present with a break frequency at $\omega = 0.5$ rad/s and, therefore, the numerator has a factor of $(2j\omega + 1)$. Two additional poles are present with break frequencies at $\omega = 2$ rad/s and $\omega = 20$ rad/s. Therefore, the composite transfer function is

$$\mathbf{G}_v(j\omega) = \frac{10(2j\omega + 1)}{(10j\omega + 1)(0.5j\omega + 1)(0.05j\omega + 1)}$$

Note carefully the ramifications of this example with regard to network design.

LEARNING ASSESSMENTS

E12.9 Determine the transfer function $\mathbf{G}(j\omega)$ if the straight-line magnitude characteristic approximation for this function is as shown in Fig. E12.9.

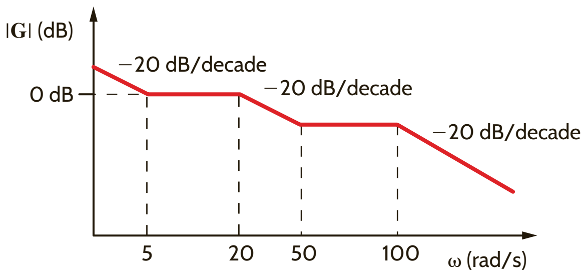


Figure E12.9

ANSWER:

$$\mathbf{G}(j\omega) = \frac{5\left(\dfrac{j\omega}{5} + 1\right)\left(\dfrac{j\omega}{50} + 1\right)}{j\omega\left(\dfrac{j\omega}{20} + 1\right)\left(\dfrac{j\omega}{100} + 1\right)}.$$

E12.10 Find $\mathbf{H}(j\omega)$ if its magnitude characteristic is shown in Fig. E12.10.

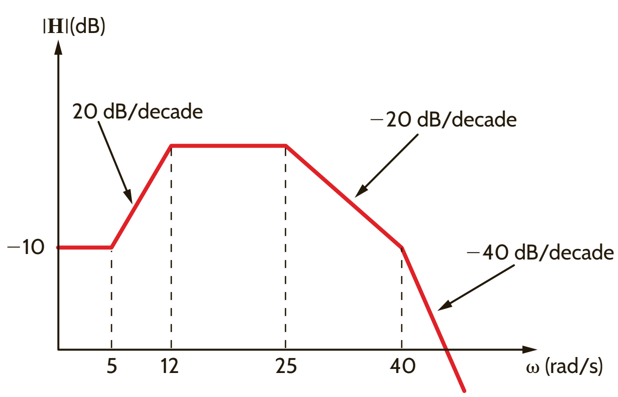


Figure E12.10

ANSWER:

$$\mathbf{H}(j\omega) = \frac{0.3162\left(\dfrac{j\omega}{5} + 1\right)}{\left(\dfrac{j\omega}{12} + 1\right)\left(\dfrac{j\omega}{25} + 1\right)\left(\dfrac{j\omega}{40} + 1\right)}.$$

12.3 Resonant Circuits

SERIES RESONANCE A circuit with extremely important frequency characteristics is shown in Fig. 12.17. The input impedance for the series *RLC* circuit is

$$\mathbf{Z}(j\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \qquad 12.10$$

The imaginary term will be zero if

$$\omega L = \frac{1}{\omega C}$$

The value of ω that satisfies this equation is

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{12.11}$$

and at this value of ω the impedance becomes

$$\mathbf{Z}(j\omega_0) = R \tag{12.12}$$

This frequency ω_0, at which the impedance of the circuit is purely real, is also called the *resonant frequency*, and the circuit itself at this frequency is said to be *in resonance*. Resonance is a very important consideration in engineering design. For example, engineers designing the attitude control system for the Saturn vehicles had to ensure that the control system frequency did not excite the body bending (resonant) frequencies of the vehicle. Excitation of the bending frequencies would cause oscillations that, if continued unchecked, would result in a buildup of stress until the vehicle would finally break apart.

Resonance is also a benefit, providing string and wind musical instruments with volume and rich tones.

At resonance the voltage and current are in phase and, therefore, the phase angle is zero and the power factor is unity. At resonance the impedance is a minimum and, therefore, the current is maximum for a given voltage. Fig. 12.18 illustrates the frequency response of the series *RLC* circuit. Note that at low frequencies the impedance of the series circuit is dominated by the capacitive term, and at high frequencies the impedance is dominated by the inductive term.

Resonance can be viewed from another perspective—that of the phasor diagram. In the series circuit the current is common to every element. Therefore, the current is employed as reference. The phasor diagram is shown in Fig. 12.19 for the three frequency values $\omega < \omega_0$, $\omega = \omega_0$, $\omega > \omega_0$.

When $\omega < \omega_0$, $\mathbf{V}_C > \mathbf{V}_L$, θ_Z is negative and the voltage $\mathbf{V}_1$ lags the current. If $\omega = \omega_0$, $\mathbf{V}_L = \mathbf{V}_C$, θ_Z is zero, and the voltage $\mathbf{V}_1$ is in phase with the current. If $\omega > \omega_0$, $\mathbf{V}_L > \mathbf{V}_C$, θ_Z is positive, and the voltage $\mathbf{V}_1$ leads the current.

For the series circuit we define what is commonly called the *quality factor Q* as

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{12.13}$$

Q is a very important factor in resonant circuits, and its ramifications will be illustrated throughout the remainder of this section.

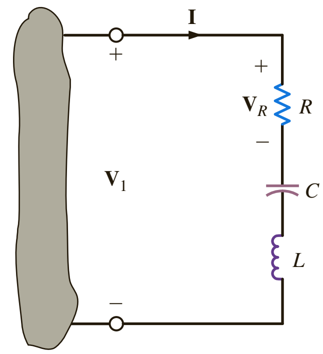


Figure 12.17

Series *RLC* circuit.

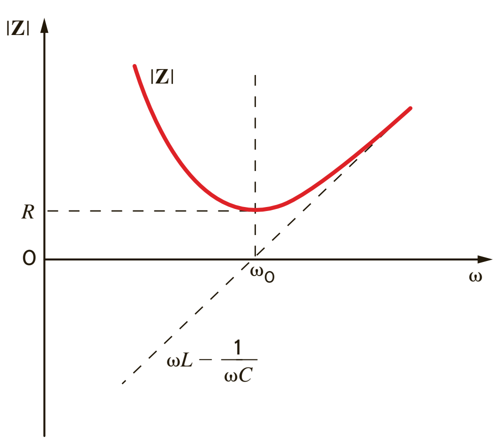


Figure 12.18

Frequency response of a series *RLC* circuit.

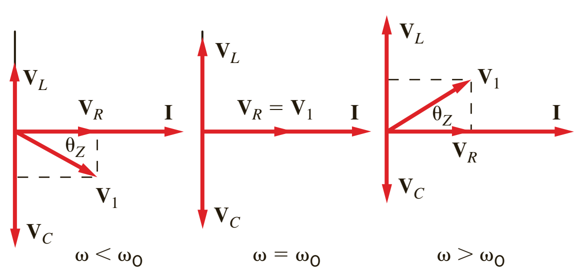


Figure 12.19

Phasor diagrams for the series *RLC* circuit.

EXAMPLE 12.7

Consider the network shown in Fig. 12.20. Let us determine the resonant frequency, the voltage across each element at resonance, and the value of the quality factor.

SOLUTION

The resonant frequency is obtained from the expression

$$\begin{aligned}\omega_0 &= \frac{1}{\sqrt{LC}}\\ &= \frac{1}{\sqrt{(25)(10^{-3})(10)(10^{-6})}}\\ &= 2000 \text{ rad/s}\end{aligned}$$

At this resonant frequency

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{\mathbf{V}}{R} = 5\angle 0° \text{ A}$$

Therefore,

$$\begin{aligned}\mathbf{V}_R &= (5\angle 0°)(2) = 10\angle 0° \text{ V}\\ \mathbf{V}_L &= j\omega_0 L\mathbf{I} = 250\angle 90° \text{ V}\\ \mathbf{V}_C &= \frac{\mathbf{I}}{j\omega_0 C} = 250\angle -90° \text{ V}\end{aligned}$$

Note the magnitude of the voltages across the inductor and capacitor with respect to the input voltage. Note also that these voltages are equal and are 180° out of phase with one another. Therefore, the phasor diagram for this condition is shown in Fig. 12.19 for $\omega = \omega_0$. The quality factor Q derived from Eq. (12.13) is

$$Q = \frac{\omega_0 L}{R} = \frac{(2)(10^3)(25)(10^{-3})}{2} = 25$$

The voltages across the inductor and capacitor can be written in terms of Q as

$$|\mathbf{V}_L| = \omega_0 L|\mathbf{I}| = \frac{\omega_0 L}{R}|\mathbf{V}_S| = Q|\mathbf{V}_S|$$

and

$$|\mathbf{V}_C| = \frac{|\mathbf{I}|}{\omega_0 C} = \frac{1}{\omega_0 CR}|\mathbf{V}_S| = Q|\mathbf{V}_S|$$

This analysis indicates that for a given current there is a resonant voltage rise across the inductor and capacitor that is equal to the product of Q and the applied voltage.

