which in mathematical form is

$$v_o(t) = 4(1 - e^{-t/0.4})[u(t) - u(t - 0.3)] + 2.11e^{-(t-0.3)/0.4}\,u(t - 0.3)\text{ V}$$

Note that the term $[u(t) - u(t - 0.3)]$ acts like a gating function that captures only the part of the step response that exists in the time interval $0 < t < 0.3$ s. The output as a function of time is shown in Fig. 7.13f.

LEARNING ASSESSMENT

E7.11 The voltage source in the network in Fig. E7.11a is shown in Fig. E7.11b. The initial current in the inductor must be zero. (Why?) Determine the output voltage $v_o(t)$ for $t > 0$.

ANSWER:
$v_o(t) = 0$ for $t < 0$, $4(1 - e^{-(3/2)t})$ V for $0 \leqq t \leqq 1$ s, and $3.11e^{-(3/2)(t-1)}$ V for $t > 1$ s.

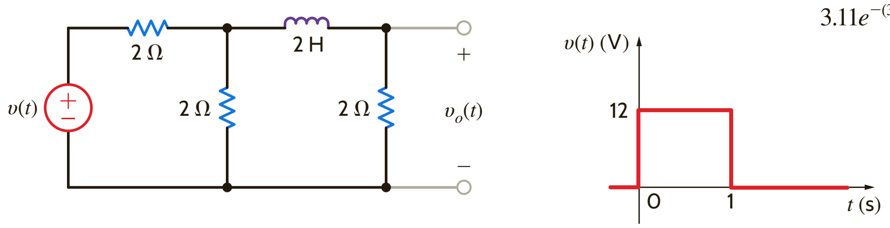


Figure E7.11 (a) (b)

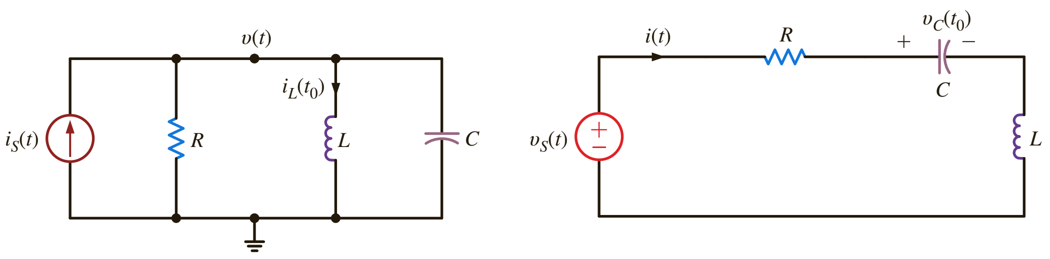
7.3 Second-Order Circuits

THE BASIC CIRCUIT EQUATION To begin our development, let us consider the two basic *RLC* circuits shown in Fig. 7.14. We assume that energy may be initially stored in both the inductor and capacitor. The node equation for the parallel *RLC* circuit is

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v(x)\,dx + i_L(t_0) + C\frac{dv}{dt} = i_S(t)$$

Similarly, the loop equation for the series *RLC* circuit is

$$Ri + \frac{1}{C}\int_{t_0}^{t} i(x)\,dx + v_C(t_0) + L\frac{di}{dt} = v_S(t)$$



(a) (b)

Figure 7.14

Parallel and series *RLC* circuits.

Note that the equation for the node voltage in the parallel circuit is of the same form as that for the loop current in the series circuit. Therefore, the solution of these two circuits is dependent on solving one equation. If the two preceding equations are differentiated with respect to time, we obtain

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt}$$

and

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv_S}{dt}$$

Since both circuits lead to a second-order differential equation with constant coefficients, we will concentrate our analysis on this type of equation.

THE RESPONSE EQUATIONS In concert with our development of the solution of a first-order differential equation that results from the analysis of either an *RL* or an *RC* circuit as outlined earlier, we will now employ the same approach here to obtain the solution of a second-order differential equation that results from the analysis of *RLC* circuits. As a general rule, for this case we are confronted with an equation of the form

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = f(t) \qquad \textbf{7.12}$$

Once again, we use the fact that if $x(t) = x_p(t)$ is a solution to Eq. (7.12), and if $x(t) = x_c(t)$ is a solution to the homogeneous equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$$

then

$$x(t) = x_p(t) + x_c(t)$$

is a solution to the original Eq. (7.12). If we again confine ourselves to a constant forcing function [i.e., $f(t) = A$], the development at the beginning of this chapter shows that the solution of Eq. (7.12) will be of the form

$$x(t) = \frac{A}{a_2} + x_c(t) \qquad \textbf{7.13}$$

Let us now turn our attention to the solution of the homogeneous equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$$

where a_1 and a_2 are constants. For simplicity we will rewrite the equation in the form

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2x(t) = 0 \qquad \textbf{7.14}$$

where we have made the following simple substitutions for the constants $a_1 = 2\zeta\omega_0$ and $a_2 = \omega_0^2$.

