E8.2 Three branch currents in a network are known to be

$$i_1(t) = 2\sin(377t + 45°) \text{ A}$$
$$i_2(t) = 0.5\cos(377t + 10°) \text{ A}$$
$$i_3(t) = -0.25\sin(377t + 60°) \text{ A}$$

Determine the phase angles by which $i_1(t)$ leads $i_2(t)$ and $i_1(t)$ leads $i_3(t)$.

ANSWER:
i_1 leads i_2 by $-55°$;
i_1 leads i_3 by $165°$.

8.2 Sinusoidal and Complex Forcing Functions

In the preceding chapters we applied a constant forcing function to a network and found that the steady-state response was also constant.

In a similar manner, if we apply a sinusoidal forcing function to a linear network, the steady-state voltages and currents in the network will also be sinusoidal. This should also be clear from the KVL and KCL equations. For example, if one branch voltage is a sinusoid of some frequency, the other branch voltages must be sinusoids of the same frequency if KVL is to apply around any closed path. This means, of course, that the forced solutions of the differential equations that describe a network with a sinusoidal forcing function are sinusoidal functions of time. For example, if we assume that our input function is a voltage $v(t)$ and our output response is a current $i(t)$, as shown in Fig. 8.4, then if $v(t) = A\sin(\omega t + \theta)$, $i(t)$ will be of the form $i(t) = B\sin(\omega t + \phi)$. The critical point here is that we know the form of the output response, and therefore the solution involves simply determining the values of the two parameters B and ϕ.

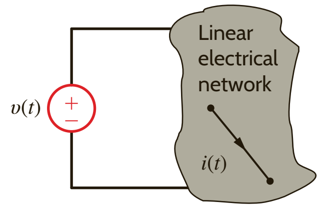

Figure 8.4 Current response to an applied voltage in an electrical network.

EXAMPLE 8.3

Consider the circuit in Fig. 8.5. Let us derive the expression for the current.

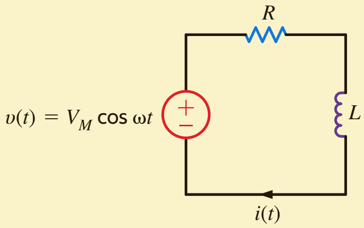

Figure 8.5 A simple *RL* circuit.

SOLUTION

The KVL equation for this circuit is

$$L\frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t$$

Since the input forcing function is $V_M \cos \omega t$, we assume that the forced response component of the current $i(t)$ is of the form

$$i(t) = A\cos(\omega t + \phi)$$

which can be written using Eq. (8.11) as

$$\begin{aligned} i(t) &= A\cos\phi\cos\omega t - A\sin\phi\sin\omega t \\ &= A_1\cos\omega t + A_2\sin\omega t \end{aligned}$$

Note that this is, as we observed in Chapter 7, of the form of the forcing function $\cos \omega t$ and its derivative $\sin \omega t$. Substituting this form for $i(t)$ into the preceding differential equation yields

$$L\frac{d}{dt}(A_1\cos\omega t + A_2\sin\omega t) + R(A_1\cos\omega t + A_2\sin\omega t) = V_M\cos\omega t$$

Evaluating the indicated derivative produces

$$-A_1\omega L \sin \omega t + A_2\omega L \cos \omega t + RA_1 \cos \omega t + RA_2 \sin \omega t = V_M \cos \omega t$$

By equating coefficients of the sine and cosine functions, we obtain

$$\begin{aligned} -A_1\omega L + A_2R &= 0 \\ A_1R + A_2\omega L &= V_M \end{aligned}$$

that is, two simultaneous equations in the unknowns A_1 and A_2. Solving these two equations for A_1 and A_2 yields

$$A_1 = \frac{RV_M}{R^2 + \omega^2L^2}$$

$$A_2 = \frac{\omega LV_M}{R^2 + \omega^2L^2}$$

Therefore,

$$i(t) = \frac{RV_M}{R^2 + \omega^2L^2} \cos \omega t + \frac{\omega LV_M}{R^2 + \omega^2L^2} \sin \omega t$$

which, using the last identity in Eq. (8.11), can be written as

$$i(t) = A \cos(\omega t + \phi)$$

where A and ϕ are determined as follows:

$$A \cos \phi = \frac{RV_M}{R^2 + \omega^2L^2}$$

$$A \sin \phi = \frac{-\omega LV_M}{R^2 + \omega^2L^2}$$

Hence,

$$\tan \phi = \frac{A \sin \phi}{A \cos \phi} = -\frac{\omega L}{R}$$

and therefore,

$$\phi = -\tan^{-1} \frac{\omega L}{R}$$

and since

$$\begin{aligned} (A \cos \phi)^2 + (A \sin \phi)^2 = A^2(\cos^2\phi + \sin^2\phi) &= A^2 \\ A^2 &= \frac{R^2V^2_M}{(R^2 + \omega^2L^2)^2} + \frac{(\omega L)^2V^2_M}{(R^2 + \omega^2L^2)^2} \\ &= \frac{V^2_M}{R^2 + \omega^2L^2} \\ A &= \frac{V_M}{\sqrt{R^2 + \omega^2L^2}} \end{aligned}$$