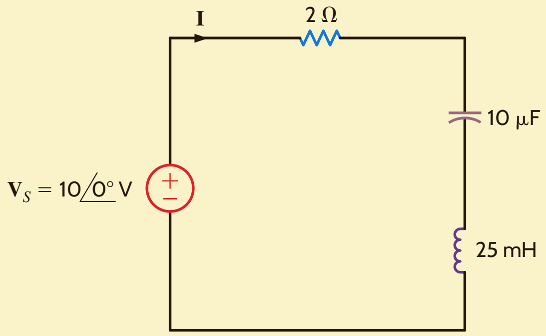


Figure 12.20 Series circuit.

EXAMPLE 12.8

In an undergraduate circuits laboratory, students are asked to construct an *RLC* network that will demonstrate resonance at $f = 1000$ Hz given a 0.02 H inductor that has a Q of 200. One student produced the circuit shown in **Fig. 12.21**, where the inductor's internal resistance is represented by R.

If the capacitor chosen to demonstrate resonance was an oil-impregnated paper capacitor rated at 300 V, let us determine the network parameters and the effect of this choice of capacitor.

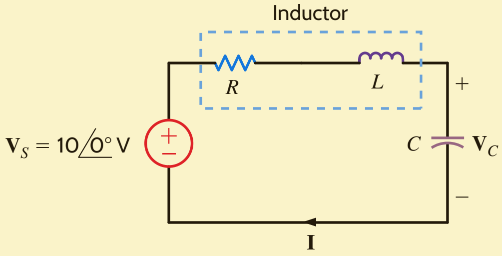


Figure 12.21 *RLC* series resonant network.

SOLUTION

For resonance at 1000 Hz, the student found the required capacitor value using the expression

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$

which yields

$$C = 1.27\ \mu\text{F}$$

The student selected an oil-impregnated paper capacitor rated at 300 V. The resistor value was found using the expression for Q

$$Q = \frac{\omega_0 L}{R} = 200$$

or

$$R = 1.59\ \Omega$$

At resonance, the current would be

$$\mathbf{I} = \frac{\mathbf{V}_S}{R}$$

or

$$\mathbf{I} = 6.28\underline{/0^\circ}\ \text{A}$$

When constructed, the current was measured to be only

$$\mathbf{I} \sim 1\underline{/0^\circ}\ \text{mA}$$

This measurement clearly indicated that the impedance seen by the source was about 10 kΩ of resistance instead of 1.59 Ω—quite a drastic difference. Suspecting that the capacitor that was selected was the source of the trouble, the student calculated what the capacitor voltage should be. If operated as designed, then at resonance,

$$\mathbf{V}_C = \frac{\mathbf{V}_S}{R}\left(\frac{1}{j\omega C}\right) = Q\mathbf{V}_S$$

or

$$\mathbf{V}_C = 2000\underline{/-90^\circ}\ \text{V}$$

which is more than six times the capacitor's rated voltage! This overvoltage had damaged the capacitor so that it did not function properly. When a new capacitor was selected and the source voltage reduced by a factor of 10, the network performed properly as a high Q circuit.

LEARNING ASSESSMENTS

E12.11 Given the network in Fig. E12.11, find the value C that will place the circuit in resonance at 1800 rad/s.

ANSWER: $C = 3.09\ \mu\text{F}$.

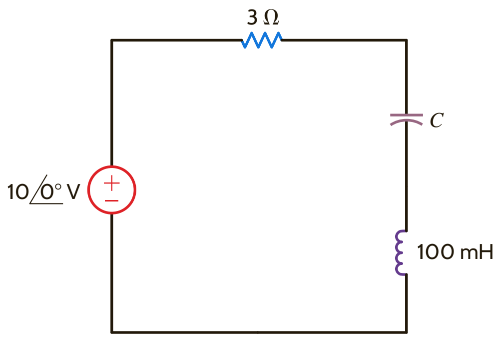


Figure E12.11

E12.12 Given the network in E12.12, determine the Q of the network and the magnitude of the voltage across the capacitor.

ANSWER: $Q = 60$; $|\mathbf{V}_C| = 600$ **V**.

E12.13 If the resonant frequency of the network in Fig. E12.13 is 10,000 rad/s, find L. Also compute the current at resonance, $\omega_0/3$, and $3\omega_0$.

ANSWER: $L = 100\ \mu\text{H}$; $6 \cos 10{,}000t$ A; $5.294 \cos(3333t + 28.07°)$ A; $5.294 \cos(30{,}000t - 28.07°)$ A.

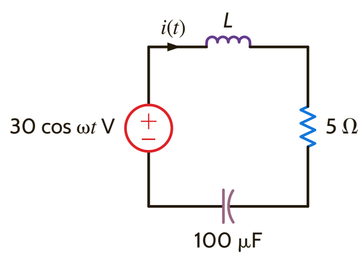


Figure E12.13

Let us develop a general expression for the ratio of $\mathbf{V}_R/\mathbf{V}_1$ for the network in Fig. 12.17 in terms of Q, ω, and ω_0. The impedance of the circuit, given by Eq. (12.10), can be used to determine the admittance, which can be expressed as

$$\begin{aligned}\mathbf{Y}(j\omega) &= \frac{1}{R[1 + j(1/R)(\omega L - 1/\omega C)]}\\ &= \frac{1}{R[1 + j(\omega L/R - 1/\omega CR)]}\\ &= \frac{1}{R[1 + jQ(\omega L/RQ - 1/\omega CRQ)]}\end{aligned} \tag{12.14}$$

Using the fact that $Q = \omega_0 L/R = 1/\omega_0 CR$, Eq. (12.14) becomes

$$\mathbf{Y}(j\omega) = \frac{1}{R[1 + jQ(\omega/\omega_0 - \omega_0/\omega)]} \tag{12.15}$$

Since $\mathbf{I} = \mathbf{Y}\mathbf{V}_1$ and the voltage across the resistor is $\mathbf{V}_R = \mathbf{I}_R$, then

$$\frac{\mathbf{V}_R}{\mathbf{V}_1} = \mathbf{G}_v(j\omega) = \frac{1}{1 + jQ(\omega/\omega_0 - \omega_0/\omega)} \tag{12.16}$$

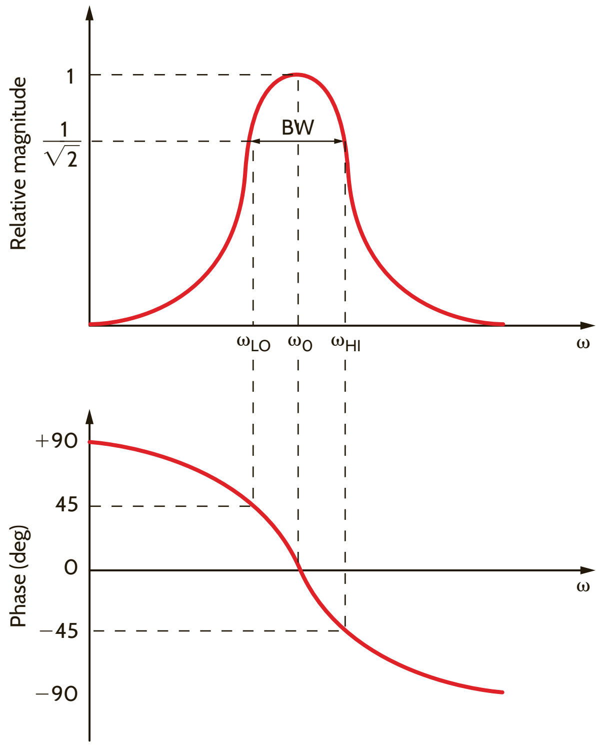


Figure 12.22

Magnitude and phase curves for Eqs. (12.17) and (12.18).