Following the development of a solution for the first-order homogeneous differential equation earlier in this chapter, the solution of Eq. (7.14) must be a function whose first- and second-order derivatives have the same form, so that the left-hand side of Eq. (7.14) will become identically zero for all *t*. Again, we assume that

$$x(t) = Ke^{st}$$

Substituting this expression into Eq. (7.14) yields

$$s^2Ke^{st} + 2\zeta\omega_0sKe^{st} + \omega_0^2Ke^{st} = 0$$

Dividing both sides of the equation by Ke^{st} yields

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \qquad \textbf{7.15}$$

This equation is commonly called the *characteristic equation*; ζ is called the exponential *damping ratio*, and ω_0 is referred to as the *undamped natural frequency*. The importance of this terminology will become clear as we proceed with the development. If this equation is satisfied, our assumed solution $x(t) = Ke^{st}$ is correct. Employing the quadratic formula, we find that Eq. (7.15) is satisfied if

$$\begin{aligned} s &= \frac{-2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2} \\ &= -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} \end{aligned} \qquad \textbf{7.16}$$

Therefore, two values of s, s_1 and s_2, satisfy Eq. (7.15):

$$\begin{aligned} s_1 &= -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{aligned} \qquad \textbf{7.17}$$

In general, then, the complementary solution of Eq. (7.14) is of the form

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \qquad \textbf{7.18}$$

K_1 and K_2 are constants that can be evaluated via the initial conditions $x(0)$ and $dx(0)/dt$. For example, since

$$x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

then

$$x(0) = K_1 + K_2$$

and

$$\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{dx(0)}{dt} = s_1 K_1 + s_2 K_2$$

Hence, $x(0)$ and $dx(0)/dt$ produce two simultaneous equations, which when solved yield the constants K_1 and K_2.

Close examination of Eqs. (7.17) and (7.18) indicates that the form of the solution of the homogeneous equation is dependent on the value ζ. For example, if $\zeta > 1$, the roots of the characteristic equation, s_1 and s_2, also called the *natural frequencies* because they determine the natural (unforced) response of the network, are real and unequal; if $\zeta < 1$, the roots are complex numbers; and finally, if $\zeta = 1$, the roots are real and equal.

Let us now consider the three distinct forms of the unforced response—that is, the response due to an initial capacitor voltage or initial inductor current.

Case 1, $\zeta > 1$ This case is commonly called *overdamped*. The natural frequencies s_1 and s_2 are real and unequal; therefore, the natural response of the network described by the second-order differential equation is of the form

$$x_c(t) = K_1 e^{-(\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1})t} + K_2 e^{-(\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1})t} \qquad \textbf{7.19}$$

where K_1 and K_2 are found from the initial conditions. This indicates that the natural response is the sum of two decaying exponentials.

Case 2, $\zeta < 1$ This case is called *underdamped*. Since $\zeta < 1$, the roots of the characteristic equation given in Eq. (7.17) can be written as

$$\begin{aligned} s_1 &= -\zeta\omega_0 + j\omega_0\sqrt{1 - \zeta^2} = -\sigma + j\omega_d \\ s_2 &= -\zeta\omega_0 - j\omega_0\sqrt{1 - \zeta^2} = -\sigma - j\omega_d \end{aligned}$$

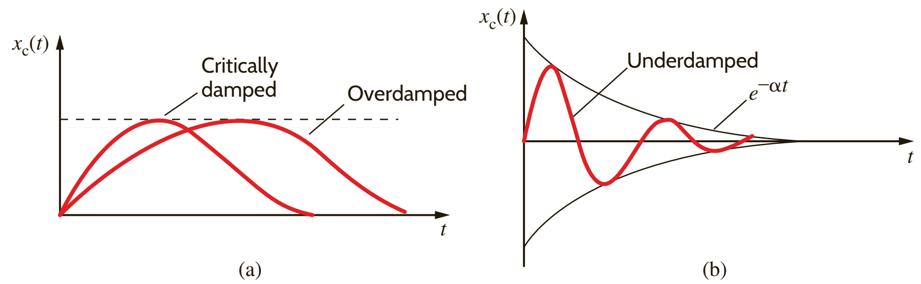


Figure 7.15
Comparison of overdamped, critically damped, and underdamped responses.

where $j = \sqrt{-1}$, $\sigma = \zeta\omega_0$, and $\omega_d = \omega_0\sqrt{1-\zeta^2}$. Thus, the natural frequencies are complex numbers (briefly discussed in the Appendix). The natural response is then of the form

$$x_c(t) = e^{-\zeta\omega_0 t}\left(A_1 \cos \omega_0\sqrt{1-\zeta^2}t + A_2 \sin \omega_0\sqrt{1-\zeta^2}t\right) \quad \textbf{7.20}$$

where A_1 and A_2, like K_1 and K_2, are constants, which are evaluated using the initial conditions $x(0)$ and $dx(0)/dt$. This illustrates that the natural response is an exponentially damped oscillatory response.

