14.4 Transfer Function

In Chapter 12 we introduced the concept of network or transfer function. It is essentially nothing more than the ratio of some output variable to some input variable. If both variables are voltages, the transfer function is a voltage gain. If both variables are currents, the transfer function is a current gain. If one variable is a voltage and the other is a current, the transfer function becomes a transfer admittance or impedance.

In deriving a transfer function, all initial conditions are set equal to zero. In addition, if the output is generated by more than one input source in a network, superposition can be employed in conjunction with the transfer function for each source.

To present this concept in a more formal manner, let us assume that the input/output relationship for a linear circuit is

$$b_n \frac{d^n y_o(t)}{dt^n} + b_{n-1} \frac{d^{n-1} y_o(t)}{dt^{n-1}} + \cdots + b_1 \frac{dy_o(t)}{dt} + b_0 y_o(t)$$
$$= a_m \frac{d^m x_i(t)}{dt^m} + a_{m-1} \frac{d^{m-1} x_i(t)}{dt^{m-1}} + \cdots + a_1 \frac{dx_i(t)}{dt} + a_0 x_i(t)$$

If all the initial conditions are zero, the transform of the equation is

$$(b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0)\mathbf{Y}_o(s) = (a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0)\mathbf{X}_i(s)$$

or

$$\frac{\mathbf{Y}_o(s)}{\mathbf{X}_i(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

This ratio of $\mathbf{Y}_o(s)$ to $\mathbf{X}_i(s)$ is called the *transfer* or *network function*, which we denote as $\mathbf{H}(s)$; that is,

$$\frac{\mathbf{Y}_o(s)}{\mathbf{X}_i(s)} = \mathbf{H}(s)$$

or

$$\mathbf{Y}_o(s) = \mathbf{H}(s)\mathbf{X}_i(s) \qquad \textbf{14.16}$$

This equation states that the output response $\mathbf{Y}_o(s)$ is equal to the network function multiplied by the input $\mathbf{X}_i(s)$. Note that if $x_i(t) = \delta(t)$ and therefore $\mathbf{X}_i(s) = 1$, the impulse response is equal to the inverse Laplace transform of the network function. This is an extremely important concept because it illustrates that if we know the impulse response of a network, we can find the response due to some other forcing function using Eq. (14.16).

At this point, it is informative to review briefly the natural response of both first-order and second-order networks. We demonstrated in Chapter 7 that if only a single storage element is present, the natural response of a network to an initial condition is always of the form

$$x(t) = X_0 e^{-t/\tau}$$

where $x(t)$ can be either $v(t)$ or $i(t)$, X_0 is the initial value of $x(t)$, and τ is the time constant of the network. We also found that the natural response of a second-order network is controlled by the roots of the *characteristic equation*, which is of the form

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

where ζ is the *damping ratio* and ω_0 is the *undamped natural frequency*. These two key factors, ζ and ω_0, control the response, and there are basically three cases of interest, illustrated in Fig. 14.8.

CASE 1, $\zeta > 1$: OVERDAMPED NETWORK The roots of the characteristic equation are $s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$ and, therefore, the network response is of the form

$$x(t) = K_1 e^{-\left(\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1}\right)t} + K_2 e^{-\left(\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1}\right)t}$$

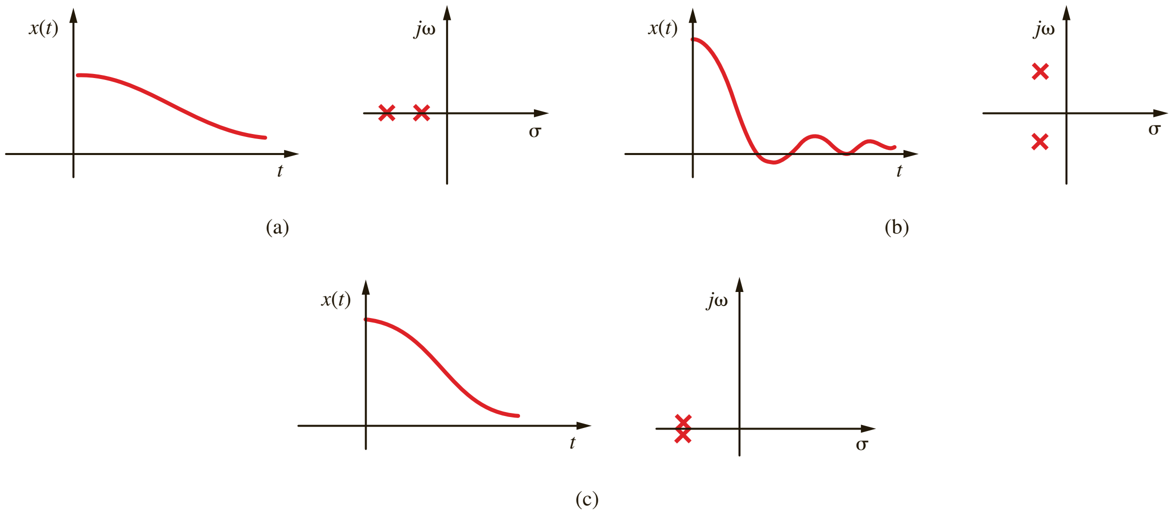


Figure 14.8

Natural response of a second-order network together with network pole locations for the three cases: (a) overdamped, (b) underdamped, and (c) critically damped.

CASE 2, $\zeta < 1$: UNDERDAMPED NETWORK The roots of the characteristic equation are $s_1, s_2 = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$ and, therefore, the network response is of the form

$$x(t) = Ke^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t + \phi\right)$$

CASE 3, $\zeta = 1$: CRITICALLY DAMPED NETWORK The roots of the characteristic equation are $s_1, s_2 = -\omega_0$ and, hence, the response is of the form

$$x(t) = K_1 te^{-\omega_0 t} + K_2 e^{-\omega_0 t}$$

The reader should note that the characteristic equation is the denominator of the transfer function $\mathbf{H}(s)$, and the roots of this equation, which are the poles of the network, determine the form of the network's natural response.

A convenient method for displaying the network's poles and zeros in graphical form is the use of a pole-zero plot. A pole-zero plot of a function can be accomplished using what is commonly called the *complex* or *s-plane*. In the complex plane the abscissa is σ and the ordinate is $j\omega$. Zeros are represented by 0's, and poles are represented by $\times$'s. Although we are concerned only with the finite poles and zeros specified by the network or response function, we should point out that a rational function must have the same number of poles and zeros. Therefore, if $n > m$, there are $n - m$ zeros at the point at infinity, and if $n < m$, there are $m - n$ poles at the point at infinity. A systems engineer can tell a lot about the operation of a network or system by simply examining its pole-zero plot.