and the magnitude and phase are

$$M(\omega) = \frac{1}{[1 + Q^2(\omega/\omega_0 - \omega_0/\omega)^2]^{1/2}} \tag{12.17}$$

and

$$\phi(\omega) = -\tan^{-1} Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \tag{12.18}$$

The sketches for these functions are shown in **Fig. 12.22**. Note that the circuit has the form of a band-pass filter. The bandwidth is defined as the difference between the two half-power frequencies. Since power is proportional to the square of the magnitude, these two frequencies may be derived by setting the magnitude $M(\omega) = 1/\sqrt{2}$; that is,

$$\left|\frac{1}{1 + jQ(\omega/\omega_0 - \omega_0/\omega)}\right| = \frac{1}{\sqrt{2}}$$

Therefore,

$$Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \pm 1 \tag{12.19}$$

Solving this equation, we obtain four frequencies,

$$\omega = \pm\frac{\omega_0}{2Q} \pm \omega_0\sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \tag{12.20}$$

Taking only the positive values, we obtain

$$\begin{aligned} \omega_{LO} &= \omega_0\left[-\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}\right] \\ \omega_{HI} &= \omega_0\left[\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}\right] \end{aligned} \tag{12.21}$$

HINT

Half-power frequencies and their dependence on ω_0 and Q are outlined in these equations.

HINT

The bandwidth is the difference between the half-power frequencies and a function of ω_0 and Q.

Subtracting these two equations yields the bandwidth as shown in Fig. 12.22:

$$\text{BW} = \omega_{HI} - \omega_{LO} = \frac{\omega_0}{Q} \tag{12.22}$$

and multiplying the two equations yields

$$\omega_0^2 = \omega_{LO}\omega_{HI} \tag{12.23}$$

which illustrates that the resonant frequency is the geometric mean of the two half-power frequencies. Recall that the half-power frequencies are the points at which the log-magnitude curve is down 3 dB from its maximum value. Therefore, the difference between the 3-dB frequencies, which is, of course, the bandwidth, is often called the 3-dB bandwidth.

LEARNING ASSESSMENT

E12.14 For the network in Fig. E12.11, compute the two half-power frequencies and the bandwidth of the network.

ANSWER:
$\omega_{HI} = 1815$ rad/s;
$\omega_{LO} = 1785$ rad/s;
BW $= 30$ rad/s.

Eq. (12.13) indicates the dependence of Q on R. A high-Q series circuit has a small value of R.

Eq. (12.22) illustrates that the bandwidth is inversely proportional to Q. Therefore, the frequency selectivity of the circuit is determined by the value of Q. A high-Q circuit has a small bandwidth and, therefore, the circuit is very selective. The manner in which Q affects the frequency selectivity of the network is graphically illustrated in Fig. 12.23. Hence, if we pass a signal with a wide frequency range through a high-Q circuit, only the frequency components within the bandwidth of the network will not be attenuated; that is, the network acts like a band-pass filter.

Q has a more general meaning that we can explore via an energy analysis of the series resonant circuit. Let's excite a series RLC circuit at its resonant frequency as shown in Fig. 12.24. Recall that the impedance of the RLC circuit at resonance is just R. Therefore, the current $i(t) = (V_m/R)\cos\omega_0 t$ A. The capacitor voltage is

$$\mathbf{V}_C = \frac{1}{j\omega_0 C}\mathbf{I} = \frac{1}{j\omega_0 C}\frac{V_m}{R}\underline{/0^\circ} = \frac{V_m}{\omega_0 RC}\underline{/-90^\circ} \tag{12.24}$$

and $v_C(t) = \dfrac{V_m}{\omega_0 RC}\cos(\omega_0 t - 90^\circ) = \dfrac{V_m}{\omega_0 RC}\sin\omega_0 t$ volts. Recall from Chapter 6 that the energy stored in an inductor is $(1/2)Li^2$ and the energy stored in a capacitor is $(1/2)Cv^2$. For the inductor:

$$w_L(t) = \frac{1}{2}Li^2(t) = \frac{1}{2}L\left(\frac{V_m}{R}\cos\omega_0 t\right)^2 = \frac{V_m^2 L}{2R^2}\cos^2\omega_0 t \text{ J} \tag{12.25}$$

and for the capacitor:

$$w_C(t) = \frac{1}{2}Cv_C^2(t) = \frac{1}{2}C\left(\frac{V_m}{\omega_0 RC}\sin\omega_0 t\right)^2 = \frac{V_m^2}{2\omega_0^2 R^2 C}\sin^2\omega_0 t \text{ J} \tag{12.26}$$

At resonance, $\omega_0^2 = 1/LC$, so the energy stored in the capacitor can be rewritten as

$$w_C(t) = \frac{V_m^2}{2\left(\frac{1}{LC}\right)R^2 C}\sin^2\omega_0 t = \frac{V_m^2 L}{2R^2}\sin^2\omega_0 t \text{ J} \tag{12.27}$$

The total energy stored in the circuit is $w_L(t) + w_C(t) = \dfrac{V_m^2 L}{2R^2}(\cos^2\omega_0 t + \sin^2\omega_0 t)$. From trigonometry, we know that $\cos^2\omega_0 t + \sin^2\omega_0 t = 1$, so the total energy stored is a constant: $\dfrac{V_m^2 L}{2R^2}$ J.

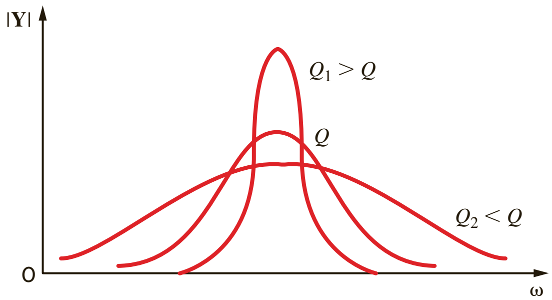


Figure 12.23

Network frequency response as a function of Q.

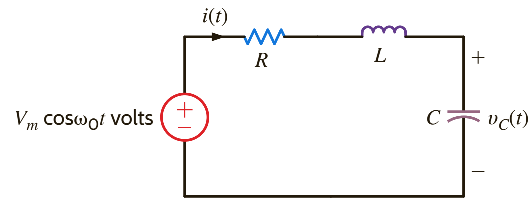


Figure 12.24

Series RLC circuit excited at its resonant frequency.

Now that we have determined that the total energy stored in the resonant circuit is a constant, let's examine the energy stored in the inductor and capacitor. Fig. 12.25 is a plot of the normalized energy stored in each element over two periods. Eq. (12.25) and (12.27) have been divided by $\frac{V_m^2 L}{2R^2}$ to yield the normalized energy. When a circuit is in resonance, there is a continuous exchange of energy between the magnetic field of the inductor and the electric field of the capacitor. This energy exchange is like the motion of a pendulum. The energy stored in the inductor starts at a maximum value, falls to zero, and then returns to a maximum; the energy stored in the capacitor starts at zero, increases to a maximum, and then returns to zero. Note that when the energy stored in the inductor is a maximum, the energy stored in the capacitor is zero and vice versa. In the first half-cycle, the capacitor absorbs energy as fast as the inductor gives it up; the opposite happens in the next half-cycle. Even though the energy stored in each element is continuously varying, the total energy stored in the resonant circuit is constant and therefore not changing with time.

