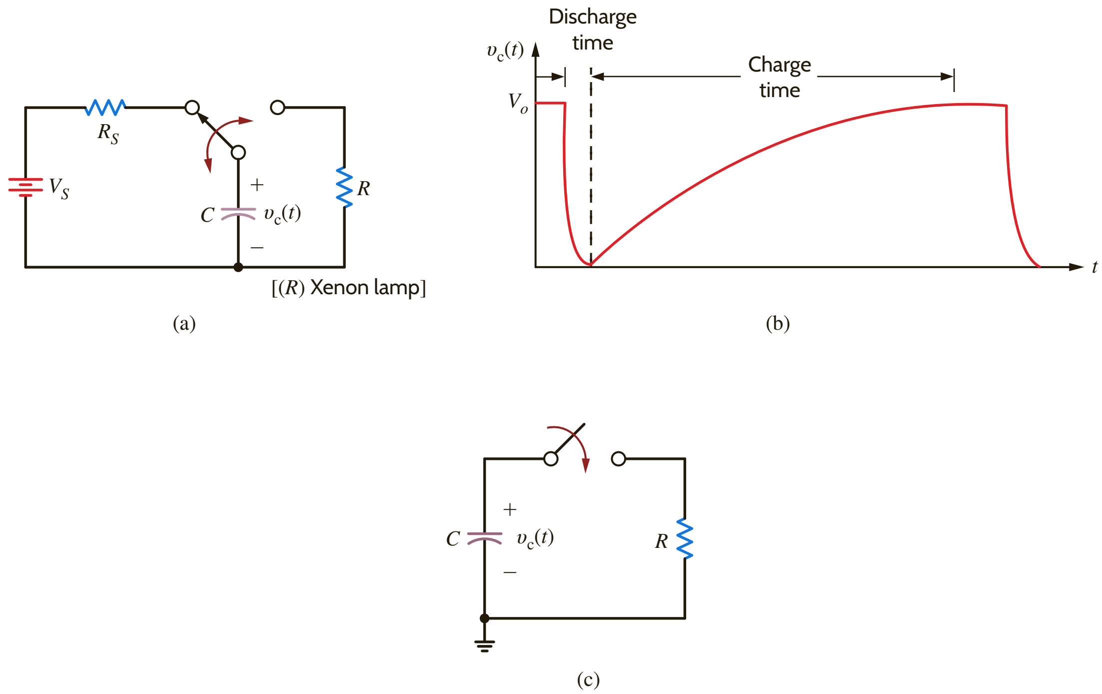

Figure 7.1

Diagrams used to describe a camera's flash circuit.

in **Fig. 7.1c** (that is, a charged capacitor that is discharged through a resistor). When the switch is closed, KCL for the circuit is

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0$$

or

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = 0$$

In the next section, we will demonstrate that the solution of this equation is

$$v_C(t) = V_o e^{-t/RC}$$

Note that this function is a decaying exponential and the rate at which it decays is a function of the values of R and C. The product RC is a very important parameter, and we will give it a special name in the following discussions.

7.2 First-Order Circuits

GENERAL FORM OF THE RESPONSE EQUATIONS In our study of first-order transient circuits we will show that the solution of these circuits (i.e., finding a voltage or current) requires us to solve a first-order differential equation of the form

$$\frac{dx(t)}{dt} + ax(t) = f(t) \tag{7.1}$$

Although a number of techniques may be used for solving an equation of this type, we will obtain a general solution that we will then employ in two different approaches to transient analysis.

A fundamental theorem of differential equations states that if $x(t) = x_p(t)$ is any solution to Eq. (7.1), and $x(t) = x_c(t)$ is any solution to the homogeneous equation

$$\frac{dx(t)}{dt} + ax(t) = 0 \tag{7.2}$$

then

$$x(t) = x_p(t) + x_c(t) \qquad 7.3$$

is a solution to the original Eq. (7.1). The term $x_p(t)$ is called the *particular integral solution*, or forced response, and $x_c(t)$ is called the *complementary solution*, or natural response.

At the present time we confine ourselves to the situation in which $f(t) = A$ (i.e., some constant). The general solution of the differential equation then consists of two parts that are obtained by solving the two equations

$$\frac{dx_p(t)}{dt} + ax_p(t) = A \qquad 7.4$$

$$\frac{dx_c(t)}{dt} + ax_c(t) = 0 \qquad 7.5$$

Since the right-hand side of Eq. (7.4) is a constant, it is reasonable to assume that the solution $x_p(t)$ must also be a constant. Therefore, we assume that

$$x_p(t) = K_1 \qquad 7.6$$

Substituting this constant into Eq. (7.4) yields

$$K_1 = \frac{A}{a} \qquad 7.7$$

Examining Eq. (7.5), we note that

$$\frac{dx_c(t)/dt}{x_c(t)} = -a \qquad 7.8$$

This equation is equivalent to

$$\frac{d}{dt}[\ln x_c(t)] = -a$$

Hence,

$$\ln x_c(t) = -at + c$$

and therefore,

$$x_c(t) = K_2 e^{-at} \qquad 7.9$$

Thus, a solution of Eq. (7.1) is

$$\begin{aligned} x(t) &= x_p(t) + x_c(t) \\ &= \frac{A}{a} + K_2 e^{-at} \qquad 7.10 \end{aligned}$$

The constant K_2 can be found if the value of the independent variable $x(t)$ is known at one instant of time.

Eq. (7.10) can be expressed in general in the form

$$x(t) = K_1 + K_2 e^{-t/\tau} \qquad 7.11$$

Once the solution in Eq. (7.11) is obtained, certain elements of the equation are given names that are commonly employed in electrical engineering. For example, the term K_1 is referred to as the *steady-state solution*: the value of the variable $x(t)$ as $t \rightarrow \infty$ when the second term becomes negligible. The constant τ is called the *time constant* of the circuit. Note that the second term in Eq. (7.11) is a decaying exponential that has a value, if $\tau > 0$, of K_2 for $t = 0$ and a value of 0 for $t = \infty$. The rate at which this exponential decays is determined by the time constant τ. A graphical picture of this effect is shown in **Fig. 7.2a**. As can be seen from the figure, the value of $x_c(t)$ has fallen from K_2 to a value of $0.368K_2$ in one time constant, a drop of 63.2%. In two time constants the value of $x_c(t)$ has fallen to $0.135K_2$, a drop of 63.2% from the value at time $t = \tau$. This means that the gap between a point on the curve

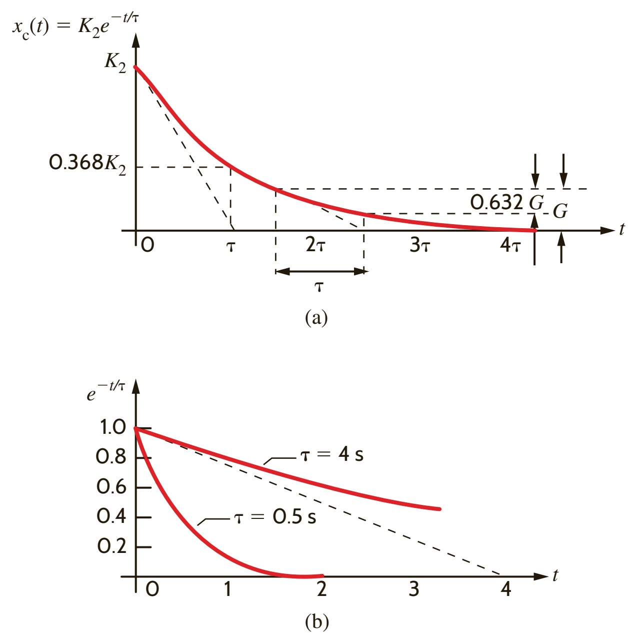

Figure 7.2

Time-constant illustrations.

and the final value of the curve is closed by 63.2% each time constant. Finally, after five time constants, $x_c(t) = 0.0067K_2$, which is less than 1%.

An interesting property of the exponential function shown in Fig. 7.2a is that the initial slope of the curve intersects the time axis at a value of $t = \tau$. In fact, we can take any point on the curve, not just the initial value, and find the time constant by finding the time required to close the gap by 63.2%. Finally, the difference between a small time constant (i.e., fast response) and a large time constant (i.e., slow response) is shown in **Fig. 7.2b**. These curves indicate that if the circuit has a small time constant, it settles down quickly to a steady-state value. Conversely, if the time constant is large, more time is required for the circuit to settle down or reach steady state. In any case, note that the circuit response essentially reaches steady state within five time constants (i.e., 5τ).

Note that the previous discussion has been very general in that no particular form of the circuit has been assumed—except that it results in a first-order differential equation.

ANALYSIS TECHNIQUES: DIFFERENTIAL EQUATIONS Eq. (7.11) defines the general form of the solution of first-order transient circuits; that is, it represents the solution of the differential equation that describes an unknown current or voltage *anywhere in the network*. One of the ways that we can arrive at this solution is to solve the equations that describe the network behavior using what is often called the *state-variable approach*. In this technique we write the equation for the voltage across the capacitor and/or the equation for the current through the inductor. Recall from Chapter 6 that these quantities cannot change instantaneously. Let us first illustrate this technique in the general sense and then examine two specific examples.

