LEARNING ASSESSMENT

E12.20 An *RLC* network has the following parameter values: $R = 10\ \Omega$, $L = 1$ H, and $C = 2$ F. Determine the values of the circuit elements if the circuit is magnitude scaled by a factor of 100 and frequency scaled by a factor of 10,000.

ANSWER: $R = 1\ \text{k}\Omega$; $L = 10$ mH; $C = 2\ \mu\text{F}$.

12.5 Filter Networks

PASSIVE FILTERS A filter network is generally designed to pass signals with a specific frequency range and reject or attenuate signals whose frequency spectrum is outside this pass-band. The most common filters are *low-pass* filters, which pass low frequencies and reject high frequencies; *high-pass* filters, which pass high frequencies and block low frequencies; *band-pass* filters, which pass some particular band of frequencies and reject all frequencies outside the range; and *band-rejection* filters, which are specifically designed to reject a particular band of frequencies and pass all other frequencies.

The ideal frequency characteristic for a low-pass filter is shown in Fig. 12.40a. Also shown is a typical or physically realizable characteristic. Ideally, we would like the low-pass filter to pass all frequencies to some frequency ω_0 and pass no frequency above that value; however, it is not possible to design such a filter with linear circuit elements. Hence, we must be content to employ filters that we can actually build in the laboratory, and these filters have frequency characteristics that are simply not ideal.

A simple low-pass filter network is shown in Fig. 12.40b. The voltage gain for the network is

$$\mathbf{G}_v(j\omega) = \frac{1}{1 + j\omega RC} \qquad \textbf{12.55}$$

which can be written as

$$\mathbf{G}_v(j\omega) = \frac{1}{1 + j\omega\tau} \qquad \textbf{12.56}$$

where $\tau = RC$, the time constant. The amplitude characteristic is

$$M(\omega) = \frac{1}{\left[1 + (\omega\tau)^2\right]^{1/2}} \qquad \textbf{12.57}$$

and the phase characteristic is

$$\phi(\omega) = -\tan^{-1}\omega\tau \qquad \textbf{12.58}$$

Note that at the break frequency, $\omega = \dfrac{1}{\tau}$, the amplitude is

$$M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}} \qquad \textbf{12.59}$$

The break frequency is also commonly called the *half-power frequency*. This name is derived from the fact that if the voltage or current is $1/\sqrt{2}$ of its maximum value, then the power, which is proportional to the square of the voltage or current, is one-half its maximum value.

The magnitude, in decibels, and phase curves for this simple low-pass circuit are shown in Fig. 12.40c. Note that the magnitude curve is flat for low frequencies and rolls off at high frequencies. The phase shifts from 0° at low frequencies to −90° at high frequencies.

The ideal frequency characteristic for a high-pass filter is shown in Fig. 12.41a, together with a typical characteristic that we could achieve with linear circuit components. Ideally, the high-pass filter passes all frequencies above some frequency ω_0 and no frequencies below that value.

A simple high-pass filter network is shown in Fig. 12.41b. This is the same network as shown in Fig. 12.40b, except that the output voltage is taken across the resistor. The voltage gain for this network is

$$\mathbf{G}_v(j\omega) = \frac{j\omega\tau}{1 + j\omega\tau} \qquad \textbf{12.60}$$

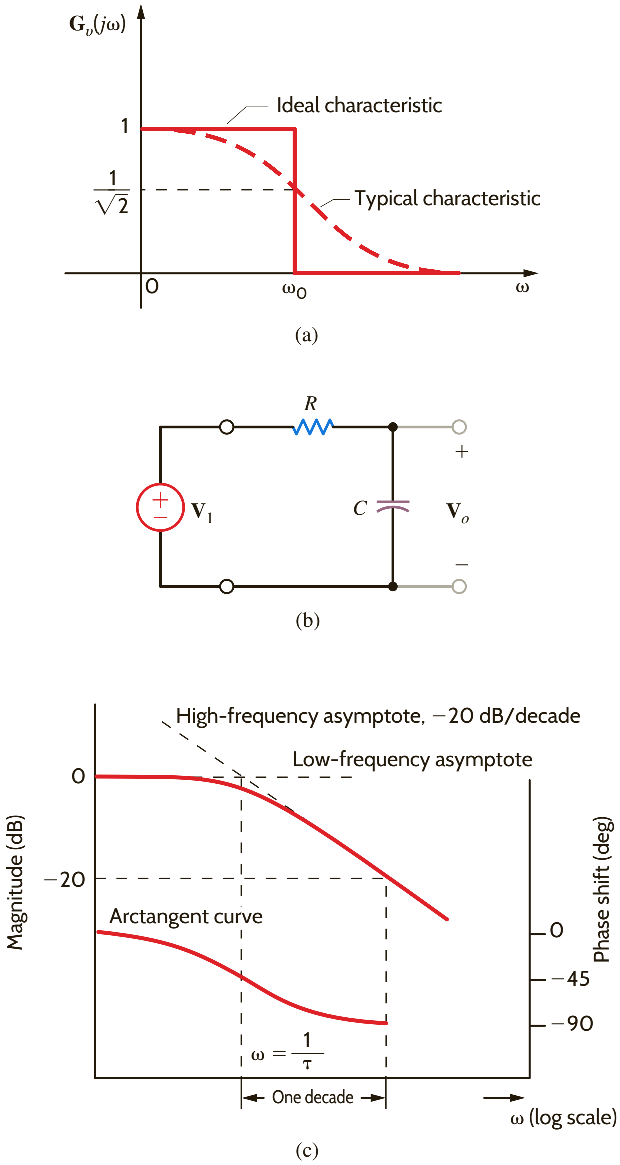


Figure 12.40

Low-pass filter circuit and its frequency characteristics.