The maximum energy stored in the RLC circuit at resonance is $W_S = \frac{V_m^2 L}{2R^2}$. Let's calculate the energy dissipated per cycle in this series resonant circuit, which is

$$W_D = \int_0^T p_R\,dt = \int_0^T i^2(t)R\,dt = \int_0^T \left(\frac{V_m}{R}\cos^2\omega_0 t\right)^2 R\,dt = \frac{V_m^2 T}{2R} \qquad \textbf{12.28}$$

The ratio of W_S to W_D is

$$\frac{W_S}{W_D} = \frac{\dfrac{V_m^2 L}{2R^2}}{\dfrac{V_m^2 T}{2R}} = \frac{L}{RT} = \frac{L}{R\dfrac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R(2\pi)} \qquad \textbf{12.29}$$

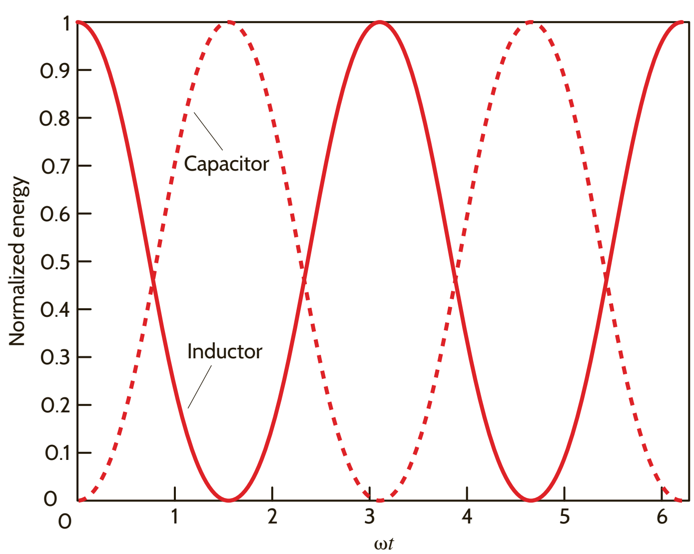


Figure 12.25

Energy transfer in a resonant circuit.

Earlier in this chapter, we defined Q to be $\omega_0 L/R$, so the equation above can be rewritten as

$$Q = 2\pi \frac{W_S}{W_D} \qquad \textbf{12.30}$$

The importance of this expression for Q stems from the fact that this expression is applicable to acoustic, electrical, and mechanical systems and therefore is generally considered to be the basic definition of Q.

EXAMPLE 12.9

Given a series circuit with $R = 2\ \Omega$, $L = 2$ mH, and $C = 5\ \mu$F, we wish to determine the resonant frequency, the quality factor, and the bandwidth for the circuit. Then we will determine the change in Q and the BW if R is changed from 2 to 0.2 Ω.

SOLUTION

Using Eq. (12.11), we have

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{[(2)(10^{-3})(5)(10^{-6})]^{1/2}}$$
$$= 10^4 \text{ rad/s}$$

and therefore, the resonant frequency is $10^4/2\pi = 1592$ Hz.

The quality factor is

$$Q = \frac{\omega_0 L}{R} = \frac{(10^4)(2)(10^{-3})}{2}$$
$$= 10$$

and the bandwidth is

$$\text{BW} = \frac{\omega_0}{Q} = \frac{10^4}{10}$$
$$= 10^3 \text{ rad/s}$$

If R is changed to $R = 0.2\ \Omega$, the new value of Q is 100 and, therefore, the new BW is 10^2 rad/s.

LEARNING ASSESSMENTS

E12.15 A series circuit is composed of $R = 2\ \Omega$, $L = 40$ mH, and $C = 100\ \mu$F. Determine the bandwidth of this circuit and its resonant frequency.

ANSWER: BW = 50 rad/s; $\omega_0 = 500$ rad/s.

E12.16 A series RLC circuit has the following properties: $R = 4\ \Omega$, $\omega_0 = 4000$ rad/s, and the BW = 100 rad/s. Determine the values of L and C.

ANSWER: $L = 40$ mH; $C = 1.56\ \mu$F.

EXAMPLE 12.10

We wish to determine the parameters R, L, and C so that the circuit shown in Fig. 12.26 operates as a band-pass filter with an ω_0 of 1000 rad/s and a bandwidth of 100 rad/s.

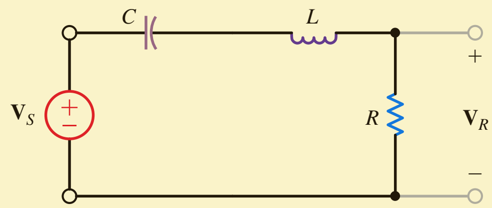


Figure 12.26
Series *RLC* circuit.

SOLUTION

The voltage gain for the network is

$$\mathbf{G}_v(j\omega) = \frac{(R/L)j\omega}{(j\omega)^2 + (R/L)j\omega + 1/LC}$$

Hence,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and since $\omega_0 = 10^3$,

$$\frac{1}{\sqrt{LC}} = 10^6$$

The bandwidth is

$$\text{BW} = \frac{\omega_0}{Q}$$

Then

$$Q = \frac{\omega_0}{\text{BW}} = \frac{1000}{100}$$
$$= 10$$

However,

$$Q = \frac{\omega_0 L}{R}$$

Therefore,

$$\frac{1000L}{R} = 10$$

Note that we have two equations in the three unknown circuit parameters R, L, and C. Hence, if we select $C = 1\ \mu\text{F}$, then

$$L = \frac{1}{10^6 C} = 1\text{ H}$$

and

$$\frac{1000(1)}{R} = 10$$

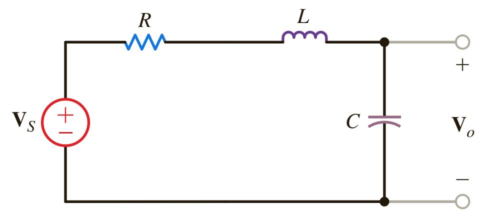
yields

$$R = 100\ \Omega$$

Therefore, the parameters $R = 100\ \Omega$, $L = 1$ H, and $C = 1\ \mu$F will produce the proper filter characteristics.

Figure 12.27

Series resonant circuit.



In Examples 12.7 and 12.8 we found that the voltage across the capacitor or inductor in the series resonant circuit could be quite high. In fact, it was equal to Q times the magnitude of the source voltage. With this in mind, let us reexamine this network as shown in Fig. 12.27. The output voltage for the network is

$$\mathbf{V}_o = \left(\frac{1/j\omega C}{R + j\omega L + 1/j\omega C}\right)\mathbf{V}_S$$

which can be written as

$$\mathbf{V}_o = \frac{\mathbf{V}_S}{1 - \omega^2 LC + j\omega CR}$$

The magnitude of this voltage can be expressed as

$$|\mathbf{V}_o| = \frac{|\mathbf{V}_S|}{\sqrt{(1 - \omega^2 LC)^2 + (\omega CR)^2}} \qquad \textbf{12.31}$$

In view of the previous discussion, we might assume that the maximum value of the output voltage would occur at the resonant frequency ω_0. Let us see whether this assumption is correct. The frequency at which $|\mathbf{V}_o|$ is maximum is the nonzero value of ω, which satisfies the equation

$$\frac{d|\mathbf{V}_o|}{d\omega} = 0 \qquad \textbf{12.32}$$

If we perform the indicated operation and solve for the nonzero ω_{max}, we obtain

$$\omega_{max} = \sqrt{\frac{1}{LC} - \frac{1}{2}\left(\frac{R}{L}\right)^2} \qquad \textbf{12.33}$$

By employing the relationships $\omega_0^2 = 1/LC$ and $Q = \omega_0 L/R$, the expression for ω_{max} can be written as

$$\begin{aligned}\omega_{max} &= \sqrt{\omega_0^2 - \frac{1}{2}\left(\frac{\omega_0}{Q}\right)^2} \\ &= \omega_0\sqrt{1 - \frac{1}{2Q^2}}\end{aligned} \qquad \textbf{12.34}$$

Clearly, $\omega_{max} \neq \omega_0$; however, ω_0 closely approximates ω_{max} if the Q is high. In addition, if we substitute Eq. (12.34) into Eq. (12.31) and use the relationships $\omega_0^2 = 1/LC$ and $\omega_0^2 C^2 R^2 = 1/Q^2$, we find that

$$|\mathbf{V}_o|_{max} = \frac{Q|\mathbf{V}_S|}{\sqrt{1 - 1/4Q^2}} \qquad \textbf{12.35}$$

Again, we see that $|\mathbf{V}_o|_{max} \approx Q|\mathbf{V}_S|$ if the network has a high Q.

EXAMPLE 12.11

Given the network in Fig. 12.27, we wish to determine ω_0 and ω_{max} for $R = 50\ \Omega$ and $R = 1\ \Omega$ if $L = 50$ mH and $C = 5\ \mu$F.

