10.1 Mutual Inductance

As we introduce this subject, we feel compelled to remind the reader, once again, that in our analyses we assume that we are dealing with "ideal" elements. For example, we ignore the resistance of the coil used to make an inductor and any stray capacitance that might exist. This approach is especially important in our discussion of mutual inductance because an exact analysis of this topic is quite involved. As is our practice, we will treat the subject in a straightforward manner and ignore issues beyond the scope of this book that only serve to complicate the presentation.

To begin our discussion of mutual inductance, we will recall two important laws: Ampère's law and Faraday's law. Ampère's law predicts that the flow of electric current will create a magnetic field. If the field links an electric circuit, and that field is time-varying, Faraday's law predicts the creation of a voltage within the linked circuit. Although this occurs to some extent in all circuits, the effect is magnified in coils because the circuit geometry amplifies the linkage effect. With these ideas in mind, consider the ideal situation in **Fig. 10.1** in which a current i flows in an N-turn coil and produces a magnetic field, represented by magnetic flux ϕ. The flux linkage for this coil is

$$\lambda = N\phi \tag{10.1}$$

For the linear systems that we are studying in this textbook, the flux linkage and current are related by

$$\lambda = Li \tag{10.2}$$

The constant of proportionality between the flux linkage and current is the inductance, which we studied in Chapter 6. Eqs. (10.1) and (10.2) can be utilized to express the magnetic flux in terms of the current:

$$\phi = \frac{L}{N} i \tag{10.3}$$

According to Faraday's law, the voltage induced in the coil is related to the time rate of change of the flux linkage λ:

$$v = \frac{d\lambda}{dt} \tag{10.4}$$

Let's substitute Eq. (10.2) into Eq. (10.4) and use the chain rule to take the derivative:

$$v = \frac{d\lambda}{dt} = \frac{d}{dt}(Li) = L\frac{di}{dt} + i\frac{dL}{dt} \tag{10.5}$$

We will not allow our inductances to vary with time, so Eq. (10.5) reduces to the defining equation for the ideal inductor, as shown in **Fig. 10.2**:

$$v = L\frac{di}{dt} \tag{10.6}$$

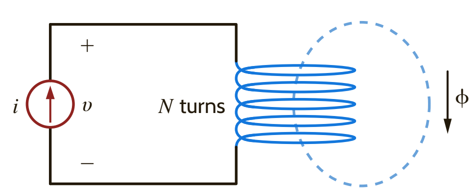


Figure 10.1
Magnetic flux ϕ linking an N-turn coil.

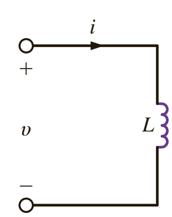


Figure 10.2
An ideal inductor.

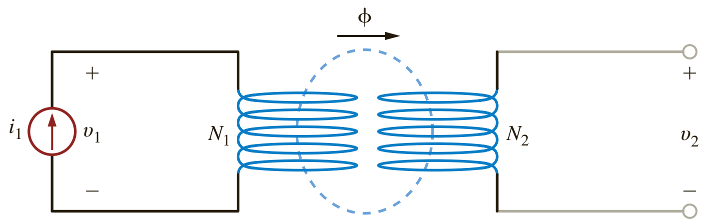


Figure 10.3

Two coils magnetically coupled.

Note that the voltage and current in this figure satisfy the passive sign convention. Eq. (10.6) tells us that a current i flowing through a coil produces a voltage v across that coil.

Now let's suppose that a second coil with N_2 turns is moved close enough to an N_1-turn coil such that the magnetic flux produced by current i_1 links the second coil. No current flows in the second coil as shown in **Fig. 10.3**. By Faraday's law, a voltage v_2 will be induced because the magnetic flux ϕ links the second coil. The flux linkage for coil 1 is

$$\lambda_i = N_1\phi = L_1 i_1 \qquad \textbf{10.7}$$

Current flowing in coil 1 produces a voltage $v_1 = \dfrac{d\lambda_1}{dt} = L_1\dfrac{di_1}{dt}$. We have been referring to L_1 as the inductance. In multiple coil systems, we will refer to L_1 as the self-inductance of coil 1. The flux linkage for coil 2 is $\lambda_2 = N_2\phi$, and from Faraday's law, the voltage v_2 is given as

$$v_2 = \frac{d\lambda_2}{dt} = \frac{d}{dt}(N_2\phi) = \frac{d}{dt}\left(N_2\left(\frac{L_1}{N_1}i_1\right)\right) = \frac{N_2}{N_1}L_1\frac{di_1}{dt} = L_{21}\frac{di_1}{dt} \qquad \textbf{10.8}$$

Note that the voltage v_2 is directly proportional to the time rate of change of i_1. The constant of proportionality, L_{21}, is defined as the mutual inductance and is given in units of henrys. We will say that the coils in Fig. 10.3 are magnetically coupled.