Consider the circuit shown in **Fig. 7.3a**. At time $t = 0$, the switch closes. The KCL equation that describes the capacitor voltage for time $t > 0$ is

$$C\frac{dv(t)}{dt} + \frac{v(t) - V_S}{R} = 0$$

or

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_S}{RC}$$

From our previous development, we assume that the solution of this first-order differential equation is of the form

$$v(t) = K_1 + K_2e^{-t/\tau}$$

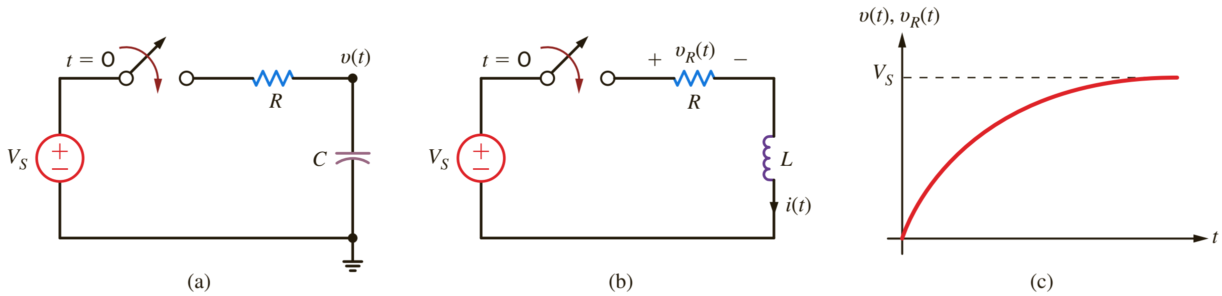

Figure 7.3
(a) *RC* circuit, (b) *RL* circuit, (c) plot of the capacitor voltage in (a) and resistor voltage in (b).

Substituting this solution into the differential equation yields

$$-\frac{K_2}{\tau}e^{-t/\tau}+\frac{K_1}{RC}+\frac{K_2}{RC}e^{-t/\tau}=\frac{V_S}{RC}$$

Equating the constant and exponential terms, we obtain

$$K_1 = V_S$$
$$\tau = RC$$

Therefore,

$$v(t) = V_S + K_2e^{-t/RC}$$

where V_S is the steady-state value and RC is the network's time constant. K_2 is determined by the initial condition of the capacitor. For example, if the capacitor is initially uncharged (that is, the voltage across the capacitor is zero at $t = 0$), then

$$0 = V_S + K_2$$

or

$$K_2 = -V_S$$

Hence, the complete solution for the voltage $v(t)$ is

$$v(t) = V_S - V_S e^{-t/RC}$$

The circuit in **Fig. 7.3b** can be examined in a similar manner. The KVL equation that describes the inductor current for $t > 0$ is

$$L\frac{di(t)}{dt} + Ri(t) = V_S$$

A development identical to that just used yields

$$i(t) = \frac{V_S}{R} + K_2e^{-\left(\frac{R}{L}\right)t}$$

where V_S/R is the steady-state value and L/R is the circuit's time constant. If there is no initial current in the inductor, then at $t = 0$

$$0 = \frac{V_S}{R} + K_2$$

and

$$K_2 = \frac{-V_S}{R}$$

Hence,

$$i(t) = \frac{V_S}{R} - \frac{V_S}{R}e^{-\frac{R}{L}t}$$

is the complete solution. Note that if we wish to calculate the voltage across the resistor, then

$$\begin{aligned} v_R(t) &= Ri(t) \\ &= V_S\left(1 - e^{-\frac{R}{L}t}\right) \end{aligned}$$

Therefore, we find that the voltage across the capacitor in the *RC* circuit and the voltage across the resistor in the *RL* circuit have the same general form. A plot of these functions is shown in Fig. 7.3c.

EXAMPLE 7.1

Consider the circuit shown in Fig. 7.4a. Assuming that the switch has been in position 1 for a long time, at time $t = 0$ the switch is moved to position 2. We wish to calculate the current $i(t)$ for $t > 0$.

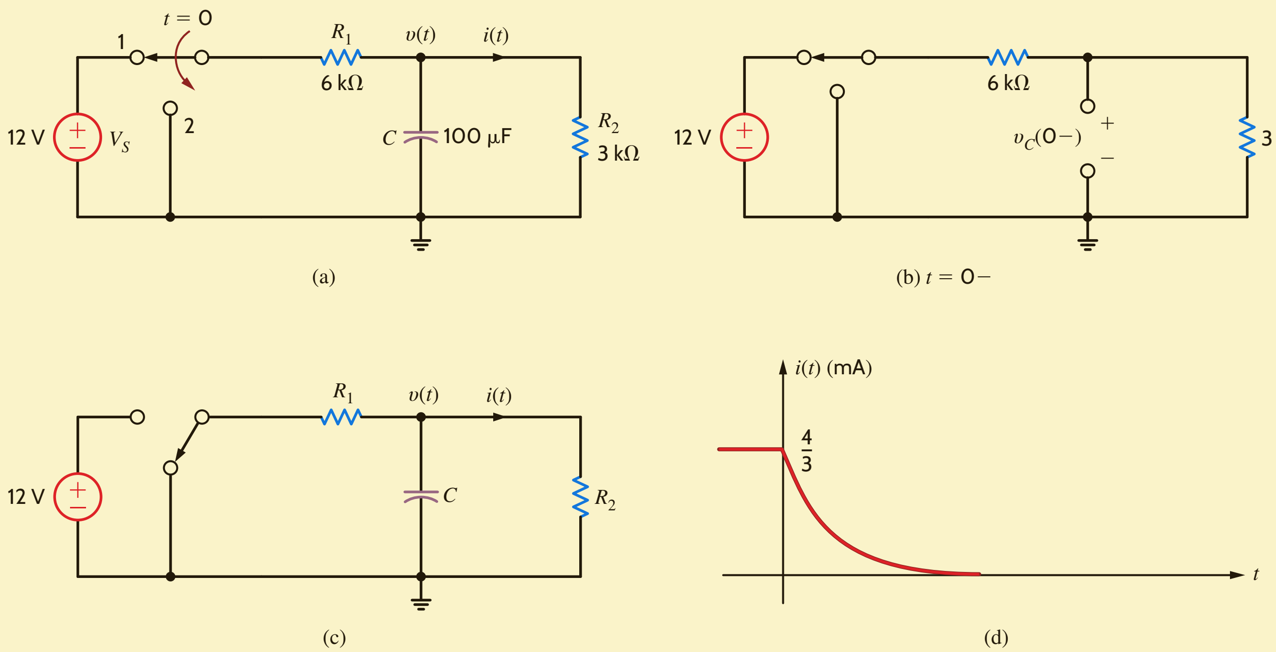

Figure 7.4
Analysis of *RC* circuits.

SOLUTION

At $t = 0-$, the capacitor is fully charged and conducts no current since the capacitor acts like an open circuit to dc. The initial voltage across the capacitor can be found using voltage division. As shown in Fig. 7.4b,

$$v_C(0-) = 12\left(\frac{3\text{k}}{6\text{k} + 3\text{k}}\right) + 3 = 4 \text{ V}$$

The network for $t > 0$ is shown in Fig. 7.4c. The KCL equation for the voltage across the capacitor is

$$\frac{v(t)}{R_1} + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2} = 0$$

Using the component values, the equation becomes

$$\frac{dv(t)}{dt} + 5v(t) = 0$$

The form of the solution to this homogeneous equation is

$$v(t) = K_2 e^{-t/\tau}$$

If we substitute this solution into the differential equation, we find that $\tau = 0.2$ s. Thus,

$$v(t) = K_2 e^{-t/0.2} \text{ V}$$

Using the initial condition $v_C(0-) = v_C(0+) = 4$ V, we find that the complete solution is

$$v(t) = 4e^{-t/0.2} \text{ V}$$

Then $i(t)$ is simply

$$i(t) = \frac{v(t)}{R_2}$$

or

$$i(t) = \frac{4}{3} e^{-t/0.2} \text{ mA}$$

EXAMPLE 7.2

The switch in the network in Fig. 7.5a opens at $t = 0$. Let us find the output voltage $v_o(t)$ for $t > 0$.

SOLUTION

At $t = 0-$ the circuit is in steady state and the inductor acts like a short circuit. The initial current through the inductor can be found in many ways; however, we will form a Thévenin equivalent for the part of the network to the left of the inductor, as shown in Fig. 7.5b. From this network we find that $I_1 = 4$ A and $V_{oc} = 4$ V. In addition, $R_{Th} = 1\ \Omega$. Hence, $i_L(0-)$ obtained from Fig. 7.5c is $i_L(0-) = 4/3$ A.