SOLUTION

The network parameters yield

$$\begin{aligned}\omega_0 &= \frac{1}{\sqrt{LC}} \\ &= \frac{1}{\sqrt{(5)(10^{-2})(5)(10^{-6})}} \\ &= 2000 \text{ rad/s}\end{aligned}$$

If $R = 50\ \Omega$, then

$$Q = \frac{\omega_0 L}{R} = \frac{(2000)(0.05)}{50} = 2$$

and

$$\omega_{\max} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} = 2000\sqrt{1 - \frac{1}{8}} = 1871 \text{ rad/s}$$

If $R = 1\ \Omega$, then $Q = 100$ and $\omega_{\max} = 2000$ rad/s.

We can plot the frequency response of the network transfer function for $R = 50\ \Omega$ and $R = 1\ \Omega$. The transfer function is

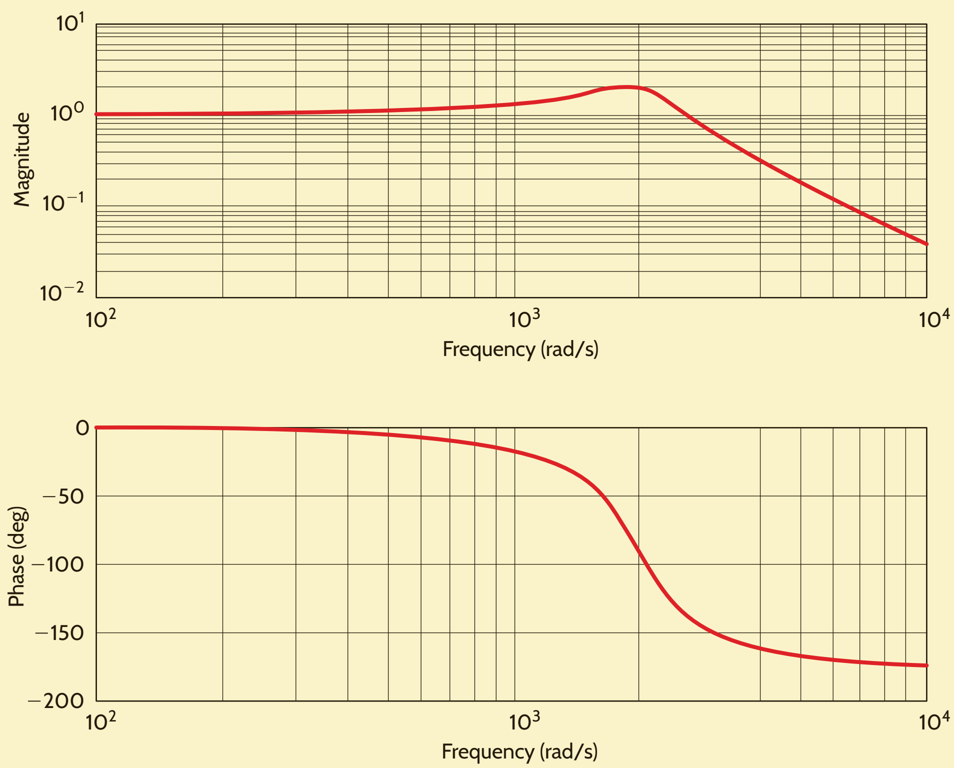
$$\frac{\mathbf{V}_o}{\mathbf{V}_S} = \frac{1}{2.5 \times 10^{-7}(j\omega)^2 + 2.5 \times 10^{-4}(j\omega) + 1}$$

for $R = 50\ \Omega$ and

$$\frac{\mathbf{V}_o}{\mathbf{V}_S} = \frac{1}{2.5 \times 10^{-7}(j\omega)^2 + 5 \times 10^{-6}(j\omega) + 1}$$

for $R = 1\ \Omega$. The magnitude and phase characteristics for the network with $R = 50\ \Omega$ and $R = 1\ \Omega$ are shown in Figs. 12.28a and b, respectively.

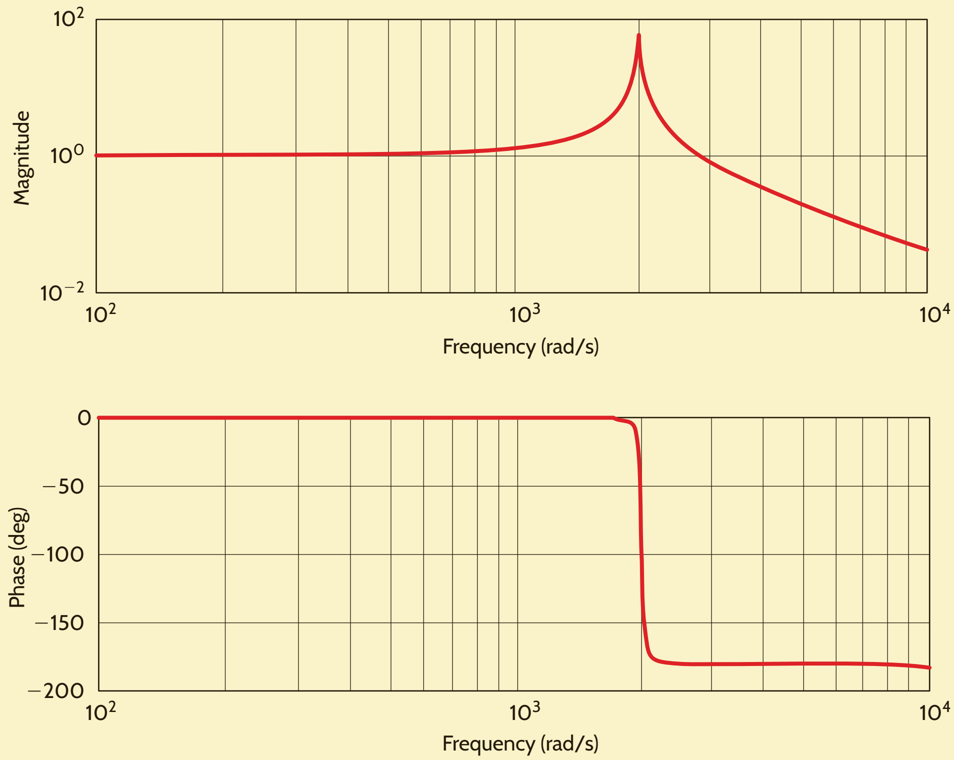
Note that when the Q of the network is small, the frequency response is not selective and $\omega_0 \neq \omega_{\max}$. However, if the Q is large, the frequency response is very selective and $\omega_0 \simeq \omega_{\max}$.



(a)

Figure 12.28 Frequency response plots for the network in Fig. 12.27 with (a) $R = 50\ \Omega$ and (b) $R = 1\ \Omega$.

Figure 12.28
(continued)



(b)

EXAMPLE 12.12

On July 1, 1940, the third longest bridge in the nation, the Tacoma Narrows Bridge, was opened to traffic across Puget Sound in Washington. On November 7, 1940, the structure collapsed in what has become the most celebrated structural failure of that century. A photograph of the bridge, taken as it swayed back and forth just before breaking apart, is shown in Fig. 12.29. Explaining the disaster in quantitative terms is a feat for civil engineers and structures experts, and several theories have been presented. However, the one common denominator in each explanation is that wind blowing across the bridge caused the entire structure to resonate to such an extent that the bridge tore itself apart. One can theorize that the wind, fluctuating at a frequency near the natural frequency of the bridge (0.2 Hz), drove the structure into resonance. Thus, the bridge can be roughly modeled as a second-order system. Let us design an *RLC* resonance network to demonstrate the bridge's vertical movement and investigate the effect of the wind's frequency.

SOLUTION

The *RLC* network shown in Fig. 12.30 is a second-order system in which $v_{in}(t)$ is analogous to vertical deflection of the bridge's roadway (1 volt = 1 foot). The values of C, L, R_A, and R_B can be derived from the data taken at the site and from scale models, as follows:

$$\text{vertical deflection at failure} \approx 4 \text{ feet}$$
$$\text{wind speed at failure} \approx 42 \text{ mph}$$
$$\text{resonant frequency} = f_0 \approx 0.2 \text{ Hz}$$

The output voltage can be expressed as

$$\mathbf{V}_o(j\omega) = \frac{j\omega\left(\dfrac{R_B}{L}\right)\mathbf{V}_{in}(j\omega)}{-\omega^2 + j\omega\left(\dfrac{R_A + R_B}{L}\right) + \dfrac{1}{LC}}$$

Figure 12.29

Tacoma Narrows Bridge on the verge of collapse (AP Photo).