Case 3, $\zeta = 1$ This case, called *critically damped*, results in

$$s_1 = s_2 = -\zeta\omega_0$$

In the case where the characteristic equation has repeated roots, the general solution is of the form

$$x_c(t) = B_1 e^{-\zeta\omega_0 t} + B_2 t e^{-\zeta\omega_0 t} \quad \textbf{7.21}$$

where B_1 and B_2 are constants derived from the initial conditions.

It is informative to sketch the natural response for the three cases we have discussed: overdamped, Eq. (7.19); underdamped, Eq. (7.20); and critically damped, Eq. (7.21). **Figure 7.15** graphically illustrates the three cases for the situations in which $x_c(0) = 0$. Note that the critically damped response peaks and decays faster than the overdamped response. The underdamped response is an exponentially damped sinusoid whose rate of decay is dependent on the factor ζ. Actually, the terms $\pm e^{-\zeta\omega_0 t}$ define what is called the *envelope* of the response, and the damped oscillations (i.e., the oscillations of decreasing amplitude) exhibited by the waveform in **Fig. 7.15b** are called *ringing*.

LEARNING ASSESSMENTS

E7.12 A parallel *RLC* circuit has the following circuit parameters: $R = 1\ \Omega$, $L = 2$ H, and $C = 2$ F. Compute the damping ratio and the undamped natural frequency of this network.

ANSWER: $\zeta = 0.5$; $\omega_0 = 0.5$ rad/s.

E7.13 A series *RLC* circuit consists of $R = 2\ \Omega$, $L = 1$ H, and a capacitor. Determine the type of response exhibited by the network if (a) $C = 1/2$ F, (b) $C = 1$ F, and (c) $C = 2$ F.

ANSWER: (a) underdamped; (b) critically damped; (c) overdamped.

THE NETWORK RESPONSE We will now analyze a number of simple *RLC* networks that contain both nonzero initial conditions and constant forcing functions. Circuits that exhibit overdamped, underdamped, and critically damped responses will be considered.

PROBLEM-SOLVING STRATEGY

SECOND-ORDER TRANSIENT CIRCUITS

STEP 1. Write the differential equation that describes the circuit.

STEP 2. Derive the characteristic equation, which can be written in the form $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$, where ζ is the damping ratio and ω_0 is the undamped natural frequency.

STEP 3. The two roots of the characteristic equation will determine the type of response. If the roots are real and unequal (i.e., $\zeta > 1$), the network response is overdamped. If the roots are real and equal (i.e., $\zeta = 1$), the network response is critically damped. If the roots are complex (i.e., $\zeta < 1$), the network response is underdamped.

STEP 4. The damping condition and corresponding response for the aforementioned three cases outlined are as follows:

Overdamped: $x(t) = K_1 e^{-(\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1})t} + K_2 e^{-(\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1})t}$

Critically damped: $x(t) = B_1 e^{-\zeta\omega_0 t} + B_2 t e^{-\zeta\omega_0 t}$

Underdamped: $x(t) = e^{-\sigma t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$, where $\sigma = \zeta\omega_0$, and

$\omega_d = \omega_0\sqrt{1 - \zeta^2}$

STEP 5. Two initial conditions, either given or derived, are required to obtain the two unknown coefficients in the response equation.

The following examples will demonstrate the analysis techniques.

EXAMPLE 7.7

Consider the parallel *RLC* circuit shown in Fig. 7.16. The second-order differential equation that describes the voltage $v(t)$ is

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

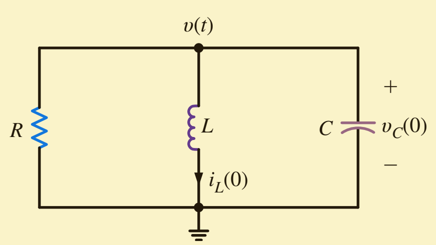


Figure 7.16 Parallel *RLC* circuit.

A comparison of this equation with Eqs. (7.14) and (7.15) indicates that for the parallel *RLC* circuit the damping term is $1/2\,RC$ and the undamped natural frequency is $1/\sqrt{LC}$. If the circuit parameters are $R = 2\ \Omega$, $C = 1/5$ F, and $L = 5$ H, the equation becomes

$$\frac{d^2v}{dt^2} + 2.5\frac{dv}{dt} + v = 0$$

Let us assume that the initial conditions on the storage elements are $i_L(0) = -1$ A and $v_C(0) = 4$ V. Let us find the node voltage $v(t)$ and the inductor current.

