

## 2.2

## Kirchhoff's Laws

The circuits we have considered previously have all contained a single resistor, and we have analyzed them using Ohm's law. At this point we begin to expand our capabilities to handle more complicated networks that result from an interconnection of two or more of these simple elements. We will assume that the interconnection is performed by electrical conductors (wires) that have zero resistance—that is, perfect conductors. Because the wires have zero resistance, the energy in the circuit is in essence lumped in each element, and we employ the term *lumped-parameter circuit* to describe the network.

To aid us in our discussion, we will define a number of terms that will be employed throughout our analysis. As will be our approach throughout this text, we will use examples to illustrate the concepts and define the appropriate terms. For example, the circuit shown in Fig. 2.5a will be used to describe the terms *node*, *loop*, and *branch*. A node is simply a point of connection of two or more circuit elements. The reader is cautioned to note that, although one node can be spread out with perfect conductors, it is still only one node. This is illustrated in Fig. 2.5b, where the circuit has been redrawn. Node 5 consists of the entire bottom connector of the circuit.

If we start at some point in the circuit and move along perfect conductors in any direction until we encounter a circuit element, the total path we cover represents a single node. Therefore, we can assume that a node is one end of a circuit element together with all the perfect conductors that are attached to it. Examining the circuit, we note that there are numerous paths through it. A *loop* is simply any *closed path* through the circuit in which no node is encountered more than once. For example, starting from node 1, one loop would contain the elements  $R_1$ ,  $v_2$ ,  $R_4$ , and  $i_1$ ; another loop would contain  $R_2$ ,  $v_1$ ,  $v_2$ ,  $R_4$ , and  $i_1$ ; and so on. However, the path  $R_1$ ,  $v_1$ ,  $R_5$ ,  $v_2$ ,  $R_3$ , and  $i_1$  is not a loop because we have encountered node 3 twice. Finally, a *branch* is a portion of a circuit containing only a single element and the nodes at each end of the element. The circuit in Fig. 2.5 contains eight branches.

Given the previous definitions, we are now in a position to consider Kirchhoff's laws, named after German scientist Gustav Robert Kirchhoff. These two laws are quite simple but extremely important. We will not attempt to prove them because the proofs are beyond our current level of understanding. However, we will demonstrate their usefulness and attempt to make the reader proficient in their use. The first law is *Kirchhoff's current law* (KCL), which states that *the algebraic sum of the currents entering any node is zero*. In mathematical form the law appears as

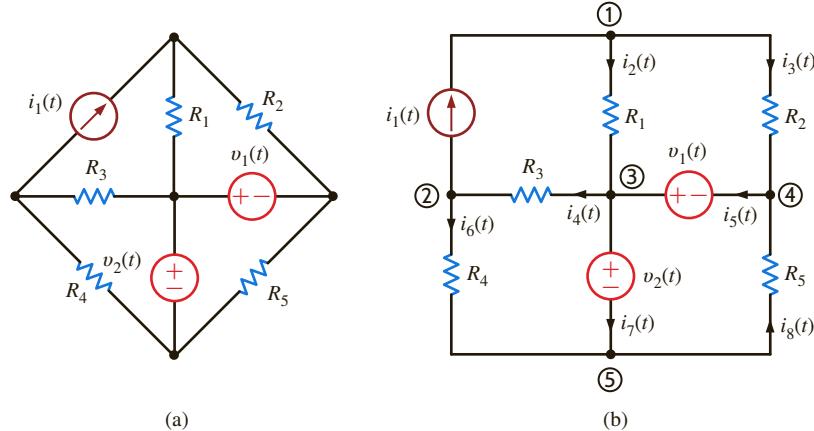
$$\sum_{j=1}^N i_j(t) = 0 \quad 2.7$$



**HINT**  
KCL is an extremely important and useful law.

**Figure 2.5**

Circuit used to illustrate KCL.



where  $i_j(t)$  is the  $j$ th current entering the node through branch  $j$  and  $N$  is the number of branches connected to the node. To understand the use of this law, consider node 3 shown in Fig. 2.5. Applying Kirchhoff's current law to this node yields

$$i_2(t) - i_4(t) + i_5(t) - i_7(t) = 0$$

We have assumed that the algebraic signs of the currents entering the node are positive and, therefore, that the signs of the currents leaving the node are negative.