Note in Fig. 14.8 that if the network poles are real and unequal, the response is slow and, therefore, $x(t)$ takes a long time to reach zero. If the network poles are complex conjugates, the response is fast; however, it overshoots and is eventually damped out. The dividing line between the overdamped and underdamped cases is the critically damped case in which the roots are real and equal. In this case, the transient response dies out as quickly as possible, with no overshoot.

EXAMPLE 14.6

If the impulse response of a network is $h(t) = e^{-t}$, let us determine the response $v_o(t)$ to an input $v_i(t) = 10e^{-2t}u(t)$ V.

SOLUTION

The transformed variables are

$$\mathbf{H}(s) = \frac{1}{s + 1}$$

$$\mathbf{V}_i(s) = \frac{10}{s + 2}$$

Therefore,

$$\mathbf{V}_o(s) = \mathbf{H}(s)\mathbf{V}_i(s)$$
$$= \frac{10}{(s + 1)(s + 2)}$$

and hence,

$$v_o(t) = 10(e^{-t} - e^{-2t})u(t) \text{ V}$$

The transfer function is important because it provides the systems engineer with a great deal of knowledge about the system's operation, since its dynamic properties are governed by the system poles.

EXAMPLE 14.7

Let us derive the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$ for the network in Fig. 14.9a.

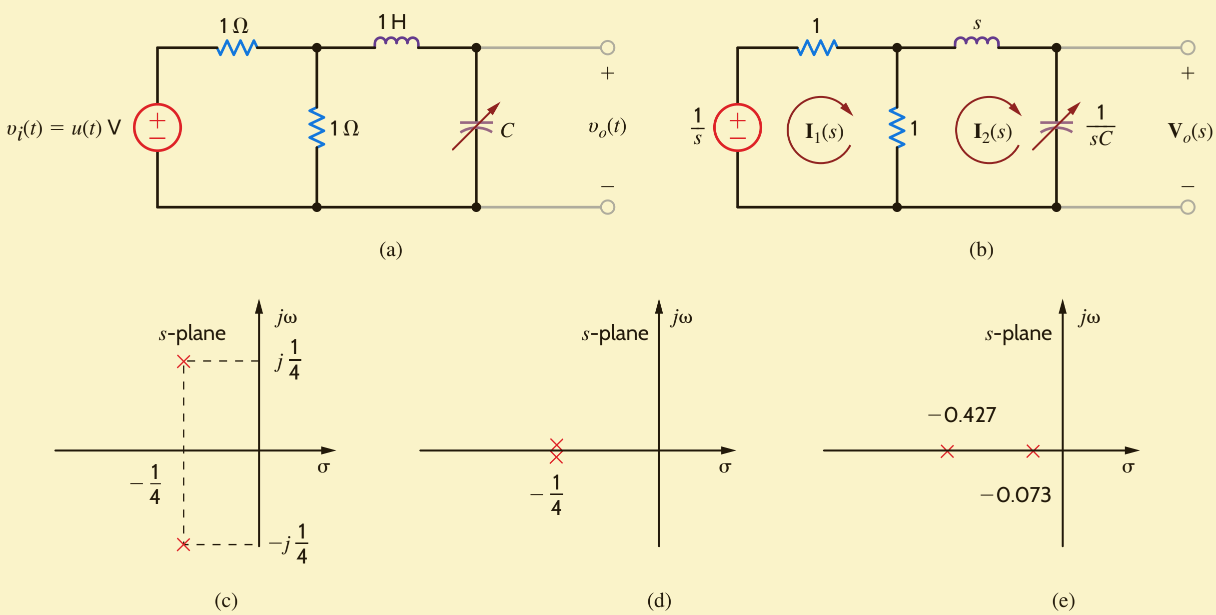


Figure 14.9

Networks and pole-zero plots used in Example 14.7.

Our output variable is the voltage across a variable capacitor, and the input voltage is a unit step. The transformed network is shown in Fig. 14.9b. The mesh equations for the network are

$$2\mathbf{I}_1(s) - \mathbf{I}_2(s) = \mathbf{V}_i(s)$$
$$-\mathbf{I}_1(s) + \left(s + \frac{1}{sC} + 1\right)\mathbf{I}_2(s) = 0$$

and the output equation is

$$\mathbf{V}_o(s) = \frac{1}{sC}\mathbf{I}_2(s)$$

From these equations we find that the transfer function is

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} = \frac{1/2C}{s^2 + \frac{1}{2}s + 1/C}$$

Since the transfer function is dependent on the value of the capacitor, let us examine the transfer function and the output response for three values of the capacitor.

a. $C = 8$ F

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} = \frac{\frac{1}{16}}{\left(s^2 + \frac{1}{2}s + \frac{1}{8}\right)} = \frac{\frac{1}{16}}{\left(s + \frac{1}{4} - j\frac{1}{4}\right)\left(s + \frac{1}{4} + j\frac{1}{4}\right)}$$

The output response is

$$\mathbf{V}_o(s) = \frac{\frac{1}{16}}{s\left(s + \frac{1}{4} - j\frac{1}{4}\right)\left(s + \frac{1}{4} + j\frac{1}{4}\right)}$$

As illustrated in Chapter 7, the poles of the transfer function, which are the roots of the characteristic equation, are complex conjugates, as shown in Fig. 14.9c; therefore, the output response will be *underdamped*. The output response as a function of time is

$$v_o(t) = \left[\frac{1}{2} + \frac{1}{\sqrt{2}}e^{-t/4}\cos\left(\frac{t}{4} + 135°\right)\right]u(t)\text{ V}$$

Note that for large values of time the transient oscillations, represented by the second term in the response, become negligible and the output settles out to a value of 1/2 V. This can also be seen directly from the circuit since for large values of time the input looks like a dc source, the inductor acts like a short circuit, the capacitor acts like an open circuit, and the resistors form a voltage divider.

b. $C = 16$ F

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} = \frac{\frac{1}{32}}{s^2 + \frac{1}{2}s + \frac{1}{16}} = \frac{\frac{1}{32}}{\left(s + \frac{1}{4}\right)^2}$$

The output response is

$$\mathbf{V}_o(s) = \frac{\frac{1}{32}}{s\left(s + \frac{1}{4}\right)^2}$$

Since the poles of the transfer function are real and equal as shown in Fig. 14.9d, the output response will be *critically damped*. $v_o(t) = \mathscr{L}^{-1}[\mathbf{V}_o(s)]$ is

$$v_o(t) = \left[\frac{1}{2} - \left(\frac{t}{8} + \frac{1}{2}\right)e^{-t/4}\right]u(t)\text{ V}$$

c. $C = 32$ F

$$\frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} = \frac{\frac{1}{64}}{s^2 + \frac{1}{2}s + \frac{1}{32}} = \frac{\frac{1}{64}}{(s + 0.427)(s + 0.073)}$$

The output response is

$$\mathbf{V}_o(s) = \frac{\frac{1}{64}}{s(s + 0.427)(s + 0.073)}$$

The poles of the transfer function are real and unequal, as shown in Fig. 14.9e and, therefore, the output response will be *overdamped*. The response as a function of time is

$$v_o(t) = \left(0.5 + 0.103e^{-0.427t} - 0.603e^{-0.073t}\right)u(t) \text{ V}$$

Although the values selected for the network parameters are not very practical, remember that both magnitude and frequency scaling, as outlined in Chapter 12, can be applied here also.

