frac{1}{2\text{k}} \\ 0 & \frac{1}{4\text{k}} + \frac{1}{4\text{k}} & -\frac{1}{4\text{k}} \\ -\frac{1}{2\text{k}} & -\frac{1}{4\text{k}} & \frac{1}{2\text{k}} + \frac{1}{4\text{k}} + \frac{1}{1\text{k}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -0.004 \\ 0.002 \\ 0 \end{bmatrix}$$

Multiplying the matrix equation by 4k yields the equation

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & -1 \\ -2 & -1 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \\ 0 \end{bmatrix}$$

The MATLAB solution is then

```
>> G = [4 0 -2; 0 2 -1; -2 -1 7]
   G =
             4      0     -2
             0      2     -1
            -2     -1      7
```

```
>> I = [-16;8;0]
   I =
       -16
         8
         0

>> V = inv(G)*I
   V =
       -4.3636
        3.6364
       -0.7273
```

LEARNING ASSESSMENTS

E3.1 Write the node equations for the circuit in Fig. E3.1.

ANSWER:

$$\frac{1}{4\text{k}}V_1 - \frac{1}{12\text{k}}V_2 = 4 \times 10^3,$$

$$\frac{-1}{12\text{k}}V_1 + \frac{1}{4\text{k}}V_2 = -2 \times 10^{-3}.$$

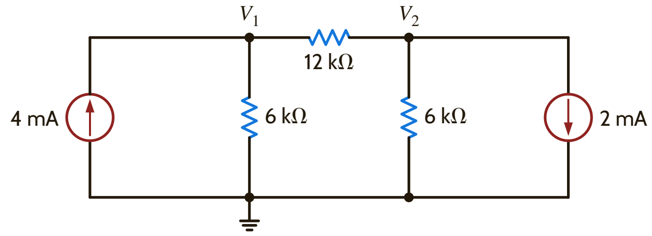


Figure E3.1

E3.2 Find all the node voltages in the network in Fig. E3.2 using MATLAB.

ANSWER:
$V_1 = 5.4286$ V,
$V_2 = 2.000$ V,
$V_3 = 3.1429$ V.

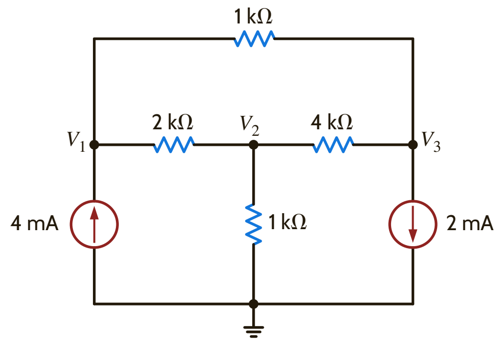


Figure E3.2

E3.3 Use nodal analysis to find V_o in Fig. E3.3.

ANSWER:
$V_o = 2.79$ V.

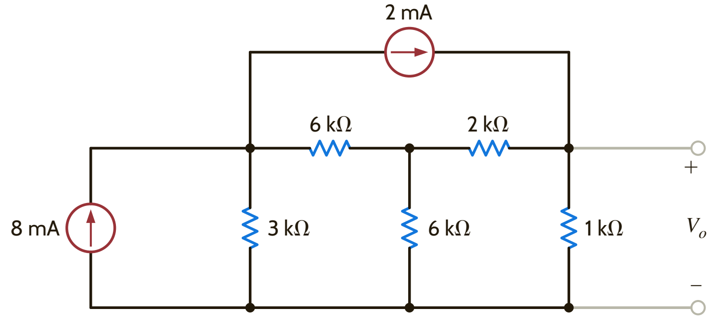


Figure E3.3

CIRCUITS CONTAINING DEPENDENT CURRENT SOURCES The presence of a dependent source may destroy the symmetrical form of the nodal equations that define the circuit. Consider the circuit shown in Fig. 3.8, which contains a current-controlled current source. The KCL equations for the nonreference nodes are

$$\beta i_o + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} = 0$$

and

$$\frac{v_2 - v_1}{R_2} + i_o - i_A = 0$$

where $i_o = v_2/R_3$. Simplifying the equations, we obtain

$$(G_1 + G_2)v_1 - (G_2 - \beta G_3)v_2 = 0$$
$$-G_2 v_1 + (G_2 + G_3)v_2 = i_A$$

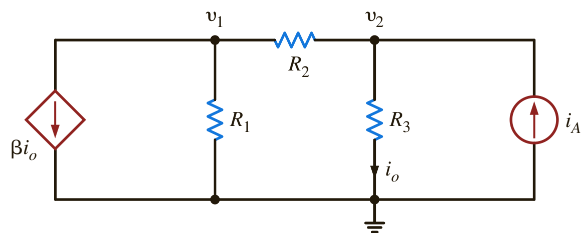
or in matrix form

$$\begin{bmatrix} (G_1 + G_2) & (-G_2 - \beta G_3) \\ -G_2 & (G_2 + G_3) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ i_A \end{bmatrix}$$

Note that the presence of the dependent source has destroyed the symmetrical nature of the node equations.