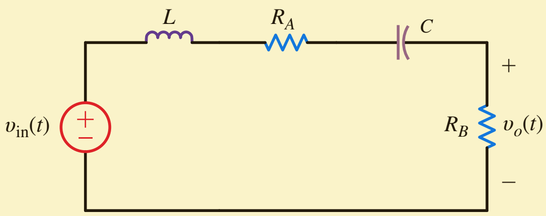


Figure 12.30

RLC resonance network for a simple Tacoma Narrows Bridge simulation.

from which we can easily extract the following expressions:

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi(0.2) \text{ rad/s}$$

$$2\zeta\omega_0 = \frac{R_A + R_B}{L}$$

and

$$\frac{\mathbf{V}_o(j\omega_0)}{\mathbf{V}_{in}(j\omega_0)} = \frac{R_B}{R_A + R_B} \approx \frac{4 \text{ feet}}{42 \text{ mph}}$$

Let us choose $R_B = 1\ \Omega$ and $R_A = 9.5\ \Omega$. Having no data for the damping ratio, ζ, we will select $L = 20$ H, which yields $\zeta = 0.209$ and $Q = 2.39$, which seem reasonable for such a large structure. Given the aforementioned choices, the required capacitor value is $C = 31.66$ mF. Using these circuit values, we now simulate the effect of 42 mph winds fluctuating at 0.05 Hz, 0.1 Hz, and 0.2 Hz using an ac analysis at the three frequencies of interest.

The results are shown in Fig. 12.31. Note that at 0.05 Hz the vertical deflection (1 ft/V) is only 0.44 feet, whereas at 0.1 Hz the bridge undulates about 1.07 feet. Finally, at the bridge's resonant frequency of 0.2 Hz, the bridge is oscillating 3.77 feet—catastrophic failure.

Clearly, we have used an extremely simplistic approach to modeling something as complicated as the Tacoma Narrows Bridge. However, we will revisit this event in Chapter 14 and examine it more closely with a more accurate model (K. Y. Billah and R. H. Scalan, "Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks," *American Journal of Physics*, 1991, vol. 59, no. 2, pp. 118–124).

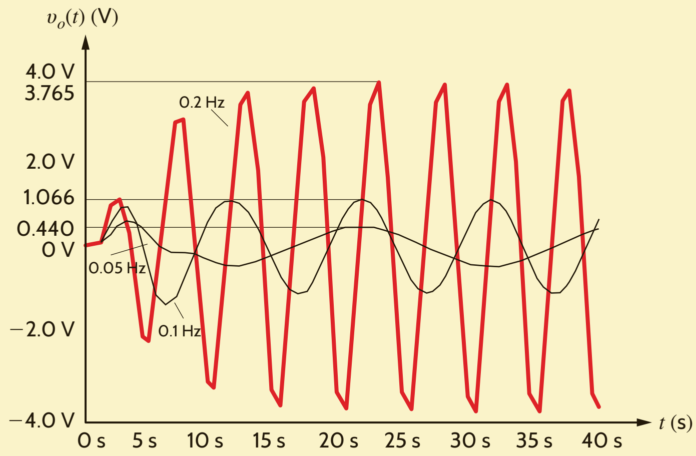


Figure 12.31

Simulated vertical deflection (1 volt = 1 foot) for the Tacoma Narrows Bridge for wind shift frequencies of 0.05, 0.1, and 0.2 Hz.

PARALLEL RESONANCE In our presentation of resonance thus far, we have focused our discussion on the series resonant circuit. Of course, resonance and all its ramifications still apply if the *RLC* elements are arranged in parallel. In fact, the series and parallel resonant circuits possess many similarities and a few differences.

Consider the network shown in **Fig. 12.32**. The source current $\mathbf{I}_S$ can be expressed as

$$\begin{aligned}\mathbf{I}_S &= \mathbf{I}_G + \mathbf{I}_C + \mathbf{I}_L \\ &= \mathbf{V}_S G + j\omega C\mathbf{V}_S + \frac{\mathbf{V}_S}{j\omega L} \\ &= \mathbf{V}_S\left[G + j\left(\omega C - \frac{1}{\omega L}\right)\right]\end{aligned}$$

When the network is in resonance,

$$\mathbf{I}_S = G\mathbf{V}_S \tag{12.36}$$

The input admittance for the parallel *RLC* circuit is

$$\mathbf{Y}(j\omega) = G + j\omega C + \frac{1}{j\omega L} \tag{12.37}$$

and the admittance of the parallel circuit, at resonance, is

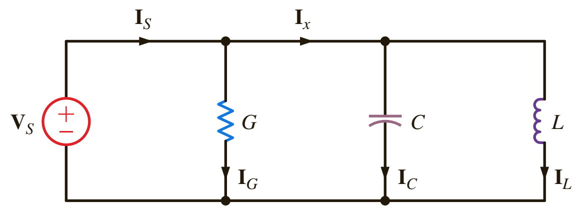
$$\mathbf{Y}(j\omega_0) = G \tag{12.38}$$

that is, all the source current flows through the conductance G. Does this mean that there is no current in L or C? Definitely not! $\mathbf{I}_C$ and $\mathbf{I}_L$ are equal in magnitude but 180° out of phase with one another. Therefore, $\mathbf{I}_x$, as shown in Fig. 12.32, is zero. In addition, if $G = 0$, the source current is zero. What is actually taking place, however, is an energy exchange between the electric field of the capacitor and the magnetic field of the inductor. As one increases, the other decreases and vice versa.

Analogous to the series resonant case, the frequency response, shown in **Fig. 12.33a**, for the parallel resonant circuit reveals that the admittance is dominated by the inductive term at low frequencies and by the capacitive term at high frequencies. Similarly, the phasor diagram

Figure 12.32

Parallel *RLC* circuit.



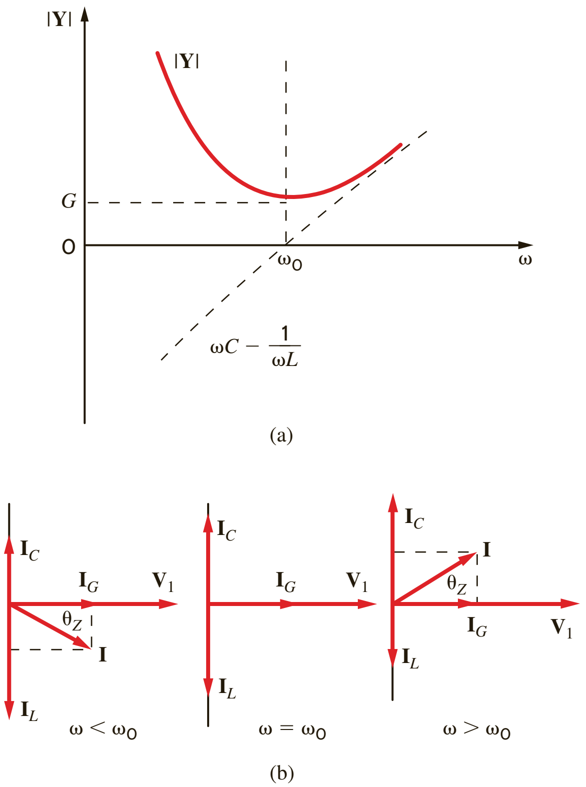


Figure 12.33
(a) The frequency plot of the admittance and (b) the phasor diagram for the parallel resonant circuit.

for the parallel resonant circuit, shown in **Fig. 12.33b**, again has much in common with that of the series circuit. For $\omega < \omega_0$, the impedance phase angle, θ_Z, is positive, again indicating that inductance dominates in the parallel circuit at low frequencies. For $\omega > \omega_0$, θ_Z is negative, and the capacitance dominates.

