15.1 Fourier Series

A periodic function is one that satisfies the relationship

$$f(t) = f(t + nT_0), \quad n = \pm 1, \pm 2, \pm 3, \ldots$$

for every value of t where T_0 is the period. As we have shown in previous chapters, the sinusoidal function is a very important periodic function. However, many other periodic functions have wide applications. For example, laboratory signal generators produce the pulse-train and square-wave signals shown in **Figs. 15.1a** and **b**, respectively, which are used for testing circuits. The oscilloscope is another laboratory instrument, and the sweep of its electron beam across the face of the cathode ray tube is controlled by a triangular signal of the form shown in **Fig. 15.1c**.

The techniques we will explore are based on the work of Jean Baptiste Joseph Fourier. Although our analyses will be confined to electric circuits, it is important to point out that the techniques are applicable to a wide range of engineering problems. In fact, it was Fourier's work in heat flow that led to the techniques that will be presented here.

In his work, Fourier demonstrated that a periodic function $f(t)$ could be expressed as a sum of sinusoidal functions. Therefore, given this fact and the fact that if a periodic function is expressed as a sum of linearly independent functions, each function in the sum must be periodic with the same period, and the function $f(t)$ can be expressed in the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n) \tag{15.1}$$

where $\omega_0 = 2\pi/T_0$ and a_0 is the average value of the waveform. An examination of this expression illustrates that all sinusoidal waveforms that are periodic with period T_0 have been included. For example, for $n = 1$, one cycle covers T_0 seconds, and $D_1 \cos(\omega_0 t + \theta_1)$ is called the *fundamental*. For $n = 2$, two cycles fall within T_0 seconds, and the term $D_2 \cos(2\omega_0 t + \theta_2)$ is called the *second harmonic*. In general, for $n = k$, k cycles fall within T_0 seconds, and $D_k \cos(k\omega_0 t + \theta_k)$ is the *kth harmonic term*.

Since the function $\cos(n\omega_0 t + \theta_k)$ can be written in exponential form using Euler's identity or as a sum of cosine and sine terms of the form $\cos n\omega_0 t$ and $\sin n\omega_0 t$ as demonstrated in Chapter 8, the series in Eq. (15.1) can be written as

$$f(t) = a_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \mathbf{c}_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t} \tag{15.2}$$

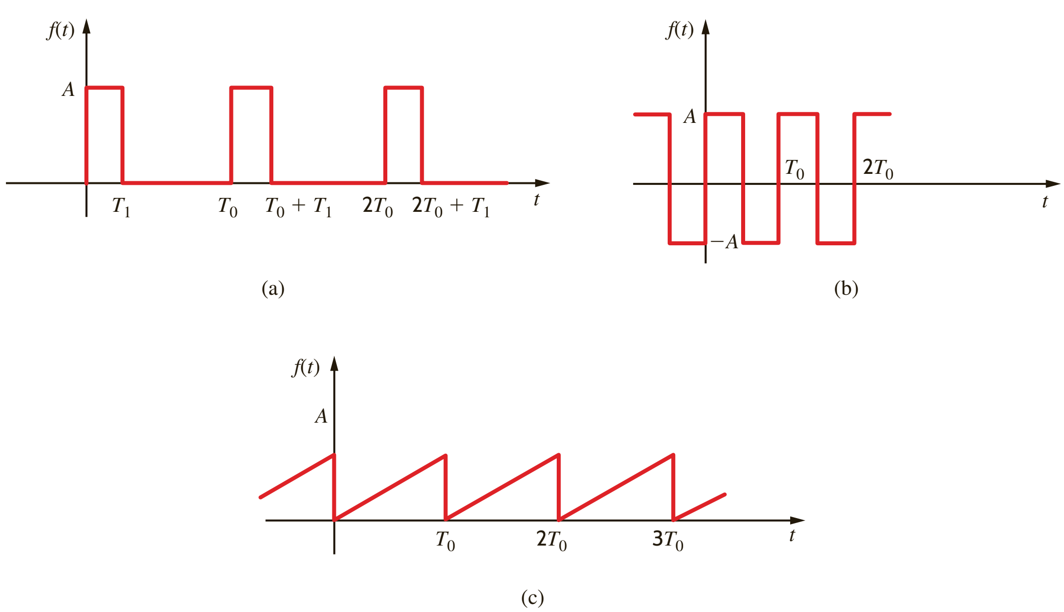


Figure 15.1
Some useful periodic signals.

Using the real-part relationship employed as a transformation between the time domain and the frequency domain, we can express $f(t)$ as

$$f(t) = a_0 + \sum_{n=1}^{\infty} \text{Re}\left[(D_n \underline{/\theta_n}) e^{jn\omega_0 t} \right] \quad \textbf{15.3}$$

$$= a_0 + \sum_{n=1}^{\infty} \text{Re}(2\mathbf{c}_n e^{jn\omega_0 t}) \quad \textbf{15.4}$$

$$= a_0 + \sum_{n=1}^{\infty} \text{Re}\left[(a_n - jb_n) e^{jn\omega_0 t} \right] \quad \textbf{15.5}$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \textbf{15.6}$$

These equations allow us to write the Fourier series in a number of equivalent forms. Note that the *phasor* for the nth harmonic is

$$D_n \underline{/\theta_n} = 2\mathbf{c}_n = a_n - jb_n \quad \textbf{15.7}$$

The approach we will take will be to represent a nonsinusoidal periodic input by a sum of complex exponential functions, which because of Euler's identity is equivalent to a sum of sines and cosines. We will then use (1) the superposition property of linear systems and (2) our knowledge that the steady-state response of a time-invariant linear system to a sinusoidal input of frequency ω_0 is a sinusoidal function of the same frequency to determine the response of such a system.

To illustrate the manner in which a nonsinusoidal periodic signal can be represented by a Fourier series, consider the periodic function shown in **Fig. 15.2a**. In **Figs. 15.2b–d** we can see the impact of using a specific number of terms in the series to represent the original function. Note that the series more closely represents the original function as we employ more and more terms.

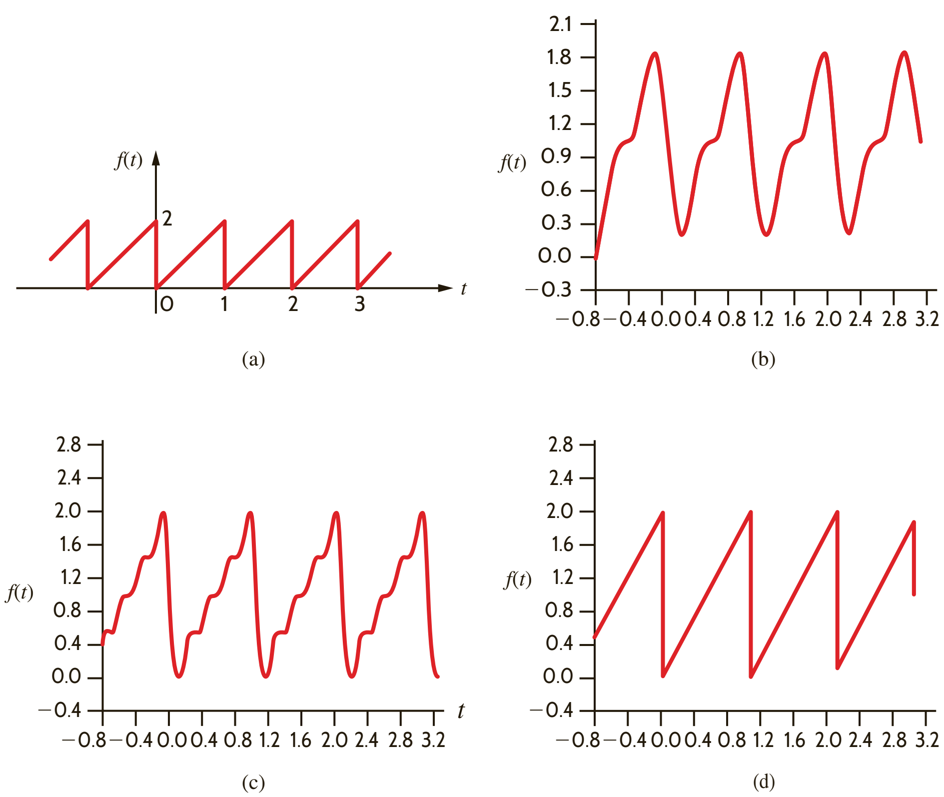


Figure 15.2

Periodic function (a) and its representation by a fixed number of Fourier series terms, (b) 2 terms, (c) 4 terms, (d) 100 terms.

EXPONENTIAL FOURIER SERIES *Any physically realizable periodic signal* may be represented over the interval $t_1 < t < t_1 + T_0$ by the *exponential Fourier series*

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t} \tag{15.8}$$

where the $\mathbf{c}_n$ are the complex (phasor) Fourier coefficients. These coefficients are derived as follows. Multiplying both sides of Eq. (15.8) by $e^{-jk\omega_0 t}$ and integrating over the interval t_1 to $t_1 + T_0$, we obtain

$$\begin{aligned}\int_{t_1}^{t_1+T_0} f(t)e^{-jk\omega_0 t}\,dt &= \int_{t_1}^{t_1+T_0}\left(\sum_{n=-\infty}^{\infty}\mathbf{c}_n e^{jn\omega_0 t}\right)e^{-jk\omega_0 t}\,dt\\ &= \mathbf{c}_k T_0\end{aligned}$$

since

$$\int_{t_1}^{t_1+T_0} e^{j(n-k)\omega_0 t}\,dt = \begin{cases} 0 & \text{for } n \neq k \\ T_0 & \text{for } n = k\end{cases}$$

Therefore, the Fourier coefficients are defined by the equation

$$\mathbf{c}_n = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)e^{-jn\omega_0 t}\,dt \tag{15.9}$$

The following example illustrates the manner in which we can represent a periodic signal by an exponential Fourier series.