If we multiply the foregoing equation by  $-1$ , we obtain the expression

$$-i_2(t) + i_4(t) - i_5(t) + i_7(t) = 0$$

which simply states that *the algebraic sum of the currents leaving a node is zero*. Alternatively, we can write the equation as

$$i_2(t) + i_5(t) = i_4(t) + i_7(t)$$

which states that *the sum of the currents entering a node is equal to the sum of the currents leaving the node*. Both of these italicized expressions are alternative forms of Kirchhoff's current law.

Once again it must be emphasized that the latter statement means that the sum of the *variables* that have been defined entering the node is equal to the sum of the *variables* that have been defined leaving the node, not the actual currents. For example,  $i_j(t)$  may be defined entering the node, but if its actual value is negative, there will be positive charge leaving the node.

Note carefully that Kirchhoff's current law states that the *algebraic* sum of the currents either entering or leaving a node must be zero. We now begin to see why we stated in Chapter 1 that it is critically important to specify both the magnitude and the direction of a current. Recall that current is charge in motion. Based on our background in physics, charges cannot be stored at a node. In other words, if we have a number of charges entering a node, then an equal number must be leaving that same node. Kirchhoff's current law is based on this principle of conservation of charge.

Finally, it is possible to generalize Kirchhoff's current law to include a closed surface. By a closed surface we mean some set of elements completely contained within the surface that are interconnected. Since the current entering each element within the surface is equal to that leaving the element (i.e., the element stores no net charge), it follows that the current entering an interconnection of elements is equal to that leaving the interconnection. Therefore, Kirchhoff's current law can also be stated as follows: *The algebraic sum of the currents entering any closed surface is zero*.

Let us write KCL for every node in the network in Fig. 2.5, assuming that the currents leaving the node are positive.

The KCL equations for nodes 1 through 5 are

$$\begin{aligned} -i_1(t) + i_2(t) + i_3(t) &= 0 \\ i_1(t) - i_4(t) + i_6(t) &= 0 \\ -i_2(t) + i_4(t) - i_5(t) + i_7(t) &= 0 \\ -i_3(t) + i_5(t) - i_8(t) &= 0 \\ -i_6(t) - i_7(t) + i_8(t) &= 0 \end{aligned}$$

Note carefully that if we add the first four equations, we obtain the fifth equation. What does this tell us? Recall that this means that this set of equations is not linearly independent. We can show that the first four equations are, however, linearly independent. Store this idea in memory because it will become very important when we learn how to write the equations necessary to solve for all the currents and voltages in a network in the following chapter.

## EXAMPLE 2.5

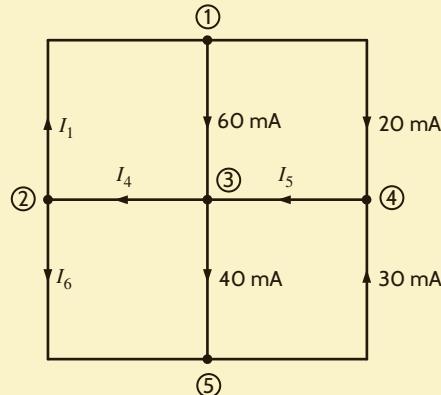
### SOLUTION

## EXAMPLE 2.6

The network in Fig. 2.5 is represented by the topological diagram shown in **Fig. 2.6**. We wish to find the unknown currents in the network.

**Figure 2.6**

Topological diagram for the circuit in Fig. 2.5.



### SOLUTION

Assuming the currents leaving the node are positive, the KCL equations for nodes 1 through 4 are

$$\begin{aligned} -I_1 + 0.06 + 0.02 &= 0 \\ I_1 - I_4 + I_6 &= 0 \\ -0.06 + I_4 - I_5 + 0.04 &= 0 \\ -0.02 + I_5 - 0.03 &= 0 \end{aligned}$$

The first equation yields  $I_1$  and the last equation yields  $I_5$ . Knowing  $I_5$ , we can immediately obtain  $I_4$  from the third equation. Then the values of  $I_1$  and  $I_4$  yield the value of  $I_6$  from the second equation. The results are  $I_1 = 80 \text{ mA}$ ,  $I_4 = 70 \text{ mA}$ ,  $I_5 = 50 \text{ mA}$ , and  $I_6 = -10 \text{ mA}$ .