SOLUTION The characteristic equation for the network is

$$s^2 + 2.5s + 1 = 0$$

and the roots are

$$s_1 = -0.5$$
$$s_2 = -2$$

Since the roots are real and unequal, the circuit is overdamped, and $v(t)$ is of the form

$$v(t) = K_1e^{-2t} + K_2e^{-0.5t}$$

The initial conditions are now employed to determine the constants K_1 and K_2. Since $v(t) = v_C(t)$,

$$v_C(0) = v(0) = 4 = K_1 + K_2$$

The second equation needed to determine K_1 and K_2 is normally obtained from the expression

$$\frac{dv(t)}{dt} = -2K_1e^{-2t} - 0.5K_2e^{-0.5t}$$

However, the second initial condition is not $dv(0)/dt$. If this were the case, we would simply evaluate the equation at $t = 0$. This would produce a second equation in the unknowns K_1 and K_2. We can, however, circumvent this problem by noting that the node equation for the circuit can be written as

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} + i_L(t) = 0$$

or

$$\frac{dv(t)}{dt} = \frac{-1}{RC}v(t) - \frac{i_L(t)}{C}$$

At $t = 0$,

$$\begin{aligned}\frac{dv(0)}{dt} &= \frac{-1}{RC}v(0) - \frac{1}{C}i_L(0)\\ &= -2.5(4) - 5(-1)\\ &= -5\end{aligned}$$

However, since

$$\frac{dv(t)}{dt} = -2K_1e^{-2t} - 0.5K_2e^{-0.5t}$$

then when $t = 0$

$$-5 = -2K_1 - 0.5K_2$$

This equation, together with the equation

$$4 = K_1 + K_2$$

produces the constants $K_1 = 2$ and $K_2 = 2$. Therefore, the final equation for the voltage is

$$v(t) = 2e^{-2t} + 2e^{-0.5t}\,\text{V}$$

Note that the voltage equation satisfies the initial condition $v(0) = 4$ V. The response curve for this voltage $v(t)$ is shown in **Fig. 7.17**.

The inductor current is related to $v(t)$ by the equation

$$i_L(t) = \frac{1}{L}\int v(t)\,dt$$

Substituting our expression for $v(t)$ yields

$$i_L(t) = \frac{1}{5}\int [2e^{-2t} + 2e^{-0.5t}]\,dt$$

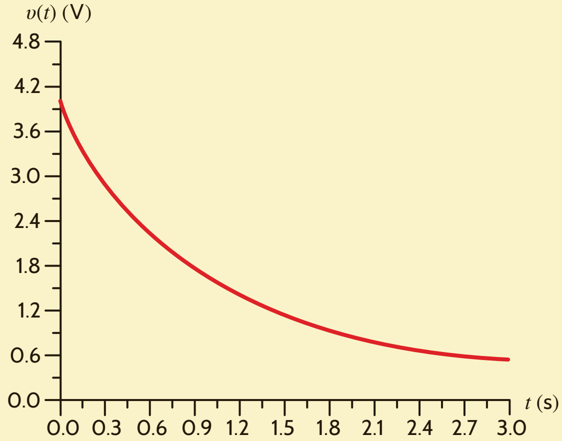


Figure 7.17
Overdamped response.

or

$$i_L(t) = -\frac{1}{5}e^{-2t} - \frac{4}{5}e^{-0.5t}\,\text{A}$$

Note that in comparison with the *RL* and *RC* circuits, the response of this *RLC* circuit is controlled by two time constants. The first term has a time constant of 1/2 s, and the second term has a time constant of 2 s.

EXAMPLE 7.8

The series *RLC* circuit shown in **Fig. 7.18** has the following parameters: $C = 0.04$ F, $L = 1$ H, $R = 6\ \Omega$, $i_L(0) = 4$ A, and $v_C(0) = -4$ V. The equation for the current in the circuit is given by the expression

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

A comparison of this equation with Eqs. (7.14) and (7.15) illustrates that for a series *RLC* circuit the damping term is $R/2L$ and the undamped natural frequency is $1/\sqrt{LC}$. Substituting the circuit element values into the preceding equation yields

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 25i = 0$$

Let us determine the expression for both the current and the capacitor voltage.

SOLUTION

The characteristic equation is then

$$s^2 + 6s + 25 = 0$$

and the roots are

$$s_1 = -3 + j4$$
$$s_2 = -3 - j4$$

Since the roots are complex, the circuit is underdamped, and the expression for $i(t)$ is

$$i(t) = K_1e^{-3t}\cos 4t + K_2e^{-3t}\sin 4t$$

Using the initial conditions, we find that

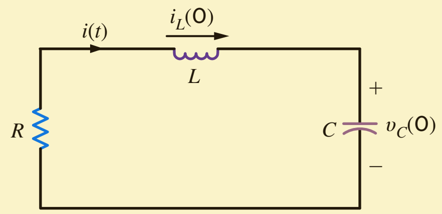
$$i(0) = 4 = K_1$$

and

$$\frac{di}{dt} = -4K_1e^{-3t}\sin 4t - 3K_1e^{-3t}\cos 4t + 4K_2e^{-3t}\cos 4t - 3K_2e^{-3t}\sin 4t$$