EXAMPLE 15.1

We wish to determine the exponential Fourier series for the periodic voltage waveform shown in Fig. 15.3.

SOLUTION

The Fourier coefficients are determined using Eq. (15.9) by integrating over one complete period of the waveform,

$$\begin{aligned}\mathbf{c}_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt\\ &= \frac{1}{T}\int_{-T/2}^{-T/4} -Ve^{-jn\omega_0 t}\,dt\\ &\quad + \int_{-T/4}^{T/4} Ve^{-jn\omega_0 t}\,dt + \int_{T/4}^{T/2} -Ve^{-jn\omega_0 t}\,dt\\ &= \frac{V}{jn\omega_0 T}\left[+e^{-jn\omega_0 t}\Big|_{-T/2}^{-T/4} - e^{-jn\omega_0 t}\Big|_{-T/4}^{T/4} + e^{-jn\omega_0 t}\Big|_{T/2}^{T/4}\right]\end{aligned}$$

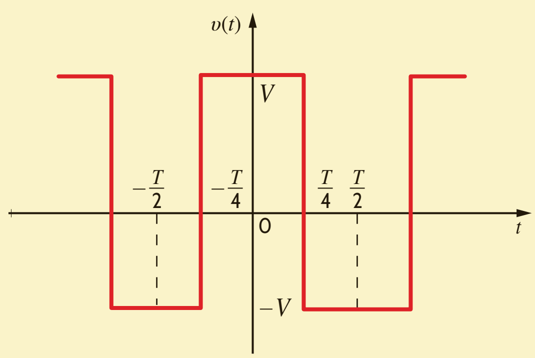


Figure 15.3 Periodic voltage waveform.

$$= \frac{V}{jn\omega_0 T}\left(2e^{jn\pi/2} - 2e^{-jn\pi/2} + e^{-jn\pi} - e^{+jn\pi}\right)$$

$$= \frac{V}{n\omega_0 T}\left[4\sin\frac{n\pi}{2} - 2\sin(n\pi)\right]$$

$$= 0 \qquad \text{for } n \text{ even}$$

$$= \frac{2V}{n\pi}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd}$$

$\mathbf{c}_0$ corresponds to the average value of the waveform. This term can be evaluated using the original equation for $\mathbf{c}_n$. Therefore,

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)\,dt$$

$$= \frac{1}{T}\left[\int_{-\frac{T}{2}}^{-\frac{T}{4}} -V\,dt + \int_{-\frac{T}{4}}^{\frac{T}{4}} V\,dt + \int_{\frac{T}{4}}^{\frac{T}{2}} -V\,dt\right]$$

$$= \frac{1}{T}\left[-\frac{VT}{4} + \frac{VT}{2} - \frac{VT}{4}\right] = 0$$

Therefore,

$$v(t) = \sum_{\substack{n=-\infty \\ n \neq 0 \\ n\,\text{odd}}}^{\infty} \frac{2V}{n\pi}\sin\frac{n\pi}{2}\,e^{jn\omega_0 t}$$

This equation can be written as

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{2V}{n\pi}\sin\frac{n\pi}{2}\,e^{jn\omega_0 t} + \sum_{\substack{n=-1 \\ n\,\text{odd}}}^{-\infty} \frac{2V}{n\pi}\sin\frac{n\pi}{2}\,e^{jn\omega_0 t}$$

$$= \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \left(\frac{2V}{n\pi}\sin\frac{n\pi}{2}\right)e^{jn\omega_0 t} + \left(\frac{2V}{n\pi}\sin\frac{n\pi}{2}\right)^{*} e^{-jn\omega_0 t}$$

Since a number plus its complex conjugate is equal to two times the real part of the number, $v(t)$ can be written as

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} 2\,\text{Re}\left(\frac{2V}{n\pi}\sin\frac{n\pi}{2}\,e^{jn\omega_0 t}\right)$$

or

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{4V}{n\pi}\sin\frac{n\pi}{2}\cos n\omega_0 t$$

Note that this same result could have been obtained by integrating over the interval $-T/4$ to $3T/4$.

LEARNING ASSESSMENTS

E15.1 Find the Fourier coefficients for the waveform in Fig. E15.1.

ANSWER:

$\mathbf{c}_n = \dfrac{1 - e^{-jn\pi}}{j2\pi n}$; $\mathbf{c}_0 = \dfrac{1}{2}$.

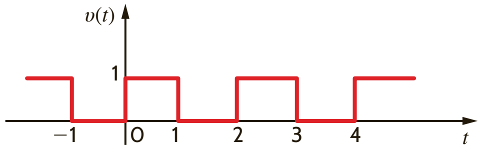


Figure E15.1

E15.2 Find the Fourier coefficients for the waveform in Fig. E15.2.

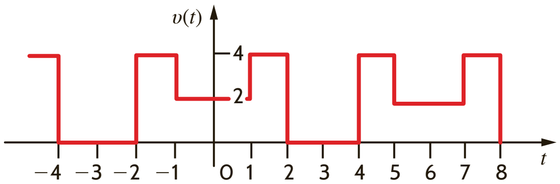


Figure E15.2

ANSWER:

$\mathbf{c}_n = \dfrac{2}{n\pi}\left(2\sin\dfrac{2\pi n}{3} - \sin\dfrac{n\pi}{3}\right)$; $\mathbf{c}_0 = 2$.

E15.3 Find the exponential Fourier series for the waveform shown in Fig. E15.3.

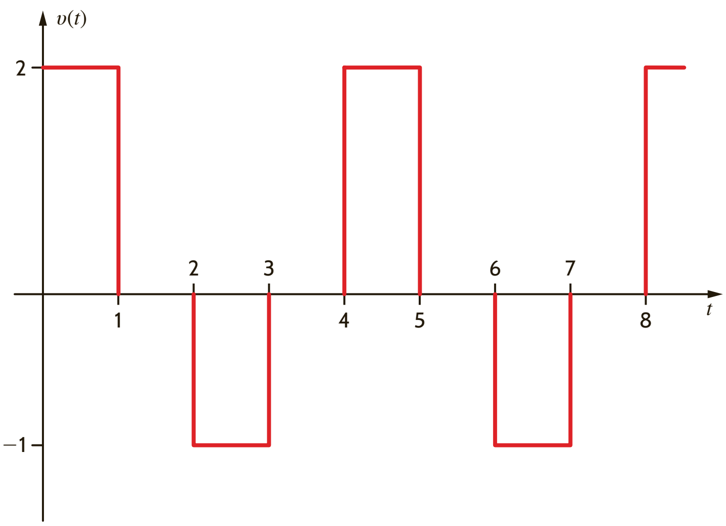


Figure E15.3

ANSWER:

$v(t) = \ldots + 0.225e^{j135°}e^{-j1.5\pi t} + 0.159e^{j90°}e^{-j\pi t} + 0.675e^{j45°}e^{-j0.5\pi t} + 0.25 + 0.675e^{-j45°}e^{j0.5\pi t} + 0.159e^{-j90°}e^{j\pi t} + 0.225e^{-j135°}e^{j1.5\pi t} + \ldots$ V.

TRIGONOMETRIC FOURIER SERIES Let us now examine another form of the Fourier series. Since

$$2\mathbf{c}_n = a_n - jb_n \tag{15.10}$$

we will examine this quantity $2\mathbf{c}_n$ and separate it into its real and imaginary parts. Using Eq. (15.9), we find that

$$2\mathbf{c}_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)e^{-jn\omega_0 t}\,dt \tag{15.11}$$

Using Euler's identity, we can write this equation in the form

$$\begin{aligned}2\mathbf{c}_n &= \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\left(\cos n\omega_0 t - j\sin n\omega_0 t\right)dt\\ &= \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\,dt - j\frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\,dt\end{aligned}$$

From Eq. (15.10) we note then that

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\,dt \tag{15.12}$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\,dt \tag{15.13}$$

These are the coefficients of the Fourier series described by Eq. (15.6), which we call the *trigonometric Fourier series*. These equations are derived directly in most textbooks using the orthogonality properties of the cosine and sine functions. Note that we can now evaluate $\mathbf{c}_n$, a_n, b_n, and since

$$2\mathbf{c}_n = D_n \underline{/\theta_n} \tag{15.14}$$

we can derive the coefficients for the *cosine Fourier series* described by Eq. (15.1). This form of the Fourier series is particularly useful because it allows us to represent each harmonic of the function as a phasor.

From Eq. (15.9) we note that $\mathbf{c}_0$, which is written as a_0, is

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1 + T_0} f(t)\,dt \tag{15.15}$$

This is the average value of the signal $f(t)$ and can often be evaluated directly from the waveform.

SYMMETRY AND THE TRIGONOMETRIC FOURIER SERIES If a signal exhibits certain symmetrical properties, we can take advantage of these properties to simplify the calculations of the Fourier coefficients. There are three types of symmetry: (1) even-function symmetry, (2) odd-function symmetry, and (3) half-wave symmetry.

Even-Function Symmetry A function is said to be even if

$$f(t) = f(-t) \tag{15.16}$$

An even function is symmetrical about the vertical axis, and a notable example is the function $\cos n\omega_0 t$. Note that the waveform in Fig. 15.3 also exhibits even-function symmetry. Let us now determine the expressions for the Fourier coefficients if the function satisfies Eq. (15.16).