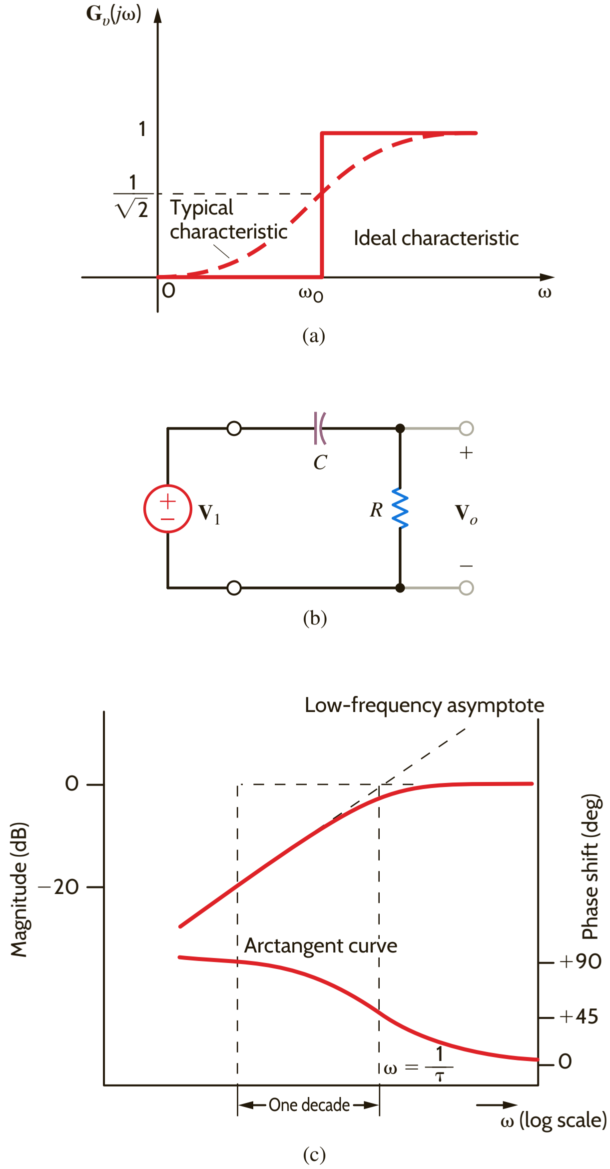


Figure 12.41

High-pass filter circuit and its frequency characteristics.

where once again $\tau = RC$. The magnitude of this function is

$$M(\omega) = \frac{\omega\tau}{[1 + (\omega\tau)^2]^{1/2}} \qquad \textbf{12.61}$$

and the phase is

$$\phi(\omega) = \frac{\pi}{2} - \tan^{-1}\omega\tau \qquad \textbf{12.62}$$

The half-power frequency is $\omega = 1/\tau$, and the phase at this frequency is 45°.

The magnitude and phase curves for this high-pass filter are shown in Fig. 12.41c. At low frequencies the magnitude curve has a slope of +20 dB/decade due to the term $\omega\tau$ in the numerator of Eq. (12.61). Then at the break frequency the curve begins to flatten out. The phase curve is derived from Eq. (12.62).

Ideal and typical amplitude characteristics for simple band-pass and band-rejection filters are shown in Figs. 12.42a and b, respectively. Simple networks that are capable of realizing the typical characteristics of each filter are shown below as characteristics in Figs. 12.42c and d. ω_0 is the center frequency of the pass or rejection band and the frequency at which the