LEARNING ASSESSMENTS

E14.12 If the unit impulse response of a network is known to be $10/9\left(e^{-t} - e^{-10t}\right)$, determine the unit step response.

ANSWER:

$x(t) = \left(1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}\right)u(t).$

E14.13 The transfer function for a network is

$$\mathbf{H}(s) = \frac{s + 10}{s^2 + 4s + 8}$$

Determine the pole-zero plot of $\mathbf{H}(s)$, the type of damping exhibited by the network, and the unit step response of the network.

ANSWER:

The network is underdamped;

$x(t) = \left[\frac{10}{8} + 1.46e^{-2t}\cos\left(2t - 210.96°\right)\right]u(t).$

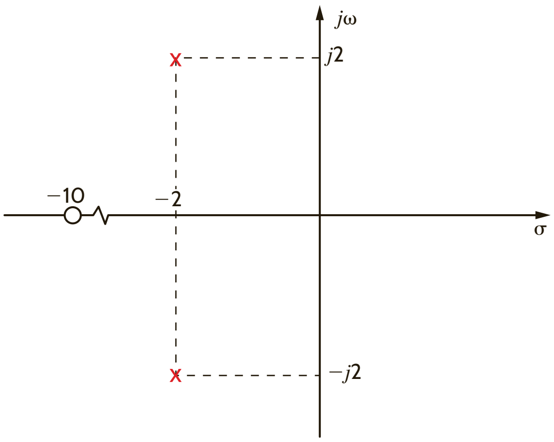


Figure E14.13

EXAMPLE 14.8

The circuit in Fig. 14.10 is an existing low-pass filter. On installation, we find that its output exhibits too much oscillation when responding to pulses. We wish to alter the filter in order to make it critically damped.

SOLUTION

First, we must determine the existing transfer function, $\mathbf{H}(s)$:

$$\mathbf{H}(s) = \frac{\mathbf{V}_O}{\mathbf{V}_S} = \frac{\dfrac{R}{1 + sRC}}{\dfrac{R}{1 + sRC} + sL} = \frac{\dfrac{1}{LC}}{s^2 + \dfrac{s}{RC} + \dfrac{1}{LC}} \qquad \textbf{14.17}$$

where the term $\dfrac{R}{1 + sRC}$ is just the parallel combination of the resistor and capacitor. Given our component values, the transfer function is

$$\mathbf{H}(s) = \frac{10^{10}}{s^2 + (5 \times 10^4)s + 10^{10}} \qquad \textbf{14.18}$$

and the resonant frequency and damping ratio are

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^5 \text{ rad/s} \quad \text{and} \quad 2\zeta\omega_0 = \frac{1}{RC} \Rightarrow \zeta = \frac{5 \times 10^4}{2\omega_0} = \frac{5 \times 10^4}{2 \times 10^5} = 0.25 \qquad \textbf{14.19}$$

The network is indeed underdamped. From Eq. (14.19), we find that raising the damping ratio by a factor of 4 to 1.0 requires that R be lowered by the same factor of 4 to 5 Ω. This can be done by adding a resistor, R_X, in parallel with R as shown in Fig. 14.11. The required resistor value can be obtained by solving Eq. (14.20) for R_X:

$$R_{eq} = 5 = \frac{RR_X}{R + R_X} = \frac{20R_X}{20 + R_X} \qquad \textbf{14.20}$$

The solution is $R_X = 6.67$ Ω.

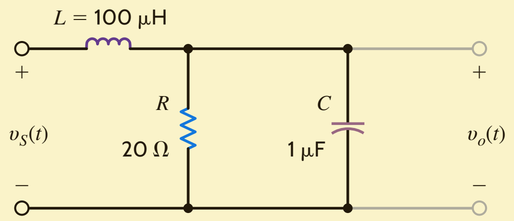


Figure 14.10 A second-order low-pass filter.

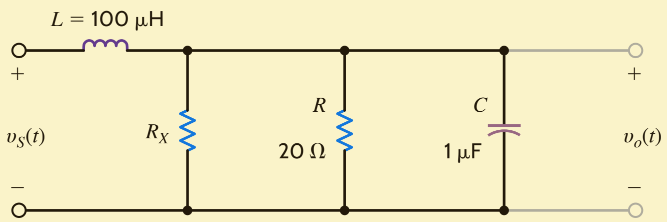


Figure 14.11 The addition of a resistor to change the damping ratio of the network.

EXAMPLE 14.9

The Recording Industry Association of America (RIAA) uses standardized recording and playback filters to improve the quality of phonographic disk recordings. This process is demonstrated in Fig. 14.12. During a recording session, the voice or music signal is passed through the recording filter, which de-emphasizes the bass content. This filtered signal is then recorded into the vinyl. On playback, the phonograph needle assembly senses the recorded message and reproduces the filtered signal, which proceeds to the playback filter. The purpose of the playback filter is to emphasize the bass content and reconstruct the original voice/music signal. Next, the reconstructed signal can be amplified and sent on to the speakers.

Let us examine the pole-zero diagrams for the record and playback filters.

SOLUTION

The transfer function for the recording filter is

$$\mathbf{G}_{vR}(s) = \frac{K(1 + s\tau_{z1})(1 + s\tau_{z2})}{1 + s\tau_p}$$

where the time constants are $\tau_{z1} = 75\ \mu\text{s}$, $\tau_{z2} = 3180\ \mu\text{s}$, and $\tau_p = 318\ \mu\text{s}$; K is a constant chosen such that $\mathbf{G}_{vR}(s)$ has a magnitude of 1 at 1000 Hz. The resulting pole and zero frequencies in radians/second are

$$\omega_{z1} = 1/\tau_{z1} = 13.33 \text{ krad/s}$$
$$\omega_{z2} = 1/\tau_{z2} = 313.46 \text{ rad/s}$$
$$\omega_p = 1/\tau_p = 3.14 \text{ krad/s}$$

Fig. 14.13a shows the pole-zero diagram for the recording filter.