Let's connect a current source to the terminals of coil 2 as shown in **Fig. 10.4**. Both currents contribute to the magnetic flux ϕ. For the coil configuration and current directions shown in this figure, the flux linkages for each coil are

$$\lambda_1 = L_1 i_1 + L_{12} i_2 \qquad \textbf{10.9}$$

$$\lambda_2 = L_{21} i_1 + L_2 i_2 \qquad \textbf{10.10}$$

Applying Faraday's law,

$$v_1 = \frac{d\lambda_1}{dt} = L_1\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \qquad \textbf{10.11}$$

$$v_2 = \frac{d\lambda_2}{dt} = L_{21}\frac{di_1}{dt} + L_2\frac{di_2}{dt} \qquad \textbf{10.12}$$

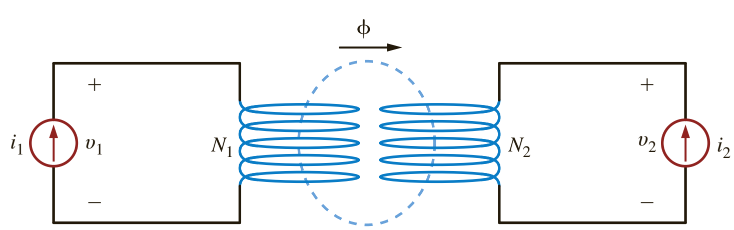


Figure 10.4

Two magnetically coupled coils driven by current sources.

Since we have limited our study to linear systems, $L_{12} = L_{21} = M$, where M is the symbol for mutual inductance. From Eqs. (10.11) and (10.12), we can see that the voltage across each coil is composed of two terms: a "self term" due to current flowing in that coil and a "mutual term" due to current flowing in the other coil.

If the direction of i_2 in Fig. 10.4 is reversed, Eqs. (10.9) through (10.12) become

$$\lambda_1 = L_1 i_1 - M i_2 \tag{10.13}$$

$$\lambda_2 = -M i_1 + L_2 i_2 \tag{10.14}$$

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \tag{10.15}$$

$$v_2 = \frac{d\lambda_2}{dt} = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \tag{10.16}$$

Eqs. (10.13)–(10.16) can also be obtained from the circuit in **Fig. 10.5**. Note that coil 2 in this figure has a different winding arrangement as compared to coil 2 in Fig. 10.4.

Our circuit diagrams will become quite complex if we have to include details of the winding configuration. The use of the dot convention permits us to maintain these details while simplifying our circuit diagrams. **Fig. 10.6a** is the circuit diagram for the magnetically coupled coils of Fig. 10.4. The coils are represented by two coupled inductors with self-inductances L_1 and L_2 and mutual inductance M. The voltage across each coil consists of two terms: a self term due to current flowing in that coil and a mutual term due to current flowing in the other coil. The self term is the same voltage that we discussed in an earlier chapter. The mutual term results from current flowing in the other coupled coil.

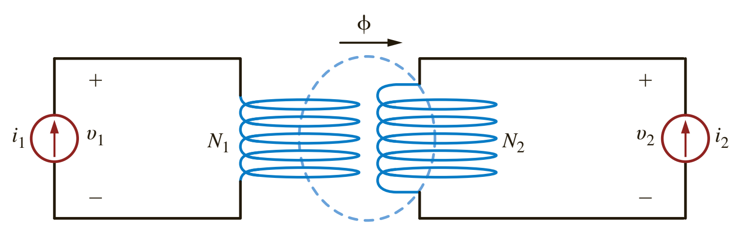


Figure 10.5

Magnetically coupled coils with different winding configuration.

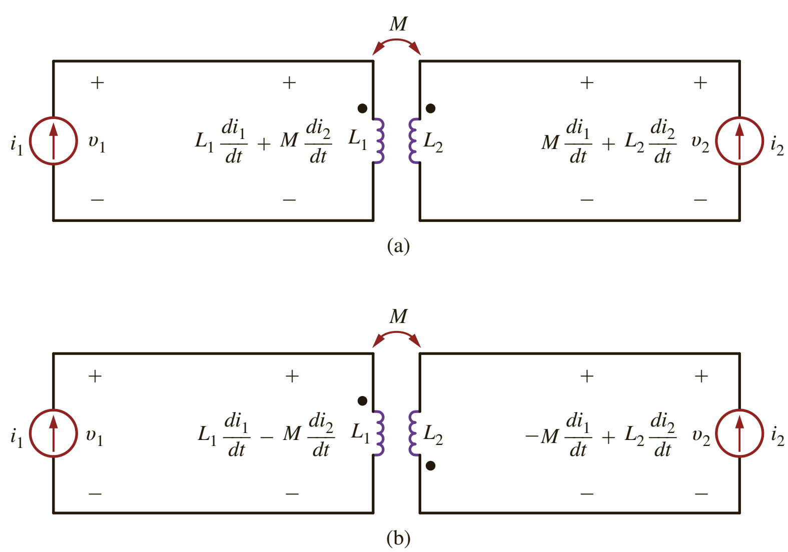


Figure 10.6

Circuit diagrams for magnetically coupled coils.