The network for $t > 0$ is shown in Fig. 7.5d. Note that the 4-V independent source and the 2-ohm resistor in series with it no longer have any impact on the resulting circuit. The KVL equation for the circuit is

$$-V_{S_1} + R_1 i(t) + L\frac{di(t)}{dt} + R_3 i(t) = 0$$

which with the component values reduces to

$$\frac{di(t)}{dt} + 2i(t) = 6$$

The solution to this equation is of the form

$$i(t) = K_1 + K_2 e^{-t/\tau}$$

which, when substituted into the differential equation, yields

$$K_1 = 3$$
$$\tau = 1/2$$

Therefore,

$$i(t) = (3 + K_2 e^{-2t}) \text{ A}$$

Evaluating this function at the initial condition, which is

$$i_L(0-) = i_L(0+) = i(0) = 4/3 \text{ A}$$

we find that

$$K_2 = \frac{-5}{3}$$

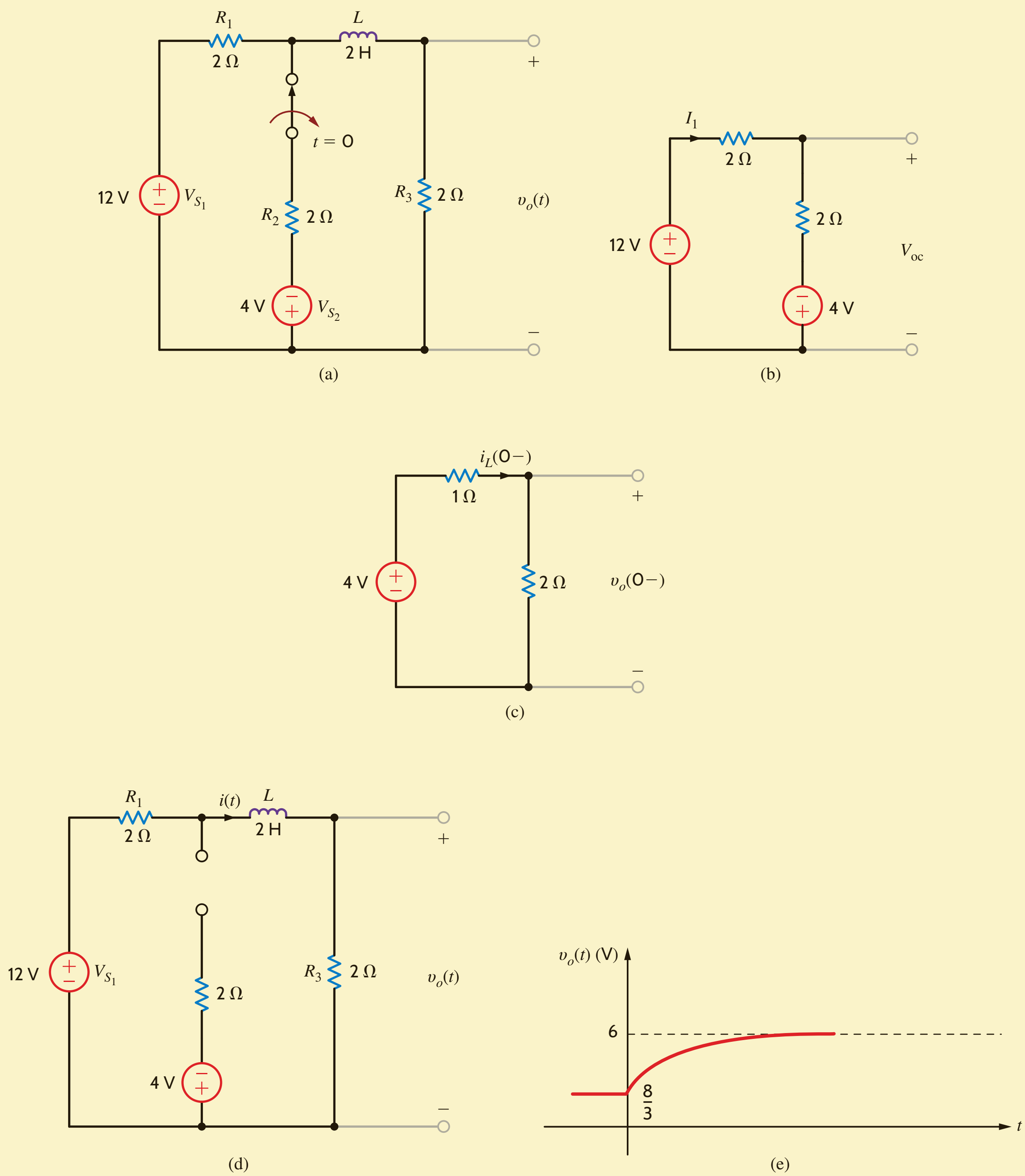

Figure 7.5

Analysis of an *RL* circuit.

Hence,

$$i(t) = \left(3 - \frac{5}{3}e^{-2t}\right) \text{A}$$

and then

$$v_o(t) = 6 - \frac{10}{3}e^{-2t} \text{ V}$$

A plot of the voltage $v_o(t)$ is shown in **Fig. 7.5e**.

LEARNING ASSESSMENTS

E7.1 Find $v_C(t)$ for $t > 0$ in the circuit shown in Fig. E7.1.

ANSWER: $v_C(t) = 8e^{-t/0.6}$ V.

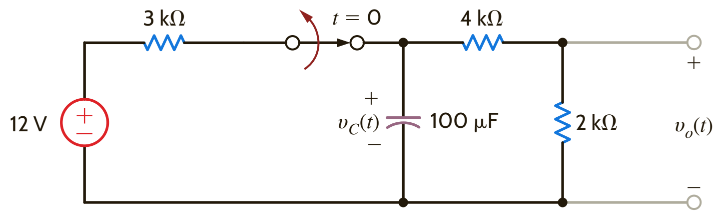

Figure E7.1

E7.2 Use the differential equation approach to find $v_o(t)$ for $t > 0$ in Fig. E7.2. Plot the response.

ANSWER: $v_o(t) = 12 - 5e^{-t/0.015}$ V.

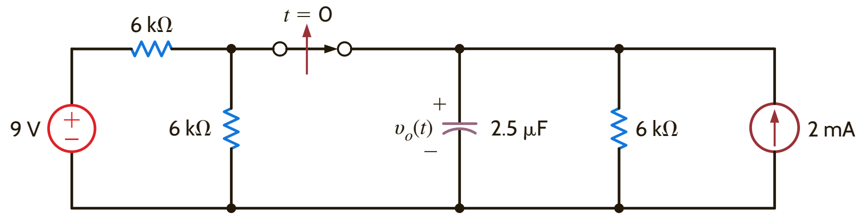

Figure E7.2

E7.3 In the circuit shown in Fig. E7.3, the switch opens at $t = 0$. Find $i_1(t)$ for $t > 0$.

ANSWER: $i_1(t) = 1e^{-9t}$ A.

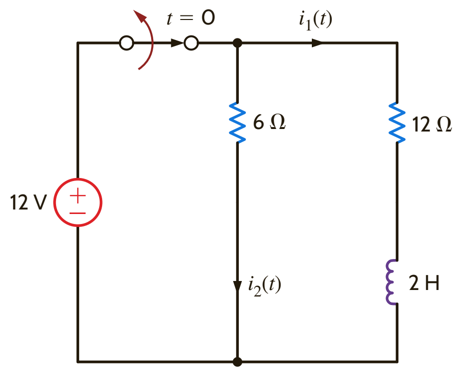

Figure E7.3

E7.4 Use the differential equation approach to find $i(t)$ for $t > 0$ in Fig. E7.4.

ANSWER: $i(t) = -2 + 6e^{-t/5 \times 10^{-6}}$ mA.

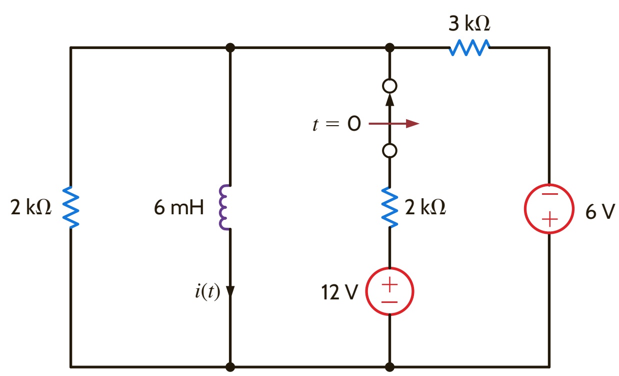

Figure E7.4

ANALYSIS TECHNIQUES: STEP BY STEP In the previous analysis technique, we derived the differential equation for the capacitor voltage or inductor current, solved the differential equation, and used the solution to find the unknown variable in the network. In the very methodical technique that we will now describe, we will use the fact that Eq. (7.11) is the form of the solution and we will employ circuit analysis to determine the constants K_1, K_2, and τ.

From Eq. (7.11) we note that as $t \to \infty$, $e^{-at} \to 0$ and $x(t) = K_1$. Therefore, if the circuit is solved for the variable $x(t)$ in steady state (i.e., $t \to \infty$) with the capacitor replaced by an open circuit [v is constant and therefore $i = C(dv/dt) = 0$] or the inductor replaced by a short circuit [i is constant and therefore $v = L(di/dt) = 0$], then the variable $x(t) = K_1$. Note that since the capacitor or inductor has been removed, the circuit is a dc circuit with constant sources and resistors, and therefore only dc analysis is required in the steady-state solution.

The constant K_2 in Eq. (7.11) can also be obtained via the solution of a dc circuit in which a capacitor is replaced by a voltage source or an inductor is replaced by a current source. The value of the voltage source for the capacitor or the current source for the inductor is a known value at one instant of time. In general, we will use the initial condition value since it is generally the one known, but the value at any instant could be used. This value can be obtained in numerous ways and is often specified as input data in a statement of the problem. However, a more likely situation is one in which a switch is thrown in the circuit and the initial value of the capacitor voltage or inductor current is determined from the previous circuit (i.e., the circuit before the switch is thrown). It is normally assumed that the previous circuit has reached steady state, and therefore the voltage across the capacitor or the current through the inductor can be found in exactly the same manner as was used to find K_1.