Figure 3.8

Circuit with a dependent source.



EXAMPLE 3.3

Let us determine the node voltages for the network in Fig. 3.8, given the following parameters:

$$\beta = 2 \qquad R_2 = 6\text{ k}\Omega \qquad i_A = 2\text{ mA}$$
$$R_1 = 12\text{ k}\Omega \qquad R_3 = 3\text{ k}\Omega$$

SOLUTION

Using these values with the equations for the network yields

$$\frac{1}{4\text{k}}V_1 + \frac{1}{2\text{k}}V_2 = 0$$
$$-\frac{1}{6\text{k}}V_1 + \frac{1}{2\text{k}}V_2 = 2 \times 10^{-3}$$

Multiplying the equations by 12k yields the equation

$$\begin{bmatrix} 3 & 6 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \end{bmatrix}$$

The MATLAB solution is then

```
>> G = [3 6; -2 6]
   G =
          3    6
         -2    6
>> I = [0;24]
   I =
          0
         24
>> V = inv(G)*I
   V =
         -4.8000
          2.4000
```

We can check these answers by determining the branch currents in the network and then using that information to test KCL at the nodes. For example, the current from top to bottom through R_3 is

$$I_o = \frac{V_2}{R_3} = \frac{12/5}{3\text{k}} = \frac{4}{5\text{k}}\text{ A}$$

Similarly, the current from right to left through R_2 is

$$I_2 = \frac{V_2 - V_1}{R_2} = \frac{12/5 - (-24/5)}{6\text{k}} = \frac{6}{5\text{k}}\text{ A}$$

All the results are shown in Fig. 3.9. Note that KCL is satisfied at every node.

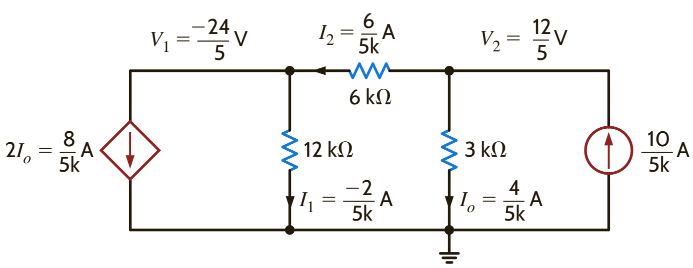


Figure 3.9
Circuit used in Example 3.3.

EXAMPLE 3.4

Let us determine the set of linearly independent equations that when solved will yield the node voltages in the network in Fig. 3.10. Then, given the following component values, we will compute the node voltages using MATLAB: $R_1 = 1$ kΩ, $R_2 = R_3 = 2$ kΩ, $R_4 = 4$ kΩ, $i_A = 2$ mA, $i_B = 4$ mA, and $\alpha = 0.002$.

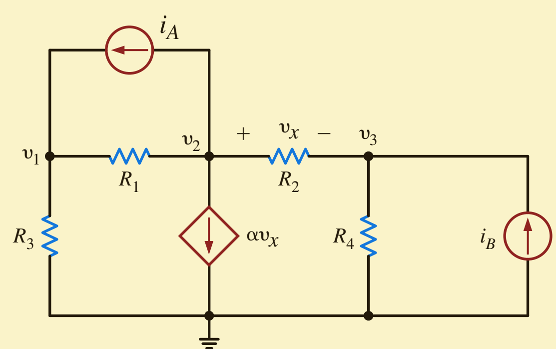


Figure 3.10
Circuit containing a voltage-controlled current source.

Applying KCL at each of the nonreference nodes yields the equations

$$\begin{aligned} G_3 v_1 + G_1(v_1 - v_2) - i_A &= 0 \\ i_A + G_1(v_2 - v_1) + \alpha v_x + G_2(v_2 - v_3) &= 0 \\ G_2(v_3 - v_2) + G_4 v_3 - i_B &= 0 \end{aligned}$$

where $v_x = v_2 - v_3$. Simplifying these equations, we obtain

$$\begin{aligned} (G_1 + G_3)v_1 - G_1 v_2 &= i_A \\ -G_1 v_1 + (G_1 + \alpha + G_2)v_2 - (\alpha + G_2)v_3 &= -i_A \\ -G_2 v_2 + (G_2 + G_4)v_3 &= i_B \end{aligned}$$