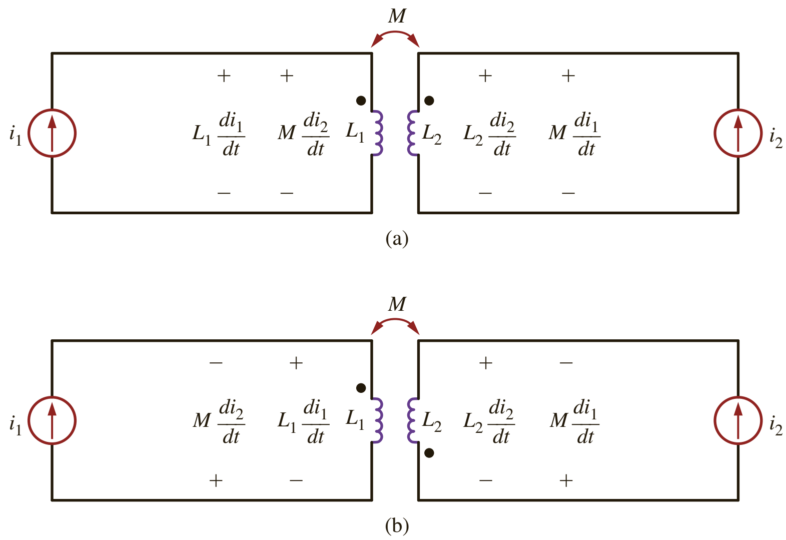


Figure 10.7

Circuit diagrams for magnetically coupled coils showing self and mutual voltage terms.

In Fig. 10.6a, the mutual terms are positive when both currents enter the dots. The opposite is true when one current enters a dot and the other current leaves a dot, as shown in Fig. 10.6b. Let's use this observation to develop a general procedure for writing circuit equations for magnetically coupled inductors. **Fig. 10.7a** is the same diagram as Fig. 10.6a except that the voltage across the inductors is broken into the self term and the mutual term. The polarity of the self terms—$L_1 di_1/dt$ and $L_2 di_2/dt$—are given by the passive sign convention used extensively throughout this text. These terms would be present even if the coils were not magnetically coupled. The mutual terms in Fig. 10.7a have the same polarity as the self terms. Note that both currents are entering the dots in Fig. 10.7a. The opposite is true in **Fig. 10.7b**. The self terms have the same polarity as before; however, the polarities for the mutual terms are different from those in Fig. 10.7a. We can now make a general statement:

> When a current is defined to enter the dotted terminal of a coil, it produces a voltage in the coupled coil, which is positive at the dotted terminal. Similarly, when a current is defined to enter the undotted terminal of a coil, it produces a voltage in the coupled coil, which is positive at the undotted terminal.

Let's illustrate the use of this statement through some examples.

PROBLEM-SOLVING STRATEGY

MAGNETICALLY COUPLED INDUCTORS

STEP 1. Assign mesh currents. It is usually much easier to write mesh equations for a circuit containing magnetically coupled inductors than nodal equations.

STEP 2. Write mesh equations by appling KVL. If a defined current enters the dotted terminal on one coil, it produces a voltage in the other coil that is positive at the dotted terminal. If a defined current enters the undotted terminal on one coil, it produces a voltage in the other coil that is positive at the undotted terminal.

STEP 3. Solve for the mesh currents.

EXAMPLE 10.1

Determine the equations for $v_1(t)$ and $v_2(t)$ in the circuit shown in Fig. 10.8a.

SOLUTION

The different voltage terms for the circuit are shown on the circuit diagram in Fig. 10.8b. The polarity of the self terms is given by the passive sign convention. For both coils, the defined currents are entering the undotted terminals on both coils. As a result, the polarity of the voltages produced by these currents is positive at the undotted terminal of the other coils. The equations for $v_1(t)$ and $v_2(t)$ are

$$v_1(t) = -L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2(t) = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

Figure 10.8a

Circuit used in Example 10.1.

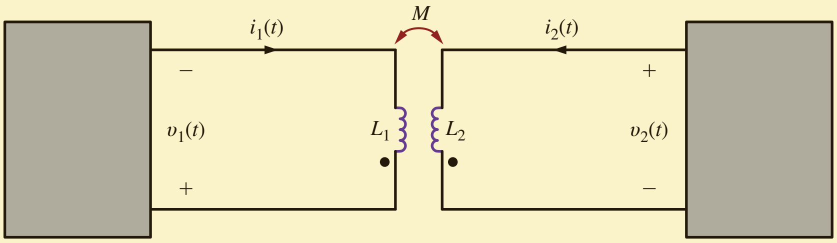
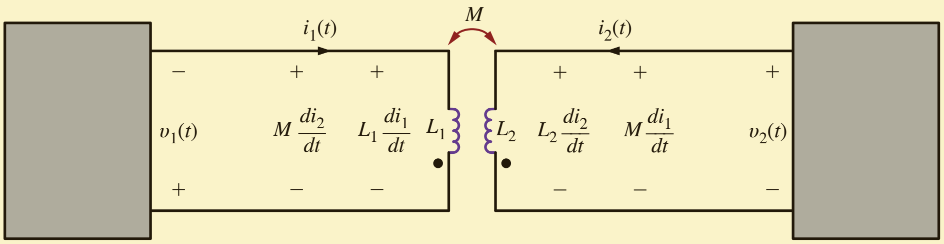


Figure 10.8b

Circuit showing self and mutual voltage terms.