Finally, the value of the time constant can be found by determining the Thévenin equivalent resistance at the terminals of the storage element. Then $\tau = R_{Th}C$ for an RC circuit, and $\tau = L/R_{Th}$ for an RL circuit.

Let us now reiterate this procedure in a step-by-step fashion.

PROBLEM-SOLVING STRATEGY

USING THE STEP-BY-STEP APPROACH

STEP 1. We assume a solution for the variable $x(t)$ of the form $x(t) = K_1 + K_2 e^{-t/\tau}$.

STEP 2. Assuming that the original circuit has reached steady state before a switch was thrown (thereby producing a new circuit), draw this previous circuit with the capacitor replaced by an open circuit or the inductor replaced by a short circuit. Solve for the voltage across the capacitor, $v_C(0-)$, or the current through the inductor, $i_L(0-)$, prior to switch action.

STEP 3. Recall from Chapter 6 that voltage across a capacitor and the current flowing through an inductor cannot change in zero time. Draw the circuit valid for $t = 0+$ with the switches in their new positions. Replace a capacitor with a voltage source $v_C(0+) = v_C(0-)$ or an inductor with a current source of value $i_L(0+) = i_L(0-)$. Solve for the initial value of the variable $x(0+)$.

STEP 4. Assuming that steady state has been reached after the switches are thrown, draw the equivalent circuit, valid for $t > 5\tau$, by replacing the capacitor by an open circuit or the inductor by a short circuit. Solve for the steady-state value of the variable

$$x(t)\big|_{t > 5\tau} \doteq x(\infty)$$

STEP 5. Since the time constant for all voltages and currents in the circuit will be the same, it can be obtained by reducing the entire circuit to a simple series circuit containing a voltage source, resistor, and a storage element (i.e., capacitor or inductor) by forming a simple Thévenin equivalent circuit at the terminals of the storage element. This Thévenin equivalent circuit is obtained by looking into the circuit from the terminals of the storage element. The time constant for a circuit containing a capacitor is $\tau = R_{Th}C$, and for a circuit containing an inductor it is $\tau = L/R_{Th}$.

STEP 6. Using the results of steps 3, 4, and 5, we can evaluate the constants in step 1 as

$$x(0+) = K_1 + K_2$$
$$x(\infty) = K_1$$

Therefore, $K_1 = x(\infty)$, $K_2 = x(0+) - x(\infty)$, and hence the solution is

$$x(t) = x(\infty) + [x(0+) - x(\infty)]e^{-t/\tau}$$

Keep in mind that this solution form applies only to a first-order circuit having dc sources. If the sources are not dc, the forced response will be different. Generally, the forced response is of the same form as the forcing functions (sources) and their derivatives.

EXAMPLE 7.3

Consider the circuit shown in Fig. 7.6a. The circuit is in steady state prior to time $t = 0$, when the switch is closed. Let us calculate the current $i(t)$ for $t > 0$.

SOLUTION

Step 1. $i(t)$ is of the form $K_1 + K_2e^{-t/\tau}$.

Step 2. The initial voltage across the capacitor is calculated from Fig. 7.6b as

$$\begin{aligned} v_C(0-) &= 36 - (2)(2) \\ &= 32 \text{ V} \end{aligned}$$

Step 3. The new circuit, valid only for $t = 0+$, is shown in Fig. 7.6c. The value of the voltage source that replaces the capacitor is $v_C(0-) = v_C(0+) = 32$ V. Hence,

$$\begin{aligned} i(0+) &= \frac{32}{6\text{k}} \\ &= \frac{16}{3} \text{ mA} \end{aligned}$$

Step 4. The equivalent circuit, valid for $t > 5\tau$, is shown in Fig. 7.6d. The current $i(\infty)$ caused by the 36-V source is

$$\begin{aligned} i(\infty) &= \frac{36}{2\text{k} + 6\text{k}} \\ &= \frac{9}{2} \text{ mA} \end{aligned}$$

Step 5. The Thévenin equivalent resistance, obtained by looking into the open-circuit terminals of the capacitor in Fig. 7.6e, is

$$R_{Th} = \frac{(2\text{k})(6\text{k})}{2\text{k} + 6\text{k}} = \frac{3}{2} \text{ k}\Omega$$

Therefore, the circuit time constant is

$$\begin{aligned} \tau &= R_{Th}C \\ &= \left(\frac{3}{2}\right)(10^3)(100)(10^{-6}) \\ &= 0.15 \text{ s} \end{aligned}$$

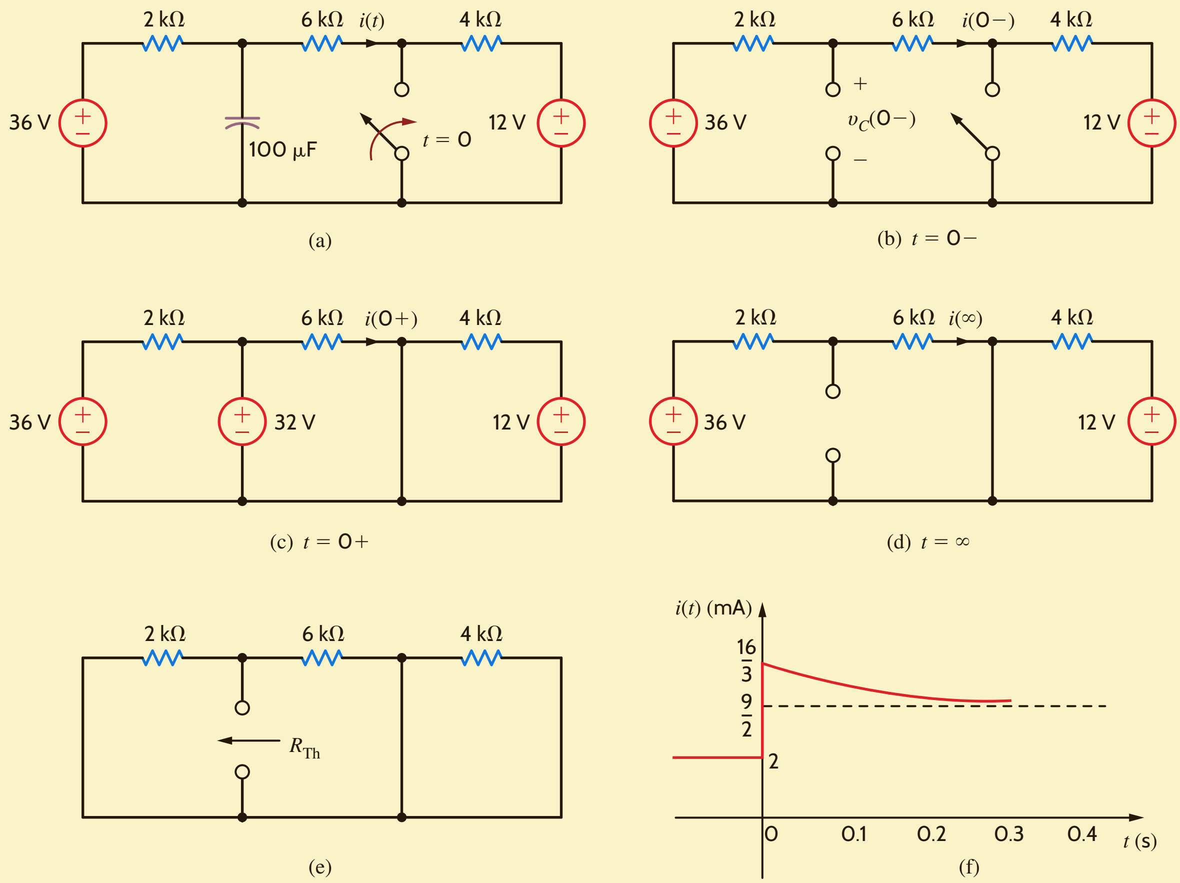

Figure 7.6

Analysis of an *RC* transient circuit with a constant forcing function.

Step 6.

$$K_1 = i(\infty) = \frac{9}{2}\text{ mA}$$

$$K_2 = i(0+) - i(\infty) = i(0+) - K_1$$

$$= \frac{16}{3} - \frac{9}{2}$$

$$= \frac{5}{6}\text{ mA}$$

Therefore,

$$i(t) = \frac{36}{8} + \frac{5}{6}e^{-t/0.15}\text{ mA}$$

The plot is shown in **Fig. 7.7** and can be compared to the sketch in **Fig. 7.6f**. Examination of Fig. 7.6f indicates once again that although the voltage across the capacitor is continuous at $t = 0$, the current $i(t)$ in the 6-kΩ resistor jumps at $t = 0$ from 2 mA to 5 1/3 mA, and finally decays to 4 1/2 mA.