Applying the general definition of resonance to the parallel resonant circuit yields an interesting result

$$Q = \frac{R}{\omega_0 L} = \frac{1}{G\omega_0 L} = R\omega_0 C = \frac{\omega_0 C}{G} \qquad \textbf{12.39}$$

This result appears to be the reciprocal of Q for the series case. However, the RLC currents in the parallel case mimic the voltages in the series case:

$$|\mathbf{I}_C| = Q|\mathbf{I}_S| \qquad \textbf{12.40}$$

and

$$|\mathbf{I}_L| = Q|\mathbf{I}_S|$$

EXAMPLE 12.13

The network in Fig. 12.32 has the following parameters:

$$\mathbf{V}_S = 120\underline{/0^\circ}\text{ V}, \qquad G = 0.01\text{ S},$$
$$C = 600\ \mu\text{F}, \quad \text{and} \quad L = 120\text{ mH}$$

If the source operates at the resonant frequency of the network, compute all the branch currents.

SOLUTION

The resonant frequency for the network is

$$\begin{aligned}\omega_0 &= \frac{1}{\sqrt{LC}}\\ &= \frac{1}{\sqrt{(120)(10^{-3})(600)(10^{-6})}}\\ &= 117.85\text{ rad/s}\end{aligned}$$

At this frequency

$$\mathbf{Y}_C = j\omega_0 C = j7.07 \times 10^{-2}\text{ S}$$

and

$$\mathbf{Y}_L = -j\left(\frac{1}{\omega_0 L}\right) = -j7.07 \times 10^{-2}\text{ S}$$

The branch currents are then

$$\begin{aligned}\mathbf{I}_G &= G\mathbf{V}_S = 1.2\underline{/0^\circ}\text{ A}\\ \mathbf{I}_C &= \mathbf{Y}_C\mathbf{V}_S = 8.49\underline{/90^\circ}\text{ A}\\ \mathbf{I}_L &= \mathbf{Y}_L\mathbf{V}_S = 8.49\underline{/-90^\circ}\text{ A}\end{aligned}$$

and

$$\begin{aligned}\mathbf{I}_S &= \mathbf{I}_G + \mathbf{I}_C + \mathbf{I}_L\\ &= \mathbf{I}_G = 1.2\underline{/0^\circ}\text{ A}\end{aligned}$$

As the analysis indicates, the source supplies only the losses in the resistive element. In addition, the source voltage and current are in phase and, therefore, the power factor is unity.

EXAMPLE 12.14

Given the parallel *RLC* circuit in Fig. 12.34,

a. Derive the expression for the resonant frequency, the half-power frequencies, the bandwidth, and the quality factor for the transfer characteristic $\mathbf{V}_{out}/\mathbf{I}_{in}$ in terms of the circuit parameters *R*, *L*, and *C*.

b. Compute the quantities in part (a) if $R = 1\text{ k}\Omega$, $L = 10\text{ mH}$, and $C = 100\ \mu\text{F}$.

SOLUTION

a. The output voltage can be written as

$$\mathbf{V}_{out} = \frac{\mathbf{I}_{in}}{\mathbf{Y}_T}$$

and, therefore, the magnitude of the transfer characteristic can be expressed as

$$\left|\frac{\mathbf{V}_{out}}{\mathbf{I}_{in}}\right| = \frac{1}{\sqrt{(1/R^2) + (\omega C - 1/\omega L)^2}}$$

The transfer characteristic is a maximum at the resonant frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \textbf{12.41}$$

and at this frequency

$$\left|\frac{\mathbf{V}_{out}}{\mathbf{I}_{in}}\right|_{max} = R \qquad \textbf{12.42}$$

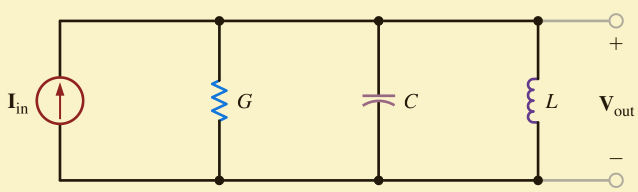


Figure 12.34

Circuit used in Example 12.14.

As demonstrated earlier, at the half-power frequencies the magnitude is equal to $1/\sqrt{2}$ of its maximum value, and hence the half-power frequencies can be obtained from the expression

$$\frac{1}{\sqrt{(1/R^2) + (\omega C - 1/\omega L)^2}} = \frac{R}{\sqrt{2}}$$

Solving this equation and taking only the positive values of ω yields

$$\omega_{LO} = -\frac{1}{2RC} + \sqrt{\frac{1}{(2RC)^2} + \frac{1}{LC}} \qquad \textbf{12.43}$$

and

$$\omega_{HI} = \frac{1}{2RC} + \sqrt{\frac{1}{(2RC)^2} + \frac{1}{LC}} \qquad \textbf{12.44}$$

Subtracting these two half-power frequencies yields the bandwidth

$$\begin{aligned} BW &= \omega_{HI} - \omega_{LO} \qquad \textbf{12.45} \\ &= \frac{1}{RC} \end{aligned}$$

Therefore, the quality factor is

$$\begin{aligned} Q &= \frac{\omega_0}{BW} \\ &= \frac{RC}{\sqrt{LC}} \qquad \textbf{12.46} \\ &= R\sqrt{\frac{C}{L}} \end{aligned}$$

Using Eqs. (12.41), (12.45), and (12.46), we can write Eqs. (12.43) and (12.44) as

$$\omega_{LO} = \omega_0\left[\frac{-1}{2Q} + \sqrt{\frac{1}{(2Q)^2} + 1}\right] \qquad \textbf{12.47}$$

$$\omega_{HI} = \omega_0\left[\frac{1}{2Q} + \sqrt{\frac{1}{(2Q)^2} + 1}\right] \qquad \textbf{12.48}$$

b. Using the values given for the circuit components, we find that

$$\omega_0 = \frac{1}{\sqrt{(10^{-2})(10^{-4})}} = 10^3 \text{ rad/s}$$

The half-power frequencies are

$$\begin{aligned} \omega_{LO} &= \frac{-1}{(2)(10^3)(10^{-4})} + \sqrt{\frac{1}{[(2)(10^{-1})]^2} + 10^6} \\ &= 995 \text{ rad/s} \end{aligned}$$

and

$$\omega_{HI} = 1005 \text{ rad/s}$$

Therefore, the bandwidth is

$$BW = \omega_{HI} = \omega_{LO} = 10 \text{ rad/s}$$

and

$$\begin{aligned} Q &= 10^3\sqrt{\frac{10^{-4}}{10^{-2}}} \\ &= 100 \end{aligned}$$

EXAMPLE 12.15

Two radio stations, WHEW and WHAT, broadcast in the same listening area: WHEW broadcasts at 100 MHz and WHAT at 98 MHz. A single-stage tuned amplifier, such as that shown in **Fig. 12.35**, can be used as a tuner to filter out one of the stations. However, single-stage tuned amplifiers have poor selectivity due to their wide bandwidths. To reduce the bandwidth (increase the quality factor) of single-stage tuned amplifiers, designers employ a technique called synchronous tuning. In this process, identical tuned amplifiers are cascaded. To demonstrate this phenomenon, let us generate a Bode plot for the amplifier shown in Fig. 12.35 when it is tuned to WHEW (100 MHz), using one, two, three, and four stages of amplification.

SOLUTION

Using the circuit for a single-stage amplifier shown in Fig. 12.35, we can cascade the stages to form a four-stage synchronously tuned amplifier. If we now plot the frequency response over the range from 90 MHz to 110 MHz, we obtain the Bode plot shown in **Fig. 12.36**.

From the Bode plot in Fig. 12.36 we see that increasing the number of stages does indeed decrease the bandwidth without altering the center frequency. As a result, the quality factor and selectivity increase. Accordingly, as we add stages, the gain at 98 MHz (WHAT's frequency) decreases, and that station is "tuned out."