The playback filter transfer function is the reciprocal of the record transfer function.

$$\mathbf{G}_{vp}(s) = \frac{1}{\mathbf{G}_{vR}(s)} = \frac{A_o(1 + s\tau_z)}{(1 + s\tau_{p1})(1 + s\tau_{p2})}$$

where the time constants are now $\tau_{p1} = 75\ \mu\text{s}$, $\tau_{p2} = 3180\ \mu\text{s}$, $\tau_z = 318\ \mu\text{s}$, and A_o is $1/K$. Pole and zero frequencies, in radians/second, are

$$\omega_{p1} = 1/\tau_{z1} = 13.33 \text{ krad/s}$$
$$\omega_{p2} = 1/\tau_{z2} = 313.46 \text{ rad/s}$$
$$\omega_z = 1/\tau_p = 3.14 \text{ krad/s}$$

which yields the pole-zero diagram in Fig. 14.13b. The voice/music signal eventually passes through both filters before proceeding to the amplifier. In the s-domain, this is equivalent to multiplying $\mathbf{V}_s(s)$ by both $\mathbf{G}_{vR}(s)$ and $\mathbf{G}_{vp}(s)$. In the pole-zero diagram, we simply superimpose the pole-zero diagrams of the two filters, as shown in Fig. 14.13c. Note that at each pole frequency there is a zero and vice versa. The pole-zero pairs cancel one another, yielding a pole-zero diagram that contains no poles and no zeros. This effect can be seen mathematically by multiplying the two transfer functions, $\mathbf{G}_{vR}(s)\mathbf{G}_{vp}(s)$, which yields a product independent of s. Thus, the original voice/music signal is reconstructed and fidelity is preserved.

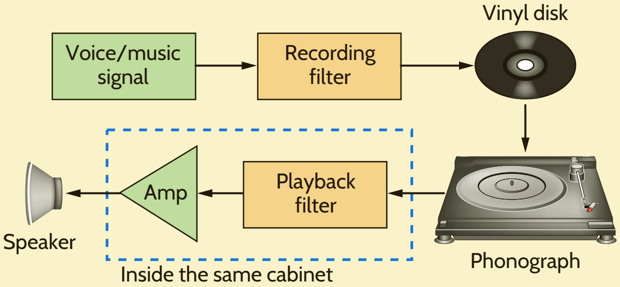


Figure 14.12
Block diagram for phonograph disk recording and playback.

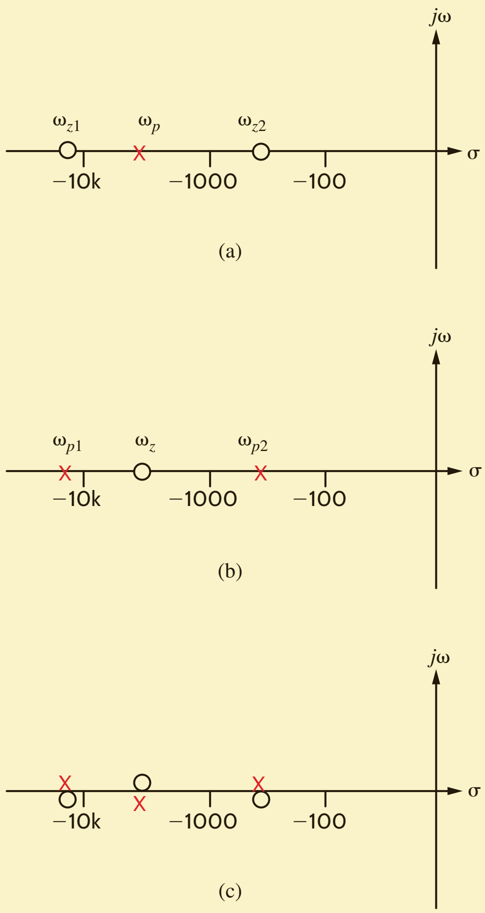


Figure 14.13

Pole-zero diagrams for RIAA phonographic filters.

EXAMPLE 14.10

In a large computer network, two computers are transferring digital data on a single wire at a rate of 1000 bits/s. The voltage waveform, v_{data}, in **Fig. 14.14** shows a possible sequence of bits alternating between "high" and "low" values. Also present in the environment is a source of 100 kHz (628 krad/s) noise, which is corrupting the data.

It is necessary to filter out the high-frequency noise without destroying the data waveform. Let us place the second-order low-pass active filter of **Fig. 14.15** in the data path so that the data and noise signals will pass through it.

SOLUTION

The filter's transfer function is found to be

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_{\text{data}}(s)} = \frac{-\left(\dfrac{R_3}{R_1}\right)\left(\dfrac{1}{R_2R_3C_1C_2}\right)}{s^2 + s\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1} + \dfrac{1}{R_3C_1}\right) + \dfrac{1}{R_2R_3C_1C_2}}$$

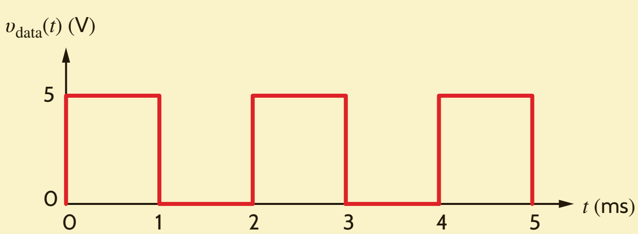


Figure 14.14

1000 bits/s digital data waveform.

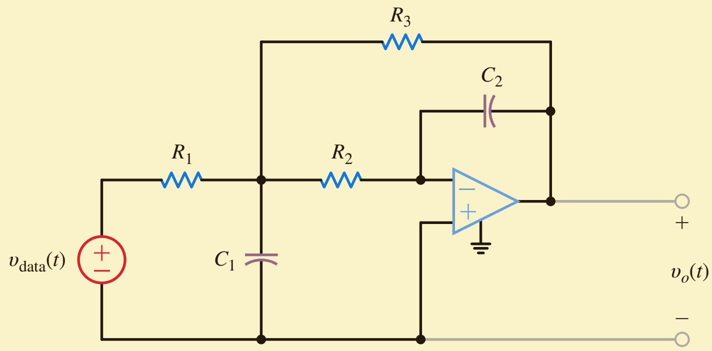


Figure 14.15
Second-order low-pass filter.

To simplify our work, let $R_1 = R_2 = R_3 = R$. From our work in Chapter 12, we know that the characteristic equation of a second-order system can be expressed as

$$s^2 + 2s\zeta\omega_0 + \omega_0^2 = 0$$

Comparing the two preceding equations, we find that

$$\omega_0 = \frac{1}{R\sqrt{C_1C_2}}$$

$$2\zeta\omega_0 = \frac{3}{RC_1}$$

and therefore,

$$\zeta = \frac{3}{2}\sqrt{\frac{C_2}{C_1}}$$

The poles of the filter are at

$$s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

To eliminate the 100-kHz noise, at least one pole should be well below 100 kHz, as shown in the Bode plot sketched in **Fig. 14.16**. By placing a pole well below 100 kHz, the gain of the filter will be quite small at 100 kHz, effectively filtering the noise.