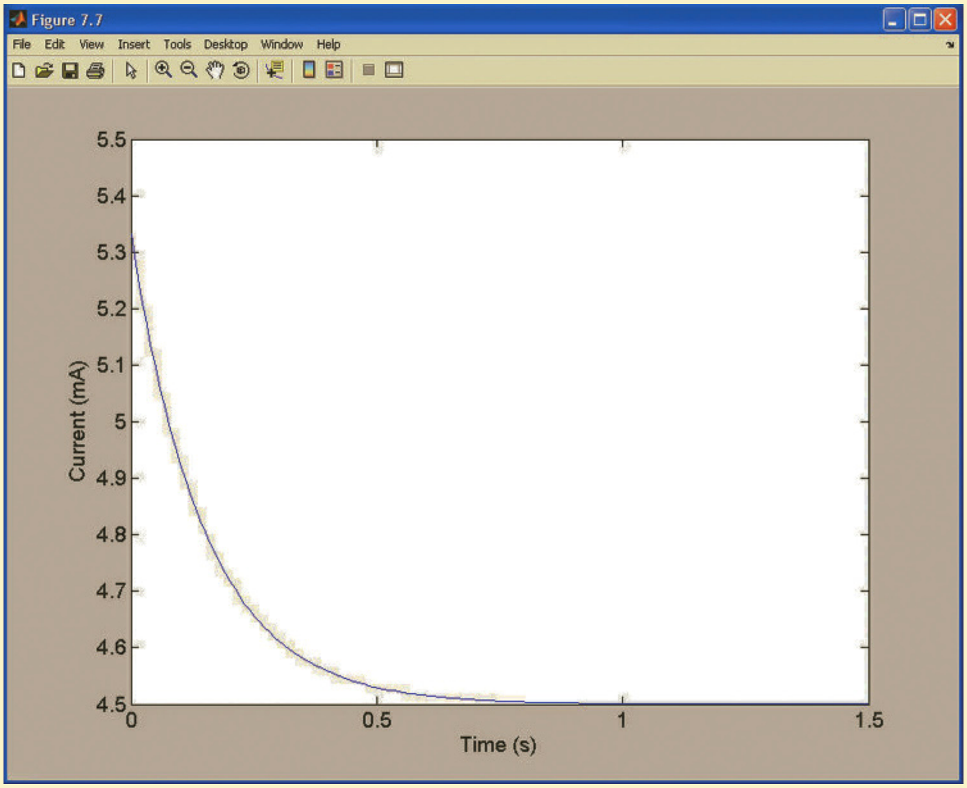

Figure 7.7
Plot for Example 7.3.

EXAMPLE 7.4

SOLUTION

The circuit shown in Fig. 7.8a is assumed to have been in a steady-state condition prior to switch closure at $t = 0$. We wish to calculate the voltage $v(t)$ for $t > 0$.

Step 1. $v(t)$ is of the form $K_1 + K_2 e^{-t/\tau}$.

Step 2. In Fig. 7.8b we see that

$$i_L(0-) = \frac{24}{4 + \frac{(6)(3)}{6+3}}\left(\frac{6}{6+3}\right)$$

$$= \frac{8}{3}\text{ A}$$

Step 3. The new circuit, valid only for $t = 0+$, is shown in Fig. 7.8c, which is equivalent to the circuit shown in Fig. 7.8d. The value of the current source that replaces the inductor is $i_L(0-) = i_L(0+) = 8/3$ A. The node voltage $v_1(0+)$ can be determined from the circuit in Fig. 7.8d using a single-node equation, and $v(0+)$ is equal to the difference between the source voltage and $v_1(0+)$. The equation for $v_1(0+)$ is

$$\frac{v_1(0+) - 24}{4} + \frac{v_1(0+)}{6} + \frac{8}{3} + \frac{v_1(0+)}{12} = 0$$

or

$$v_1(0+) = \frac{20}{3}\text{ V}$$

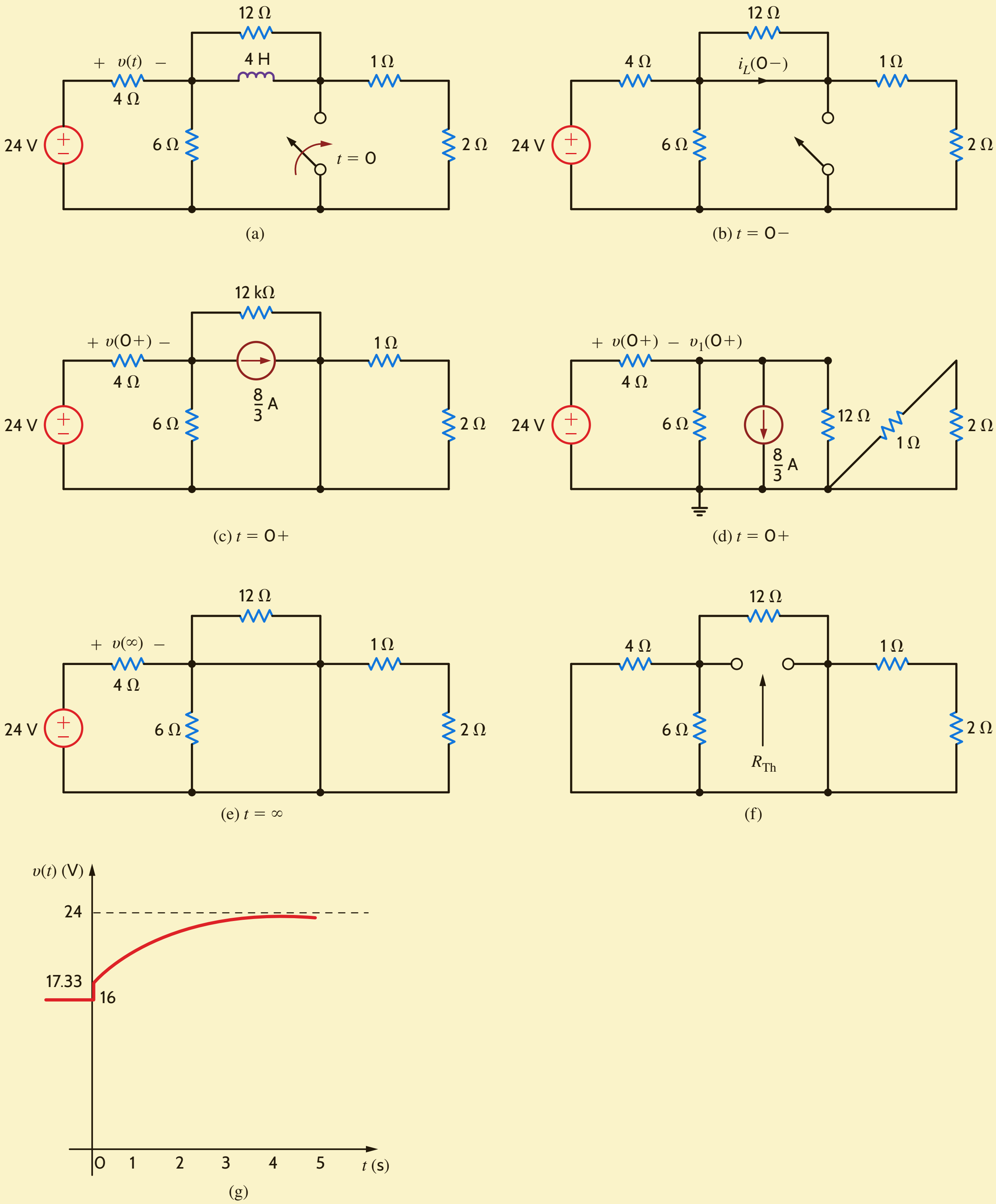

Figure 7.8

Analysis of an *RL* transient circuit with a constant forcing function.

Then

$$v(0+) = 24 - v_1(0+)$$
$$= \frac{52}{3}\text{ V}$$

Step 4. The equivalent circuit for the steady-state condition after switch closure is given in **Fig. 7.8e**. Note that the 6-, 12-, 1-, and 2-Ω resistors are shorted, and therefore $v(\infty) = 24$ V.

Step 5. The Thévenin equivalent resistance is found by looking into the circuit from the inductor terminals. This circuit is shown in **Fig. 7.8f**. Note carefully that R_{Th} is equal to the 4-, 6-, and 12-Ω resistors in parallel. Therefore, $R_{Th} = 2\ \Omega$, and the circuit time constant is

$$\tau = \frac{L}{R_{Th}} = \frac{4}{2} = 2\text{ s}$$

Step 6. From the previous analysis we find that

$$K_1 = v(\infty) = 24$$
$$K_2 = v(0+) - v(\infty) = -\frac{20}{3}$$

and hence that

$$v(t) = 24 - \frac{20}{3}e^{-t/2}\text{ V}$$

From Fig. 7.8b we see that the value of $v(t)$ before switch closure is 16 V. This value jumps to 17.33 V at $t = 0$.

A plot of this function for $t > 0$ is shown in **Fig. 7.9**.