If we let $t_1 = -T_0/2$ in Eq. (15.15), we obtain

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt$$

which can be written as

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{0} f(t)\,dt + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt$$

If we now change the variable on the first integral (i.e., let $t = -x$), then $f(-x) = f(x)$, $dt = -dx$, and the range of integration is from $x = T_0/2$ to 0. Therefore, the preceding equation becomes

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_{T_0/2}^{0} f(x)(-dx) + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt \\ &= \frac{1}{T_0}\int_{0}^{T_0/2} f(x)\,dx + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt \\ &= \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\,dt \end{aligned} \tag{15.17}$$

The other Fourier coefficients are derived in a similar manner. The a_n coefficient can be written

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{0} f(t)\cos n\omega_0 t\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos n\omega_0 t\,dt$$

Employing the change of variable that led to Eq. (15.17), we can express the preceding equation as

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_{T_0/2}^{0} f(x)\cos(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos n\omega_0 t\,dt \\ &= \frac{2}{T_0}\int_{0}^{T_0/2} f(x)\cos n\omega_0 x\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos n\omega_0 t\,dt \end{aligned}$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\, dt \qquad \textbf{15.18}$$

Once again, following the preceding development, we can write the equation for the b_n coefficient as

$$b_n = \frac{2}{T_0}\left[\int_{-T_0/2}^{0} f(t)\sin n\omega_0 t\, dt + \int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt\right]$$

The variable change employed previously yields

$$b_n = \frac{2}{T_0}\int_{T_0/2}^{0} f(x)\sin(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{-2}{T_0}\int_0^{T_0/2} f(x)\sin n\omega_0 x\, dx + \frac{2}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt$$

$$b_n = 0 \qquad \textbf{15.19}$$

The preceding analysis indicates that the Fourier series for an even periodic function consists only of a constant term and cosine terms. Therefore, if $f(t)$ is even, $b_n = 0$ and from Eqs. (15.10) and (15.14), $\mathbf{c}_n$ are real and θ_n are multiples of 180°.

Odd-Function Symmetry A function is said to be odd if

$$f(t) = -f(-t) \qquad \textbf{15.20}$$

An example of an odd function is sin $n\omega_0 t$. Another example is the waveform in Fig. 15.4a. Following the mathematical development that led to Eqs. (15.17) to (15.19), we can show that for an odd function the Fourier coefficients are

$$a_0 = 0 \qquad \textbf{15.21}$$

$$a_n = 0 \quad \text{for all } n > 0 \qquad \textbf{15.22}$$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt \qquad \textbf{15.23}$$

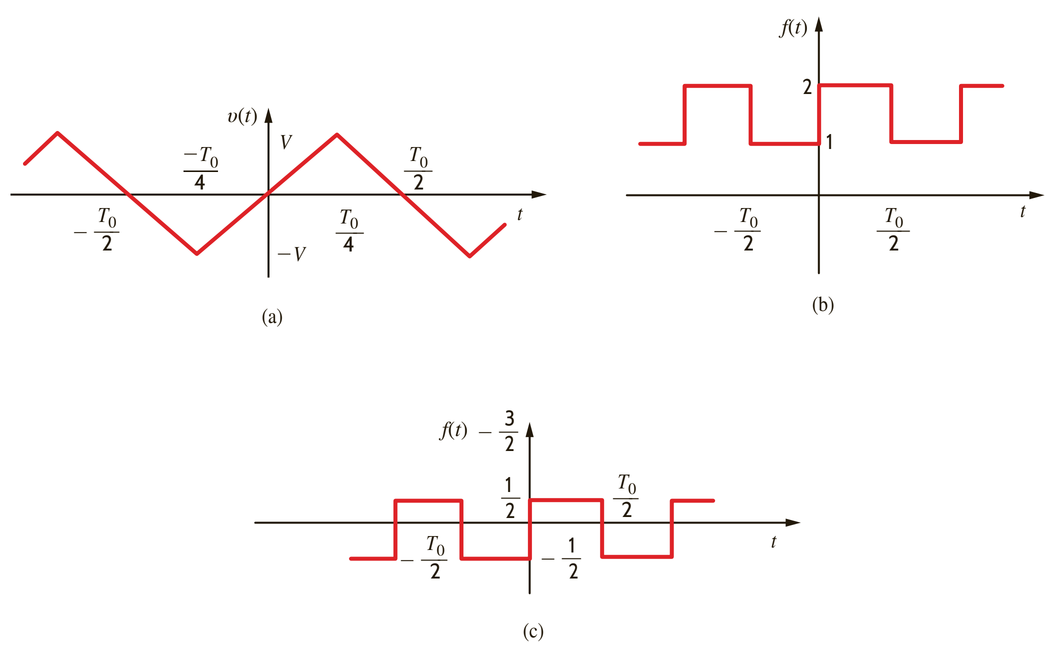


Figure 15.4
Three waveforms; (a) and (c) possess half-wave symmetry.

Therefore, if $f(t)$ is odd, $a_n = 0$ and, from Eqs. (15.10) and (15.14), $\mathbf{c}_n$ are pure imaginary and θ_n are odd multiples of 90°.

Half-Wave Symmetry A function is said to possess *half-wave symmetry* if

$$f(t) = -f\left(t - \frac{T_0}{2}\right) \qquad \textbf{15.24}$$

Basically, this equation states that each half-cycle is an inverted version of the adjacent half-cycle; that is, if the waveform from $-T_0/2$ to 0 is inverted, it is identical to the waveform from 0 to $T_0/2$. The waveforms shown in Figs. 15.4a and c possess half-wave symmetry.

Once again we can derive the expressions for the Fourier coefficients, in this case by repeating the mathematical development that led to the equations for even-function symmetry using the change of variable $t = x + T_0/2$ and Eq. (15.24). The results of this development are the following equations:

$$a_0 = 0 \qquad \textbf{15.25}$$

$$a_n = b_n = 0 \qquad \text{for } n \text{ even} \qquad \textbf{15.26}$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\, dt \qquad \text{for } n \text{ odd} \qquad \textbf{15.27}$$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt \qquad \text{for } n \text{ odd} \qquad \textbf{15.28}$$

The following equations are often useful in the evaluation of the trigonometric Fourier series coefficients:

$$\begin{aligned}
\int \sin ax\, dx &= -\frac{1}{a}\cos ax \\
\int \cos ax\, dx &= \frac{1}{a}\sin ax \\
\int x \sin ax\, dx &= \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax \\
\int x \cos ax\, dx &= \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax
\end{aligned} \qquad \textbf{15.29}$$

EXAMPLE 15.2

We wish to find the trigonometric Fourier series for the periodic signal in Fig. 15.3.

SOLUTION

The waveform exhibits even-function symmetry and therefore

$$a_0 = 0$$

$$b_n = 0 \qquad \text{for all } n$$

The waveform exhibits half-wave symmetry and therefore

$$a_n = 0 \qquad \text{for } n \text{ even}$$

Hence,

$$\begin{aligned}
a_n &= \frac{4}{T_0}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt \qquad \text{for } n \text{ odd} \\
&= \frac{4}{T}\left(\int_0^{T/4} V\cos n\omega_0 t\, dt - \int_{T/4}^{T/2} V\cos n\omega_0 t\, dt\right) \\
&= \frac{4V}{n\omega_0 T}\left(\sin n\omega_0 t\Big|_0^{T/4} - \sin n\omega_0 t\Big|_{T/4}^{T/2}\right)
\end{aligned}$$

$$= \frac{4V}{n\omega_0 T}\left(\sin\frac{n\pi}{2} - \sin n\pi + \sin\frac{n\pi}{2}\right)$$

$$= \frac{8V}{n2\pi}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd}$$

$$= \frac{4V}{n\pi}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd}$$

The reader should compare this result with that obtained in Example 15.1.

EXAMPLE 15.3

Let us determine the trigonometric Fourier series expansion for the waveform shown in Fig. 15.4a.

SOLUTION

The function not only exhibits odd-function symmetry, but it possesses half-wave symmetry as well. Therefore, it is necessary to determine only the coefficients b_n for n odd. Note that

$$v(t) = \begin{cases} \dfrac{4Vt}{T_0} & 0 \le t \le T_0/4 \\ 2V - \dfrac{4Vt}{T_0} & T_0/4 < t \le T_0/2 \end{cases}$$

The b_n coefficients are then

$$b_n = \frac{4}{T_0}\int_0^{T_0/4} \frac{4Vt}{T_0}\sin n\omega_0 t\, dt + \frac{4}{T_0}\int_{T_0/4}^{T_0/2}\left(2V - \frac{4Vt}{T_0}\right)\sin n\omega_0 t\, dt$$

The evaluation of these integrals is tedious but straightforward and yields

$$b_n = \frac{8V}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd}$$

Hence, the Fourier series expansion is

$$v(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8V}{n^2\pi^2}\sin\frac{n\pi}{2}\sin n\omega_0 t$$

EXAMPLE 15.4

We wish to find the trigonometric Fourier series expansion of the waveform in Fig. 15.4b.

SOLUTION

Note that this waveform has an average value of 3/2. Therefore, instead of determining the Fourier series expansion of $f(t)$, we will determine the Fourier series for $f(t) - 3/2$, which is the waveform shown in Fig. 15.4c. The latter waveform possesses half-wave symmetry. The function is also odd and therefore

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} \frac{1}{2}\sin n\omega_0 t\, dt$$

$$= \frac{2}{T_0}\left(\frac{-1}{n\omega_0}\cos n\omega_0 t \Big|_0^{T_0/2}\right)$$

$$= \frac{-2}{n\omega_0 T_0}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi} \qquad \text{for } n \text{ odd}$$

Therefore, the Fourier series expansion for $f(t) - 3/2$ is

$$f(t) - \frac{3}{2} = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{2}{n\pi} \sin n\omega_0 t$$

or

$$f(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{2}{n\pi} \sin n\omega_0 t$$

EXAMPLE 15.5

Electrical sources such as batteries, solar panels, and fuel cells produce a dc output voltage. An electrical load requiring an ac voltage can be powered from a dc source using a device called an inverter, which converts a dc voltage to an ac voltage. Inverters can produce single-phase or three-phase ac voltages. Single-phase inverters are often classified as pure or true sine wave inverters or modified sine wave inverters. The output from a pure sine wave inverter is shown in Fig. 15.5. This waveform was discussed in Chapter 8 and could be described by $v(t) = 170 \sin 377t$ volts. Fig. 15.6 is the output voltage from a modified sine wave inverter. Note that this waveform is more square wave than sine wave.