Given the component values, the equations become

$$\begin{bmatrix} \frac{1}{1k}+\frac{1}{2k} & -\frac{1}{k} & 0 \\ -\frac{1}{k} & \frac{1}{k}+\frac{2}{k}+\frac{1}{2k} & -\left(\frac{2}{k}+\frac{1}{2k}\right) \\ 0 & -\frac{1}{2k} & \frac{1}{2k}+\frac{1}{4k} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.002 \\ -0.002 \\ 0.004 \end{bmatrix}$$

Multiplying the equations by 4k yields the equation

$$\begin{bmatrix} 6 & -4 & 0 \\ -4 & 14 & -10 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 16 \end{bmatrix}$$

The MATLAB solution is then

```
>> G = [6 -4 0;-4 14 -10;0 -2 3]
   G =
           6    -4     0
          -4    14   -10
           0    -2     3
>> I = [8;-8;16]
   I =
           8
          -8
          16
>> V = inv(G)*I
   V =
           8.5714
          10.8571
          12.5714
```

LEARNING ASSESSMENTS

E3.4 Find the node voltages in the circuit in Fig. E3.4.

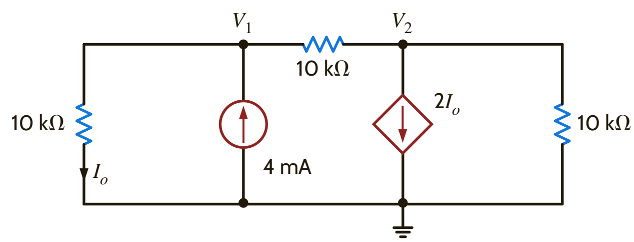


Figure E3.4

ANSWER:
$V_1 = 16$ V,
$V_2 = -8$ V.

E3.5 Find the voltage V_o in the network in Fig. E3.5.

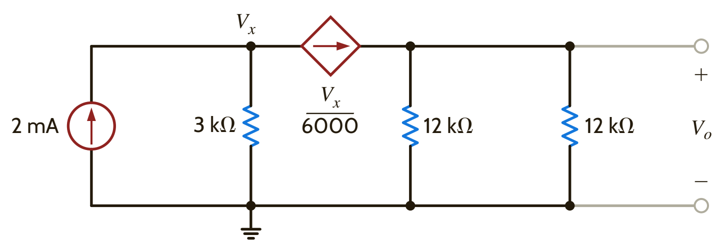


Figure E3.5

ANSWER:
$V_o = 4$ V.

E3.6 Find V_o in Fig. E3.6 using nodal analysis.

ANSWER:
$V_o = 0.952$ V.

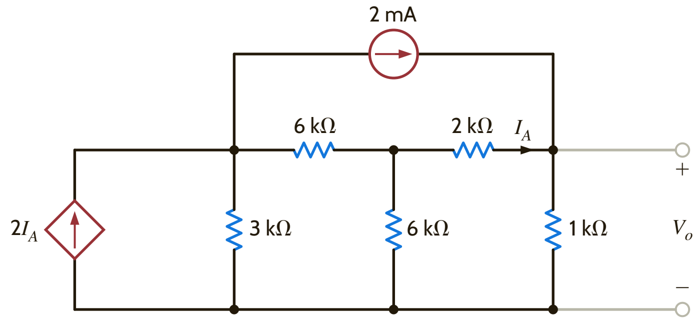


Figure E3.6

CIRCUITS CONTAINING INDEPENDENT VOLTAGE SOURCES As is our practice, in our discussion of this topic we will proceed from the simplest case to more complicated cases. The simplest case is that in which an independent voltage source is connected to the reference node. The following example illustrates this case.

EXAMPLE 3.5

Consider the circuit shown in Fig. 3.11a. Let us determine all node voltages and branch currents.

SOLUTION

This network has three nonreference nodes with labeled node voltages V_1, V_2, and V_3. Based on our previous discussions, we would assume that in order to find all the node voltages we would need to write a KCL equation at each of the nonreference nodes. The resulting three linearly independent simultaneous equations would produce the unknown node voltages. However, note that V_1 and V_3 are known quantities because an independent voltage source is connected directly between the reference node and each of these nodes. Therefore, $V_1 = 12$ V and $V_3 = -6$ V. Furthermore, note that the current through the 9-kΩ resistor is $[12 - (-6)]/9\text{k} = 2$ mA from left to right. We do not know V_2 or the current in the remaining resistors. However, since only one node voltage is unknown, a single-node equation will produce it. Applying KCL to this center node yields

$$\frac{V_2 - V_1}{12\text{k}} + \frac{V_2 - 0}{6\text{k}} + \frac{V_2 - V_3}{12\text{k}} = 0 \qquad \text{or} \qquad \frac{V_2 - 12}{12\text{k}} + \frac{V_2}{6\text{k}} + \frac{V_2 - (-6)}{12\text{k}} = 0$$

from which we obtain

$$V_2 = \frac{3}{2}\text{ V}$$

HINT
Any time an independent voltage source is connected between the reference node and a nonreference node, the nonreference node voltage is known.