If we arbitrarily choose an overdamped system with $\omega_0 = 25$ krad/s and $\zeta = 2$, the resulting filter is overdamped with poles at $s_1 = -6.7$ krad/s and $s_2 = -93.3$ krad/s. The pole-zero diagram for the filter is shown in **Fig. 14.17**.

If we let $R = 40\ \text{k}\Omega$, then we may write

$$\omega_0 = 25{,}000 = \frac{1}{40{,}000\sqrt{C_1C_2}}$$

or

$$C_1C_2 = 10^{-18}$$

Also,

$$\zeta = 2 = \frac{3}{2}\sqrt{\frac{C_2}{C_1}}$$

Figure 14.16
Bode plot sketch for a second-order low-pass filter.

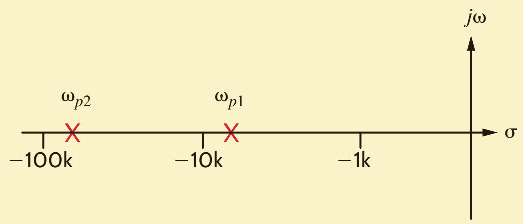


Figure 14.17
Pole-zero diagram for low-pass filter.

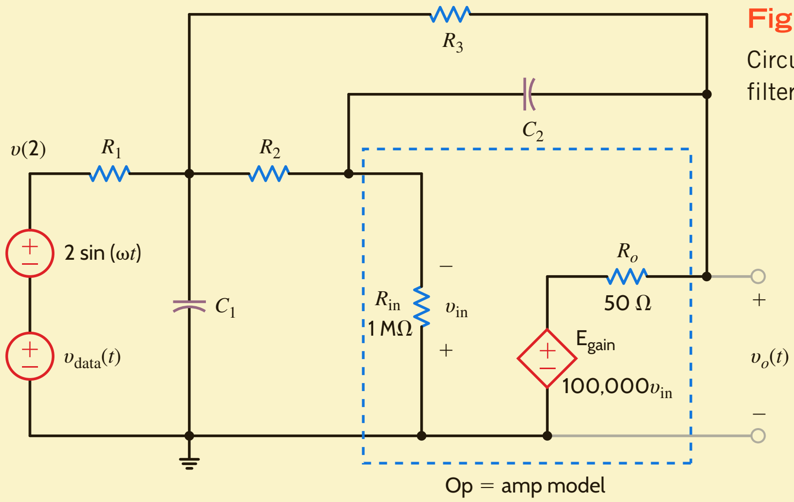


Figure 14.18
Circuit for second-order filter.

which can be expressed as

$$\frac{C_2}{C_1} = \frac{16}{9}$$

Solving for C_1 and C_2 yields

$$C_1 = 0.75 \text{ nF}$$
$$C_2 = 1.33 \text{ nF}$$

The circuit used to simulate the filter is shown in **Fig. 14.18**. The sinusoidal source has a frequency of 100 kHz and is used to represent the noise source.

Plots for the input to the filter and the output voltage for 2 ms are shown in **Fig. 14.19**. Note that output indeed contains much less of the 100-kHz noise. Also, the fast rise and fall times of the data signal are slower in the output voltage. Despite this slower response, the output voltage is fast enough to keep pace with the 1000-bits/s transfer rate.

Let us now increase the data transfer rate from 1000 to 25,000 bits/s, as shown in **Fig. 14.20**. The total input and output signals are plotted in **Fig. 14.21** for 200 μs. Now the output cannot keep pace with the input, and the data information is lost. Let us investigate why this occurs. We know that the filter is second order with poles at s_1 and s_2. If we represent the data input as a 5-V step function, the output voltage is

$$\mathbf{V}_o(s) = \mathbf{G}_v(s)\left(\frac{5}{s}\right) = \frac{K}{(s + s_1)(s + s_2)}\left(\frac{5}{s}\right)$$

where K is a constant. Since the filter is overdamped, s_1 and s_2 are real and positive. A partial fraction expansion of $\mathbf{V}_o(s)$ is of the form

$$\mathbf{V}_o(s) = \frac{K_1}{s} + \frac{K_2}{(s + s_1)} + \frac{K_3}{(s + s_2)}$$

yielding the time-domain expression

$$v_o(t) = \left[K_1 + K_2 e^{-s_1 t} + K_3 e^{-s_2 t}\right] u(t) \text{ V}$$

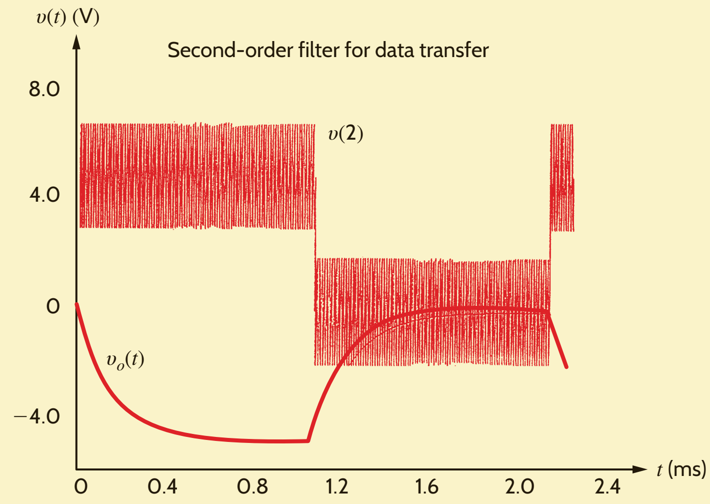


Figure 14.19
Simulation output for node 2 and $v_o(t)$.

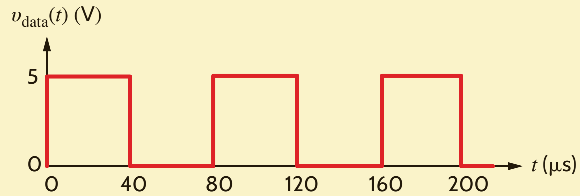


Figure 14.20
25,000-bits/s digital data waveform.