Figure 7.18

Series *RLC* circuit.



and thus

$$\frac{di(0)}{dt} = -3K_1 + 4K_2$$

Although we do not know $di(0)/dt$, we can find it via KVL. From the circuit we note that

$$Ri(0) + L\frac{di(0)}{dt} + v_C(0) = 0$$

or

$$\begin{aligned}\frac{di(0)}{dt} &= -\frac{R}{L}i(0) - \frac{v_C(0)}{L}\\ &= -\frac{6}{1}(4) + \frac{4}{1}\\ &= -20\end{aligned}$$

Therefore,

$$-3K_1 + 4K_2 = -20$$

and since $K_1 = 4$, $K_2 = -2$, the expression then for $i(t)$ is

$$i(t) = 4e^{-3t}\cos 4t - 2e^{-3t}\sin 4t \text{ A}$$

Note that this expression satisfies the initial condition $i(0) = 4$. The voltage across the capacitor could be determined via KVL using this current:

$$Ri(t) + L\frac{di(t)}{dt} + v_C(t) = 0$$

or

$$v_C(t) = -Ri(t) - L\frac{di(t)}{dt}$$

Substituting the preceding expression for $i(t)$ into this equation yields

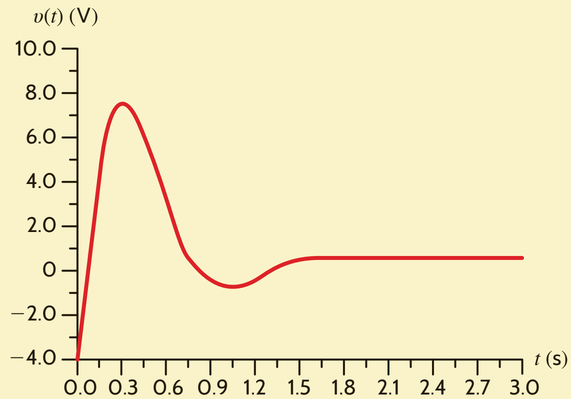
$$v_C(t) = -4e^{-3t}\cos 4t + 22e^{-3t}\sin 4t \text{ V}$$

Note that this expression satisfies the initial condition $v_C(0) = -4$ V.

A plot of the function is shown in **Fig. 7.19**:

Figure 7.19

Underdamped response.



EXAMPLE 7.9

Let us examine the circuit in Fig. 7.20, which is slightly more complicated than the two we have already considered.

SOLUTION

The two equations that describe the network are

$$L\frac{di(t)}{dt} + R_1 i(t) + v(t) = 0$$

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

Substituting the second equation into the first yields

$$\frac{d^2v}{dt^2} + \left(\frac{1}{R_2C} + \frac{R_1}{L}\right)\frac{dv}{dt} + \frac{R_1 + R_2}{R_2LC}v = 0$$

If the circuit parameters and initial conditions are

$$R_1 = 10\ \Omega \qquad C = \frac{1}{8}\text{ F} \qquad v_C(0) = 1\text{ V}$$

$$R_2 = 8\ \Omega \qquad L = 2\text{ H} \qquad i_L(0) = \frac{1}{2}\text{ A}$$

the differential equation becomes

$$\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 9v = 0$$

We wish to find expressions for the current $i(t)$ and the voltage $v(t)$.

The characteristic equation is then

$$s^2 + 6s + 9 = 0$$

and hence the roots are

$$s_1 = -3$$

$$s_2 = -3$$

Since the roots are real and equal, the circuit is critically damped. The term $v(t)$ is then given by the expression

$$v(t) = K_1e^{-3t} + K_2te^{-3t}$$

Since $v(t) = v_C(t)$,

$$v(0) = v_C(0) = 1 = K_1$$

In addition,

$$\frac{dv(t)}{dt} = -3K_1e^{-3t} + K_2e^{-3t} - 3K_2te^{-3t}$$

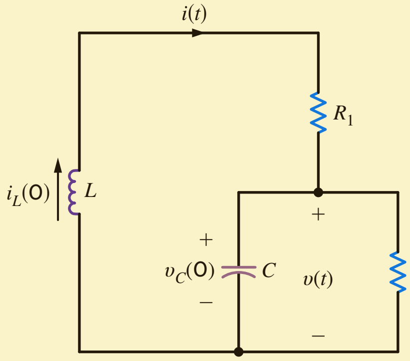


Figure 7.20

Series-parallel RLC circuit.

However,

$$\frac{dv(t)}{dt} = \frac{i(t)}{C} - \frac{v(t)}{R_2C}$$

Setting these two expressions equal to one another and evaluating the resultant equation at $t = 0$ yields

$$\frac{1/2}{1/8} - \frac{1}{1} = -3K_1 + K_2$$

$$3 = -3K_1 + K_2$$

$K_1 = 1$, $K_2 = 6$, and the expression for $v(t)$ is

$$v(t) = e^{-3t} + 6te^{-3t}\text{ V}$$

Note that the expression satisfies the initial condition $v(0) = 1$.