EXAMPLE 10.2

Write mesh equations for the circuit of Fig. 10.9a using the assigned mesh currents.

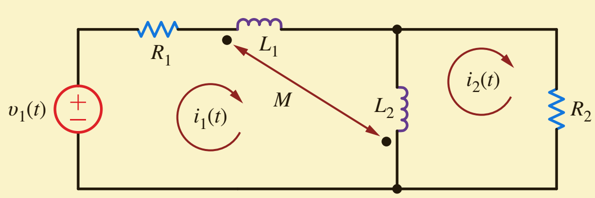
SOLUTION

The circuit in Fig. 10.9b shows the voltage terms for mesh 1. The polarity of the self terms for L_1 and L_2 is determined by the passive sign convention. The current $(i_2 - i_1)$ enters the dotted terminal of inductor L_2. This current produces the mutual term shown across inductor L_1. Current i_1 enters the dotted terminal of L_1 and produces a voltage across L_2 that is positive at its dotted terminal. The equation for this mesh is

$$v_1(t) = R_1 i_1(t) + L_1\frac{di_1}{dt} + M\frac{d}{dt}(i_2 - i_1) + L_2\frac{d}{dt}(i_1 - i_2) - M\frac{di_1}{dt}$$

Figure 10.9a

Circuit used in Example 10.2.



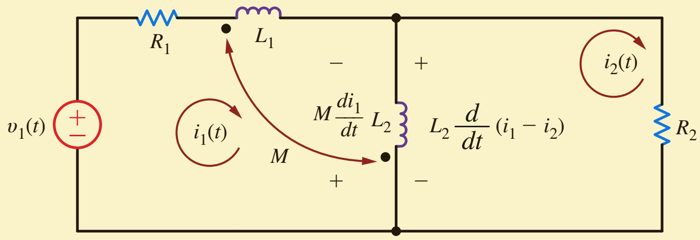


Figure 10.9b

Circuit showing voltage terms for mesh 1.

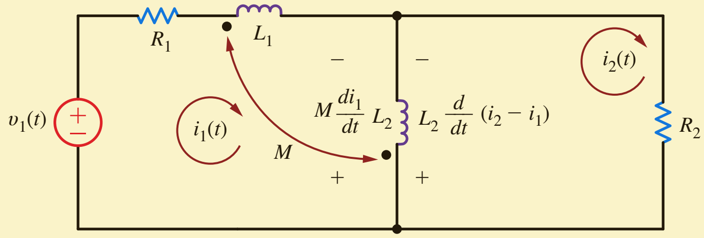


Figure 10.9c

Circuit showing voltage terms for mesh 2.

The voltage terms for the second mesh are shown in Fig. 10.9c. The equation for mesh 2 is

$$R_2 i_2(t) + L_2\frac{d}{dt}(i_2 - i_1) + M\frac{di_1}{dt} = 0$$

LEARNING ASSESSMENT

E10.1 Write the equations for $v_1(t)$ and $v_2(t)$ in the circuit in Fig. E10.1.

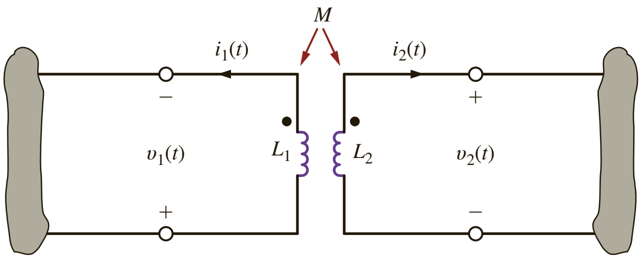


Figure E10.1

ANSWER:

$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt},$$

$$v_2(t) = -L_2\frac{di_2(t)}{dt} - M\frac{di_1(t)}{dt}.$$

Assume that the coupled circuit in Fig. 10.10 is excited with a sinusoidal source. The voltages will be of the form $\mathbf{V}_1 e^{j\omega t}$ and $\mathbf{V}_2 e^{j\omega t}$, and the currents will be of the form $\mathbf{I}_1 e^{j\omega t}$ and $\mathbf{I}_2 e^{j\omega t}$, where $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$, and $\mathbf{I}_2$ are phasors. Substituting these voltages and currents into Eqs. (10.11) and (10.12), and using the fact that $L_{12} = L_{21} = M$, we obtain

$$\begin{aligned}\mathbf{V}_1 &= j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2 \\ \mathbf{V}_2 &= j\omega L_2\mathbf{I}_2 + j\omega M\mathbf{I}_1\end{aligned} \qquad \textbf{10.17}$$

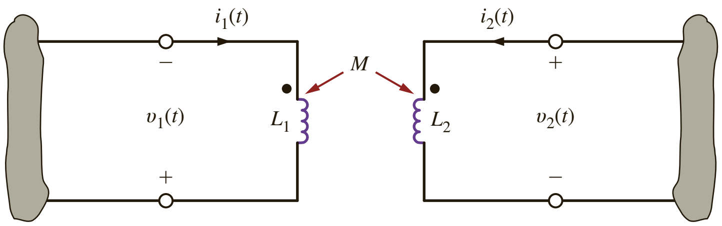


Figure 10.10
Mutually coupled coils.