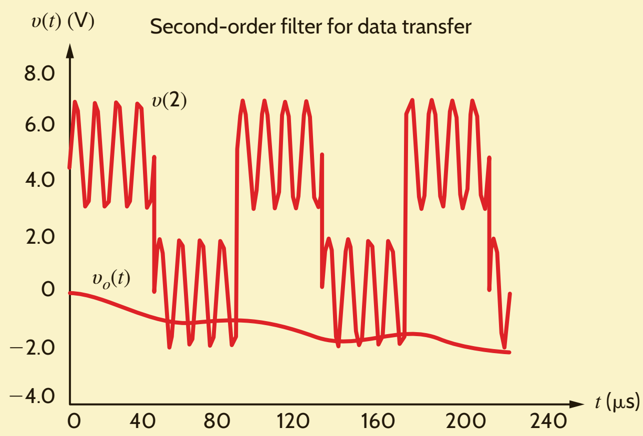


Figure 14.21
Simulation output for node 2 and $v_o(t)$ with 25,000-bits/s data transfer rate.

where K_1, K_2, and K_3 are real constants. The exponential time constants are the reciprocals of the pole frequencies.

$$\tau_1 = \frac{1}{s_1} = \frac{1}{6.7\text{k}} = 149\ \mu\text{s}$$
$$\tau_2 = \frac{1}{s_2} = \frac{1}{93.3\text{k}} = 10.7\ \mu\text{s}$$

Since exponentials reach steady state in roughly 5τ, the exponential associated with τ_2 affects the output for about 50 μs and the τ_1 exponential will reach steady state after about 750 μs. From Fig. 14.20 we see that at a 25,000-bits/s data transfer rate, each bit (a "high" or "low" voltage value) occupies a 40-μs time span. Therefore, the exponential associated with s_1, and thus $v_o(t)$, is still far from its steady-state condition when the next bit is transmitted. In short, s_1 is too small.

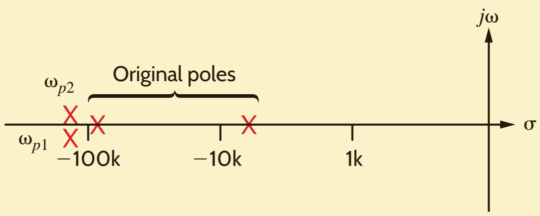


Figure 14.22
Pole-zero diagram for both original and critically damped systems.

Let us remedy this situation by increasing the pole frequencies and changing to a critically damped system, $\zeta = 1$. If we select $\omega_0 = 125$ krad/s, the poles will be at $s_1 = s_2 = -125$ krad/s or 19.9 kHz—both below the 100-kHz noise we wish to filter out. Fig. 14.22 shows the new pole positions moved to the left of their earlier positions, which we expect will result in a quicker response to the v_{data} pulse train.

Now the expressions for ω_0 and ζ are

$$\omega_0 = 125{,}000 = \frac{1}{40{,}000\sqrt{C_1 C_2}}$$

or

$$C_1 C_2 = 4 \times 10^{-20}$$

Also,

$$\zeta = 1 = \frac{3}{2}\sqrt{\frac{C_2}{C_1}}$$

which can be expressed as

$$\frac{C_2}{C_1} = \frac{4}{9}$$

Solving for C_1 and C_2 yields

$$C_1 = 300 \text{ pF}$$
$$C_2 = 133.3 \text{ pF}$$

A simulation using these new capacitor values produces the input–output data shown in Fig. 14.23. Now the output voltage just reaches the "high" and "low" levels just before v_{data} makes its next transition and the 100-kHz noise is still much reduced.

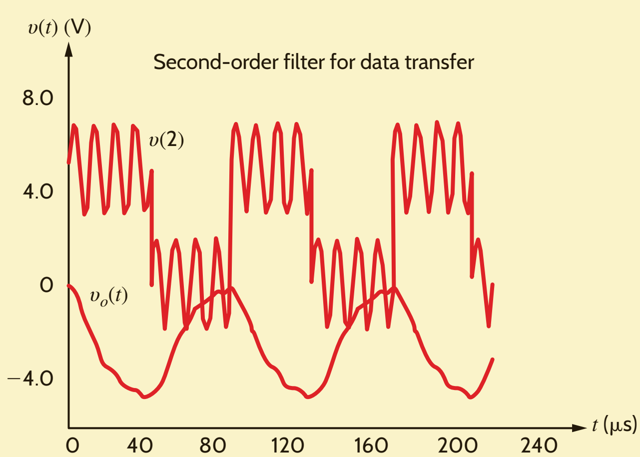


Figure 14.23
Simulation outputs for node 2 and $v_o(t)$ for the critically damped system.

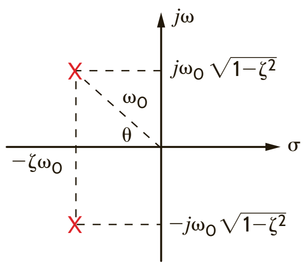


Figure 14.24

Pole locations for a second-order underdamped network.

Recall from our previous discussion that if a second-order network is underdamped, the characteristic equation of the network is of the form

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

and the roots of this equation, which are the network poles, are of the form

$$s_1, s_2 = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$$

The roots s_1 and s_2, when plotted in the s-plane, generally appear as shown in **Fig. 14.24**, where

$$\zeta = \text{damping ratio}$$
$$\omega_0 = \text{undamped natural frequency}$$

and as shown in Fig. 14.24,

$$\zeta = \cos\theta$$

The damping ratio and the undamped natural frequency are exactly the same quantities as those employed in Chapter 12 when determining a network's frequency response. We find that these same quantities govern the network's transient response.

EXAMPLE 14.11

Let us examine the effect of pole position in the s-plane on the transient response of the second-order *RLC* series network shown in **Fig. 14.25**.

SOLUTION

The voltage gain transfer function is

$$\mathbf{G}_v(s) = \frac{\dfrac{1}{LC}}{s^2 + s\left(\dfrac{R}{L}\right) + \dfrac{1}{LC}} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

For this analysis we will let $\omega_0 = 2000$ rad/s for $\zeta = 0.25, 0.50, 0.75$, and 1.0. From the preceding equation we see that

$$LC = \frac{1}{\omega_0^2} = 2.5 \times 10^{-7}$$

and

$$R = 2\zeta\sqrt{\frac{L}{C}}$$

If we arbitrarily let $L = 10$ mH, then $C = 25\ \mu$F. Also, for $\zeta = 0.25, 0.50, 0.75$, and 1.0, $R = 10\ \Omega$, $20\ \Omega$, $30\ \Omega$, and $40\ \Omega$, respectively. Over the range of ζ values, the network ranges from underdamped to critically damped. Since poles are complex for underdamped systems, the real and imaginary components and the magnitude of the poles of $\mathbf{G}_v(s)$ are given in Table 14.1 for the ζ values listed previously.

Fig. 14.26 shows the pole-zero diagrams for each value of ζ. Note first that all the poles lie on a circle; thus, the pole magnitudes are constant, consistent with Table 14.1. Second, as

Figure 14.25

RLC series network.

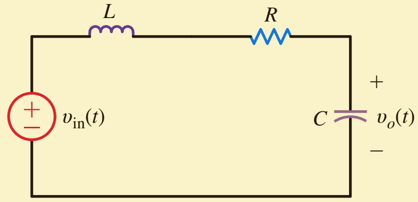


TABLE 14.1 Pole locations for $\zeta = 0.25$ to 1.0

DAMPING RATIO	REAL	IMAGINARY	MAGNITUDE
1.00	2000.0	0.0	2000.0
0.75	1500.0	1322.9	2000.0
0.50	1000.0	1732.1	2000.0
0.25	500.0	1936.5	2000.0

ζ decreases, the real part of the pole decreases while the imaginary part increases. In fact, when ζ goes to zero, the poles become imaginary.