Hence, the final expression for $i(t)$ is

$$i(t) = \frac{V_M}{\sqrt{R^2 + \omega^2L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

The preceding analysis indicates that ϕ is zero if $L = 0$ and hence $i(t)$ is in phase with $v(t)$. If $R = 0$, $\phi = -90°$, and the current lags the voltage by 90°. If L and R are both present, the current lags the voltage by some angle between 0° and 90°.

This example illustrates an important point: solving even a simple one-loop circuit containing one resistor and one inductor is very complicated compared to the solution of a single-loop circuit containing only two resistors. Imagine for a moment how laborious it would be to solve a more complicated circuit using the procedure employed in Example 8.3. To circumvent this approach, we will establish a correspondence between sinusoidal time functions and complex numbers. We will then show that this relationship leads to a set of algebraic equations for currents and voltages in a network (e.g., loop currents or node voltages) in which the coefficients of the variables are complex numbers. Hence, once again we will find that determining the currents or voltages in a circuit can be accomplished by solving a set of algebraic equations; however, in this case, their solution is complicated by the fact that variables in the equations have complex, rather than real, coefficients.

The vehicle we will employ to establish a relationship between time-varying sinusoidal functions and complex numbers is Euler's equation, which for our purposes is written as

$$e^{j\omega t} = \cos \omega t + j \sin \omega t \tag{8.12}$$

This complex function has a real part and an imaginary part:

$$\begin{aligned} \operatorname{Re}\left(e^{j\omega t}\right) &= \cos \omega t \\ \operatorname{Im}\left(e^{j\omega t}\right) &= \sin \omega t \end{aligned} \tag{8.13}$$

where Re(·) and Im(·) represent the real part and the imaginary part, respectively, of the function in the parentheses. Recall that $j = \sqrt{-1}$.

Now suppose that we select as our forcing function in Fig. 8.4 the nonrealizable voltage

$$v(t) = V_M e^{j\omega t} \tag{8.14}$$

which, because of Euler's identity, can be written as

$$v(t) = V_M \cos \omega t + jV_M \sin \omega t \tag{8.15}$$

The real and imaginary parts of this function are each realizable. We think of this complex forcing function as two forcing functions, a real one and an imaginary one, and as a consequence of linearity, the principle of superposition applies and thus the current response can be written as

$$i(t) = I_M \cos(\omega t + \phi) + jI_M \sin(\omega t + \phi) \tag{8.16}$$

where $I_M \cos(\omega t + \phi)$ is the response due to $V_M \cos \omega t$ and $jI_M \sin(\omega t + \phi)$ is the response due to $jV_M \sin \omega t$. This expression for the current containing both a real and an imaginary term can be written via Euler's equation as

$$i(t) = I_M e^{j(\omega t + \phi)} \tag{8.17}$$

Because of the preceding relationships, we find that rather than apply the forcing function $V_M \cos \omega t$ and calculate the response $I_M \cos(\omega t + \phi)$, we can apply the complex forcing function $V_M e^{j\omega t}$ and calculate the response $I_M e^{j(\omega t + \phi)}$, the real part of which is the desired response $I_M \cos(\omega t + \phi)$. Although this procedure may initially appear to be more complicated, it is not. It is through this technique that we will convert the differential equation to an algebraic equation that is much easier to solve.

EXAMPLE 8.4

Once again, let us determine the current in the *RL* circuit examined in Example 8.3. However, rather than apply $V_M \cos \omega t$, we will apply $V_M e^{j\omega t}$.

SOLUTION

The forced response will be of the form

$$i(t) = I_M e^{j(\omega t + \phi)}$$

where only I_M and ϕ are unknown. Substituting $v(t)$ and $i(t)$ into the differential equation for the circuit, we obtain

$$RI_M e^{j(\omega t + \phi)} + L \frac{d}{dt}\left(I_M e^{j(\omega t + \phi)}\right) = V_M e^{j\omega t}$$