The current $i(t)$ can be determined from the nodal analysis equation at $v(t)$:

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

Substituting $v(t)$ from the preceding equation, we find

$$i(t) = \frac{1}{8}[-3e^{-3t} + 6e^{-3t} - 18te^{-3t}] + \frac{1}{8}[e^{-3t} + 6te^{-3t}]$$

or

$$i(t) = \frac{1}{2}e^{-3t} - \frac{3}{2}te^{-3t}\text{ A}$$

If this expression for the current is employed in the circuit equation,

$$v(t) = -L\frac{di(t)}{dt} - R_1i(t)$$

we obtain

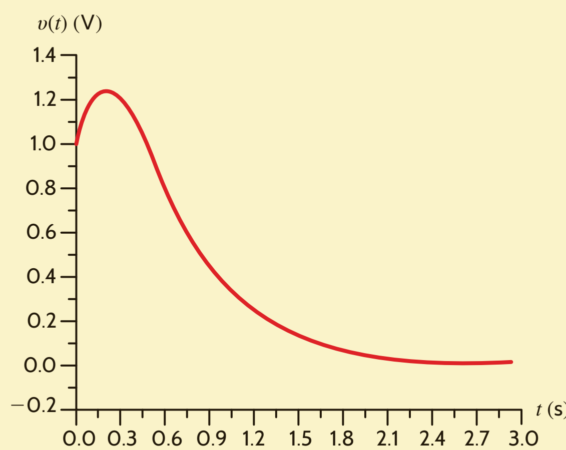
$$v(t) = e^{-3t} + 6te^{-3t}\text{ V}$$

which is identical to the expression derived earlier.

A plot of this critically damped function is shown in **Fig. 7.21**.

Figure 7.21

Critically damped response.



LEARNING ASSESSMENTS

E7.14 The switch in the network in Fig. E7.14 opens at $t = 0$. Find $i(t)$ for $t > 0$.

ANSWER:
$i(t) = -2e^{-t/2} + 4e^{-t}$ A.

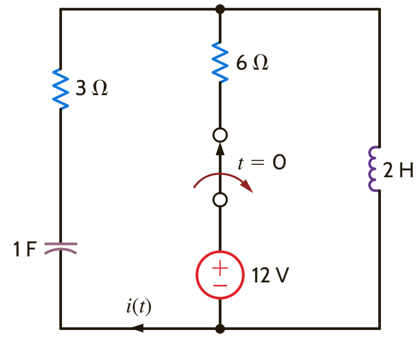


Figure E7.14

E7.15 The switch in the network in Fig. E7.15 moves from position 1 to position 2 at $t = 0$. Find $v_o(t)$ for $t > 0$.

ANSWER:
$v_o(t) = 2(e^{-t} - 3e^{-3t})$ V.

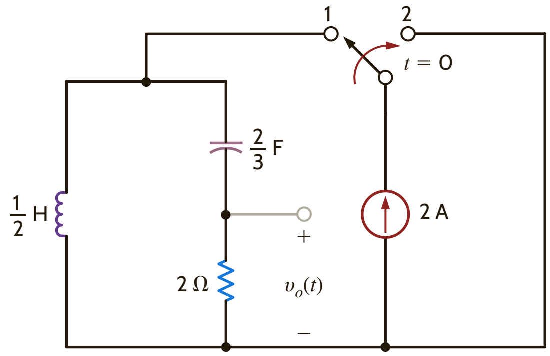


Figure E7.15

E7.16 Find $v_C(t)$ for $t > 0$ in Fig. E7.16.

ANSWER:
$v_C(t) = -2e^{-2t}$
$\cos t - 1.5e^{-2t} \sin t + 24$ V.

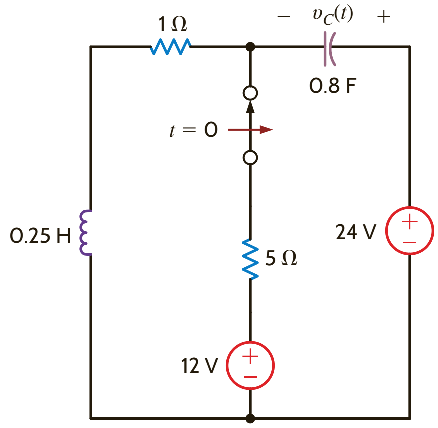


Figure E7.16

EXAMPLE 7.10

Consider the circuit shown in Fig. 7.22. This circuit is the same as the one analyzed in Example 7.8, except that a constant forcing function is present. The circuit parameters are the same as those used in Example 7.8:

$$C = 0.04\ \text{F} \qquad i_L(0) = 4\ \text{A}$$
$$L = 1\ \text{H} \qquad v_C(0) = -4\ \text{V}$$
$$R = 6\ \Omega$$

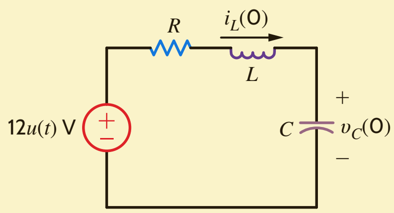


Figure 7.22

Series *RLC* circuit with a step function input.