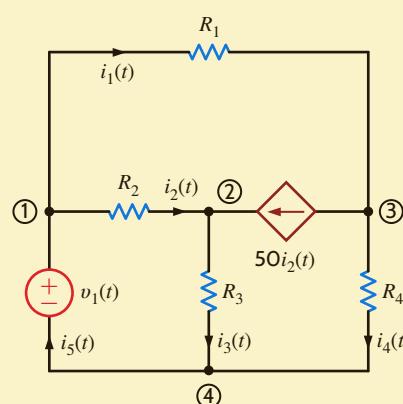
As indicated earlier, dependent or controlled sources are very important because we encounter them when analyzing circuits containing active elements such as transistors. The following example presents a circuit containing a current-controlled current source.

## EXAMPLE 2.7

Let us write the KCL equations for the circuit shown in **Fig. 2.7**.

**Figure 2.7**

Circuit containing a dependent current source.



The KCL equations for nodes 1 through 4 follow:

$$\begin{aligned} i_1(t) + i_2(t) - i_5(t) &= 0 \\ -i_2(t) + i_3(t) - 50i_2(t) &= 0 \\ -i_1(t) + 50i_2(t) + i_4(t) &= 0 \\ i_5(t) - i_3(t) - i_4(t) &= 0 \end{aligned}$$

If we added the first three equations, we would obtain the negative of the fourth. What does this tell us about the set of equations?

## SOLUTION

Kirchhoff's second law, called *Kirchhoff's voltage law* (KVL), states that the *algebraic sum of the voltages around any loop is zero*. As was the case with Kirchhoff's current law, we will defer the proof of this law and concentrate on understanding how to apply it. Once again the reader is cautioned to remember that we are dealing only with lumped-parameter circuits. These circuits are conservative, meaning that the work required to move a unit charge around any loop is zero.

In Chapter 1, we related voltage to the difference in energy levels within a circuit and talked about the energy conversion process in a flashlight. Because of this relationship between voltage and energy, Kirchhoff's voltage law is based on the conservation of energy.

Recall that in Kirchhoff's current law, the algebraic sign was required to keep track of whether the currents were entering or leaving a node. In Kirchhoff's voltage law, the algebraic sign is used to keep track of the voltage polarity. In other words, as we traverse the circuit, it is necessary to sum to zero the increases and decreases in energy level. Therefore, it is important we keep track of whether the energy level is increasing or decreasing as we go through each element.

Let us find  $I_4$  and  $I_1$  in the network represented by the topological diagram in Fig. 2.6.

## EXAMPLE 2.8

### SOLUTION

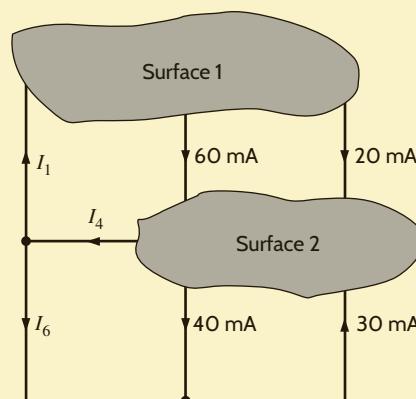
This diagram is redrawn in Fig. 2.8; node 1 is enclosed in surface 1, and nodes 3 and 4 are enclosed in surface 2. A quick review of the previous example indicates that we derived a value for  $I_4$  from the value of  $I_5$ . However,  $I_5$  is now completely enclosed in surface 2. If we apply KCL to surface 2, assuming the currents out of the surface are positive, we obtain

$$I_4 - 0.06 - 0.02 - 0.03 + 0.04 = 0$$

or

$$I_4 = 70 \text{ mA}$$

which we obtained without any knowledge of  $I_5$ . Likewise for surface 1, what goes in must come out and, therefore,  $I_1 = 80 \text{ mA}$ . The reader is encouraged to cut the network in Fig. 2.6 into two pieces in any fashion and show that KCL is always satisfied at the boundaries.



**Figure 2.8**

Diagram used to demonstrate KCL for a surface.