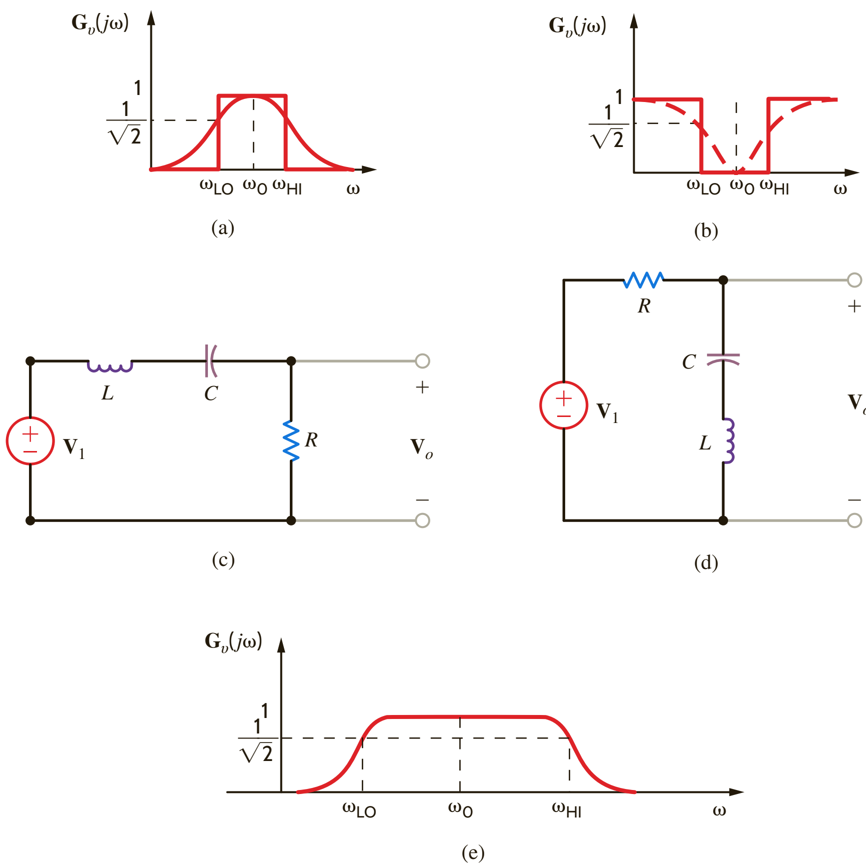


Figure 12.42

Band-pass and band-rejection filters and characteristics.

maximum or minimum amplitude occurs. ω_{LO} and ω_{HI} are the lower and upper break frequencies or *cutoff frequencies*, where the amplitude is $1/\sqrt{2}$ of the maximum value. The width of the pass or rejection band is called *bandwidth*, and hence

$$\text{BW} = \omega_{HI} - \omega_{LO} \tag{12.63}$$

To illustrate these points, let us consider the band-pass filter. The voltage transfer function is

$$\mathbf{G}_v(j\omega) = \frac{R}{R + j(\omega L - 1/\omega C)}$$

and, therefore, the amplitude characteristic is

$$M(\omega) = \frac{RC\omega}{\sqrt{(RC\omega)^2 + (\omega^2 LC - 1)^2}}$$

At low frequencies

$$M(\omega) \approx \frac{RC\omega}{1} \approx 0$$

At high frequencies

$$M(\omega) \approx \frac{RC\omega}{\omega^2 LC} \approx \frac{R}{\omega L} \approx 0$$

In the midfrequency range $(RC\omega)^2 \gg (\omega^2 LC - 1)^2$, and thus $M(\omega) \approx 1$. Therefore, the frequency characteristic for this filter is shown in **Fig. 12.42e**. The center frequency is $\omega_0 = 1/\sqrt{LC}$. At the lower cutoff frequency

$$\omega^2 LC - 1 = -RC\omega$$

or

$$\omega^2 + \frac{R\omega}{L} - \omega_0^2 = 0$$

Solving this expression for ω_{LO}, we obtain

$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

At the upper cutoff frequency

$$\omega^2 LC - 1 = +RC\omega$$

or

$$\omega^2 - \frac{R}{L}\omega - \omega_0^2 = 0$$

Solving this expression for ω_{HI}, we obtain

$$\omega_{HI} = \frac{+(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

Therefore, the bandwidth of the filter is

$$\text{BW} = \omega_{HI} - \omega_{LO} = \frac{R}{L}$$

EXAMPLE 12.18

Consider the frequency-dependent network in Fig. 12.43. Given the following circuit parameter values: $L = 159\ \mu\text{H}$, $C = 159\ \mu\text{F}$, and $R = 10\ \Omega$, let us demonstrate that this one network can be used to produce a low-pass, high-pass, or band-pass filter.

SOLUTION

The voltage gain $\mathbf{V}_R/\mathbf{V}_S$ is found by voltage division to be

$$\frac{\mathbf{V}_R}{\mathbf{V}_S} = \frac{R}{j\omega L + R + 1/(j\omega C)} = \frac{j\omega\left(\frac{R}{L}\right)}{(j\omega)^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{LC}}$$

$$= \frac{(62.9 \times 10^3)j\omega}{-\omega^2 + (62.9 \times 10^3)j\omega + 39.6 \times 10^6}$$

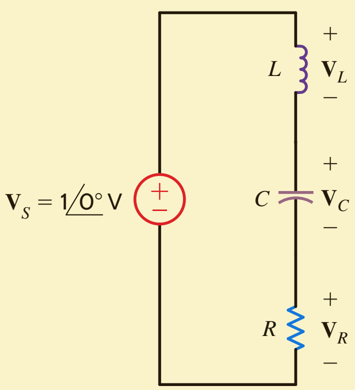


Figure 12.43 Circuit used in Example 12.18.

which is the transfer function for a band-pass filter. At resonance, $\omega^2 = 1/LC$, and hence

$$\frac{\mathbf{V}_R}{\mathbf{V}_S} = 1$$

Now consider the gain $\mathbf{V}_L/\mathbf{V}_S$:

$$\frac{\mathbf{V}_L}{\mathbf{V}_S} = \frac{j\omega L}{j\omega L + R + 1/(j\omega C)} = \frac{-\omega^2}{(j\omega)^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{LC}}$$

$$= \frac{-\omega^2}{-\omega^2 + (62.9 \times 10^3)j\omega + 39.6 \times 10^6}$$

which is a second-order high-pass filter transfer function. Again, at resonance,

$$\frac{\mathbf{V}_L}{\mathbf{V}_S} = \frac{j\omega L}{R} = jQ = j0.1$$

Similarly, the gain $\mathbf{V}_C/\mathbf{V}_S$ is

$$\frac{\mathbf{V}_C}{\mathbf{V}_S} = \frac{1/(j\omega C)}{j\omega L + R + 1/(j\omega C)} = \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{LC}}$$

$$= \frac{39.6 \times 10^6}{-\omega^2 + (62.9 \times 10^3)j\omega + 39.6 \times 10^6}$$

which is a second-order low-pass filter transfer function. At the resonant frequency,

$$\frac{\mathbf{V}_C}{\mathbf{V}_S} = \frac{1}{j\omega CR} = -jQ = -j0.1$$

Thus, one circuit produces three different filters depending on where the output is taken. This can be seen in the Bode plot for each of the three voltages in Fig. 12.44, where $\mathbf{V}_S$ is set to $1\underline{/0^\circ}$ V.