Once all the node voltages are known, Ohm's law can be used to find the branch currents shown in Fig. 3.11b. The diagram illustrates that KCL is satisfied at every node.

Note that the presence of the voltage sources in this example has simplified the analysis, since two of the three linear independent equations are $V_1 = 12$ V and $V_3 = -6$ V. We will find that as a general rule, whenever voltage sources are present between nodes, the node voltage equations that describe the network will be simpler.

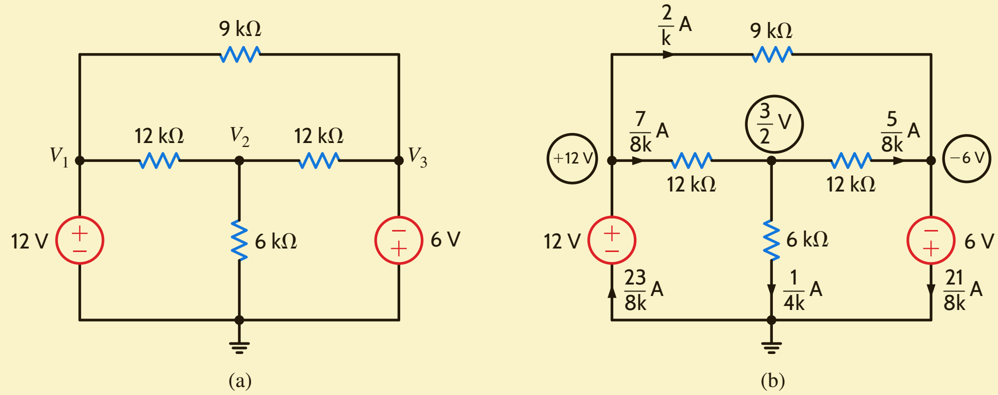


Figure 3.11
Circuit used in Example 3.5.

LEARNING ASSESSMENTS

E3.7 Use nodal analysis to find the current I_o in the network in Fig. E3.7.

ANSWER: $I_o = \frac{3}{4}$ mA.

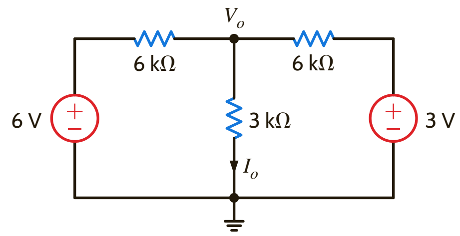


Figure E3.7

E3.8 Find V_o in Fig. E3.8 using nodal analysis.

ANSWER: $V_o = 3.89$ V.

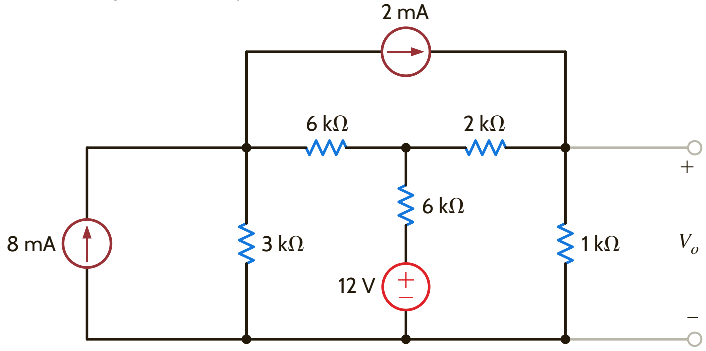


Figure E3.8

Next let us consider the case in which an independent voltage source is connected between two nonreference nodes.

EXAMPLE 3.6

Suppose we wish to find the currents in the two resistors in the circuit of Fig. 3.12a.

SOLUTION

If we try to attack this problem in a brute-force manner, we immediately encounter a problem. Thus far, branch currents were either known source values or could be expressed as the branch voltage divided by the branch resistance. However, the branch current through the 6-V source is certainly not known and cannot be directly expressed using Ohm's law. We can, of course, give this current a name and write the KCL equations at the two nonreference nodes in terms of this current. However, this approach is no panacea because this technique will result in *two* linearly independent simultaneous equations in terms of *three* unknowns—that is, the two node voltages and the current in the voltage source.

To solve this dilemma, we recall that $N - 1$ linearly independent equations are required to determine the $N - 1$ nonreference node voltages in an N-node circuit. Since our network has three nodes, we need two linearly independent equations. Now note that if somehow one of the node voltages is known, we immediately know the other; that is, if V_1 is known, then $V_2 = V_1 - 6$. If V_2 is known, then $V_1 = V_2 + 6$. Therefore, the difference in potential between the two nodes is *constrained* by the voltage source and, hence,

$$V_1 - V_2 = 6$$

This constraint equation is one of the two linearly independent equations needed to determine the node voltages.