The model of the coupled circuit in the frequency domain is identical to that in the time domain except for the way the elements and variables are labeled. The sign on the mutual terms is handled in the same manner as is done in the time domain.

EXAMPLE 10.3

The two mutually coupled coils in Fig. 10.11a can be interconnected in four possible ways. We wish to determine the equivalent inductance of each of the four possible interconnections.

SOLUTION

Case 1 is shown in Fig. 10.11b. In this case

$$\begin{aligned}\mathbf{V} &= j\omega L_1\mathbf{I} + j\omega M\mathbf{I} + j\omega L_2\mathbf{I} + j\omega M\mathbf{I}\\ &= j\omega L_{eq}\mathbf{I}\end{aligned}$$

where $L_{eq} = L_1 + L_2 + 2M$.

Case 2 is shown in Fig. 10.11c. Using KVL, we obtain

$$\begin{aligned}\mathbf{V} &= j\omega L_1\mathbf{I} - j\omega M\mathbf{I} + j\omega L_2\mathbf{I} - j\omega M\mathbf{I}\\ &= j\omega L_{eq}\mathbf{I}\end{aligned}$$

where $L_{eq} = L_1 + L_2 - 2M$.

Case 3 is shown in Fig. 10.11d and redrawn in Fig. 10.11e. The two KVL equations are

$$\mathbf{V} = j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2$$
$$\mathbf{V} = j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I}_2$$

Solving these equations for $\mathbf{I}_1$ and $\mathbf{I}_2$ yields

$$\mathbf{I}_1 = \frac{\mathbf{V}(L_2 - M)}{j\omega(L_1L_2 - M^2)}$$
$$\mathbf{I}_2 = \frac{\mathbf{V}(L_1 - M)}{j\omega(L_1L_2 - M^2)}$$

Using KCL gives us

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{V}(L_1 + L_2 - 2M)}{j\omega(L_1L_2 - M^2)} = \frac{\mathbf{V}}{j\omega L_{eq}}$$

where

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

Case 4 is shown in Fig. 10.11f. The voltage equations in this case will be the same as those in case 3 except that the signs of the mutual terms will be negative. Therefore,

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$

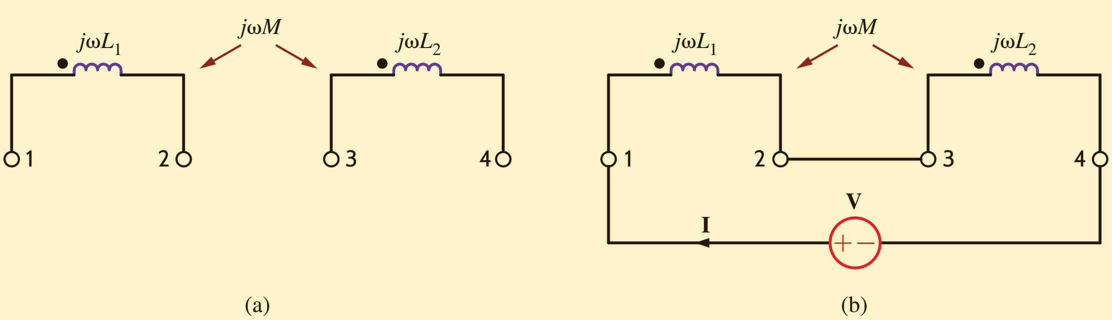


Figure 10.11
Circuits used in Example 10.3.