We want to find an expression for $v_C(t)$ for $t > 0$.

SOLUTION

From our earlier mathematical development we know that the general solution of this problem will consist of a particular solution plus a complementary solution. From Example 7.8 we know that the complementary solution is of the form $K_3e^{-3t}\cos 4t + K_4e^{-3t}\sin 4t$. The particular solution is a constant, since the input is a constant and therefore the general solution is

$$v_C(t) = K_3e^{-3t}\cos 4t + K_4e^{-3t}\sin 4t + K_5$$

An examination of the circuit shows that in the steady state, the final value of $v_C(t)$ is 12 V, since in the steady-state condition, the inductor is a short circuit and the capacitor is an open circuit. Thus, $K_5 = 12$. The steady-state value could also be immediately calculated from the differential equation. The form of the general solution is then

$$v_C(t) = K_3e^{-3t}\cos 4t + K_4e^{-3t}\sin 4t + 12$$

The initial conditions can now be used to evaluate the constants K_3 and K_4:

$$v_C(0) = -4 = K_3 + 12$$
$$-16 = K_3$$

Since the derivative of a constant is zero, the results of Example 7.8 show that

$$\frac{dv_C(0)}{dt} = \frac{i(0)}{C} = 100 = -3K_3 + 4K_4$$

and since $K_3 = -16$, $K_4 = 13$. Therefore, the general solution for $v_C(t)$ is

$$v_C(t) = 12 - 16e^{-3t}\cos 4t + 13e^{-3t}\sin 4t\ \text{V}$$

Note that this equation satisfies the initial condition $v_C(0) = -4$ and the final condition $v_C(\infty) = 12$ V.

EXAMPLE 7.11

Let us examine the circuit shown in Fig. 7.23. A close examination of this circuit will indicate that it is identical to that shown in Example 7.9 except that a constant forcing function is present. We assume the circuit is in steady state at $t = 0-$. The equations that describe the circuit for $t > 0$ are

$$L\frac{di(t)}{dt} + R_1 i(t) + v(t) = 24$$

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

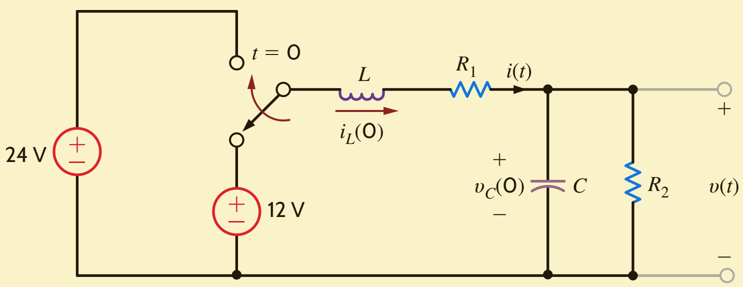


Figure 7.23
Series-parallel *RLC* circuit with a constant forcing function.

Combining these equations, we obtain

$$\frac{d^2v(t)}{dt^2} + \left(\frac{1}{R_2C} + \frac{R_1}{L}\right)\frac{dv(t)}{dt} + \frac{R_1 + R_2}{R_2LC}v(t) = \frac{24}{LC}$$

If the circuit parameters are $R_1 = 10\ \Omega$, $R_2 = 2\ \Omega$, $L = 2$ H, and $C = 1/4$ F, the differential equation for the output voltage reduces to

$$\frac{d^2v(t)}{dt^2} + 7\frac{dv(t)}{dt} + 12v(t) = 48$$

Let us determine the output voltage $v(t)$.

The characteristic equation is

$$s^2 + 7s + 12 = 0$$

and hence the roots are

$$s_1 = -3$$
$$s_2 = -4$$

The circuit response is overdamped, and therefore the general solution is of the form

$$v(t) = K_1e^{-3t} + K_2e^{-4t} + K_3$$

The steady-state value of the voltage, K_3, can be computed from **Fig. 7.24a**. Note that

$$v(\infty) = 4\text{ V} = K_3$$

The initial conditions can be calculated from **Figs. 7.24b** and **c**, which are valid at $t = 0-$ and $t = 0+$, respectively. Note that $v(0+) = 2$ V and, hence, from the response equation