Next consider the network in Fig. 3.12b, in which the 6-V source is completely enclosed within the dashed surface. The constraint equation governs this dashed portion of the network. The remaining equation is obtained by applying KCL to this dashed surface, which is commonly called a *supernode*. Recall that in Chapter 2 we demonstrated that KCL must hold for a surface, and this technique eliminates the problem of dealing with a current through a voltage source. KCL for the supernode is

$$-6 \times 10^{-3} + \frac{V_1}{6k} + \frac{V_2}{12k} + 4 \times 10^{-3} = 0$$

Solving these equations yields $V_1 = 10$ V and $V_2 = 4$ V and, hence, $I_1 = 5/3$ mA and $I_2 = 1/3$ mA. A quick check indicates that KCL is satisfied at every node.

Note that applying KCL at the reference node yields the same equation as shown above. The student might think that the application of KCL at the reference node saves one from having to deal with supernodes. Recall that we do not apply KCL at any node—even the reference node—that contains an independent voltage source. This idea can be illustrated with the circuit in the next example.

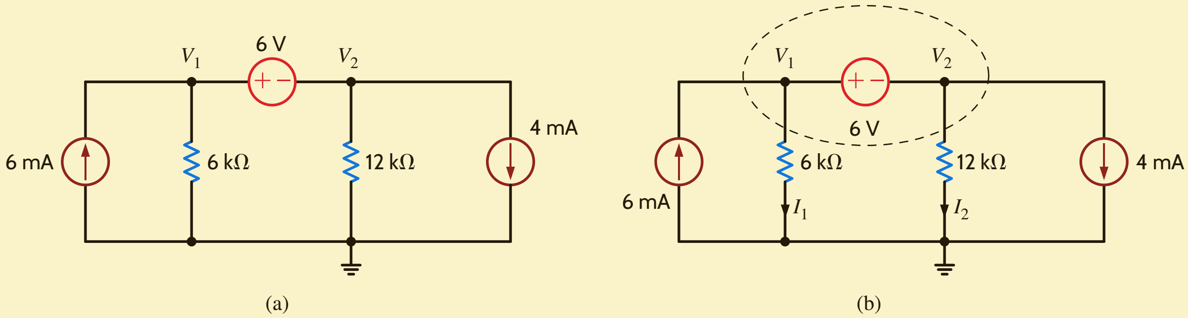


Figure 3.12 Circuits used in Example 3.6.

EXAMPLE 3.7

Let us determine the current I_o in the network in **Fig. 3.13a**.

SOLUTION

Examining the network, we note that node voltages V_2 and V_4 are known and the node voltages V_1 and V_3 are constrained by the equation

$$V_1 - V_3 = 12$$

The network is redrawn in **Fig. 3.13b**.

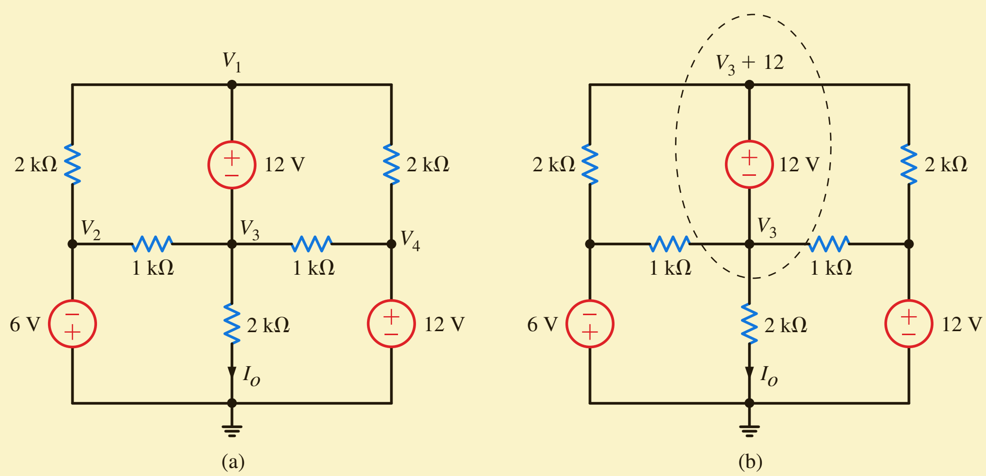


Figure 3.13 Example circuit with supernodes.