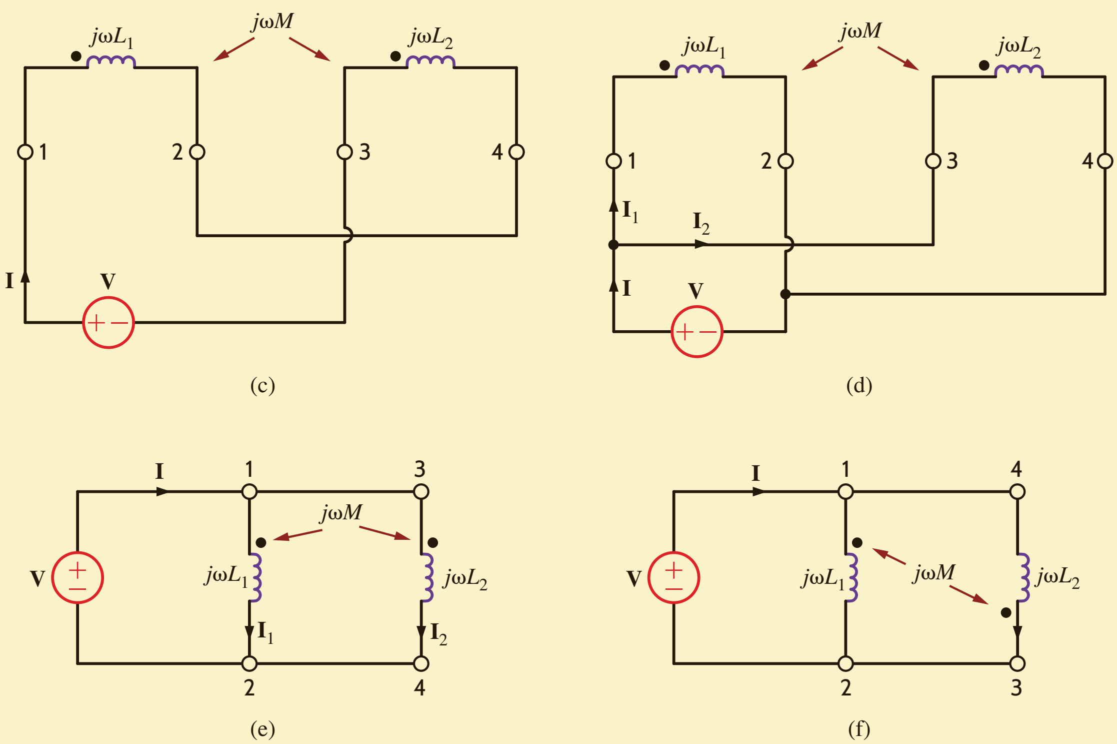


Figure 10.11
(continued)

EXAMPLE 10.4

We wish to determine the output voltage $\mathbf{V}_o$ in the circuit in Fig. 10.12.

SOLUTION

The two KVL equations for the network are

$$(2 + j4)\mathbf{I}_1 - j2\mathbf{I}_2 = 24\underline{/30^\circ}$$
$$-j2\mathbf{I}_1 + (2 + j6 - j2)\mathbf{I}_2 = 0$$

Solving the equations yields

$$\mathbf{I}_2 = 2.68\underline{/3.43^\circ}\text{ A}$$

Therefore,

$$\mathbf{V}_o = 2\mathbf{I}_2$$
$$= 5.36\underline{/3.43^\circ}\text{ V}$$

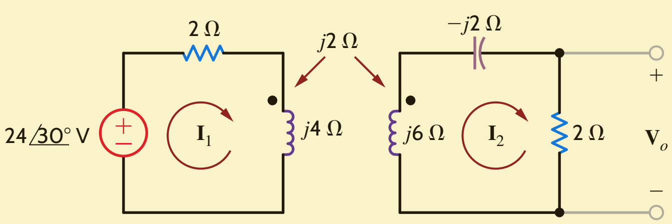


Figure 10.12
Example of a magnetically coupled circuit.

Let us now consider a more complicated example involving mutual inductance.

EXAMPLE 10.5

Consider the circuit in Fig. 10.13. We wish to write the mesh equations for this network.

SOLUTION

Because of the multiple currents that are present in the coupled inductors, we must be very careful in writing the circuit equations.

The mesh equations for the phasor network are

$$\mathbf{I}_1 R_1 + j\omega L_1(\mathbf{I}_1 - \mathbf{I}_2) + j\omega M(\mathbf{I}_2 - \mathbf{I}_3) + \frac{1}{j\omega C_1}(\mathbf{I}_1 - \mathbf{I}_2) = \mathbf{V}$$

$$\frac{1}{j\omega C_1}(\mathbf{I}_2 - \mathbf{I}_1) + j\omega L_1(\mathbf{I}_2 - \mathbf{I}_1) + j\omega M(\mathbf{I}_3 - \mathbf{I}_2) + R_2\mathbf{I}_2$$
$$+ j\omega L_2(\mathbf{I}_2 - \mathbf{I}_3) + j\omega M(\mathbf{I}_1 - \mathbf{I}_2) + R_3(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$R_3(\mathbf{I}_3 - \mathbf{I}_2) + j\omega L_2(\mathbf{I}_3 - \mathbf{I}_2) + j\omega M(\mathbf{I}_2 - \mathbf{I}_1)$$
$$+ \frac{1}{j\omega C_2}\mathbf{I}_3 + R_4\mathbf{I}_3 = 0$$

which can be rewritten in the form

$$\left(R_1 + j\omega L_1 + \frac{1}{j\omega C_1}\right)\mathbf{I}_1 - \left(j\omega L_1 + \frac{1}{j\omega C_1} - j\omega M\right)\mathbf{I}_2 - j\omega M\mathbf{I}_3 = \mathbf{V}$$

$$-\left(j\omega L_1 + \frac{1}{j\omega C_1} - j\omega M\right)\mathbf{I}_1$$
$$+\left(\frac{1}{j\omega C_1} + j\omega L_1 + R_2 + j\omega L_2 + R_3 - j2\omega M\right)\mathbf{I}_2$$
$$-(j\omega L_2 + R_3 - j\omega M)\mathbf{I}_3 = 0$$

$$-j\omega M\mathbf{I}_1 - (R_3 + j\omega L_2 - j\omega M)\mathbf{I}_2$$
$$+\left(R_3 + j\omega L_2 + \frac{1}{j\omega C_2} + R_4\right)\mathbf{I}_3 = 0$$

Note the symmetrical form of these equations.