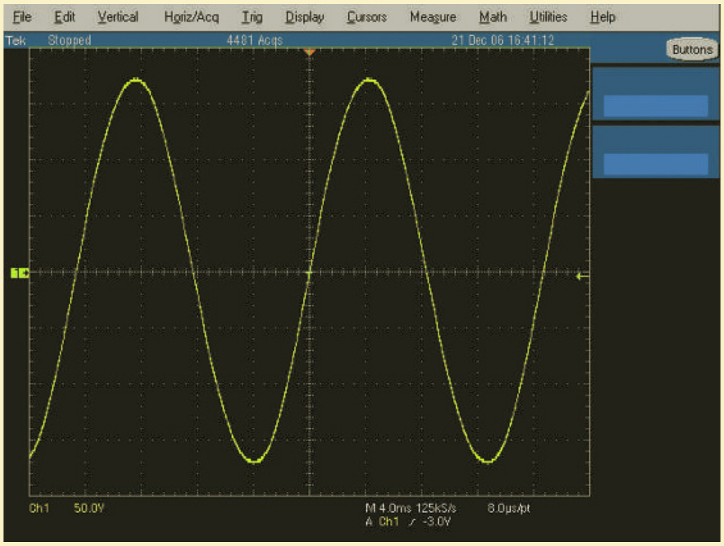


Figure 15.5 Output voltage for a pure sine wave inverter.

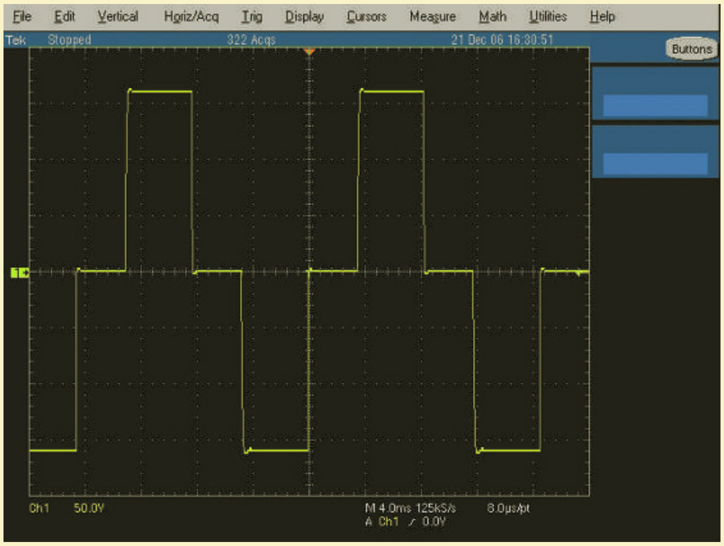


Figure 15.6 Output voltage for a modified sine wave inverter.

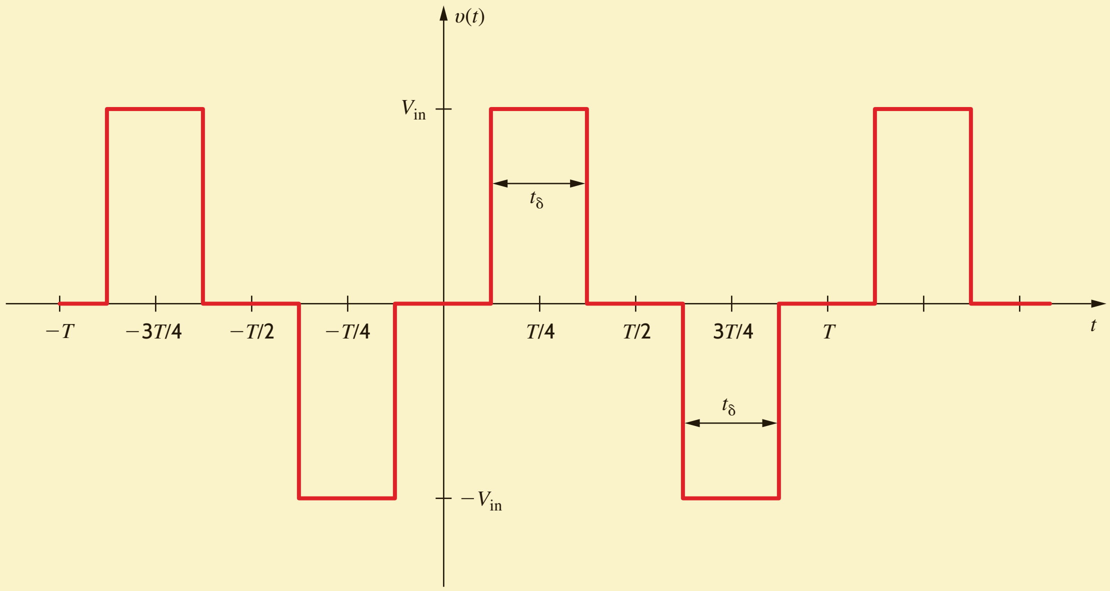


Figure 15.7

Waveform for determining Fourier components of the modified sine wave inverter output voltage.

Let's determine the Fourier components of the modified sine wave inverter output voltage using the waveform in Fig. 15.7. Note that this waveform consists of one positive pulse of width t_δ centered about $T/4$ and a negative pulse of the same width centered about $3T/4$. Close examination of this waveform reveals that it is an odd function with half-wave symmetry. Therefore,

$$a_0 = 0$$
$$a_n = 0 \text{ for all } n$$
$$b_n = 0 \text{ for } n \text{ even}$$

We can find b_n for n odd using

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\, dt$$

The waveform has a value of V_{in} between $t = T/4 - \delta/2$ and $t = T/4 + \delta/2$ and zero elsewhere over the interval from 0 to $T/2$. Therefore,

$$b_n = \frac{4}{T}\int_{T/4 - t_\delta/2}^{T/4 + t_\delta/2} V_{in}\sin n\omega_0 t\, dt$$

$$b_n = \frac{4V_{in}}{T}\int_{T/4 - t_\delta/2}^{T/4 + t_\delta/2} \sin n\omega_0 t\, dt$$

Integrating yields

$$b_n = \frac{4V_{in}}{n\omega_0 T}\Big[-\cos n\omega_0 t\Big]_{T/4 - t_\delta/2}^{T/4 + t_\delta/2}$$

Recalling that $\omega_0 T = 2\pi$ and evaluating the function at the limits produces

$$b_n = \frac{2V_{in}}{\pi}\left[-\cos\left(\frac{n\omega_0 T}{4} + \frac{n\omega_0 t_\delta}{2}\right) + \cos\left(\frac{n\omega_0 T}{4} - \frac{n\omega_0 t_\delta}{2}\right)\right]$$

The expression in brackets is $-\cos(\alpha + \beta) + \cos(\alpha - \beta)$. Using the appropriate trigonometric identities, we have

$$-\cos(\alpha + \beta) + \cos(\alpha - \beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$-\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\sin\alpha\sin\beta$$

The expression for b_n, which is valid for n odd, becomes

$$b_n = \frac{4V_{\text{in}}}{n\pi}\left[\sin\left(\frac{n\omega_0 T}{4}\right)\sin\left(\frac{n\omega_0 t_\delta}{2}\right)\right]$$

Let's define $\omega_0 t_\delta = \delta$ and again utilize $\omega_0 T = 2\pi$:

$$b_n = \frac{4V_{\text{in}}}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\delta}{2}\right)\right]$$

Using this expression,

$$b_1 = \frac{4V_{\text{in}}}{\pi}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\delta}{2}\right) = \frac{4V_{\text{in}}}{\pi}\sin\left(\frac{\delta}{2}\right)$$
$$b_3 = \frac{4V_{\text{in}}}{3\pi}\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\delta}{2}\right) = -\frac{4V_{\text{in}}}{3\pi}\sin\left(\frac{3\delta}{2}\right)$$
$$b_5 = \frac{4V_{\text{in}}}{5\pi}\sin\left(\frac{5\pi}{2}\right)\sin\left(\frac{5\delta}{2}\right) = \frac{4V_{\text{in}}}{5\pi}\sin\left(\frac{5\delta}{2}\right)$$

Now let's plot the absolute value of b_1, b_3, and b_5 as δ varies between 0º and 180º for $V_{\text{in}} = 1$ volt as shown in **Fig. 15.8**. Note that b_1—the coefficient of the first harmonic or fundamental—is zero for $\delta = 0°$ and reaches a maximum value of $4/\pi = 1.273$ volts for $\delta = 180°$. Examination of this plot reveals that the absolute value of the third harmonic is zero for $\delta = 120°$. The expression for b_3 contains the term $\sin(3\delta/2)$, which has a value of zero for $\delta = 120°$. If we chose $\delta = 72°$, the amplitude of the fifth harmonic would be zero. This example illustrates that it is possible to eliminate one harmonic from the Fourier series for the output voltage by proper selection of the angle δ.

