9.2 Average Power

The average value of any periodic waveform (e.g., a sinusoidal function) can be computed by integrating the function over a complete period and dividing this result by the period. Therefore, if the voltage and current are given by Eqs. (9.1) and (9.2), respectively, the average power is

$$\begin{aligned} P &= \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\,dt \\ &= \frac{1}{T}\int_{t_0}^{t_0+T} V_M I_M \cos(\omega t+\theta_v)\cos(\omega t+\theta_i)\,dt \end{aligned} \qquad \textbf{9.6}$$

where t_0 is arbitrary, $T = 2\pi/\omega$ is the period of the voltage or current, and P is measured in watts. Actually, we may average the waveform over any integral number of periods so that Eq. (9.6) can also be written as

$$P = \frac{1}{nT}\int_{t_0}^{t_0+nT} V_M I_M \cos(\omega t+\theta_v)\cos(\omega t+\theta_i)\,dt \qquad \textbf{9.7}$$

where n is a positive integer.

Employing Eq. (9.5) for the expression in (9.6), we obtain

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_M I_M}{2}\left[\cos(\theta_v-\theta_i)+\cos(2\omega t+\theta_v+\theta_i)\right]dt \qquad \textbf{9.8}$$

We could, of course, plod through the indicated integration; however, with a little forethought we can determine the result by inspection. The first term is independent of t, and therefore a constant in the integration. Integrating the constant over the period and dividing by the period simply results in the original constant. The second term is a cosine wave. It is well known that the average value of a cosine wave over one complete period or an integral number of periods is zero, and therefore the second term in Eq. (9.8) vanishes. In view of this discussion, Eq. (9.8) reduces to

HINT

A frequently used equation for calculating the average power.

$$P = \frac{1}{2}V_M I_M \cos(\theta_v-\theta_i) \qquad \textbf{9.9}$$

Note that since $\cos(-\theta) = \cos(\theta)$, the argument for the cosine function can be either $\theta_v - \theta_i$ or $\theta_i - \theta_v$. In addition, note that $\theta_v - \theta_i$ is the angle of the circuit impedance, as shown in Fig. 9.1. Therefore, *for a purely resistive circuit*,

$$P = \frac{1}{2}V_M I_M \qquad \textbf{9.10}$$

and *for a purely reactive circuit*,

$$\begin{aligned} P &= \frac{1}{2}V_M I_M \cos(90^\circ) \\ &= 0 \end{aligned}$$

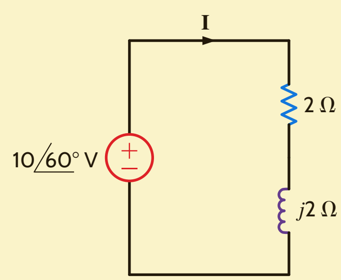
Because purely reactive impedances absorb no average power, they are often called *lossless elements*. The purely reactive network operates in a mode in which it stores energy over one part of the period and releases it over another.

EXAMPLE 9.2

We wish to determine the average power absorbed by the impedance shown in Fig. 9.3.

Figure 9.3

Example *RL* circuit.



SOLUTION

From the figure we note that

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_M \angle \theta_v}{2 + j2} = \frac{10\angle 60°}{2.83\angle 45°} = 3.53\angle 15° \text{ A}$$

Therefore,

$$I_M = 3.53 \text{ A} \quad \text{and} \quad \theta_i = 15°$$

Hence,

$$\begin{aligned} P &= \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i) \\ &= \frac{1}{2}(10)(3.53)\cos(60° - 15°) \\ &= 12.5 \text{ W} \end{aligned}$$

Since the inductor absorbs no power, we can employ Eq. (9.10), provided that V_M in that equation is the voltage across the resistor. Using voltage division, we obtain

$$\mathbf{V}_R = \frac{(10\angle 60°)(2)}{2 + j2} = 7.07\angle 15° \text{ V}$$

and therefore,

$$\begin{aligned} P &= \frac{1}{2}(7.07)(3.53) \\ &= 12.5 \text{ W} \end{aligned}$$

In addition, using Ohm's law, we could also employ the expressions

$$P = \frac{1}{2}\frac{V_M^2}{R}$$

or

$$P = \frac{1}{2} I_M^2 R$$

where once again we must be careful that the V_M and I_M in these equations refer to the voltage across the resistor and the current through it, respectively.