Since we want to find the current I_o, V_1 (in the supernode containing V_1 and V_3) is written as $V_3 + 12$. The KCL equation at the supernode is then

$$\frac{V_3 + 12 - (-6)}{2k} + \frac{V_3 + 12 - 12}{2k} + \frac{V_3 - (-6)}{1k} + \frac{V_3 - 12}{1k} + \frac{V_3}{2k} = 0$$

Solving the equation for V_3 yields

$$V_3 = -\frac{6}{7}\text{ V}$$

I_o can then be computed immediately as

$$I_o = \frac{-\frac{6}{7}}{2\text{k}} = -\frac{3}{7}\text{ mA}$$

LEARNING ASSESSMENTS

E3.9 Use nodal analysis to find I_o in the network in Fig. E3.9.

ANSWER: $I_o = 3.8$ mA.

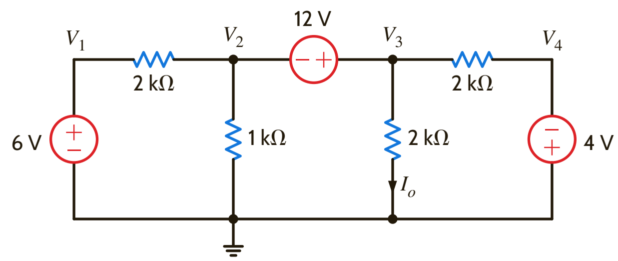


Figure E3.9

E3.10 Find V_o in Fig. E3.10 using nodal analysis.

ANSWER: $V_o = 5.6$ V.

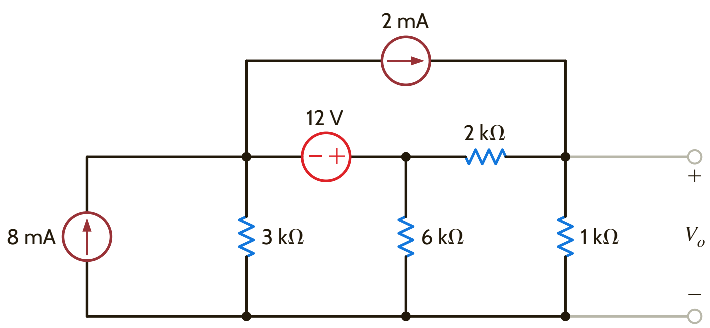


Figure E3.10

CIRCUITS CONTAINING DEPENDENT VOLTAGE SOURCES As the following examples will indicate, networks containing dependent (controlled) sources are treated in the same manner as described earlier.

EXAMPLE 3.8

We wish to find I_o in the network in Fig. 3.14.

SOLUTION

Since the dependent voltage source is connected between the node labeled V_1 and the reference node,

$$V_1 = 2\text{k}I_x$$

KCL at the node labeled V_2 is

$$\frac{V_2 - V_1}{2\text{k}} - \frac{4}{\text{k}} + \frac{V_2}{1\text{k}} = 0$$

where

$$I_x = \frac{V_2}{1\text{k}}$$

Solving these equations yields $V_2 = 8$ V and $V_1 = 16$ V. Therefore,

$$I_o = \frac{V_1 - V_2}{2\text{k}}$$
$$= 4 \text{ mA}$$

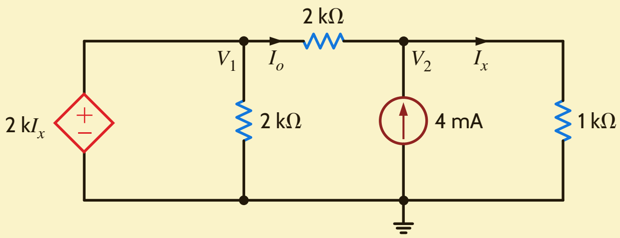


Figure 3.14
Circuits used in Example 3.8.

EXAMPLE 3.9

Let us find the current I_o in the network in **Fig. 3.15**.

SOLUTION

This circuit contains both an independent voltage source and a voltage-controlled voltage source. Note that $V_3 = 6$ V, $V_2 = V_x$, and a supernode exists between the nodes labeled V_1 and V_2.

Applying KCL to the supernode, we obtain

$$\frac{V_1 - V_3}{6\text{k}} + \frac{V_1}{12\text{k}} + \frac{V_2}{6\text{k}} + \frac{V_2 - V_3}{12\text{k}} = 0$$

where the constraint equation for the supernode is

$$V_1 - V_2 = 2V_x$$

The final equation is

$$V_3 = 6$$

Solving these equations, we find that

$$V_1 = \frac{9}{2} \text{ V}$$

and, hence,

$$I_o = \frac{V_1}{12\text{k}} = \frac{3}{8} \text{ mA}$$

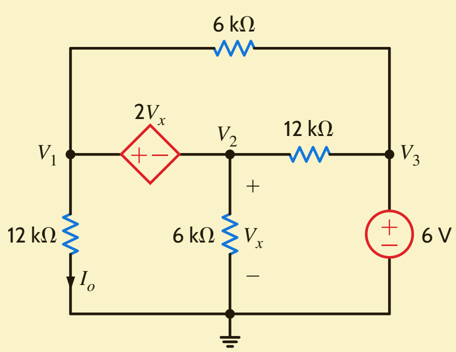


Figure 3.15
Circuit used in Example 3.9.