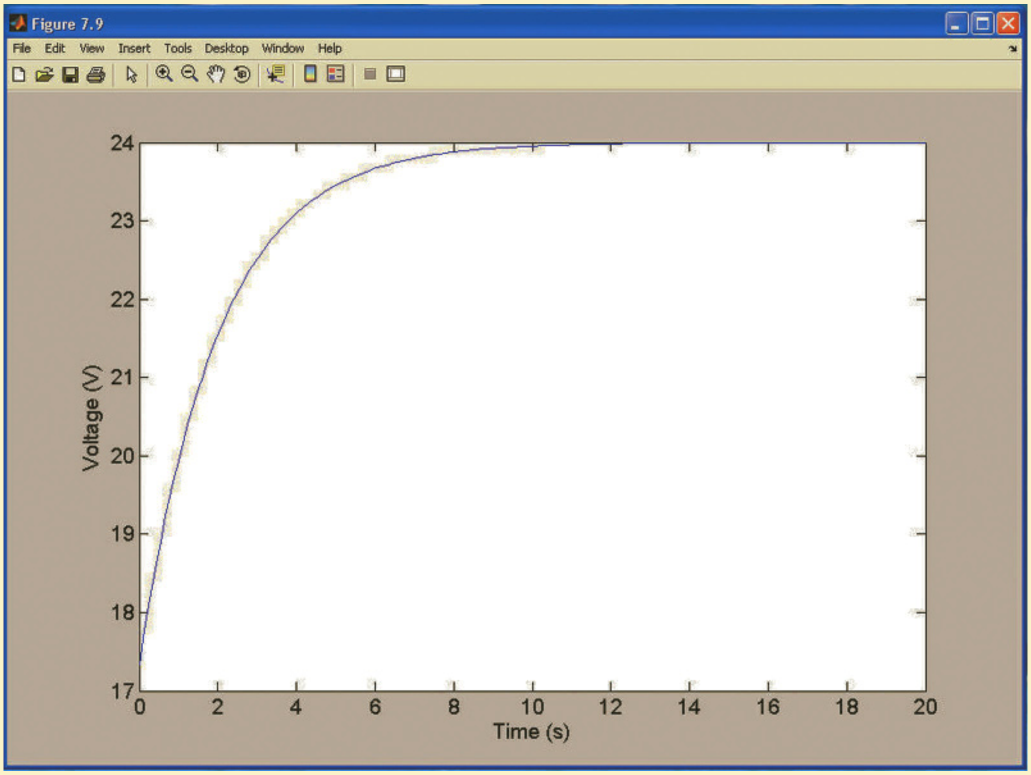

Figure 7.9
Plot for Example 7.4.

LEARNING ASSESSMENTS

E7.5 Consider the network in Fig. E7.5. The switch opens at $t = 0$. Find $v_o(t)$ for $t > 0$.

ANSWER:
$v_o(t) = \frac{24}{5} + \frac{1}{5}e^{-(5/8)t}$ V.

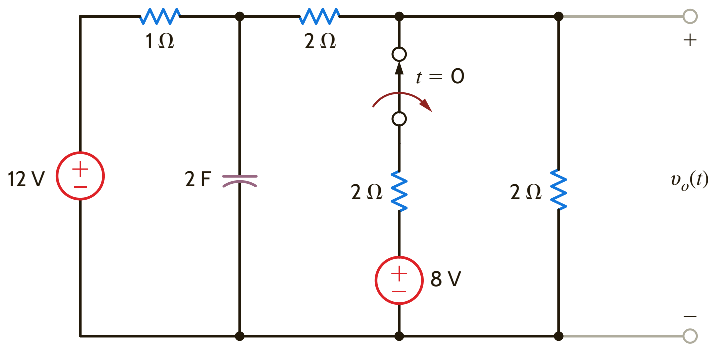

Figure E7.5

E7.6 Consider the network in Fig. E7.6. If the switch opens at $t = 0$, find the output voltage $v_o(t)$ for $t > 0$.

ANSWER:
$v_o(t) = 6 - \frac{10}{3}e^{-2t}$ V.

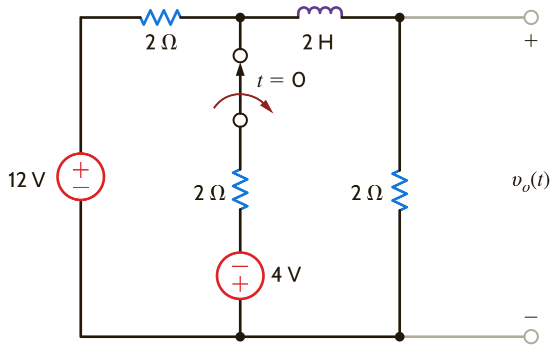

Figure E7.6

E7.7 Find $v_o(t)$ for $t > 0$ in Fig. E7.7 using the step-by-step method.

ANSWER:
$v_o(t) = -3.33e^{-t/0.06}$ V.

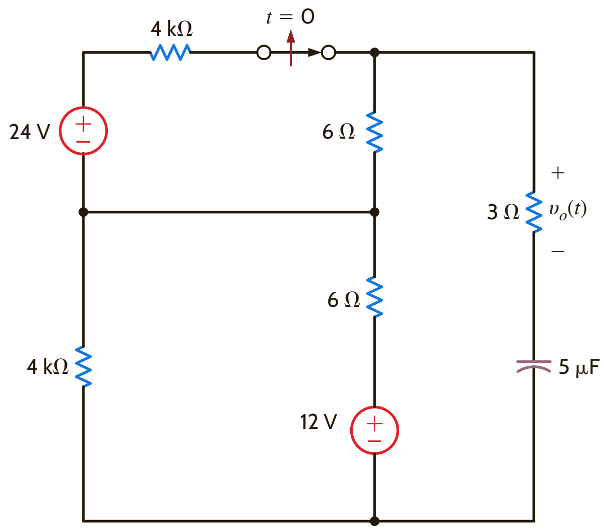

Figure E7.7

E7.8 Find $i_o(t)$ for $t > 0$ in Fig. E7.8 using the step-by-step method.

ANSWER:
$i_o(t) = 2.1 - 0.6e^{-t/0.001}$ A.

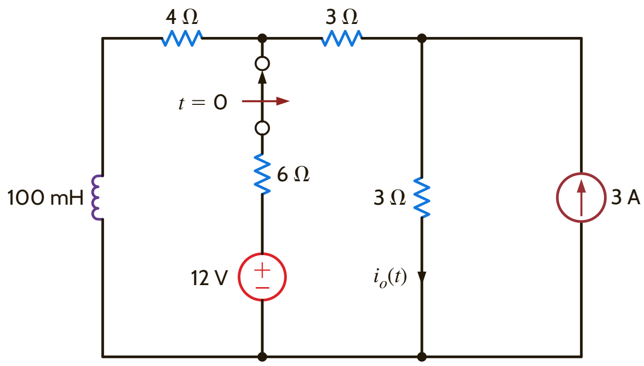

Figure E7.8

EXAMPLE 7.5

SOLUTION

The circuit shown in Fig. 7.10a has reached steady state with the switch in position 1. At time $t = 0$ the switch moves from position 1 to position 2. We want to calculate $v_o(t)$ for $t > 0$.

Step 1. $v_o(t)$ is of the form $K_1 + K_2e^{-t/\tau}$.

Step 2. Using the circuit in Fig. 7.10b, we can calculate $i_L(0-)$:

$$i_A = \frac{12}{4} = 3 \text{ A}$$

Then

$$i_L(0-) = \frac{12 + 2i_A}{6} = \frac{18}{6} = 3 \text{ A}$$

Step 3. The new circuit, valid only for $t = 0+$, is shown in Fig. 7.10c. The value of the current source that replaces the inductor is $i_L(0-) = i_L(0+) = 3$ A. Because of the current source

$$v_o(0+) = (3)(6) = 18 \text{ V}$$

Step 4. The equivalent circuit, for the steady-state condition after switch closure, is given in Fig. 7.10d. Using the voltages and currents defined in the figure, we can compute $v_o(\infty)$ in a variety of ways. For example, using node equations, we can find $v_o(\infty)$ from

$$\frac{v_B - 36}{2} + \frac{v_B}{4} + \frac{v_B + 2i_A'}{6} = 0$$
$$i_A' = \frac{v_B}{4}$$
$$v_o(\infty) = v_B + 2i_A'$$

or, using loop equations,

$$36 = 2(i_1 + i_2) + 4i_1$$
$$36 = 2(i_1 + i_2) + 6i_2 - 2i_1$$
$$v_o(\infty) = 6i_2$$

Using either approach, we find that $v_o(\infty) = 27$ V.