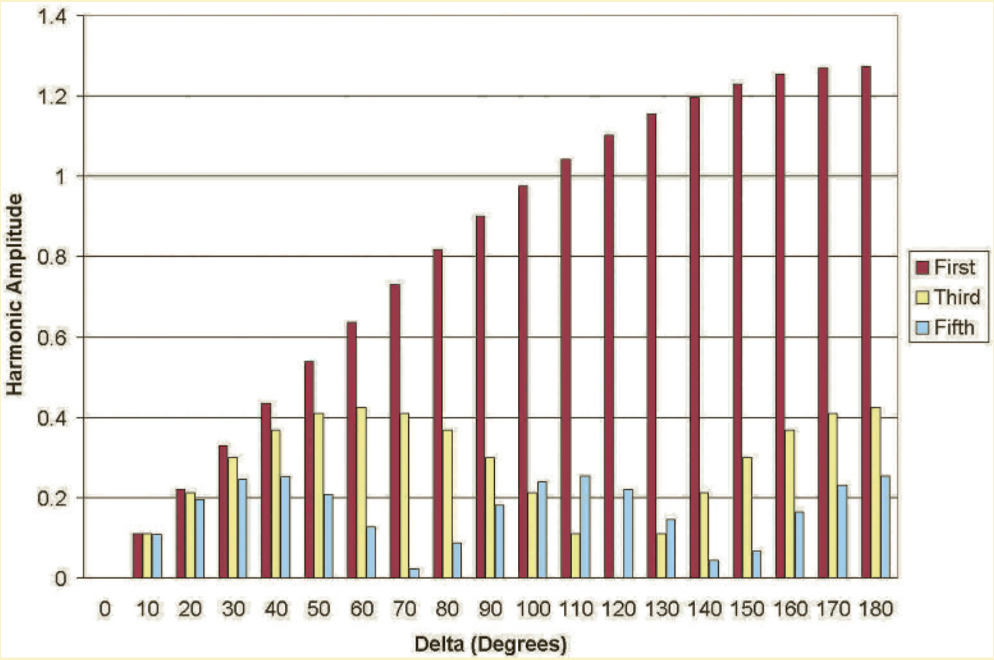


Figure 15.8

Plot of harmonic amplitude versus the angle δ.

LEARNING ASSESSMENTS

E15.4 Determine the type of symmetry exhibited by the waveform in Figs. E15.2 and E15.4.

ANSWER: Fig. E15.2, even symmetry; Fig. E15.4, half-wave symmetry.

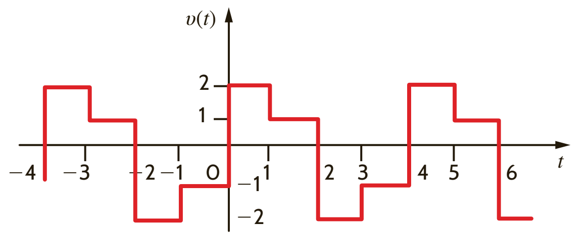


Figure E15.4

E15.5 Find the trigonometric Fourier series for the voltage waveform in Fig. E15.2.

ANSWER:
$$v(t) = 2 + \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(2 \sin \frac{2\pi n}{3} - \sin \frac{n\pi}{3}\right) \cos \frac{n\pi}{3} t.$$

E15.6 Find the trigonometric Fourier series for the voltage waveform in Fig. E15.4.

ANSWER:
$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} t + \frac{2}{n\pi}(2 - \cos n\pi) \sin \frac{n\pi}{2} t.$$

E15.7 Determine the trigonometric Fourier series for the waveform shown in Fig. E15.3.

ANSWER:
$v(t) = 0.25 + 0.955 \cos(0.5\pi t) + 0.955 \sin(0.5\pi t) + 0.318 \sin(\pi t) - 0.318 \cos(1.5\pi t) + 0.318 \sin(1.5\pi t) + \ldots$ V.

TIME-SHIFTING Let us now examine the effect of time-shifting a periodic waveform $f(t)$ defined by the equation

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t}$$

Note that

$$f(t - t_0) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0(t - t_0)}$$

$$f(t - t_0) = \sum_{n=-\infty}^{\infty} \left(\mathbf{c}_n e^{-jn\omega_0 t_0}\right) e^{jn\omega_0 t} \qquad \textbf{15.30}$$

Since $e^{-jn\omega_0 t_0}$ corresponds to a phase shift, the Fourier coefficients of the time-shifted function are the Fourier coefficients of the original function, with the angle shifted by an amount directly proportional to frequency. Therefore, time shift in the time domain corresponds to phase shift in the frequency domain.

EXAMPLE 15.6

Let us time-delay the waveform in Fig. 15.3 by a quarter period and compute the Fourier series.

SOLUTION

The waveform in Fig. 15.3 time-delayed by $T_0/4$ is shown in Fig. 15.9. Since the time delay is $T_0/4$,

$$n\omega_0 t_d = n\frac{2\pi}{T_0}\frac{T_0}{4} = n\frac{\pi}{2} = n90^\circ$$

Therefore, using Eq. (15.30) and the results of Example 15.1, the Fourier coefficients for the time-shifted waveform are

$$\mathbf{c}_n = \frac{2V}{n\pi}\sin\frac{n\pi}{2}\underline{/-n90^\circ} \qquad n\,\text{odd}$$

and therefore,

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{4V}{n\pi}\sin\frac{n\pi}{2}\cos(n\omega_0 t - n90^\circ)$$

If we compute the Fourier coefficients for the time-shifted waveform in Fig. 15.9, we obtain

$$\begin{aligned}
\mathbf{c}_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-jn\omega_0 t}\,dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{0} -Ve^{-jn\omega_0 t}\,dt + \frac{1}{T_0}\int_{0}^{T_0/2} Ve^{-jn\omega_0 t}\,dt \\
&= \frac{2V}{jn\pi} \qquad \text{for } n\,\text{odd}
\end{aligned}$$

Therefore,

$$\mathbf{c}_n = \frac{2V}{n\pi}\underline{/-90^\circ} \qquad n\,\text{odd}$$

Since n is odd, we can show that this expression is equivalent to the one obtained earlier.

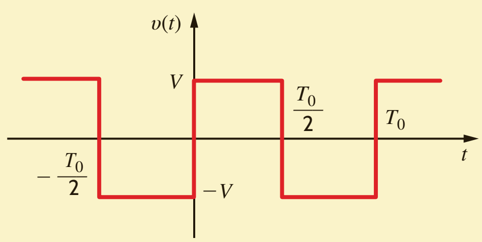


Figure 15.9
Waveform in Fig. 15.3 time-shifted by $T_0/4$.

In general, we can compute the phase shift in degrees using the expression

$$\text{phase shift(deg)} = \omega_0 t_d = (360^\circ)\frac{t_d}{T_0} \qquad \textbf{15.31}$$

so that a time shift of one-quarter period corresponds to a 90° phase shift.

As another interesting facet of the time shift, consider a function $f_1(t)$ that is nonzero in the interval $0 \leq t \leq T_0/2$ and is zero in the interval $T_0/2 < t \leq T_0$. For purposes of illustration, let us assume that $f_1(t)$ is the triangular waveform shown in **Fig. 15.10a**. $f_1(t - T_0/2)$ is then shown in **Fig. 15.10b**. Then the function $f(t)$ defined as

$$f(t) = f_1(t) - f_1\left(t - \frac{T_0}{2}\right) \qquad \textbf{15.32}$$

is shown in **Fig. 15.10c**. Note that $f(t)$ has half-wave symmetry. In addition, note that if

$$f_1(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{-jn\omega_0 t}$$

then

$$\begin{aligned}
f(t) = f_1(t) - f_1\left(t - \frac{T_0}{2}\right) &= \sum_{n=-\infty}^{\infty} \mathbf{c}_n(1 - e^{-jn\pi})e^{jn\omega_0 t} \\
&= \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} 2\mathbf{c}_n e^{jn\omega_0 t} & n\,\text{odd} \\ 0 & n\,\text{even} \end{cases} \qquad \textbf{15.33}
\end{aligned}$$

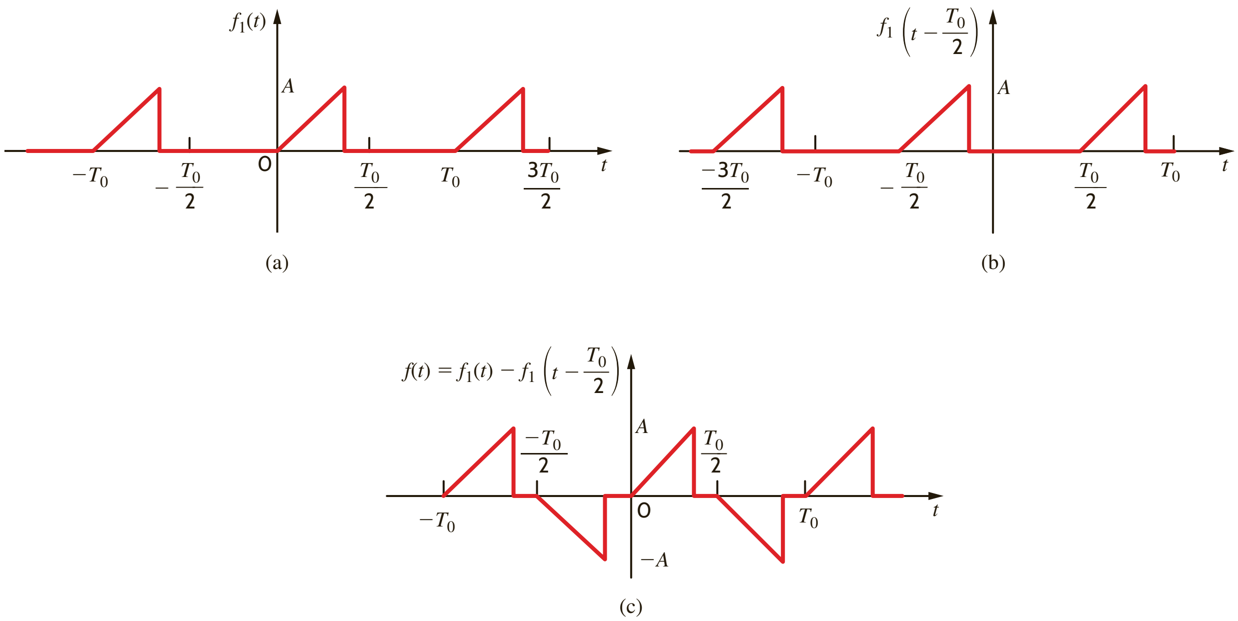


Figure 15.10

Waveforms that illustrate the generation of half-wave symmetry.