We know that Kirchhoff's voltage law must be satisfied at all times. Note from the Bode plot that $\mathbf{V}_R + \mathbf{V}_C + \mathbf{V}_L$ also equals $\mathbf{V}_S$ at all frequencies! Finally, let us demonstrate KVL by adding $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$:

$$\mathbf{V}_L + \mathbf{V}_R + \mathbf{V}_C = \frac{\left((j\omega)^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{\sqrt{LC}}\right)\mathbf{V}_S}{(j\omega)^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{\sqrt{LC}}} = \mathbf{V}_S$$

Thus, even though $\mathbf{V}_S$ is distributed between the resistor, capacitor, and inductor based on frequency, the sum of the three voltages completely reconstructs $\mathbf{V}_S$.

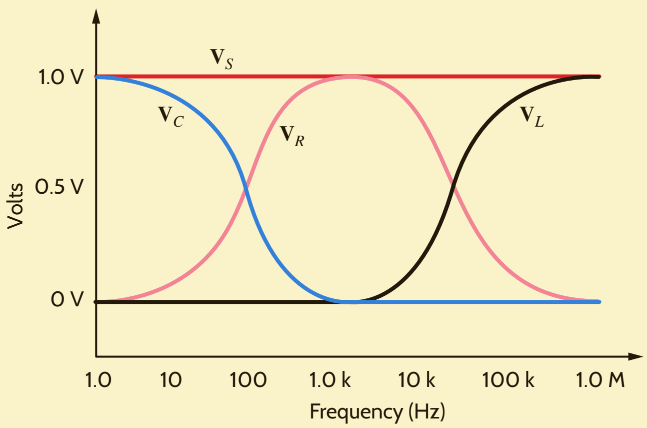


Figure 12.44

Bode plots for network in Fig. 12.43.

EXAMPLE 12.19

A telephone transmission system suffers from 60-Hz interference caused by nearby power utility lines. Let us use the network in Fig. 12.45 to design a simple notch filter to eliminate the 60-Hz interference.

SOLUTION

The resistor R_{eq} represents the equivalent resistance of the telephone system to the right of the LC combination. The LC parallel combination has an equivalent impedance of

$$\mathbf{Z} = (j\omega L)//(1/j\omega C) = \frac{(L/C)}{j\omega L + 1/(j\omega C)}$$

Now the voltage transfer function is

$$\frac{\mathbf{V}_o}{\mathbf{V}_{\text{in}}} = \frac{R_{\text{eq}}}{R_{\text{eq}} + \mathbf{Z}} = \frac{R_{\text{eq}}}{R_{\text{eq}} + \dfrac{(L/C)}{j\omega L + (1/j\omega C)}}$$

which can be written

$$\frac{\mathbf{V}_o}{\mathbf{V}_{\text{in}}} = \frac{(j\omega)^2 + \frac{1}{LC}}{(j\omega)^2 + \left(\frac{j\omega}{R_{\text{eq}}C}\right) + \frac{1}{LC}}$$

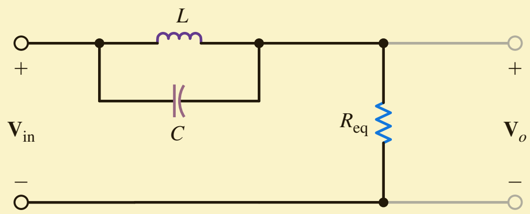


Figure 12.45
Circuit used in Example 12.19.

Note that at resonance, the numerator and thus $\mathbf{V}_o$ go to zero. We want resonance to occur at 60 Hz. Thus,

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi(60) = 120\pi$$

If we select $C = 100\ \mu\text{F}$, then the required value for L is 70.3 mH—both are reasonable values. To demonstrate the effectiveness of the filter, let the input voltage consist of a 60-Hz sinusoid and a 1000-Hz sinusoid of the form

$$v_{in}(t) = 1\ \sin[(2\pi)60t] + 0.2\ \sin[(2\pi)1000t]\ \text{V}$$

The input and output waveforms are both shown in **Fig. 12.46**. Note that the output voltage, as desired, contains none of the 60-Hz interference.