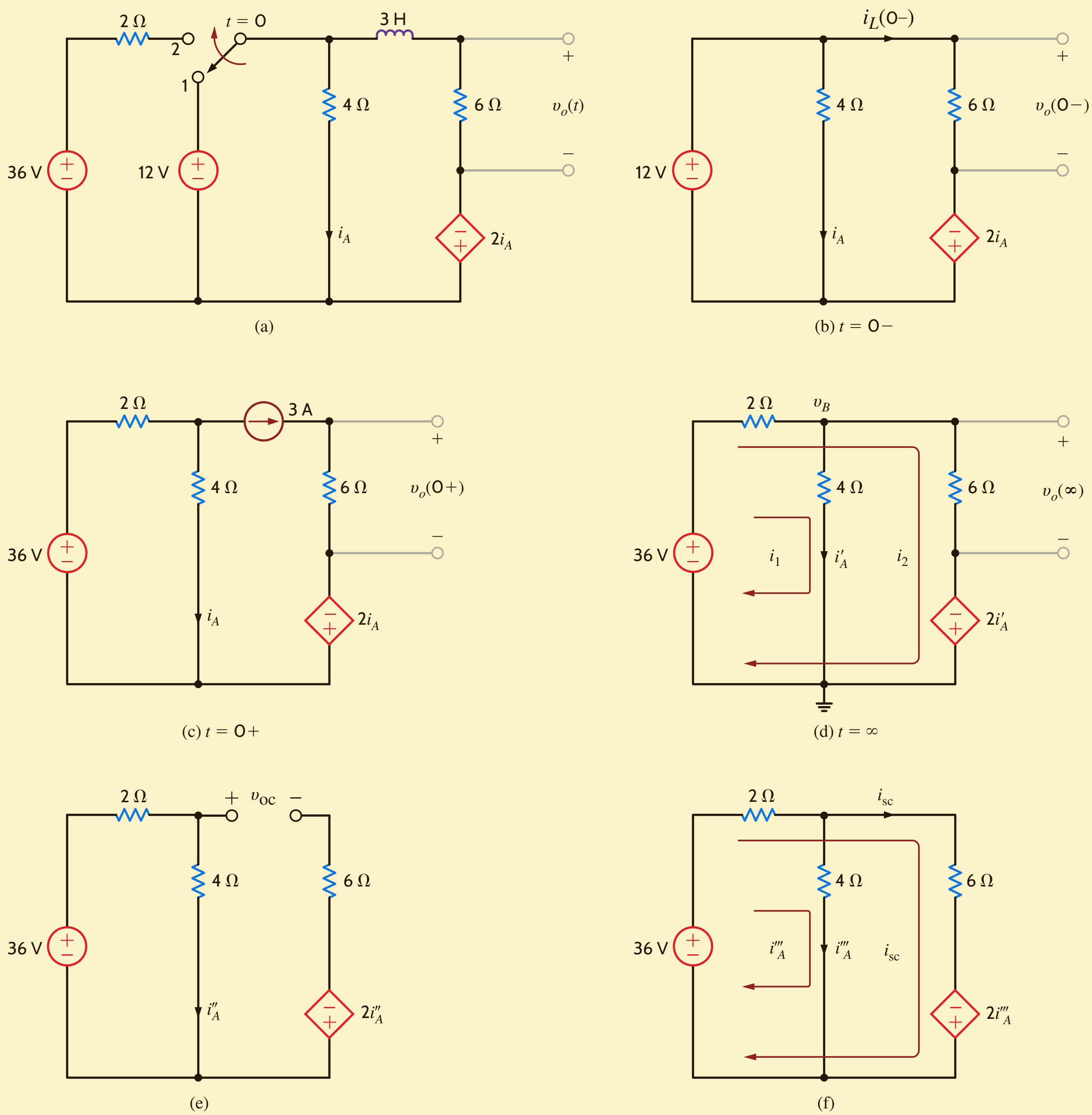

Figure 7.10

Analysis of an *RL* transient circuit containing a dependent source.

Step 5. The Thévenin equivalent resistance can be obtained via v_{oc} and i_{sc} because of the presence of the dependent source. From **Fig. 7.10e** we note that

$$i''_A = \frac{36}{2 + 4} = 6 \text{ A}$$

Therefore,

$$\begin{aligned} v_{oc} &= (4)(6) + 2(6) \\ &= 36 \text{ V} \end{aligned}$$

From **Fig. 7.10f** we can write the following loop equations (identical to those in step 4).

$$36 = 2(i_A''' + i_{sc}) + 4i_A'''$$
$$36 = 2(i_A''' + i_{sc}) + 6i_{sc} - 2i_A'''$$

Solving these equations for i_{sc} yields

$$i_{sc} = \frac{9}{2}\text{ A}$$

Therefore,

$$R_{Th} = \frac{v_{oc}}{i_{sc}} = \frac{36}{9/2} = 8\ \Omega$$

Hence, the circuit time constant is

$$\tau = \frac{L}{R_{Th}} = \frac{3}{8}\text{ s}$$

Step 6. Using the information just computed, we can derive the final equation for $v_o(t)$:

$$K_1 = v_o(\infty) = 27$$
$$K_2 = v_o(0+) - v_o(\infty) = 18 - 27 = -9$$

Therefore,

$$v_o(t) = 27 - 9e^{-t/(3/8)}\text{ V}$$

LEARNING ASSESSMENTS

E7.9 If the switch in the network in Fig. E7.9 closes at $t = 0$, find $v_o(t)$ for $t > 0$.

ANSWER:
$v_o(t) = 24 + 36e^{-(t/12)}$ V.

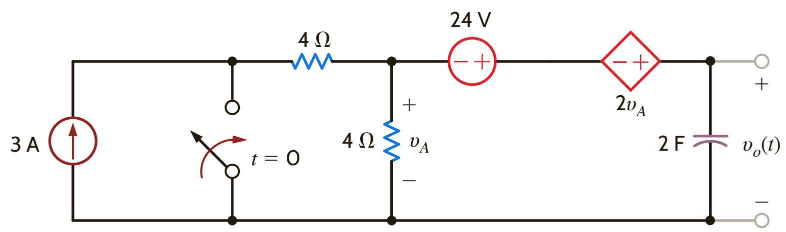

Figure E7.9

E7.10 Find $i_o(t)$ for $t > 0$ in Fig. E7.10 using the step-by-step method.

ANSWER:
$i_o(t) = 1.5 + 0.2143\,e^{-(t/0.7)}$ mA.

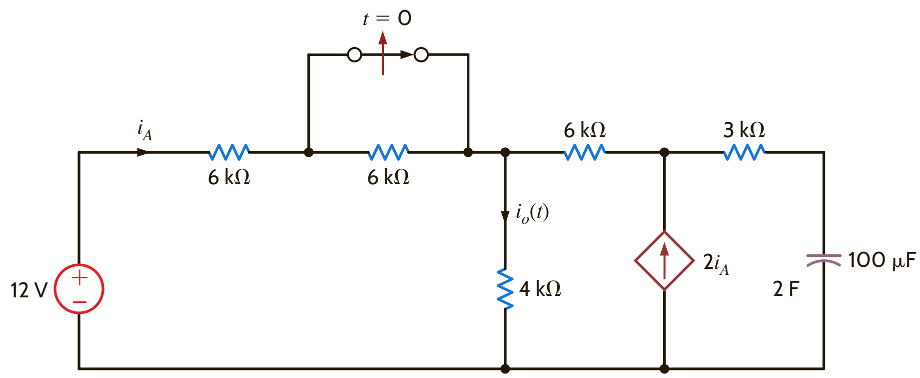

Figure E7.10

At this point, it is appropriate to state that not all switch action will always occur at time $t = 0$. It may occur at any time t_0. In this case the results of the step-by-step analysis yield the following equations:

$$x(t_0) = K_1 + K_2$$
$$x(\infty) = K_1$$

and

$$x(t) = x(\infty) + [x(t_0) - x(\infty)]e^{-(t - t_0)/\tau} \qquad t > t_0$$

The function is essentially time-shifted by t_0 seconds.

Finally, note that if more than one independent source is present in the network, we can simply employ superposition to obtain the total response.

PULSE RESPONSE Thus far we have examined networks in which a voltage or current source is suddenly applied. As a result of this sudden application of a source, voltages or currents in the circuit are forced to change abruptly. A forcing function whose value changes in a discontinuous manner or has a discontinuous derivative is called a *singular function*. Two such singular functions that are very important in circuit analysis are the unit impulse function and the unit step function. We will defer a discussion of the unit impulse function until a later chapter and concentrate on the unit step function.

The *unit step function* is defined by the following mathematical relationship:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

In other words, this function, which is dimensionless, is equal to zero for negative values of the argument and equal to 1 for positive values of the argument. It is undefined for a zero argument where the function is discontinuous. A graph of the unit step is shown in Fig. 7.11a.

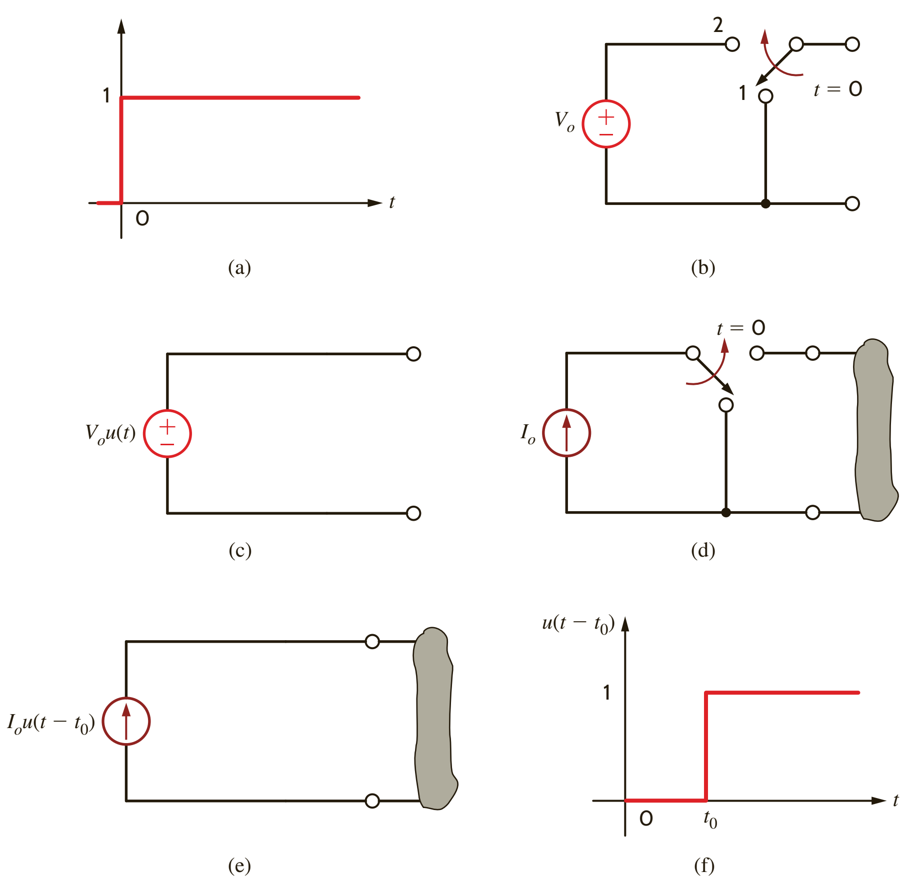

Figure 7.11
Graphs and models of the unit step function.