EXAMPLE 9.3

For the circuit shown in Fig. 9.4, we wish to determine both the total average power absorbed and the total average power supplied.

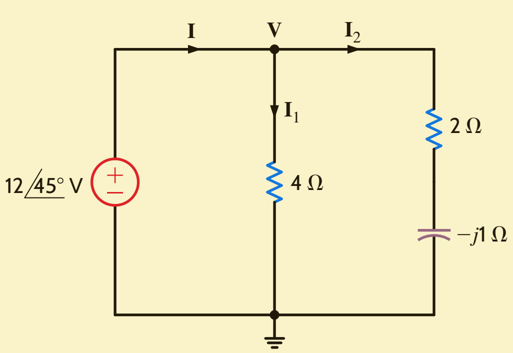


Figure 9.4

Example circuit for illustrating a power balance.

SOLUTION From the figure we note that

$$\mathbf{I}_1 = \frac{12\angle 45^\circ}{4} = 3\angle 45^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{12\angle 45^\circ}{2 - j1} = \frac{12\angle 45^\circ}{2.24\angle -26.57^\circ} = 5.36\angle 71.57^\circ \text{ A}$$

and therefore,

$$\begin{aligned}\mathbf{I} &= \mathbf{I}_1 + \mathbf{I}_2\\ &= 3\angle 45^\circ + 5.36\angle 71.57^\circ\\ &= 8.15\angle 62.10^\circ \text{ A}\end{aligned}$$

The average power absorbed in the 4-Ω resistor is

$$P_4 = \frac{1}{2} V_M I_M = \frac{1}{2}(12)(3) = 18 \text{ W}$$

The average power absorbed in the 2-Ω resistor is

$$P_2 = \frac{1}{2} I_M^2 R = \frac{1}{2}(5.34)^2(2) = 28.7 \text{ W}$$

Therefore, the total average power absorbed is

$$P_A = 18 + 28.7 = 46.7 \text{ W}$$

Note that we could have calculated the power absorbed in the 2-Ω resistor using $1/2V_M^2/R$ if we had first calculated the voltage across the 2-Ω resistor.

The total average power supplied by the source is

$$\begin{aligned}P_S &= \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)\\ &= \frac{1}{2}(12)(8.15)\cos(45^\circ - 62.10^\circ)\\ &= 46.7 \text{ W}\end{aligned}$$

Thus, the total average power supplied is, of course, equal to the total average power absorbed.

LEARNING ASSESSMENTS

E9.1 Find the average power absorbed by each resistor in the network in Fig. E9.1.

ANSWER: $P_{2\Omega} = 7.20$ W; $P_{4\Omega} = 7.20$ W.

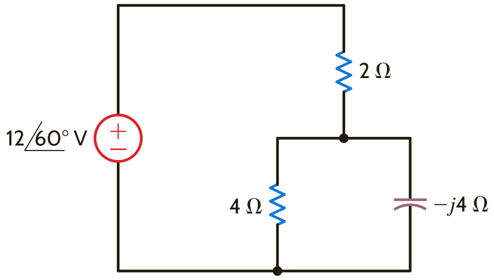


Figure E9.1

E9.2 Given the network in Fig. E9.2, find the average power absorbed by each passive circuit element and the total average power supplied by the current source.

ANSWER:
$P_{3\Omega} = 56.60$ W;
$P_{4\Omega} = 33.96$ W;
$P_L = 0$; $P_{CS} = 90.50$ W.

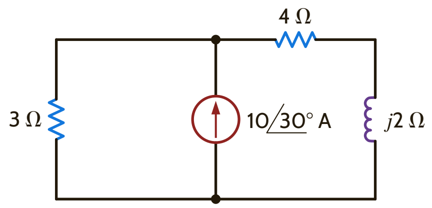


Figure E9.2

E9.3 Find the power supplied and the power absorbed by each element in Fig. E9.3.

ANSWER:
$P_C = 0$ W; $P_L = 0$ W;
$P_{4\Omega} = 1.78$ W; $P_{2\Omega} = 2.22$ W;
$P_{CS} = -4$ W.