## LEARNING ASSESSMENTS

**E2.4** Given the networks in Fig. E2.3, find (a)  $I_1$  in Fig. E2.4a and (b)  $I_T$  in Fig. E2.4b.

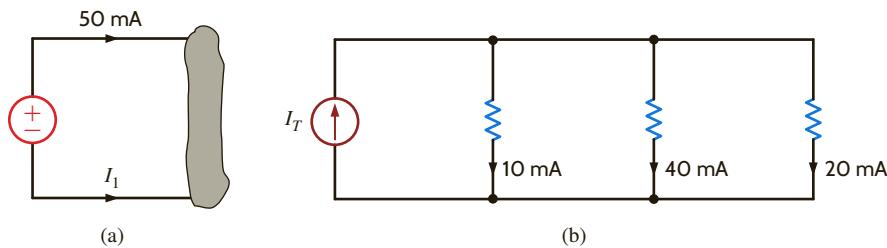


Figure E2.4

**E2.5** Find (a)  $I_1$  in the network in Fig. E2.5a and (b)  $I_1$  and  $I_2$  in the circuit in Fig. E2.5b.

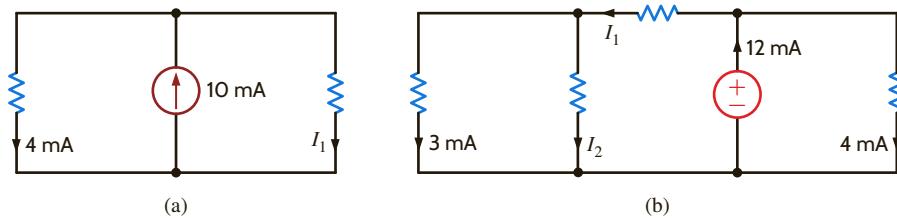


Figure E2.5

**E2.6** Find the current  $i_x$  in the circuits in Fig. E2.6.

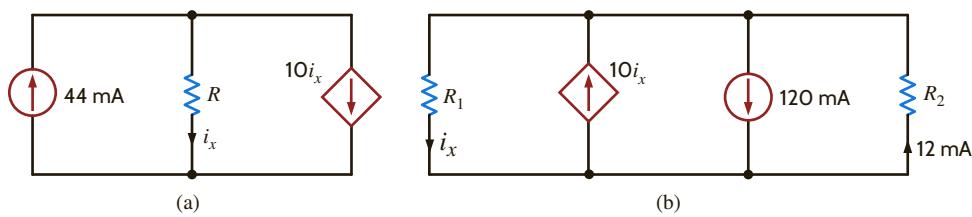


Figure E2.6

### ANSWER:

- (a)  $I_1 = -50 \text{ mA}$ ;  
(b)  $I_T = 70 \text{ mA}$ .

### ANSWER:

- (a)  $I_1 = 6 \text{ mA}$ ;  
(b)  $I_1 = 8 \text{ mA}$  and  $I_2 = 5 \text{ mA}$ .

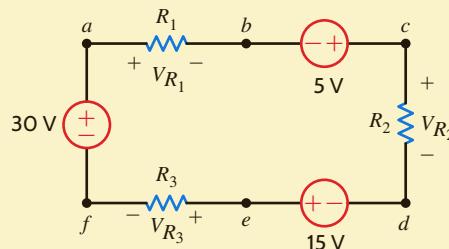
### ANSWER:

- (a)  $i_x = 4 \text{ mA}$ ;  
(b)  $i_x = 12 \text{ mA}$ .

In applying KVL, we must traverse any loop in the circuit and sum to zero the increases and decreases in energy level. At this point, we have a decision to make. Do we want to consider a decrease in energy level as positive or negative? We will adopt a policy of considering a decrease in energy level as positive and an increase in energy level as negative. As we move around a loop, we encounter the plus sign first for a decrease in energy level and a negative sign first for an increase in energy level.

Finally, we employ the convention  $V_{ab}$  to indicate the voltage of point  $a$  with respect to point  $b$ : that is, the variable for the voltage between point  $a$  and point  $b$ , with point  $a$  considered positive relative to point  $b$ . Since the potential is measured between two points, it is convenient to use an arrow between the two points, with the head of the arrow located at the positive node. Note that the double-subscript notation, the + and - notation, and the single-headed arrow notation are all the same if the head of the arrow is pointing toward the

Consider the circuit shown in **Fig. 2.9**. If  $V_{R_1}$  and  $V_{R_2}$  are known quantities, let us find  $V_{R_3}$ .



**Figure 2.9**

Circuit used to illustrate KVL.