Finally, let us consider two additional circuits that, for purposes of comparison, we will examine using more than one method.

EXAMPLE 3.10

Let us find V_o in the network in Fig. 3.16a. Note that the circuit contains two voltage sources, one of which is a controlled source, and two independent current sources. The circuit is redrawn in Fig. 3.16b in order to label the nodes and identify the supernode surrounding the controlled source. Because of the presence of the independent voltage source, the voltage at node 4 is known to be 4 V. We will use this knowledge in writing the node equations for the network.

Since the network has five nodes, four linear independent equations are sufficient to determine all the node voltages. Within the supernode, the defining equation is

$$V_1 - V_2 = 2V_x$$

where

$$V_2 = V_x$$

and thus

$$V_1 = 3V_x$$

Furthermore, we know that one additional equation is

$$V_4 = 4$$

Thus, given these two equations, only two more equations are needed in order to solve for the unknown node voltages. These additional equations result from applying KCL at the supernode and at the node labeled V_3. The equations are

$$-\frac{2}{\text{k}} + \frac{V_x}{1\text{k}} + \frac{V_x - V_3}{1\text{k}} + \frac{3V_x - V_3}{1\text{k}} + \frac{3V_x - 4}{1\text{k}} = 0$$

$$\frac{V_3 - 3V_x}{1\text{k}} + \frac{V_3 - V_x}{1\text{k}} = \frac{2}{\text{k}}$$

Combining the equations yields the two equations

$$8V_x - 2V_3 = 6$$
$$-4V_x + 2V_3 = 2$$

Solving these equations, we obtain

$$V_x = 2\text{ V} \quad \text{and} \quad V_3 = 5\text{ V}$$
$$V_o = 3V_x - V_3 = 1\text{ V}$$

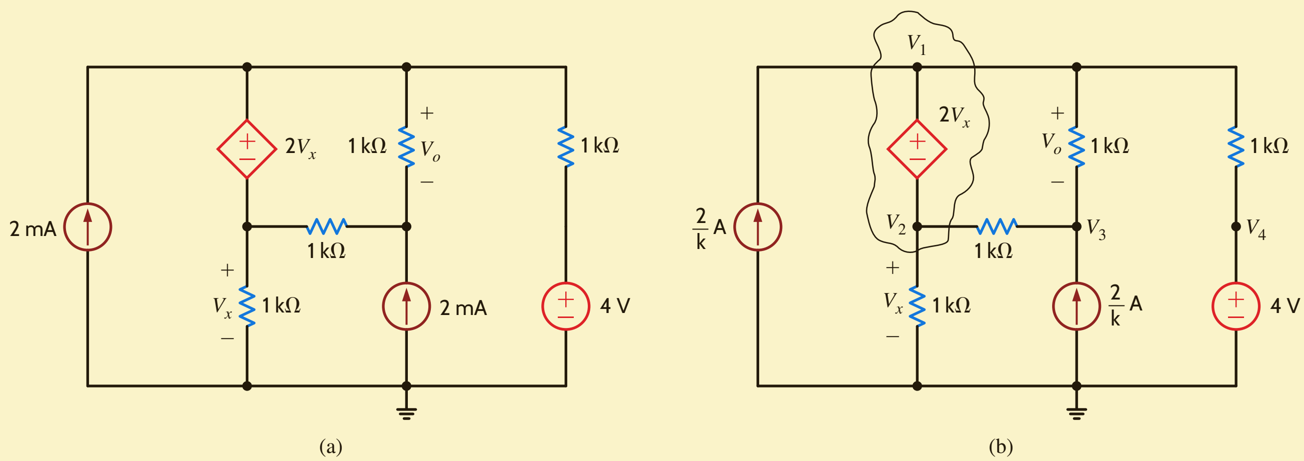


Figure 3.16
Circuit used in Example 3.10.

EXAMPLE 3.11

We wish to find I_o in the network in Fig. 3.17a. Note that this circuit contains three voltage sources, one of which is a controlled source, and another is a controlled current source. Because two of the voltage sources are connected to the reference node, one node voltage is known directly and one is specified by the dependent source. Furthermore, the difference in voltage between two nodes is defined by the 6-V independent source.

The network is redrawn in Fig. 3.17b in order to label the nodes and identify the supernode. Since the network has six nodes, five linear independent equations are needed to determine the unknown node voltages.

