13.4 Properties of the Transform

A number of useful theorems describe important properties of the Laplace transform. We will first demonstrate a couple of these theorems, provide a concise listing of a number of them, and, finally, illustrate their usefulness via several examples.

The *time-scaling theorem* states that

$$\mathcal{L}[f(at)] = \frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right) \qquad a > 0 \tag{13.5}$$

The *Laplace transform* of $f(at)$ is

$$\mathcal{L}[f(at)] = \int_0^\infty f(at)e^{-st}\,dt$$

Now let $\lambda = at$ and $d\lambda = a\,dt$. Then

$$\begin{aligned}\mathcal{L}[f(at)] &= \int_0^\infty f(\lambda)e^{-(\lambda/a)s}\frac{d\lambda}{a}\\ &= \frac{1}{a}\int_0^\infty f(\lambda)e^{-(s/a)\lambda}\,d\lambda\\ &= \frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right) \qquad a > 0\end{aligned}$$

The *time-shifting theorem* states that

$$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}\mathbf{F}(s) \qquad t_0 \geq 0 \tag{13.6}$$

This theorem is illustrated as follows:

$$\begin{aligned}\mathcal{L}[f(t-t_0)u(t-t_0)] &= \int_0^\infty f(t-t_0)u(t-t_0)\,e^{-st}\,dt\\ &= \int_{t_0}^\infty f(t-t_0)e^{-st}\,dt\end{aligned}$$

If we now let $\lambda = t - t_0$ and $d\lambda = dt$, then

$$\begin{aligned}\mathcal{L}[f(t-t_0)u(t-t_0)] &= \int_0^\infty f(\lambda)\,e^{-s(\lambda+t_0)}\,d\lambda\\ &= e^{-t_0 s}\int_0^\infty f(\lambda)\,e^{-s\lambda}\,d\lambda\\ &= e^{-t_0 s}\mathbf{F}(s) \qquad t_0 \geq 0\end{aligned}$$

The *frequency-shifting*, or *modulation*, *theorem* states that

$$\mathcal{L}[e^{-at}f(t)] = \mathbf{F}(s+a) \tag{13.7}$$

By definition,

$$\begin{aligned}\mathcal{L}[e^{-at}f(t)] &= \int_0^\infty e^{-at}f(t)e^{-st}\,dt\\ &= \int_0^\infty f(t)\,e^{-(s+a)t}\,dt\\ &= \mathbf{F}(s+a)\end{aligned}$$

The three theorems we have demonstrated, together with a number of other important properties, are listed in a concise manner in Table 13.2. Let us now provide several simple examples that illustrate how these properties can be used.

TABLE 13.2 Some useful properties of the Laplace transform

PROPERTY NUMBER	$f(t)$	$\mathbf{F}(s)$
1. Magnitude scaling	$Af(t)$	$A\mathbf{F}(s)$
2. Addition/subtraction	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
3. Time scaling	$f(at)$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right),\ a > 0$
4. Time shifting	$f(t - t_0)u(t - t_0),\ t \geq 0$	$e^{-t_0 s}\mathbf{F}(s)$
	$f(t)u(t - t_0)$	$e^{-t_0 s}\mathscr{L}[f(t + t_0)]$
5. Frequency shifting	$e^{-at}f(t)$	$\mathbf{F}(s + a)$
6. Differentiation	$\frac{d^n f(t)}{dt^n}$	$s^n\mathbf{F}(s) - s^{n-1}f(0) - s^{n-2}f^1(0) \cdots - s^0 f^{n-1}(0)$
7. Multiplication by t	$tf(t)$	$-\frac{d\mathbf{F}(s)}{ds}$
	$t^n f(t)$	$(-1)^n \frac{d^n\mathbf{F}(s)}{ds^n}$
8. Division by t	$\frac{f(t)}{t}$	$\int_s^\infty \mathbf{F}(\lambda)\, d\lambda$
9. Integration	$\int_0^t f(\lambda)\, d\lambda$	$\frac{1}{s}\mathbf{F}(s)$
10. Convolution	$\int_0^t f_1(\lambda) f_2(t - \lambda)\, d\lambda$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$

EXAMPLE 13.5

Use the Laplace transform of $\cos \omega t$ to find the Laplace transform of $e^{-at} \cos \omega t$.

SOLUTION

Since the Laplace transform of $\cos \omega t$ is known to be

$$\mathscr{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

then using property number 5,

$$\mathscr{L}[e^{-at} \cos \omega t] = \frac{s + a}{(s + a)^2 + \omega^2}$$

EXAMPLE 13.6

Let us demonstrate property number 8.

SOLUTION

If $f(t) = te^{-at}$, then

$$\mathbf{F}(\lambda) = \frac{1}{(\lambda + a)^2}$$

Therefore,

$$\int_s^\infty \mathbf{F}(\lambda)\, d\lambda = \int_s^\infty \frac{1}{(\lambda + a)^2}\, d\lambda = \frac{-1}{\lambda + a}\bigg|_s^\infty = \frac{1}{s + a}$$

Hence,

$$f_1(t) = \frac{f(t)}{t} = \frac{te^{-at}}{t} = e^{-at} \qquad \text{and} \qquad \mathbf{F}_1(s) = \frac{1}{s + a}$$