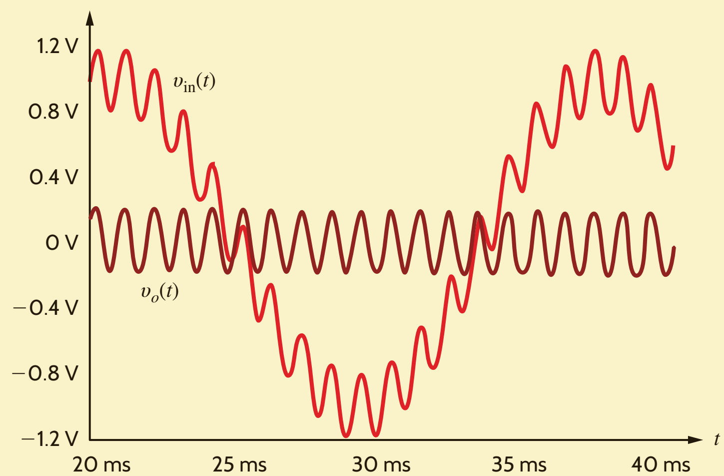


Figure 12.46
Transient analysis of the network in Fig. 12.45.

LEARNING ASSESSMENTS

E12.21 Given the filter network shown in Fig. E12.21, sketch the magnitude characteristic of the Bode plot for $\mathbf{G}_v(j\omega)$.

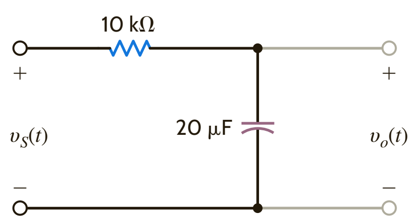
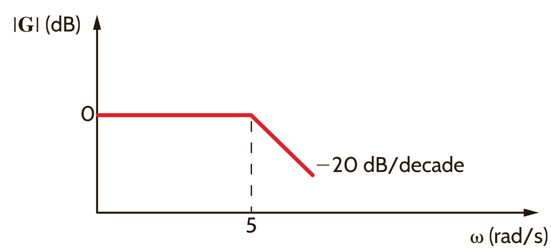


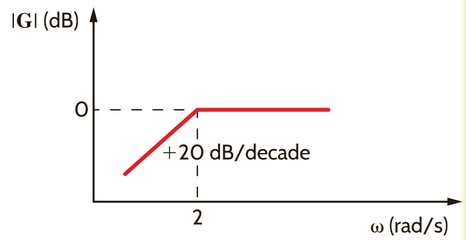
Figure E12.21

ANSWER:



E12.22 Given the filter network in Fig. E12.22, sketch the magnitude characteristic of the Bode plot for $\mathbf{G}_v(j\omega)$.

ANSWER:



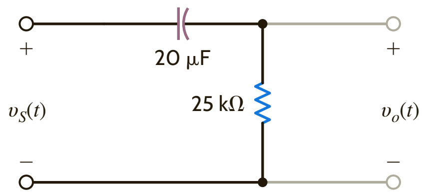
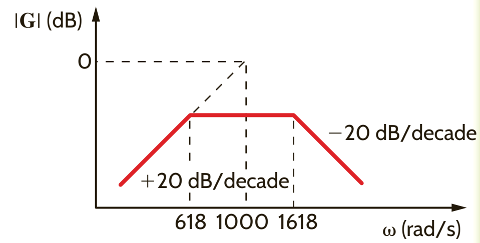


Figure E12.22

E12.23 A band-pass filter network is shown in Fig. E12.23. Sketch the magnitude characteristic of the Bode plot for $\mathbf{G}_v(j\omega)$.

ANSWER:



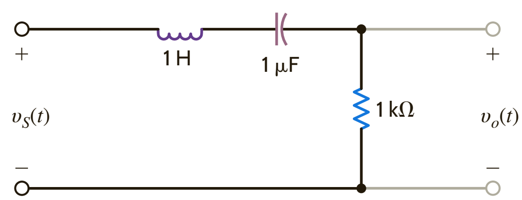


Figure E12.23

E12.24 Determine what type of filter the network shown in Fig. E12.24 represents by determining the voltage transfer function.

ANSWER:

$$\frac{s\left(\frac{L}{R_1}\right)}{s\left[\frac{(R_1 + R_2)L}{R_1R_2}\right] + 1}$$; This is a high-pass filter.

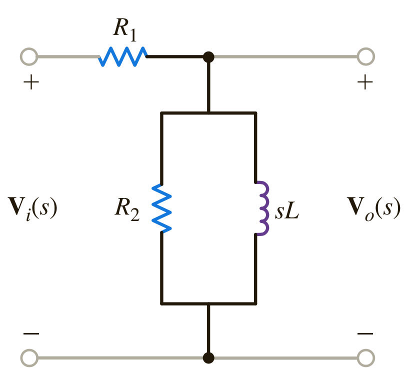


Figure E12.24

EXAMPLE 12.20

The ac-dc converter in Fig. 12.47a is designed for use with a hand-held calculator. Ideally, the circuit should convert a 120-V rms sinusoidal voltage to a 9-V dc output. In actuality, the output is

$$v_o(t) = 9 + 0.5 \sin 377t \text{ V}$$

Let us use a low-pass filter to reduce the 60-Hz component of $v_o(t)$.

SOLUTION

The Thévenin equivalent circuit for the converter is shown in Fig. 12.47b. By placing a capacitor across the output terminals, as shown in Fig. 12.47c, we create a low-pass filter at the output. The transfer function of the filtered converter is

$$\frac{\mathbf{V}_{OF}}{\mathbf{V}_{Th}} = \frac{1}{1 + sR_{Th}C}$$

which has a pole at a frequency of $f = 1/2\pi R_{Th}C$. To obtain significant attenuation at 60 Hz, we choose to place the pole at 6 Hz, yielding the equation

$$\frac{1}{2\pi R_{Th}C} = 6$$

or

$$C = 53.05\ \mu\text{F}$$

A transient simulation of the converter is used to verify performance.