The two equations for the supernode are

$$V_1 - V_4 = -6$$

$$\frac{V_1 - 12}{1\text{k}} + \frac{V_1 - V_3}{1\text{k}} + 2I_x + \frac{V_4 - V_3}{1\text{k}} + \frac{V_4}{1\text{k}} + \frac{V_4 - V_5}{1\text{k}} = 0$$

The three remaining equations are

$$V_2 = 12$$

$$V_3 = 2V_x$$

$$\frac{V_5 - V_4}{1\text{k}} + \frac{V_5}{1\text{k}} = 2I_x$$

The equations for the control parameters are

$$V_x = V_1 - 12$$

$$I_x = \frac{V_4}{1\text{k}}$$

Combining these equations yields the following set of equations:

$$-2V_1 + 5V_4 - V_5 = -36$$

$$V_1 - V_4 = -6$$

$$-3V_4 + 2V_5 = 0$$

In matrix form, the equations are

$$\begin{bmatrix} -2 & 5 & -1 \\ 1 & -1 & 0 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -36 \\ -6 \\ 0 \end{bmatrix}$$

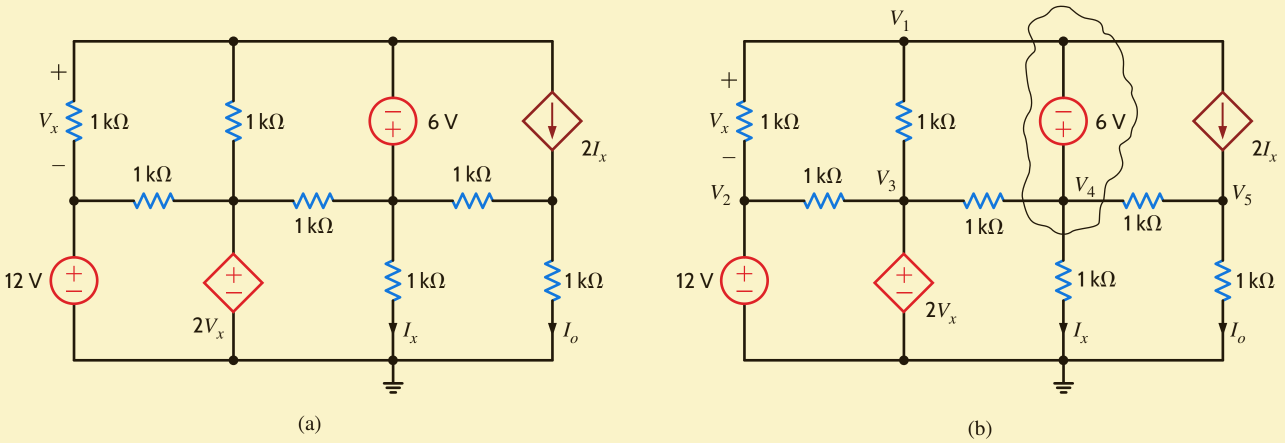


Figure 3.17

Circuit used in Example 3.11.

The MATLAB solution is then

```
>> G = [-2 5 -1;1 -1 0;0 -3 2]
   G =
       -2    5   -1
        1   -1    0
        0   -3    2

>> I = [-36;-6;0]
   I =
       -36
        -6
         0

>> V = inv(G)*I
   V =
       -38.0000
       -32.0000
       -48.0000
```

Then, since $V_3 = 2V_x$, $V_3 = -100$ V. I_o is -48 mA. The reader is encouraged to verify that KCL is satisfied at every node.

PROBLEM-SOLVING STRATEGY

NODAL ANALYSIS

STEP 1. Determine the number of nodes in the circuit. Select one node as the reference node. Assign a node voltage between each nonreference node and the reference node. All node voltages are assumed positive with respect to the reference node. For an N-node circuit, there are $N - 1$ node voltages. As a result, $N - 1$ linearly independent equations must be written to solve for the node voltages.

STEP 2. Write a constraint equation for each voltage source—independent or dependent—in the circuit in terms of the assigned node voltages using KVL. Each constraint equation represents one of the necessary linearly independent equations, and N_v voltage sources yield N_v linearly independent equations. For each dependent voltage source, express the controlling variable for that source in terms of the node voltages.

A voltage source—independent or dependent—may be connected between a nonreference node and the reference node or between two nonreference nodes. A supernode is formed by a voltage source and its two connecting nonreference nodes.

STEP 3. Use KCL to formulate the remaining $N - 1 - N_v$ linearly independent equations. First, apply KCL at each nonreference node not connected to a voltage source. Second, apply KCL at each supernode. Treat dependent current sources like independent current sources when formulating the KCL equations. For each dependent current source, express the controlling variable in terms of the node voltages.