Therefore, we see that any function with half-wave symmetry can be expressed in the form of Eq. (15.32), where the Fourier series is defined by Eq. (15.33), and $\mathbf{c}_n$ is the Fourier coefficient for $f_1(t)$.

LEARNING ASSESSMENT

E15.8 If the waveform in Fig. E15.1 is time-delayed 1 s, we obtain the waveform in Fig. E15.8. Compute the exponential Fourier coefficients for the waveform in Fig. E15.8 and show that they differ from the coefficients for the waveform in Fig. E15.1 by an angle $n(180°)$.

ANSWER:

$\mathbf{c}_0 = \frac{1}{2};\ \mathbf{c}_n = -\left(\frac{1 - e^{-jn\pi}}{j2\pi n}\right).$

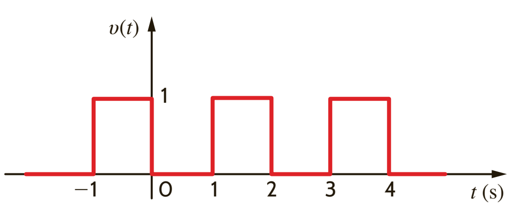


Figure E15.8

WAVEFORM GENERATION The magnitude of the harmonics in a Fourier series is independent of the time scale for a given waveshape. Therefore, the equations for a variety of waveforms can be given in tabular form without expressing a specific time scale. Table 15.1 is a set of commonly occurring periodic waves where the advantage of symmetry has been used to simplify the coefficients. These waveforms can be used to generate other waveforms. The level of a wave can be adjusted by changing the average value component; the time can be shifted by adjusting the angle of the harmonics; and two waveforms can be added to produce a third waveform. For example, the waveforms in Figs. 15.11a and b can be added to produce the waveform in Fig. 15.11c.

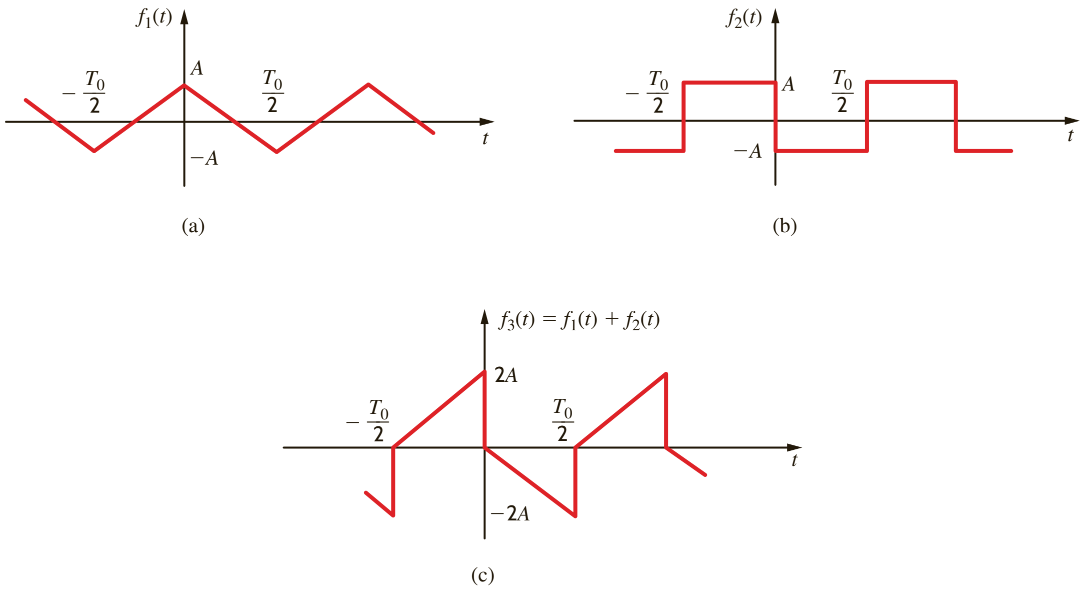
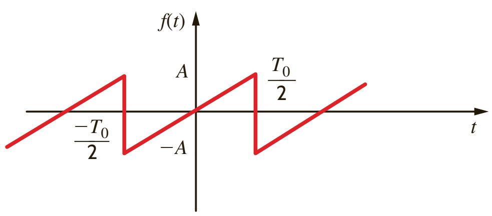
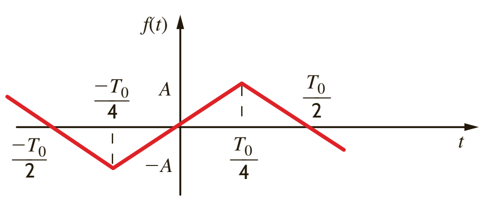
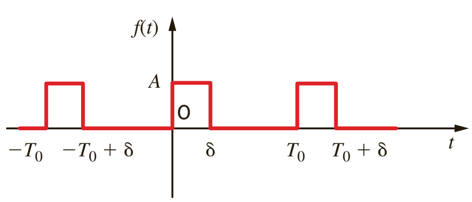
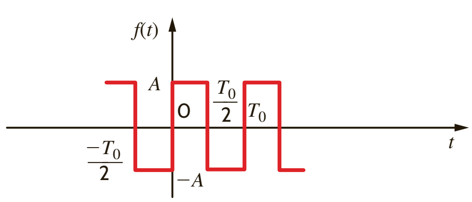


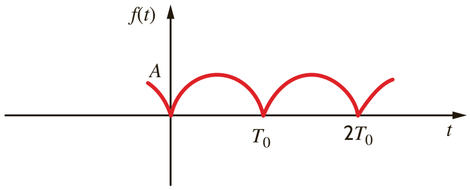
Figure 15.11 Example of waveform generation.

TABLE 15.1 Fourier series for some common waveforms

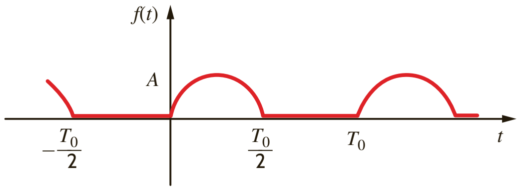
	$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2A}{n\pi} \sin n\omega_0 t$
	$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8A}{n^2\pi^2} \sin \frac{n\pi}{2} \sin n\omega_0 t$
	$f(t) = \sum_{n=-\infty}^{\infty} \frac{A}{n\pi} \sin \frac{n\pi\delta}{T_0} e^{jn\omega_0[t-(\delta/2)]}$
	$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4A}{n\pi} \sin n\omega_0 t$

(*Continues on the next page*)

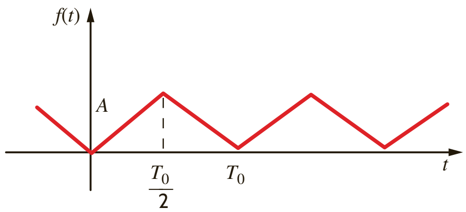
TABLE 15.1 (*Continued*)



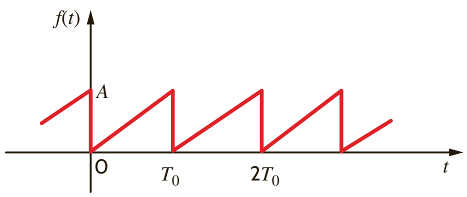
$$f(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi(1-4n^2)} \cos n\omega_0 t$$



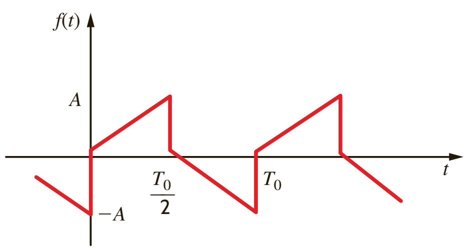
$$f(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega_0 t + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{2A}{\pi(1-n^2)} \cos n\omega_0 t$$



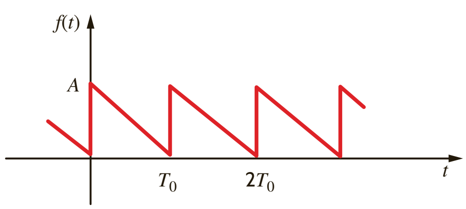
$$f(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n \neq 0 \\ n \text{ odd}}}^{\infty} \frac{-2A}{n^2\pi^2} e^{jn\omega_0 t}$$



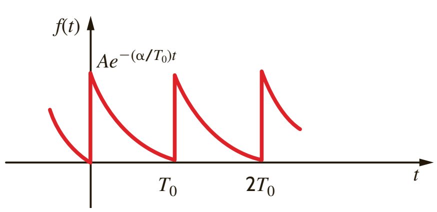
$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-A}{n\pi} \sin n\omega_0 t$$



$$f(t) = \sum_{n=1}^{\infty} \frac{-4A}{\pi^2 n^2} \cos n\omega_0 t + \frac{2A}{\pi n} \sin n\omega_0 t$$



$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{\pi n} \sin n\omega_0 t$$



$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A(1-e^{-\alpha})}{\alpha + j2\pi n} e^{jn\omega_0 t}$$

LEARNING ASSESSMENT

E15.9 Two periodic waveforms are shown in Fig. E15.9. Compute the exponential Fourier series for each waveform, and then add the results to obtain the Fourier series for the waveform in Fig. E15.2.