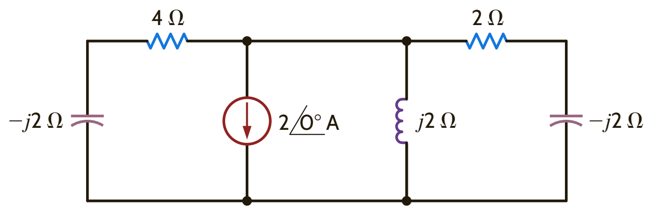


Figure E9.3

When determining average power, if more than one source is present in a network, we can use any of our network analysis techniques to find the necessary voltage and/or current to compute the power. However, we must remember that, in general, we cannot apply superposition to power.

HINT
Superposition is not applicable to power. Why not?

EXAMPLE 9.4

Consider the network shown in Fig. 9.5. We wish to determine the total average power absorbed and supplied by each element.

SOLUTION

From the figure we note that

$$\mathbf{I}_2 = \frac{12\angle 30^\circ}{2} = 6\angle 30^\circ \text{ A}$$

and

$$\mathbf{I}_3 = \frac{12\angle 30^\circ - 6\angle 0^\circ}{j1} = \frac{4.39 + j6}{j1} = 7.44\angle -36.21^\circ \text{ A}$$

The power absorbed by the 2-Ω resistor is

$$P_2 = \frac{1}{2}V_M I_M = \frac{1}{2}(12)(6) = 36 \text{ W}$$

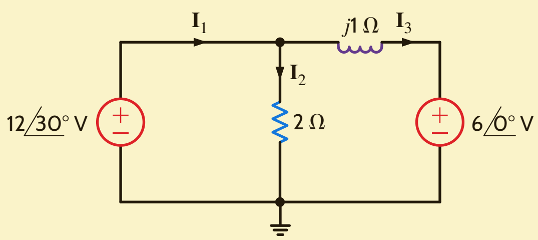
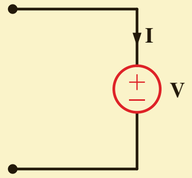


Figure 9.5 Example *RL* circuit with two sources.

HINT

Under the following condition:



If $P = \mathbf{IV}$ is positive, power is being absorbed.

If $P = \mathbf{IV}$ is negative, power is being generated.

According to the direction of $\mathbf{I}_3$, the $6\underline{/0^\circ}$-V source is absorbing power. The power it absorbs is given by

$$P_{6\underline{/0^\circ}} = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$$

$$= \frac{1}{2}(6)(7.44)\cos[0^\circ - (-36.21^\circ)]$$

$$= 18 \text{ W}$$

At this point an obvious question arises: How do we know whether the $6\underline{/0^\circ}$-V source is supplying power to the remainder of the network or absorbing it? The answer to this question is actually straightforward. If we employ our passive sign convention that was adopted in the earlier chapters—that is, if the current reference direction enters the positive terminal of the source and the answer is positive—the source is absorbing power. If the answer is negative, the source is supplying power to the remainder of the circuit. A generator sign convention could have been used, and under this condition the interpretation of the sign of the answer would be reversed. Note that once the sign convention is adopted and used, the sign for average power will be negative only if the angle difference is greater than 90° (i.e., $|\theta_v - \theta_i| > 90^\circ$).

To obtain the power supplied to the network, we compute $\mathbf{I}_1$ as

$$\mathbf{I}_1 = \mathbf{I}_2 + \mathbf{I}_3$$

$$= 6\underline{/30^\circ} + 7.44\underline{/-36.21^\circ}$$

$$= 11.29\underline{/-7.10^\circ} \text{ A}$$

Therefore, the power supplied by the $12\underline{/30^\circ}$-V source using the generator sign convention is

$$P_S = \frac{1}{2}(12)(11.29)\cos(30^\circ + 7.10^\circ)$$

$$= 54 \text{ W}$$

and hence the power absorbed is equal to the power supplied.

LEARNING ASSESSMENTS

E9.4 Determine the total average power absorbed and supplied by each element in the network in Fig. E9.4.

ANSWER:
$P_{CS} = -69.4$ W;
$P_{VS} = 19.8$ W;
$P_{4\Omega} = 49.6$ W; $P_C = 0$.

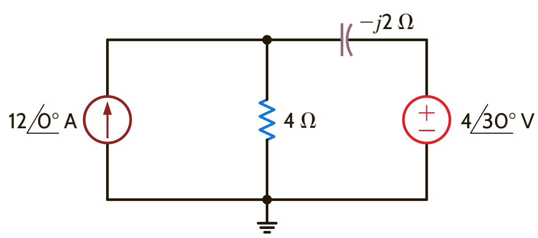


Figure E9.4