Fig. 12.47d shows the output without filtering, $v_o(t)$, and with filtering, $v_{OF}(t)$. The filter has successfully reduced the unwanted 60-Hz component by a factor of roughly six.

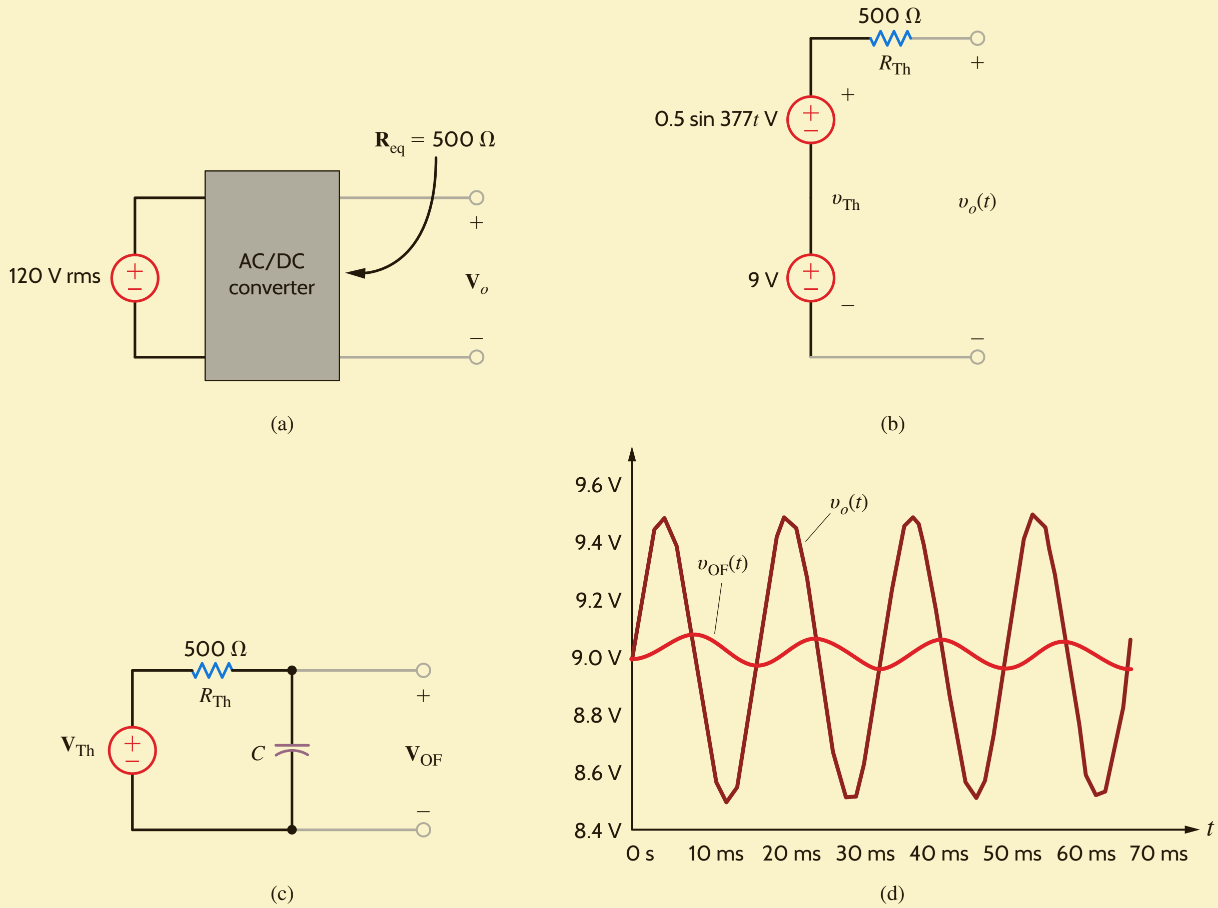


Figure 12.47

Circuits and output plots for ac/dc converter.

EXAMPLE 12.21

The antenna of an FM radio picks up stations across the entire FM frequency range—approximately 87.5 MHz to 108 MHz. The radio's circuitry must have the capability to first reject all of the stations except the one that the listener wants to hear and then to boost the minute antenna signal. A tuned amplifier incorporating parallel resonance can perform both tasks simultaneously.

The network in Fig. 12.48a is a circuit model for a single-stage tuned transistor amplifier where the resistor, capacitor, and inductor are discrete elements. Let us find the transfer function $\mathbf{V}_o(s)/\mathbf{V}_A(s)$, where $\mathbf{V}_A(s)$ is the antenna voltage and the value of C for maximum gain at 91.1 MHz. Finally, we will simulate the results.