$$v(0+) = 2\text{ V} = K_1 + K_2 + 4$$
$$-2 = K_1 + K_2$$

Fig. 7.24c illustrates that $i(0+) = 1$. From the response equation we see that

$$\frac{dv(0)}{dt} = -3K_1 - 4K_2$$

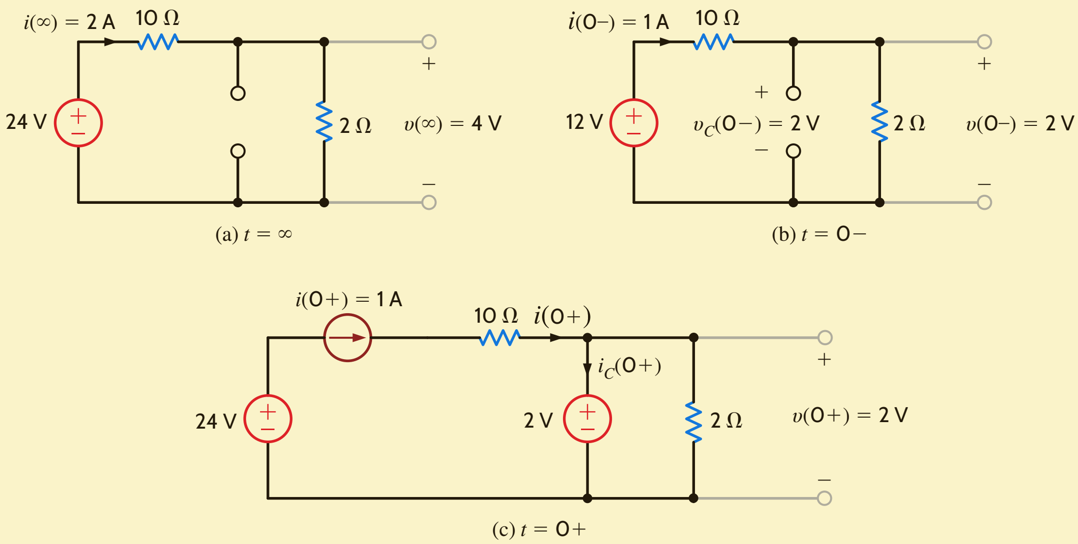


Figure 7.24
Equivalent circuits at $t = \infty$, $t = 0-$, and $t = 0+$ for the circuit in Fig. 7.23.

Since

$$\frac{dv(0)}{dt} = \frac{i(0)}{C} - \frac{v(0)}{R_2C}$$
$$= 4 - 4$$
$$= 0$$

then

$$0 = -3K_1 - 4K_2$$

Solving the two equations for K_1 and K_2 yields $K_1 = -8$ and $K_2 = 6$. Therefore, the general solution for the voltage response is

$$v(t) = 4 - 8e^{-3t} + 6e^{-4t}\text{ V}$$

Note that this equation satisfies both the initial and final values of $v(t)$.

LEARNING ASSESSMENTS

E7.17 The switch in the network in Fig. E7.17 moves from position 1 to position 2 at $t = 0$. Compute $i_o(t)$ for $t > 0$ and use this current to determine $v_o(t)$ for $t > 0$.

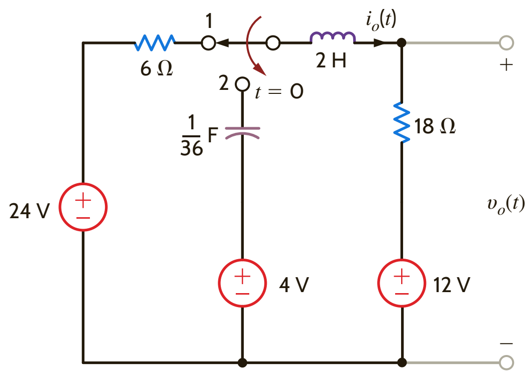


Figure E7.17

ANSWER:
$i_o(t) = -\frac{11}{6}e^{-3t} + \frac{14}{6}e^{-6t}$A;
$v_o(t) = 12 + 18i_o(t)$ V.

E7.18 Find $i(t)$ for $t > 0$ in Fig. E7.18.

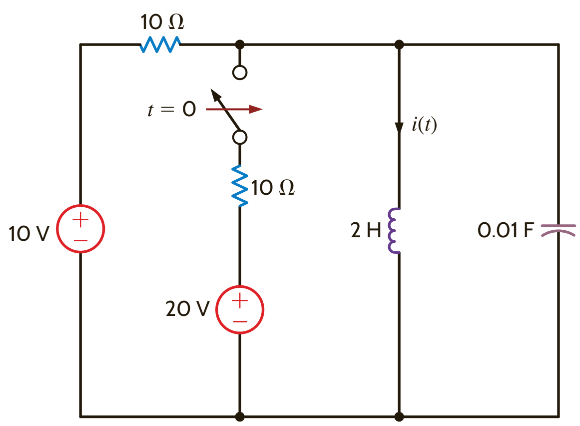


Figure E7.18

ANSWER:
$i(t) = 0.4144e^{-17.07t}$
$- 2.414e^{-2.93t} + 3$ A.