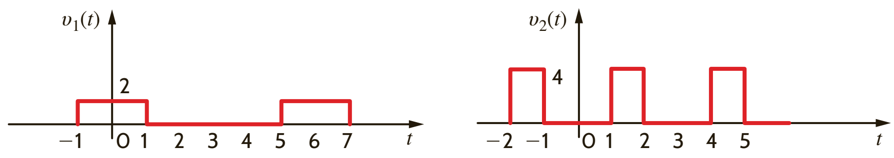


Figure E15.9

ANSWER:

$$v_1(t) = \frac{2}{3} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{3} e^{jn\omega_0 t};$$

$$v_2(t) = \frac{4}{3} + \sum_{n=-\infty}^{\infty} -\frac{4}{n\pi} \left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right) e^{jn\omega_0 t}.$$

FREQUENCY SPECTRUM The *frequency spectrum* of the function $f(t)$ expressed as a Fourier series consists of a plot of the amplitude of the harmonics versus frequency, which we call the *amplitude spectrum*, and a plot of the phase of the harmonics versus frequency, which we call the *phase spectrum*. Since the frequency components are discrete, the spectra are called *line spectra*. Such spectra illustrate the frequency content of the signal. Plots of the amplitude and phase spectra are based on Eqs. (15.1), (15.3), and (15.7) and represent the amplitude and phase of the signal at specific frequencies.

EXAMPLE 15.7

The Fourier series for the triangular-type waveform shown in Fig. 15.11c with $A = 5$ is given by the equation

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \left(\frac{20}{n\pi} \sin n\omega_0 t - \frac{40}{n^2\pi^2} \cos n\omega_0 t\right)$$

We wish to plot the first four terms of the amplitude and phase spectra for this signal.

SOLUTION

Since $D_n \underline{/\theta_n} = a_n - jb_n$, the first four terms for this signal are

$$D_1 \underline{/\theta_1} = -\frac{40}{\pi^2} - j\frac{20}{\pi} = 7.5\underline{/-122^\circ}$$

$$D_3 \underline{/\theta_3} = -\frac{40}{9\pi^2} - j\frac{20}{3\pi} = 2.2\underline{/-102^\circ}$$

$$D_5 \underline{/\theta_5} = -\frac{40}{25\pi^2} - j\frac{20}{5\pi} = 1.3\underline{/-97^\circ}$$

$$D_7 \underline{/\theta_7} = -\frac{40}{49\pi^2} - j\frac{20}{7\pi} = 0.91\underline{/-95^\circ}$$

Therefore, the plots of the amplitude and phase versus ω are as shown in Fig. 15.12.

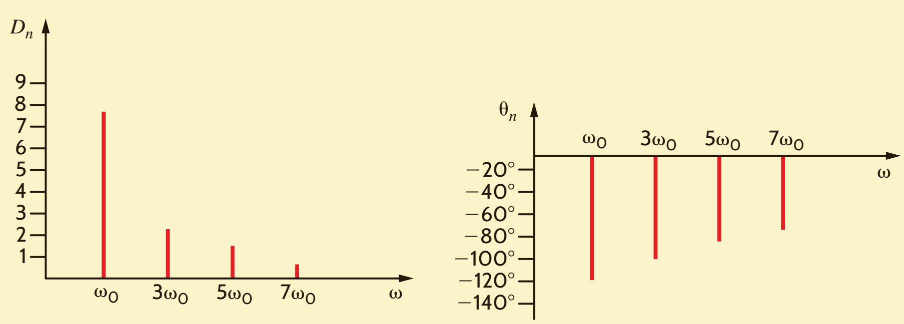


Figure 15.12
Amplitude and phase spectra.

EXAMPLE 15.8

The circuit shown in **Fig. 15.13** is a notch filter. At its resonant frequency, the *L-C* series circuit has zero effective impedance and, as a result, any signal at that frequency is short-circuited. For this reason, the filter is often referred to as a trap.

Consider the following scenario. A system operating at 1 kHz has picked up noise at a fundamental frequency of 10 kHz, as well as some second- and third-harmonic junk. Given this information, we wish to design a filter that will eliminate both the noise and its attendant harmonics.

SOLUTION

The key to the trap is setting the resonant frequency of the *L-C* series branch to the frequency we wish to eliminate. Since we have three frequency components to remove, 10 kHz, 20 kHz, and 30 kHz, we will simply use three different *L-C* branches as shown in **Fig. 15.14** and set L_1C_1 to trap at 10 kHz, L_2C_2 at 20 kHz, and L_3C_3 at 30 kHz. If we arbitrarily set the value of all inductors to 10 μH and calculate the value of each capacitor, we obtain

$$C_1 = \frac{1}{(2\pi)^2 f^2 L} = \frac{1}{(2\pi)^2(10^8)(10^{-5})} = 25.3\ \mu\text{F}$$

$$C_2 = \frac{1}{(2\pi)^2(4 \times 10^8)(10^{-5})} = 6.34\ \mu\text{F}$$

$$C_3 = \frac{1}{(2\pi)^2(9 \times 10^8)(10^{-5})} = 2.81\ \mu\text{F}$$

The three traps shown in Fig. 15.14 should eliminate the noise and its harmonics.

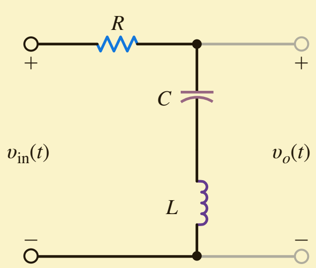


Figure 15.13
A notch filter, or trap, utilizing a series *L-C* branch.

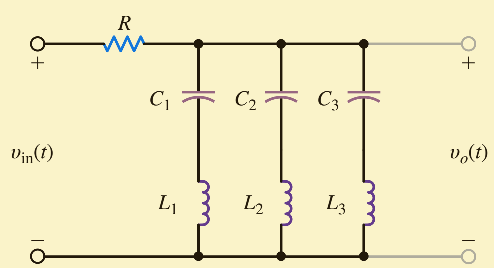


Figure 15.14
The notch filter in Fig. 15.13 expanded to remove three different frequency components.

LEARNING ASSESSMENTS

E15.10 Determine the trigonometric Fourier series for the voltage waveform in Fig. E15.10 and plot the first four terms of the amplitude and phase spectra for this signal.

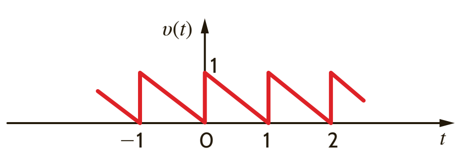


Figure E15.10

ANSWER:
$a_0 = 1/2$;
$D_1 = -j(1/\pi)$;
$D_2 = -j(1/2\pi)$;
$D_3 = -j(1/3\pi)$;
$D_4 = -j(1/4\pi)$.

E15.11 The discrete line spectrum for a periodic function is shown in Fig. E15.11. Determine the expression for $f(t)$.

ANSWER:
$f(t) = 0.25 + 1.35\cos(40\pi t - 135°) + \cos(80\pi t - 90°) + 0.5\cos(120\pi t - 45°) + 0.35\cos(160\pi t - 90°)$.

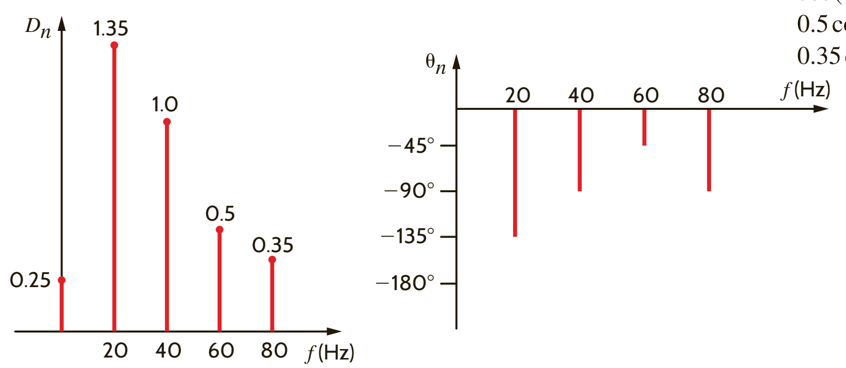


Figure E15.11

STEADY-STATE NETWORK RESPONSE If a periodic signal is applied to a network, the steady-state voltage or current response at some point in the circuit can be found in the following manner. First, we represent the periodic forcing function by a Fourier series. If the input forcing function for a network is a voltage, the input can be expressed in the form

$$v(t) = v_0 + v_1(t) + v_2(t) + \cdots$$

and therefore represented in the time domain as shown in **Fig. 15.15**. Each source has its own amplitude and frequency. Next we determine the response due to each component of the input Fourier series; that is, we use phasor analysis in the frequency domain to determine the network response due to each source. The network response due to each source in the frequency domain is then transformed to the time domain. Finally, we add the time-domain solutions due to each source using the Principle of Superposition to obtain the Fourier series for the total *steady-state* network response.

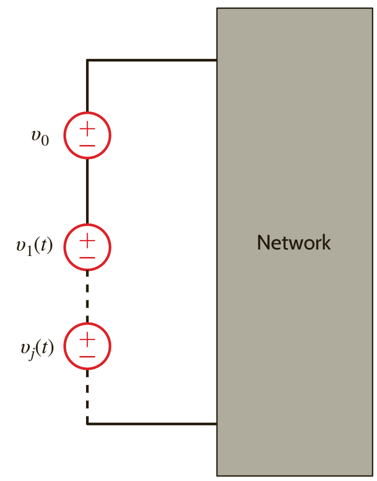


Figure 15.15

Network with a periodic voltage forcing function.