The unit step is dimensionless, and therefore a voltage step of V_o volts or a current step of I_o amperes is written as $V_o u(t)$ and $I_o u(t)$, respectively. Equivalent circuits for a voltage step are shown in **Figs. 7.11b** and **c**. Equivalent circuits for a current step are shown in **Figs. 7.11d** and **e**. If we use the definition of the unit step, it is easy to generalize this function by replacing the argument t by $t - t_0$. In this case

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

A graph of this function is shown in **Fig. 7.11f**. Note that $u(t - t_0)$ is equivalent to delaying $u(t)$ by t_0 seconds, so that the abrupt change occurs at time $t = t_0$.

Step functions can be used to construct one or more pulses. For example, the voltage pulse shown in **Fig. 7.12a** can be formulated by initiating a unit step at $t = 0$ and subtracting one that starts at $t = T$, as shown in **Fig. 7.12b**. The equation for the pulse is

$$v(t) = A[u(t) - u(t - T)]$$

If the pulse is to start at $t = t_0$ and have width T, the equation would be

$$v(t) = A\{u(t - t_0) - u[t - (t_0 + T)]\}$$

Using this approach, we can write the equation for a pulse starting at any time and ending at any time. Similarly, using this approach, we could write the equation for a series of pulses, called a *pulse train*, by simply forming a summation of pulses constructed in the manner illustrated previously.

The following example will serve to illustrate many of the concepts we have just presented.

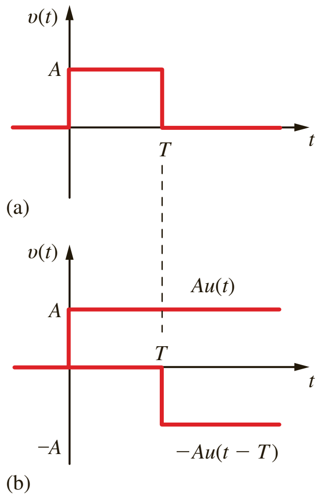

Figure 7.12

Construction of a pulse via two step functions.

EXAMPLE 7.6

Consider the circuit shown in **Fig. 7.13a**. The input function is the voltage pulse shown in **Fig. 7.13b**. Since the source is zero for all negative time, the initial conditions for the

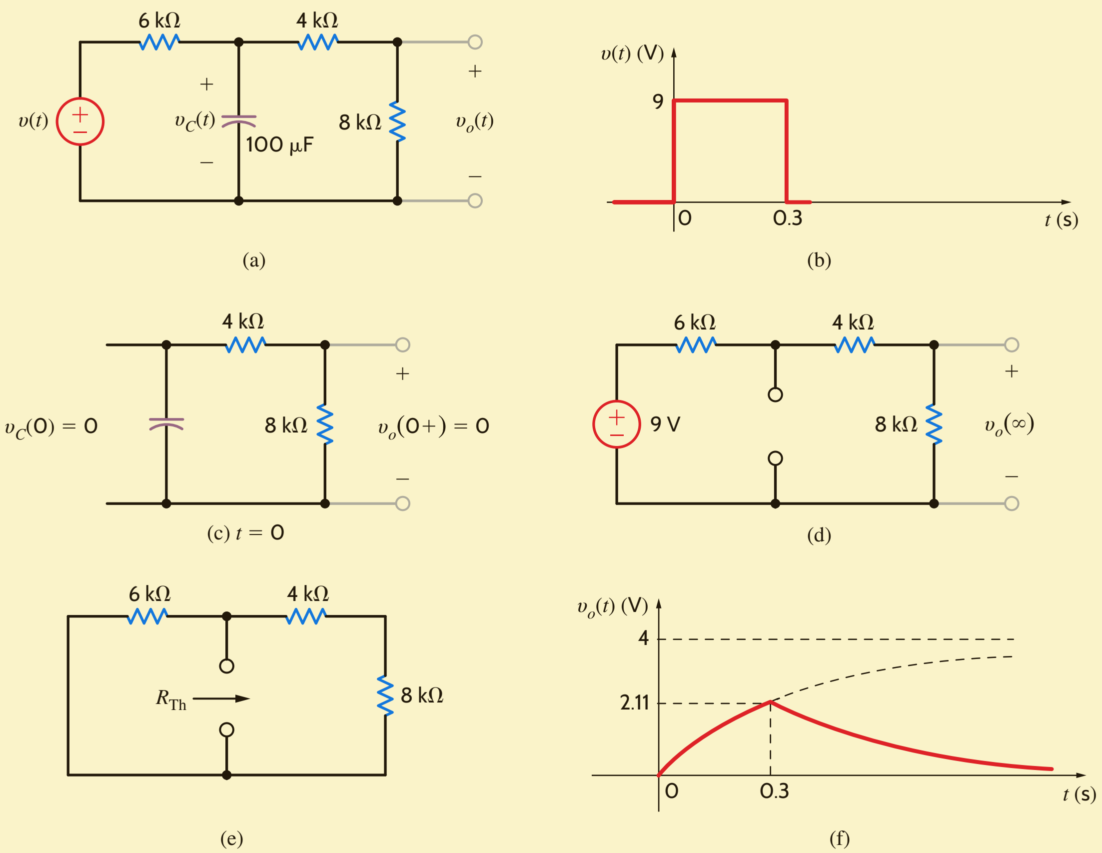

Figure 7.13

Pulse response of a network.

network are zero [i.e., $v_C(0-) = 0$]. The response $v_o(t)$ for $0 < t < 0.3$ s is due to the application of the constant source at $t = 0$ and is not influenced by any source changes that will occur later. At $t = 0.3$ s the forcing function becomes zero, and therefore $v_o(t)$ for $t > 0.3$ s is the source-free or natural response of the network.

Let us determine the expression for the voltage $v_o(t)$.

SOLUTION

Since the output voltage $v_o(t)$ is a voltage division of the capacitor voltage, and the initial voltage across the capacitor is zero, we know that $v_o(0+) = 0$, as shown in **Fig. 7.13c**.

If no changes were made in the source after $t = 0$, the steady-state value of $v_o(t)$ [i.e., $v_o(\infty)$] due to the application of the unit step at $t = 0$ would be

$$\begin{aligned} v_o(\infty) &= \frac{9}{6\text{k} + 4\text{k} + 8\text{k}}(8\text{k}) \\ &= 4\text{ V} \end{aligned}$$

as shown in **Fig. 7.13d**.

The Thévenin equivalent resistance is

$$\begin{aligned} R_{\text{Th}} &= \frac{(6\text{k})(12\text{k})}{6\text{k} + 12\text{k}} \\ &= 4\text{ k}\Omega \end{aligned}$$

as illustrated in **Fig. 7.13e**.

The circuit time constant τ is

$$\begin{aligned} \tau &= R_{\text{Th}}C \\ &= (4)(10^3)(100)(10^{-6}) \\ &= 0.4\text{ s} \end{aligned}$$

Therefore, the response $v_o(t)$ for the period $0 < t < 0.3$ s is

$$v_o(t) = 4 - 4e^{-t/0.4}\text{ V} \qquad 0 < t < 0.3\text{ s}$$

The capacitor voltage can be calculated by realizing that using voltage division, $v_o(t) = 2/3\ v_C(t)$:

$$v_C(t) = \frac{3}{2}\left(4 - 4e^{-t/0.4}\right)\text{ V}$$

Since the capacitor voltage is continuous,

$$v_C(0.3-) = v_C(0.3+)$$

then

$$\begin{aligned} v_o(0.3+) &= \frac{2}{3}v_C(0.3-) \\ &= 4\left(1 - e^{-0.3/0.4}\right) \\ &= 2.11\text{ V} \end{aligned}$$

Since the source is zero for $t > 0.3$ s, the final value for $v_o(t)$ as $t \to \infty$ is zero. Thus, the expression for $v_o(t)$ for $t > 0.3$ s is

$$v_o(t) = 2.11e^{-(t-0.3)/0.4}\text{ V} \qquad t > 0.3\text{ s}$$

The term $e^{-(t-0.3)/0.4}$ indicates that the exponential decay starts at $t = 0.3$ s. The complete solution can be written by means of superposition as

$$v_o(t) = 4(1 - e^{-t/0.4})u(t) - 4(1 - e^{-(t-0.3)/0.4})u(t - 0.3)\text{ V}$$

or, equivalently, the complete solution is

$$v_o(t) = \left\{\begin{array}{ll} 0 & t < 0 \\ 4(1 - e^{-t/0.4})\text{ V} & 0 < t < 0.3\text{ s} \\ 2.11e^{-(t-0.3)/0.4}\text{ V} & 0.3\text{ s} < t \end{array}\right\}$$