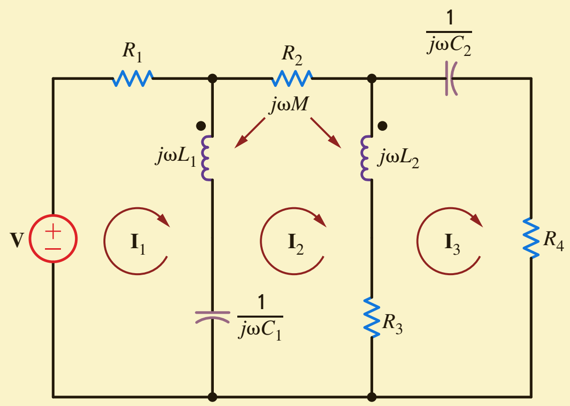


Figure 10.13 Example of a magnetically coupled circuit.

LEARNING ASSESSMENTS

E10.2 Find the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ and the output voltage $\mathbf{V}_o$ in the network in Fig. E10.2.

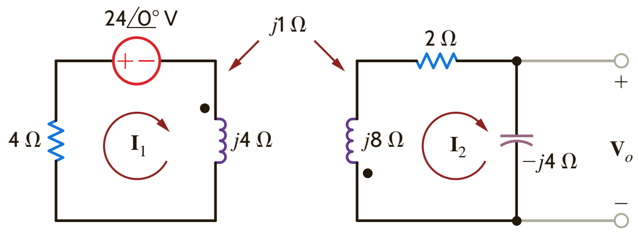


Figure E10.2

ANSWER:
$\mathbf{I}_1 = +4.29\underline{/137.2^\circ}$ A;
$\mathbf{I}_2 = 0.96\underline{/-16.26^\circ}$ A;
$\mathbf{V}_o = 3.84\underline{/-106.26^\circ}$ V.

E10.3 Write the KVL equations in standard form for the network in Fig. E10.3.

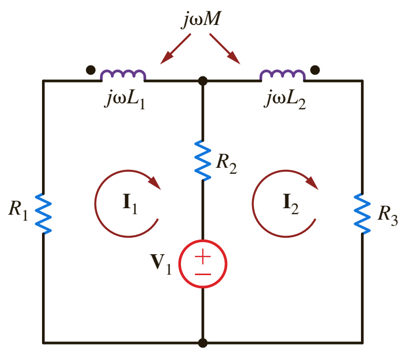


Figure E10.3

ANSWER:
$(R_1 + j\omega L_1 + R_2)\mathbf{I}_1$
$-(R_2 + j\omega M)\mathbf{I}_2 = -\mathbf{V}_1;$
$-(R_2 + j\omega M)\mathbf{I}_1$
$+(R_2 + j\omega L_2 + R_3)\mathbf{I}_2 = \mathbf{V}_1.$

E10.4 Find $\mathbf{V}_o$ in Fig. E10.4.

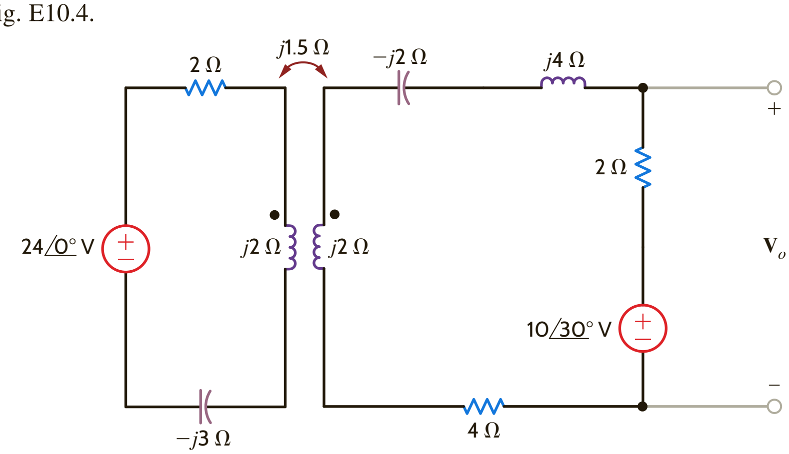


Figure E10.4

ANSWER:
$\mathbf{V}_o = 11.2\underline{/53.5^\circ}$ V.

E10.5 Find $\mathbf{V}_o$ in Fig. E10.5.