AVERAGE POWER We have shown that when a linear network is forced with a nonsinusoidal periodic signal, voltages and currents throughout the network are of the form

$$v(t) = V_{DC} + \sum_{n=1}^{\infty} V_n \cos\left(n\omega_0 t - \theta_{v_n}\right)$$

and

$$i(t) = I_{DC} + \sum_{n=1}^{\infty} I_n \cos\left(n\omega_0 t - \theta_{i_n}\right)$$

If we employ the passive sign convention and assume that the voltage across an element and the current through it are given by the preceding equations, then from Eq. (9.6),

$$\begin{aligned} P &= \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\,dt \\ &= \frac{1}{T}\int_{t_0}^{t_0+T} v(t)i(t)\,dt \end{aligned} \qquad \textbf{15.34}$$

Note that the integrand involves the product of two infinite series. However, the determination of the average power is actually easier than it appears. First, note that the product $V_{DC}I_{DC}$ when integrated over a period and divided by the period is simply $V_{DC}I_{DC}$. Second, the product of V_{DC} and any harmonic of the current or I_{DC} and any harmonic of the voltage when integrated over a period yields zero. Third, the product of any two *different* harmonics of the voltage and the current when integrated over a period yields zero. Finally, nonzero terms result only from the products of voltage and current at the *same* frequency. Hence, using the mathematical development that follows Eq. (9.6), we find that

$$P = V_{DC}I_{DC} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2}\cos\left(\theta_{v_n} - \theta_{i_n}\right) \qquad \textbf{15.35}$$

EXAMPLE 15.9

We wish to determine the steady-state voltage $v_o(t)$ in Fig. 15.16 if the input voltage $v(t)$ is given by the expression

$$v(t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \left(\frac{20}{n\pi} \sin 2nt - \frac{40}{n^2\pi^2} \cos 2nt \right) \text{V}$$

SOLUTION

Note that this source has no constant term, and therefore its dc value is zero. The amplitude and phase for the first four terms of this signal are given in Example 15.7, and therefore the signal $v(t)$ can be written as

$$\begin{aligned} v(t) &= 7.5 \cos(2t - 122°) + 2.2 \cos(6t - 102°) \\ &\quad + 1.3 \cos(10t - 97°) + 0.91 \cos(14t - 95°) + \cdots \end{aligned}$$

From the network we find that

$$\mathbf{I} = \frac{\mathbf{V}}{2 + \dfrac{2/j\omega}{2 + 1/j\omega}} = \frac{\mathbf{V}(1 + 2j\omega)}{4 + 4j\omega}$$

$$\mathbf{I}_1 = \frac{\mathbf{I}(1/j\omega)}{2 + 1/j\omega} = \frac{\mathbf{I}}{1 + 2j\omega}$$

$$\mathbf{V}_o = (1)\mathbf{I}_1 = 1 \cdot \frac{\mathbf{V}(1 + 2j\omega)}{4 + 4j\omega} \frac{1}{1 + 2j\omega} = \frac{\mathbf{V}}{4 + 4j\omega}$$

Therefore, since $\omega_0 = 2$,

$$\mathbf{V}_o(n) = \frac{\mathbf{V}(n)}{4 + j8n}$$

The individual components of the output due to the components of the input source are then

$$\begin{aligned} \mathbf{V}_o(\omega_0) &= \frac{7.5\angle -122°}{4 + j8} = 0.84\angle -185.4° \text{ V} \\ \mathbf{V}_o(3\omega_0) &= \frac{2.2\angle -102°}{4 + j24} = 0.09\ \angle -182.5° \text{ V} \\ \mathbf{V}_o(5\omega_0) &= \frac{1.3\angle -97°}{4 + j40} = 0.03\ \angle -181.3° \text{ V} \\ \mathbf{V}_o(7\omega_0) &= \frac{0.91\angle -95°}{4 + j56} = 0.016\ \angle -181° \text{ V} \end{aligned}$$

Hence, the steady-state output voltage $v_o(t)$ can be written as

$$\begin{aligned} v_o(t) &= 0.84 \cos(2t - 185.4°) + 0.09 \cos(6t - 182.5°) \\ &\quad + 0.03 \cos(10t - 181.3°) + 0.016 \cos(14t - 181°) + \cdots \text{V} \end{aligned}$$

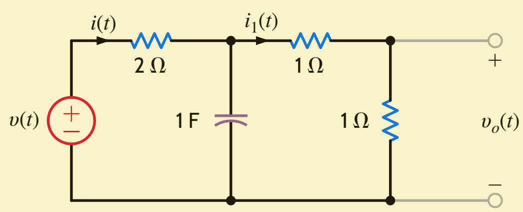


Figure 15.16

RC circuit employed in Example 15.9.

EXAMPLE 15.10

In the network in Fig. 15.17, $v(t) = 42 + 16\cos(377t + 30°) + 12\cos(754t - 20°)$ V. We wish to compute the current $i(t)$ and determine the average power absorbed by the network.

SOLUTION

The capacitor acts as an open circuit to dc, and therefore $I_{DC} = 0$. At $\omega = 377$ rad/s,

$$\frac{1}{j\omega C} = \frac{1}{j(377)(100)(10)^{-6}} = -j26.53\ \Omega$$

$$j\omega L = j(377)(20)10^{-3} = j7.54\ \Omega$$

Hence,

$$\mathbf{I}_{377} = \frac{16\underline{/30°}}{16 + j7.54 - j26.53} = 0.64\underline{/79.88°}\text{ A}$$

At $\omega = 754$ rad/s,

$$\frac{1}{j\omega C} = \frac{1}{j(754)(100)(10)^{-6}} = -j13.26\ \Omega$$

$$j\omega L = j(754)(20)10^{-3} = j15.08\ \Omega$$

Hence,

$$\mathbf{I}_{754} = \frac{12\underline{/20°}}{16 + j15.08 - j13.26} = 0.75\ \underline{/-26.49°}\text{ A}$$

Therefore, the current $i(t)$ is

$$i(t) = 0.64\cos(377t + 79.88°) + 0.75\cos(754t + 26.49°)\text{ A}$$

and the average power absorbed by the network is

$$P = (42)(0) + \frac{(16)(0.64)}{2}\cos(30° - 79.88°) + \frac{(12)(0.75)}{2}\cos(-20° + 26.49°) = 7.77\text{ W}$$

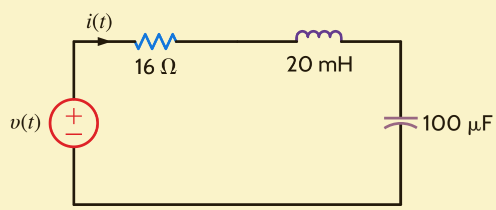


Figure 15.17 Network used in Example 15.10.

PROBLEM-SOLVING STRATEGY

STEADY-STATE RESPONSE TO PERIODIC FORCING FUNCTIONS

STEP 1. Determine the Fourier series for the periodic forcing function, which is now expressed as a summation of harmonically related sinusoidal functions.

STEP 2. Use phasor analysis to determine the network response due to each sinusoidal function acting alone.

STEP 3. Use the Principle of Superposition to add the time-domain solution from each source acting alone to determine the total steady-state network response.

STEP 4. If you need to calculate the average power dissipated in a network element, determine the average power dissipated in that element due to each source acting alone and then sum these for the total power dissipation from the periodic forcing function.

LEARNING ASSESSMENTS

E15.12 Determine the expression for the steady-state current $i(t)$ in Fig. E15.12 if the input voltage $v_S(t)$ is given by the expression

$$v_S(t) = \frac{20}{\pi} + \sum_{n=1}^{\infty} \frac{-40}{\pi(4n^2 - 1)} \cos 2nt \text{ V}$$

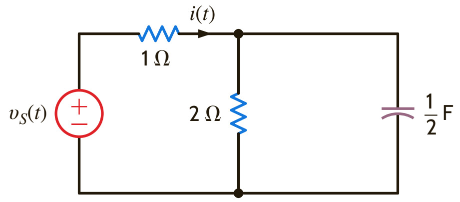


Figure E15.12

ANSWER:
$i(t) = 2.12 + \sum_{n=1}^{\infty} \frac{-40}{\pi(4n^2 - 1)} \frac{1}{A_n} \cos(2nt - \theta_n)$ A.

E15.13 At the input terminals of a network, the voltage $v(t)$ and the current $i(t)$ are given by the following expressions:

$$v(t) = 64 + 36\cos(377t + 60^\circ) - 24\cos(754t + 102^\circ) \text{ V}$$
$$i(t) = 1.8\cos(377t + 45^\circ) + 1.2\cos(754t + 100^\circ) \text{ A}$$

Find the average power absorbed by the network.

ANSWER:
$P = 16.91$ W.

E15.14 Determine the first three terms of the steady-state current $i(t)$ in Fig. E15.14 if the input voltage is given by

$$v(t) = \frac{30}{\pi} + 15\sin 10t + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{60}{\pi(1 - n^2)} \cos 10nt \text{ V}.$$

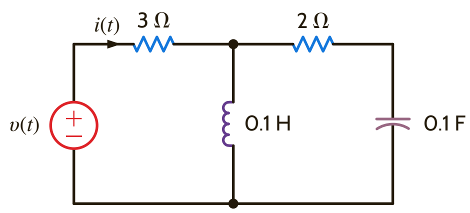


Figure E15.14

ANSWER:
$i(t) = 3.18 + 4.12\cos(10t + 106^\circ) + 1.45\cos(20t + 166^\circ)$ A.

E15.15 Find the average power absorbed by the network in Fig. E15.15 if

$$v(t) = 20 + 5\cos 377t + 3.5\cos(754t - 20^\circ) \text{ V and}$$
$$i(t) = 1.2\cos(377t - 30^\circ) + 0.8\cos(754t + 45^\circ) \text{ A}$$

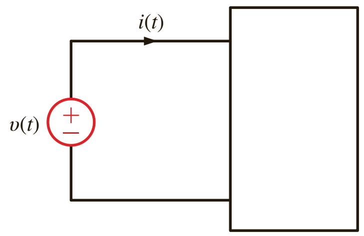


Figure E15.15

ANSWER:
$P = 3.19$ W.