A simulation of a unit step transient excitation for all four values of R is shown in Fig. 14.27. We see that as ζ decreases, the overshoot in the output voltage increases. Furthermore, when the network is critically damped ($\zeta = 1$), there is no overshoot at all. In most applications, excessive overshoot is not desired. To correct this, the damping ratio, ζ, should be increased, which for this circuit would require an increase in the resistor value.

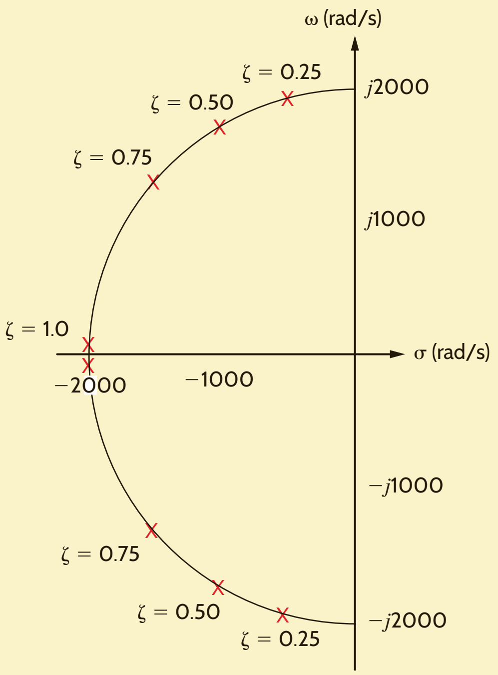


Figure 14.26

Pole-zero diagrams for $\zeta = 0.25$ to 1.0.

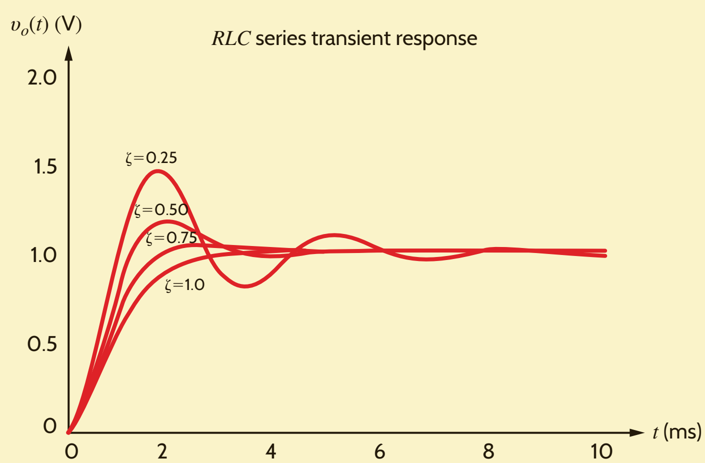


Figure 14.27

Transient response output for $\zeta = 0.25$ to 1.0.

EXAMPLE 14.12

Let us revisit the Tacoma Narrows Bridge disaster examined in Example 12.12. A photograph of the bridge as it collapsed is shown in Fig. 14.28.

In Chapter 12 we assumed that the bridge's demise was brought on by winds oscillating back and forth at a frequency near that of the bridge (0.2 Hz). We found that we could create an *RLC* circuit, shown in Fig. 12.30, that resonates at 0.2 Hz and has an output voltage consistent with the vertical deflection of the bridge. This kind of forced resonance never happened at Tacoma Narrows. The real culprit was not so much wind fluctuations but the bridge itself. This is thoroughly explained in the paper "Resonance, Tacoma Narrows Bridge

Figure 14.28

Tacoma Narrows Bridge as it collapsed on November 7, 1940 (AP Photo/nap).

Failure, and Undergraduate Physics Textbooks," by K. Y. Billah and R. H. Scalan published in the *American Journal of Physics*, vol. 59, no. 2 (1991), pp. 118–124, in which the authors determined that changes in wind speed affected the coefficients of the second-order differential equation that models the resonant behavior. In particular, the damping ratio, ζ, was dependent on the wind speed and is roughly given as

$$\zeta = 0.00460 - 0.00013U \qquad \textbf{14.21}$$

where U is the wind speed in mph. Note, as shown in **Fig. 14.29**, that ζ becomes negative at wind speeds in excess of 35 mph—a point we will demonstrate later. Furthermore, Billah and Scalan report that the bridge resonated in a twisting mode, which can be easily seen in Fig. 12.29 and is described by the differential equation

$$\frac{d^2\theta(t)}{dt^2} + 2\zeta\omega_0\frac{d\theta(t)}{dt} + \omega_0^2\theta(t) = 0$$

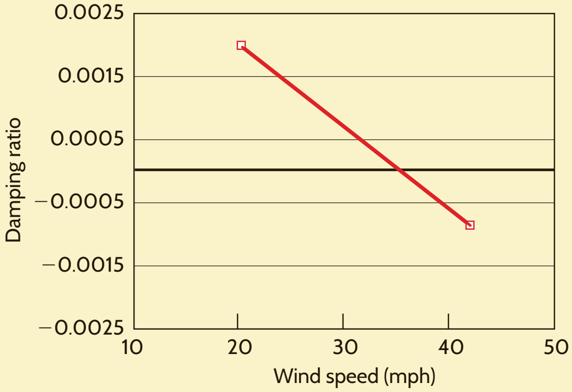
or

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + \omega_0^2\theta = 0 \qquad \textbf{14.22}$$

where $\theta(t)$ is the angle of twist in degrees and wind speed is implicit in ζ through Eq. (14.21). Billah and Scalan list the following data obtained either by direct observation at the bridge

Figure 14.29

Damping ratio versus wind speed for the second-order twisting model of the Tacoma Narrows Bridge.



site or through scale model experiments afterward:

$$\text{Wind speed at failure} \approx 42 \text{ mph}$$
$$\text{Twist at failure} \approx \pm 12°$$
$$\text{Time to failure} \approx 45 \text{ minutes}$$

We will start the twisting oscillations using an initial condition on $\theta(0)$ and see whether the bridge oscillations decrease or increase over time. Let us now design a network that will simulate the true Tacoma Narrows disaster.

SOLUTION

First, we solve for $\ddot{\theta}(t)$ in Eq. (14.22):

$$\ddot{\theta} = -2\zeta\omega_0\dot{\theta} - \omega_0^2\theta$$
$$\ddot{\theta} = -2(2\pi)(0.2)(0.0046 - 0.00013U)\dot{\theta} - [2(2\pi)(0.2)]^2\theta \qquad \textbf{14.23}$$

or

$$\ddot{\theta} = -(0.01156 - 0.00033U)\dot{\theta} - 1.579\theta$$

We now wish to model this equation to produce a voltage proportional to $\ddot{\theta}(t)$. We can accomplish this using the op-amp integrator circuit shown in **Fig. 14.30**.