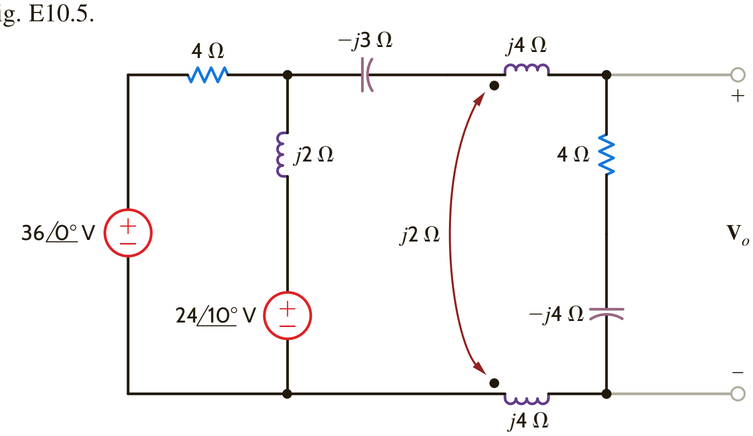


Figure E10.5

ANSWER:
$\mathbf{V}_o = 32.8\underline{/-12.14^\circ}$ V.

E10.6 Find $\mathbf{V}_o$ in Fig. E10.6.

ANSWER: $\mathbf{V}_o = 11.4\angle 0.334^\circ$ V.

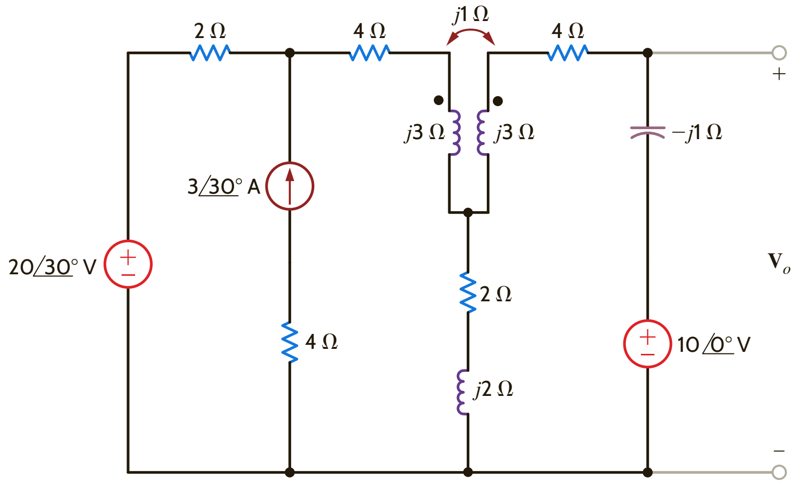


Figure E10.6

EXAMPLE 10.6

Given the network in Fig. 10.14 with the parameters $\mathbf{Z}_S = 3 + j1\ \Omega$, $j\omega L_1 = j2\ \Omega$, $j\omega L_2 = j2\ \Omega$, $j\omega M = j1\ \Omega$, and $\mathbf{Z}_L = 1 - j1\ \Omega$, determine the impedance seen by the source $\mathbf{V}_S$.

SOLUTION

The mesh equations for the network are

$$\mathbf{V}_S = (\mathbf{Z}_S + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$
$$0 = -j\omega M\mathbf{I}_1 + (j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2$$

If we now define $\mathbf{Z}_{11} = \mathbf{Z}_S + j\omega L_1$ and $\mathbf{Z}_{22} = j\omega L_2 + \mathbf{Z}_L$, then the second equation yields

$$\mathbf{I}_2 = \frac{j\omega M}{\mathbf{Z}_{22}}\mathbf{I}_1$$

If this secondary mesh equation is substituted into the primary mesh equation, we obtain

$$\mathbf{V}_S = \mathbf{Z}_{11}\mathbf{I}_1 + \frac{\omega^2 M^2}{\mathbf{Z}_{22}}\mathbf{I}_1$$

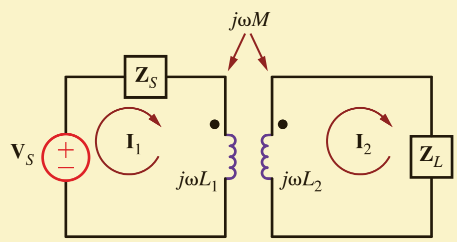


Figure 10.14 Circuit employed in Example 10.6.

and therefore

$$\frac{\mathbf{V}_S}{\mathbf{I}_1} = \mathbf{Z}_{11} + \frac{\omega^2 M^2}{\mathbf{Z}_{22}}$$

which is the impedance seen by $\mathbf{V}_S$. Note that the mutual term is squared, and therefore the impedance is independent of the location of the dots.

Using the values of the circuit parameters, we find that

$$\frac{\mathbf{V}_S}{\mathbf{I}_1} = (3 + j1 + j2) + \frac{1}{j2 + 1 - j1}$$
$$= 3 + j3 + 0.5 - j0.5$$
$$= 3.5 + j2.5\ \Omega$$