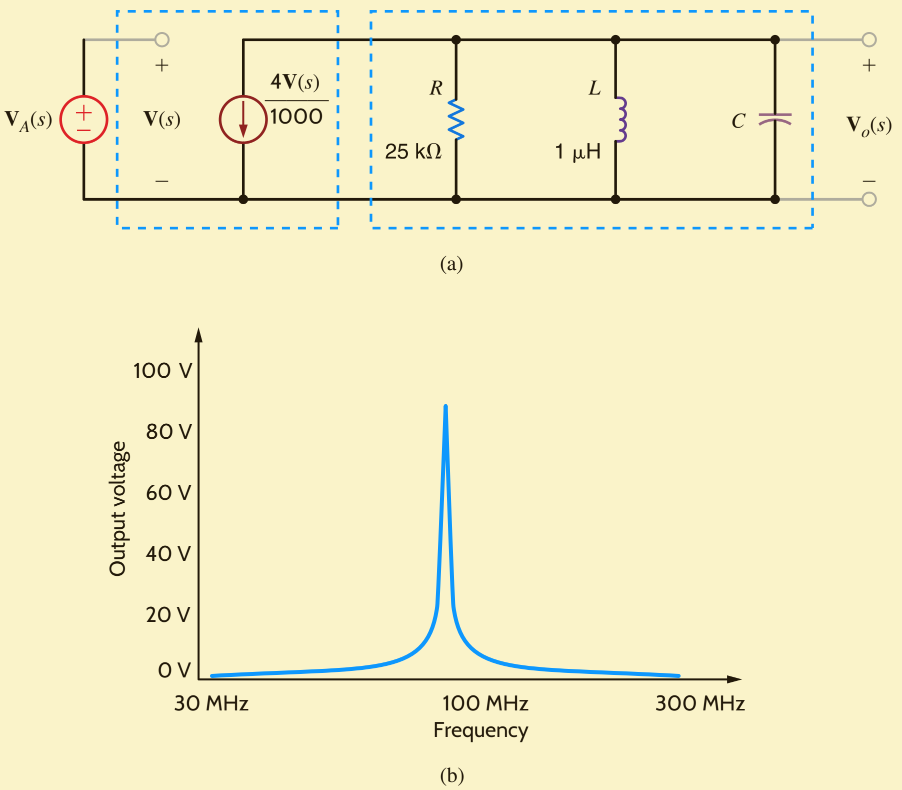


Figure 12.48
Circuit and Bode plot for the parallel resonant tuned amplifier.

SOLUTION

Since $\mathbf{V}(s) = \mathbf{V}_A(s)$, the transfer function is

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_A(s)} = -\frac{4}{1000}\left[R//sL//\frac{1}{sC}\right]$$

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_A(s)} = -\frac{4}{1000}\left[\frac{s/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}\right]$$

The parallel resonant network is actually a band-pass filter. Maximum gain occurs at the center frequency, f_0. This condition corresponds to a minimum value in the denominator. Isolating the denominator polynomial, $D(s)$, and letting $s = j\omega$, we have

$$\mathbf{D}(j\omega) = \frac{1}{LC} - \omega^2 + \frac{j\omega}{RC}$$

which has a minimum value when the real part goes to zero, or

$$\frac{1}{LC} - \omega_0^2 = 0$$

yielding a center frequency of

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Thus, for a center frequency of 91.1 MHz, we have

$$2\pi(91.1 \times 10^6) = \frac{1}{\sqrt{LC}}$$

and the required capacitor value is

$$C = 3.05 \text{ pF}$$

The Bode plot for the tuned amplifier, as shown in Fig. 12.48b, confirms the design, since the center frequency is 91.1 MHz, as specified.

EXAMPLE 12.22

The circuit in Fig. 12.49a is called a notch filter. From a sketch of its Bode plot in Fig. 12.49b, we see that at the notch frequency, f_n, the transfer function gain is zero, while at frequencies above and below f_n the gain is unity. Let us design a notch filter to remove an annoying 60-Hz hum from the output voltage of a cassette tape player and generate its Bode plot.

SOLUTION

Fig. 12.49c shows a block diagram for the filter implementation. The tape output contains both the desired music and the undesired hum. After filtering, the voltage $\mathbf{V}_{\text{amp}}$ will have no 60-Hz component as well as some attenuation at frequencies around 60 Hz. An equivalent circuit for the block diagram including a Thévenin equivalent for the tape deck and an equivalent resistance for the power amp is shown in Fig. 12.49d. Applying voltage division, we find the transfer function to be

$$\frac{\mathbf{V}_{\text{amp}}}{\mathbf{V}_{\text{tape}}} = \frac{R_{\text{amp}}}{R_{\text{amp}} + R_{\text{tape}} + \left(sL//\dfrac{1}{Cs}\right)}$$

After some manipulation, the transfer function can be written as

$$\frac{\mathbf{V}_{\text{amp}}}{\mathbf{V}_{\text{tape}}} = \frac{R_{\text{amp}}}{R_{\text{amp}} + R_{\text{tape}}}\left[\frac{s^2LC + 1}{s^2LC + s\left(\dfrac{L}{R_{\text{tape}} + R_{\text{amp}}}\right) + 1}\right]$$

We see that the transfer function contains two zeros and two poles. Letting $s = j\omega$, the zero frequencies, ω_z, are found to be at

$$\omega_z = \pm\frac{1}{\sqrt{LC}}$$

Obviously, we would like the zero frequencies to be at 60 Hz. If we arbitrarily choose $C = 10\ \mu\text{F}$, then $L = 0.704$ H.

The Bode plot, shown in Fig. 12.49e, confirms that there is indeed zero transmission at 60 Hz.