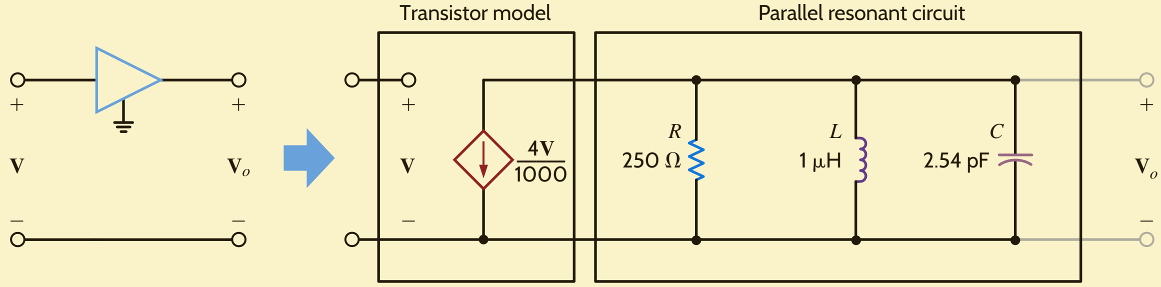
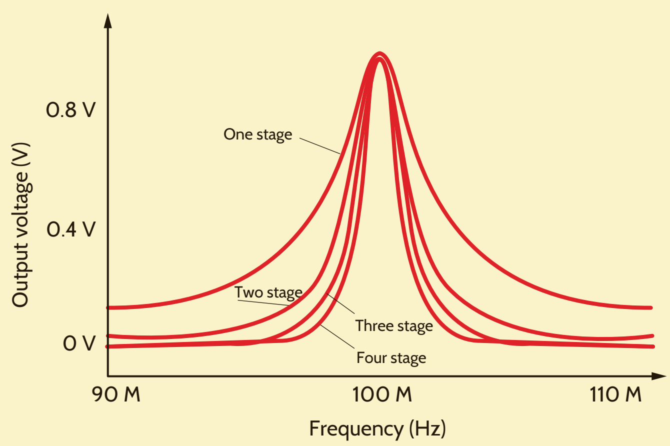


Figure 12.35
Single-stage tuned amplifier.

Figure 12.36
Bode plots for one-, two-, three-, and four-stage tuned amplifiers.



LEARNING ASSESSMENTS

E12.17 A parallel *RLC* circuit has the following parameters: $R = 2\ \text{k}\Omega$, $L = 20$ mH, and $C = 150\ \mu\text{F}$. Determine the resonant frequency, the Q, and the bandwidth of the circuit.

ANSWER:
$\omega_0 = 577$ rad/s; $Q = 173$; BW $= 3.33$ rad/s.

E12.18 A parallel *RLC* circuit has the following parameters: $R = 6\ \text{k}\Omega$, BW $= 1000$ rad/s, and $Q = 120$. Determine the values of L, C, and ω_0.

ANSWER:
$L = 417.5\ \mu\text{H}$; $C = 0.167\ \mu\text{F}$; $\omega_0 = 119{,}760$ rad/s.

E12.19 The parallel *RLC* resonant circuit in Fig. E12.19 has a resonant frequency of 12,000 rad/s and an admittance of 5 mS at resonance. Find R and C.

ANSWER:
$R = 200\ \Omega$; $C = 69.44$ nF.

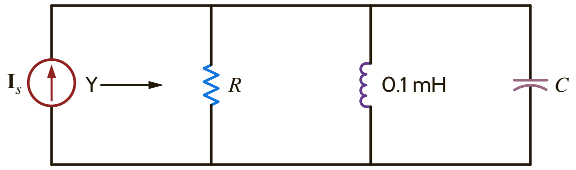


Figure E12.19

In general, the resistance of the winding of an inductor cannot be neglected, and hence a more practical parallel resonant circuit is the one shown in Fig. 12.37. The input admittance of this circuit is

$$\begin{aligned}\mathbf{Y}(j\omega) &= j\omega C + \frac{1}{R + j\omega L}\\ &= j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2}\\ &= \frac{R}{R^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)\end{aligned}$$

The frequency at which the admittance is purely real is

$$\omega_r C - \frac{\omega_r L}{R^2 + \omega_r^2 L^2} = 0$$

$$\omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad \textbf{12.49}$$

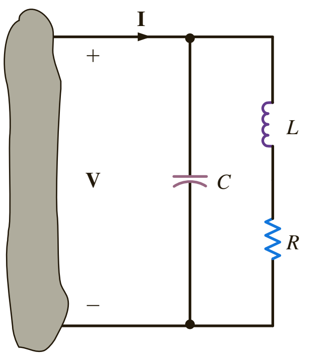


Figure 12.37
Practical parallel resonant circuit.

EXAMPLE 12.16

Given the tank circuit in Fig. 12.38, let us determine ω_0 and ω_r for $R = 50\ \Omega$ and $R = 5\ \Omega$.

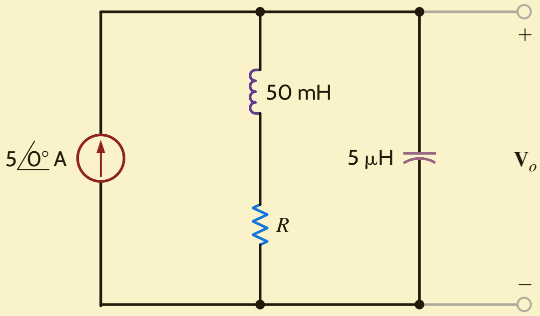


Figure 12.38
Tank circuit used in Example 12.16.

SOLUTION Using the network parameter values, we obtain

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{(0.05)(5)(10^{-6})}}$$
$$= 2000 \text{ rad/s}$$
$$f_0 = 318.3 \text{ Hz}$$

If $R = 50\ \Omega$, then

$$\omega_r = \sqrt{\frac{1}{\sqrt{LC}} - \frac{R^2}{L^2}}$$
$$= \sqrt{\frac{1}{(0.05)(5)(10^{-6})} - \left(\frac{50}{0.05}\right)^2}$$
$$= 1732 \text{ rad/s}$$
$$f_r = 275.7 \text{ Hz}$$

If $R = 5\ \Omega$, then

$$\omega_r = \sqrt{\frac{1}{(0.05)(5)(10^{-6})} - \left(\frac{5}{0.05}\right)^2}$$
$$= 1997 \text{ rad/s}$$
$$f_r = 317.9 \text{ Hz}$$

Note that as $R \to 0$, $\omega_r \to \omega_0$. This fact is also illustrated in the frequency-response curves in **Figs. 12.39a** and **b**, where we have plotted $|\mathbf{V}_o|$ versus frequency for $R = 50\ \Omega$ and $R = 5\ \Omega$, respectively.

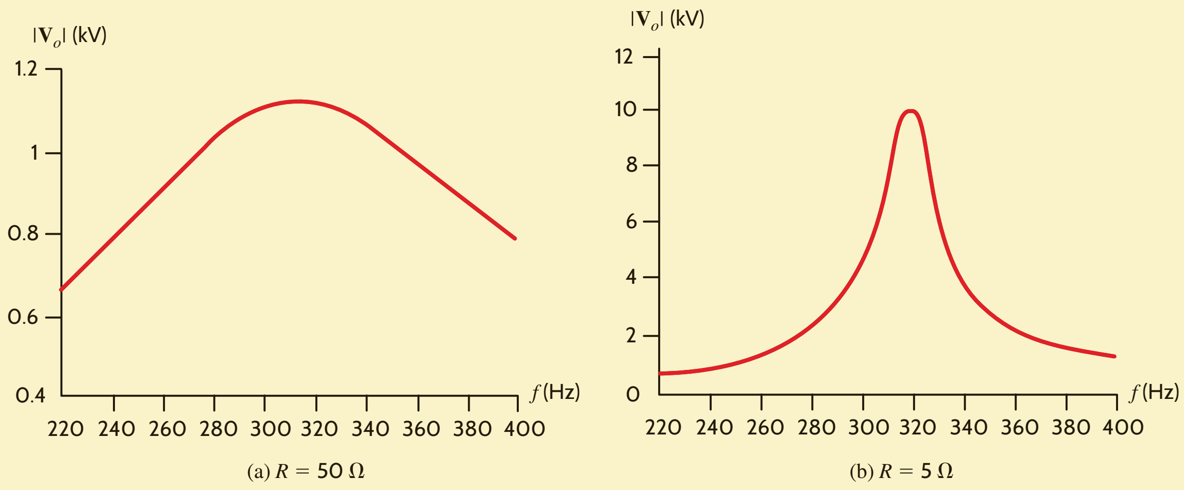


Figure 12.39

Frequency-response curves for Example 12.16.

Let us now try to relate some of the things we have learned about resonance to the Bode plots we presented earlier. The admittance for the series resonant circuit is

$$\mathbf{Y}(j\omega) = \frac{1}{R + j\omega L + 1/j\omega C}$$
$$= \frac{j\omega C}{(j\omega)^2 LC + j\omega CR + 1} \qquad \textbf{12.50}$$