The circuit's operation can perhaps be best understood by first assigning the voltage v_α to be proportional to $\ddot{\theta}(t)$, where 1 V represents 1 deg/s^2. Thus, the output of the first integrator, v_ω, must be

$$v_\omega = -\frac{1}{R_\omega C_\omega}\int v_\alpha\, dt$$

or, since $R_\omega = 1\ \Omega$ and $C_\omega = 1$ F,

$$v_\omega = -\int v_\alpha\, dt$$

So v_ω is proportional to $-\dot{\theta}(t)$ and 1 V equals -1 deg/s. Similarly, the output of the second integrator must be

$$v_\theta = -\int v_\omega\, dt$$

where $v_\theta(t)$ is proportional to $\theta(t)$ and 1 V equals 1 degree. The outputs of the integrators are then fed back as inputs to the summing op-amp. Note that the dependent sources, E_ω and E_{wind}, re-create the coefficient on $\dot{\theta}(t)$ in Eq. (14.21); that is,

$$2\zeta\omega_0 = (2)(0.2)(2\pi)\zeta = 0.01156 - 0.00033U$$

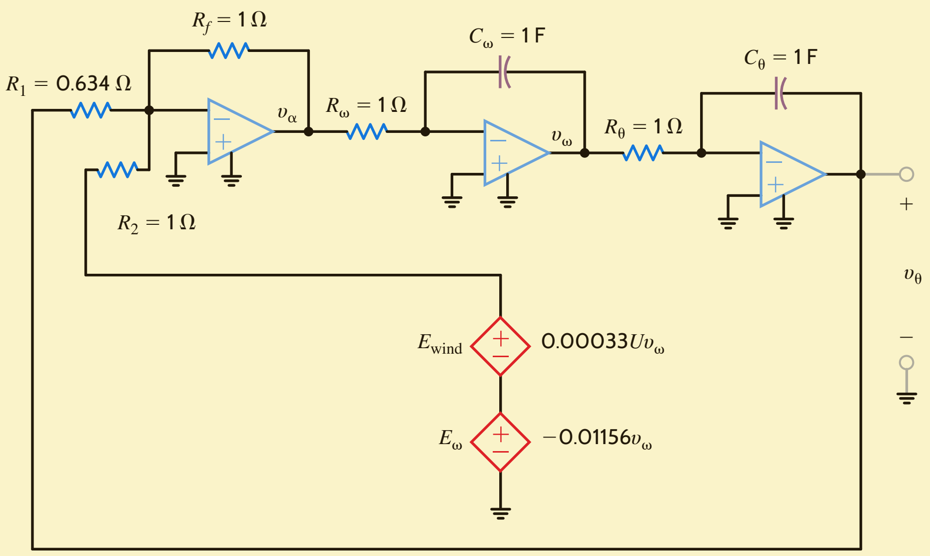


Figure 14.30 Circuit diagram for Tacoma Narrows Bridge simulations.

To simulate various wind speeds, we need only change the gain factor of E_{wind}. Finally, we can solve the circuit for $v_\alpha(t)$:

$$v_\alpha(t) = -\left(\frac{R_f}{R_2}\right)(E_\omega - E_{\text{wind}}) - \left(\frac{R_f}{R_1}\right)v_\theta$$

which matches Eq. (14.23) if

$$\frac{R_f}{R_1} = \omega_0^2 = [2\pi(0.2)]^2 = 1.579$$

and

$$\frac{R_f}{R_2}[E_\omega - E_{\text{wind}}] = 2\zeta\omega_0$$

or

$$\frac{R_f}{R_2} = 1$$

Thus, if $R_f = R_2 = 1\ \Omega$ and $R_1 = 0.634\ \Omega$, the circuit will simulate the bridge's twisting motion. We will start the twisting oscillations using an initial condition $\theta(0)$ and see whether the bridge oscillations decrease or increase over time.

The first simulation is for a wind speed of 20 mph and one degree of twist. The corresponding output voltage is shown in Fig. 14.31. The bridge twists at a frequency of 0.2 Hz and the oscillations decrease exponentially, indicating a nondestructive situation.

Fig. 14.32 shows the output for 35-mph winds and an initial twist of one degree. Notice that the oscillations neither increase nor decrease. This indicates that the damping ratio is zero.

Finally, the simulation at a wind speed of 42 mph and one degree initial twist is shown in Fig. 14.33. The twisting becomes worse and worse until after 45 minutes, the bridge is twisting ± 12.5 degrees, which matches values reported by Billah and Scalan for collapse.

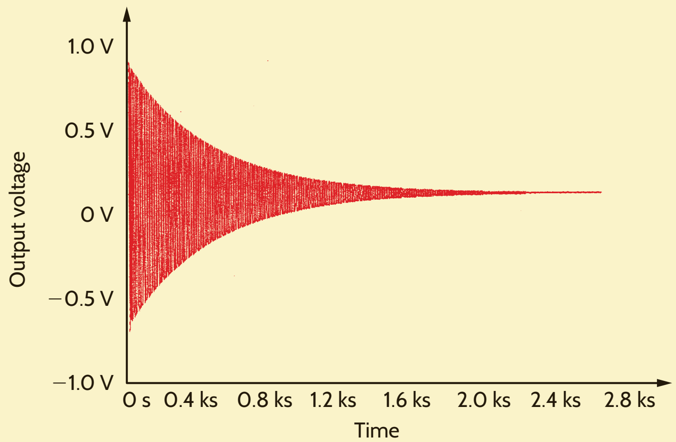


Figure 14.31

Tacoma Narrows Bridge simulation at 20-mph wind speed and one degree twist initial condition.

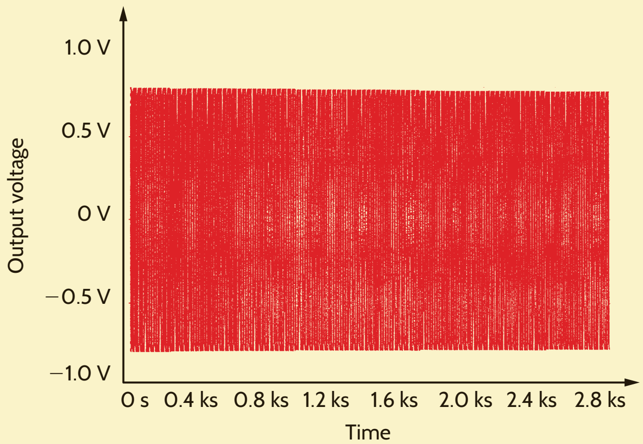


Figure 14.32

Tacoma Narrows Bridge simulation at 35-mph winds and one degree of initial twist.