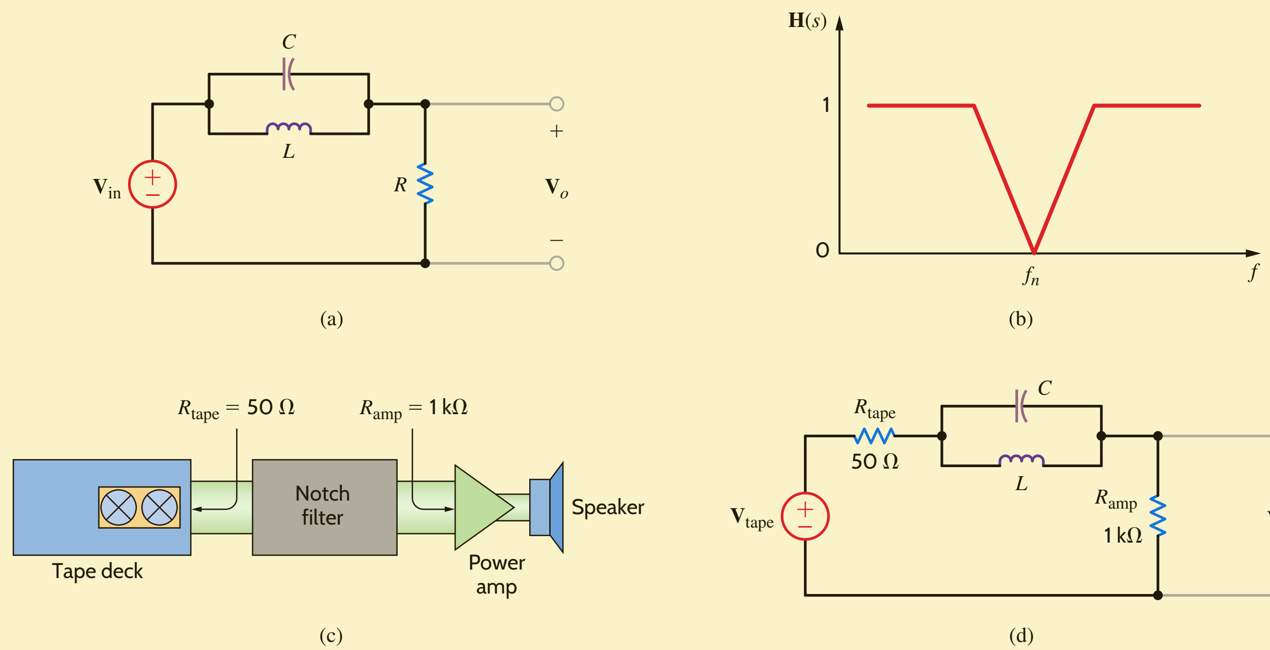


Figure 12.49

Circuits and Bode plots for 60-Hz notch filter.

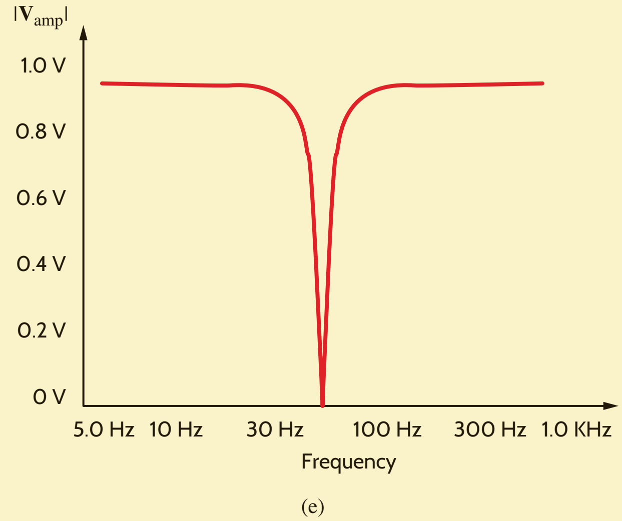


Figure 12.49
(continued)

EXAMPLE 12.23

A fast-growing field within electrical engineering is mixed-mode circuitry, which combines digital and analog networks to create a larger system. A key component in these systems is the analog-to-digital converter, or ADC. It "measures" an analog voltage and converts it to a digital representation. If these conversions are done quickly enough, the result is a sequence of data points, as shown in **Fig. 12.50a**. Connecting the dots reveals the original analog signal, $v_A(t)$. Unfortunately, as seen in **Fig. 12.50b**, undesired signals such as $v_B(t)$ at higher frequencies can also have the same set of data points. This phenomenon is called aliasing and can be avoided by employing a low-pass filter, called an anti-aliasing filter, before the ADC as shown in **Fig. 12.50c**. In general, the half-power frequency of the filter should be greater than the frequency of the signals you wish to convert but less than those you want to reject.

We wish to design an anti-aliasing filter, with a half-power frequency at 100 Hz, that will permit us to acquire a 60-Hz signal. In this design we will assume the ADC has infinite input resistance.

SOLUTION

Assuming the ADC has infinite input resistance, we find that the transfer function for the filter is quite simple:

$$\frac{\mathbf{V}_o}{\mathbf{V}_{\text{in}}} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

The half-power frequency is

$$f_P = \frac{1}{2\pi RC} = 100 \text{ Hz}$$

If we somewhat arbitrarily choose C at 100 nF, a little larger than the resistor but smaller than the ADC integrated circuit in size, the resulting resistor value is 15.9 kΩ.