## EXAMPLE 2.9

Starting at point *a* in the network and traversing it in a clockwise direction, we obtain the equation

$$+V_{R_1} - 5 + V_{R_2} - 15 + V_{R_3} - 30 = 0$$

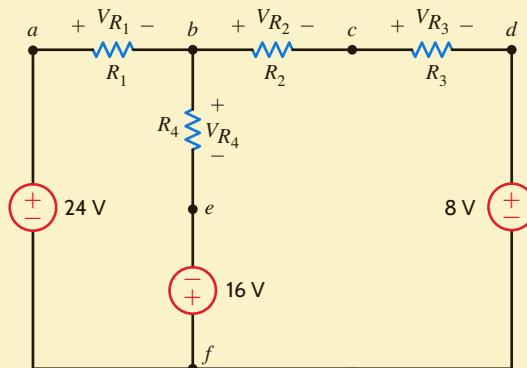
which can be written as

$$\begin{aligned} +V_{R_1} + V_{R_2} + V_{R_3} &= 5 + 15 + 30 \\ &= 50 \end{aligned}$$

Now suppose that  $V_{R_1}$  and  $V_{R_2}$  are known to be 18 V and 12 V, respectively. Then  $V_{R_3} = 20$  V.

## SOLUTION

Consider the network in **Fig. 2.10**.



**Figure 2.10**

Circuit used to explain KVL.

## EXAMPLE 2.10

Let us demonstrate that only two of the three possible loop equations are linearly independent.

Note that this network has three closed paths: the left loop, right loop, and outer loop. Applying our policy for writing KVL equations and traversing the left loop starting at point *a*, we obtain

$$V_{R_1} + V_{R_4} - 16 - 24 = 0$$

The corresponding equation for the right loop starting at point *b* is

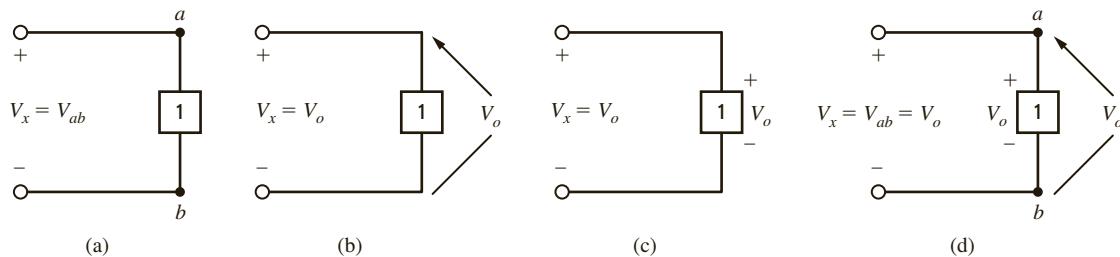
$$V_{R_2} + V_{R_3} + 8 + 16 - V_{R_4} = 0$$

The equation for the outer loop starting at point *a* is

$$V_{R_1} + V_{R_2} + V_{R_3} + 8 - 24 = 0$$

Note that if we add the first two equations, we obtain the third equation. Therefore, as we indicated in Example 2.5, the three equations are not linearly independent. Once again, we will address this issue in the next chapter and demonstrate that we need only the first two equations to solve for the voltages in the circuit.

## SOLUTION

**Figure 2.11**

Equivalent forms for labeling voltage.

positive terminal and the first subscript in the double-subscript notation. All of these equivalent forms for labeling voltages are shown in **Fig. 2.11**. The usefulness of the arrow notation stems from the fact that we may want to label the voltage between two points that are far apart in a network. In this case, the other notations are often confusing.

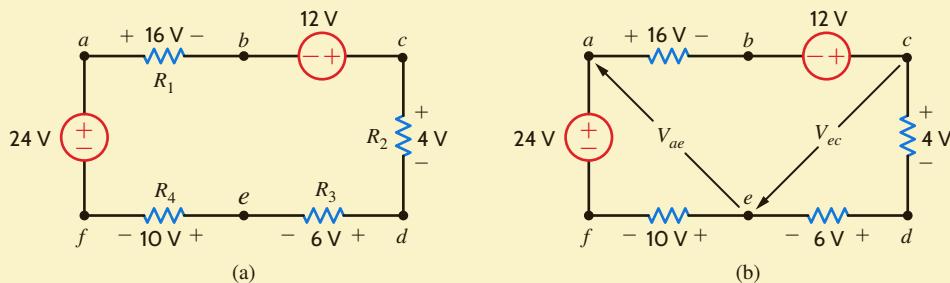
## EXAMPLE 2.11



Consider the network in **Fig. 2.12a**. Let us apply KVL to determine the voltage between two points. Specifically, in terms of the double-subscript notation, let us find  $V_{ae}$  and  $V_{ec}$ .

**Figure 2.12**

Network used in Example 2.11.



### SOLUTION

The circuit is redrawn in **Fig. 2.12b**. Since points  $a$  and  $e$  as well as  $e$  and  $c$  are not physically close, the arrow notation is very useful. Our approach to determining the unknown voltage is to apply KVL with the unknown voltage in the closed path. Therefore, to determine  $V_{ae}$  we can use the path  $aefaa$  or  $abcdea$ . The equations for the two paths in which  $V_{ae}$  is the only unknown are

$$V_{ae} + 10 - 24 = 0$$

and

$$16 - 12 + 4 + 6 - V_{ae} = 0$$

Note that both equations yield  $V_{ae} = 14$  V. Even before calculating  $V_{ae}$ , we could calculate  $V_{ec}$  using the path  $cdec$  or  $cefabc$ . However, since  $V_{ae}$  is now known, we can also use the path  $ceabc$ . KVL for each of these paths is

$$\begin{aligned} 4 + 6 + V_{ec} &= 0 \\ -V_{ec} + 10 - 24 + 16 - 12 &= 0 \end{aligned}$$

and

$$-V_{ec} - V_a + 16 - 12 = 0$$

Each of these equations yields  $V_{ec} = -10$  V.

In general, the mathematical representation of Kirchhoff's voltage law is

$$\sum_{j=1}^N v_j(t) = 0 \quad 2.8$$

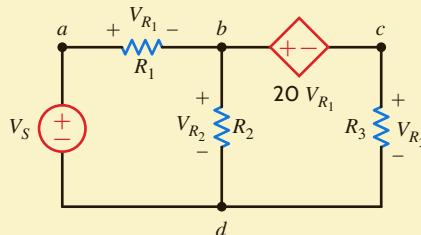
where  $v_j(t)$  is the voltage across the  $j$ th branch (with the proper reference direction) in a loop containing  $N$  voltages. This expression is analogous to Eq. (2.7) for Kirchhoff's current law.



**KVL** is an extremely important and useful law.

Given the network in Fig. 2.13 containing a dependent source, let us write the KVL equations for the two closed paths  $abda$  and  $bcd$ .

## EXAMPLE 2.12



**Figure 2.13**

Network containing a dependent source.

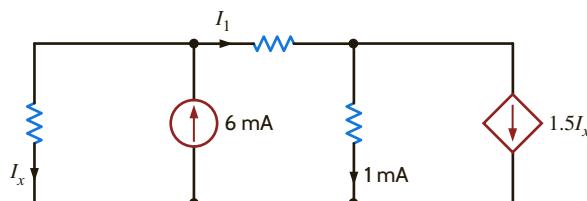
The two KVL equations are

$$\begin{aligned} V_{R_1} + V_{R_2} - V_S &= 0 \\ 20V_{R_1} + V_{R_3} - V_{R_2} &= 0 \end{aligned}$$

## SOLUTION

## LEARNING ASSESSMENTS

**E2.7** Find  $I_x$  and  $I_1$  in Fig. E2.7.

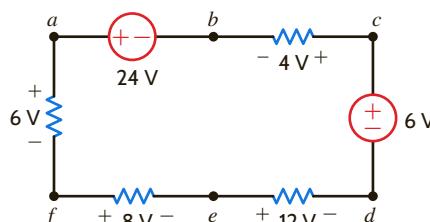


**Figure E2.7**

## ANSWER:

$$\begin{aligned} I_x &= 2 \text{ mA;} \\ I_1 &= 4 \text{ mA.} \end{aligned}$$

**E2.8** Find  $V_{ad}$  and  $V_{eb}$  in the network in Fig. E2.8.



**Figure E2.8**

## ANSWER:

$$\begin{aligned} V_{ad} &= 26 \text{ V;} \\ V_{eb} &= 10 \text{ V.} \end{aligned}$$