

## LEARNING ASSESSMENT

**E10.7** Find the impedance seen by the source in the circuit in Fig. E10.7.

**ANSWER:**

$$Z_S = 2.25/20.9^\circ \Omega$$

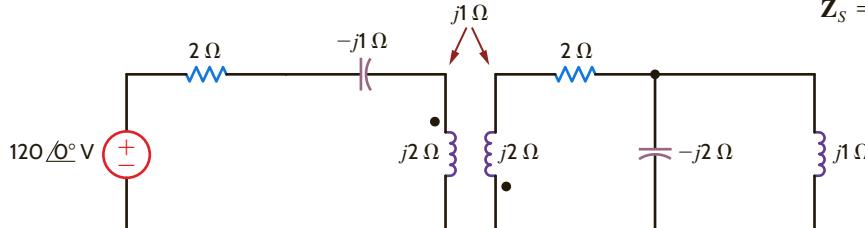


Figure E10.7

We now perform an energy analysis on a pair of mutually coupled inductors, which will yield some interesting relationships for the circuit elements. Our analysis will involve the performance of an experiment on the network shown in Fig. 10.15. Before beginning the experiment, we set all voltages and currents in the circuit equal to zero. Once the circuit is quiescent, we begin by letting the current  $i_1(t)$  increase from zero to some value  $I_1$  with the right-side terminals open circuited. Since the right-side terminals are open,  $i_2(t) = 0$ , and therefore the power entering these terminals is zero. The instantaneous power entering the left-side terminals is

$$p(t) = v_1(t)i_1(t) = \left[ L_1 \frac{di_1(t)}{dt} \right] i_1(t)$$

The energy stored within the coupled circuit at  $t_1$  when  $i_1(t) = I_1$  is then

$$\int_0^{t_1} v_1(t)i_1(t) dt = \int_0^{t_1} L_1 i_1(t) di_1(t) = \frac{1}{2} L_1 I_1^2$$

Continuing our experiment, starting at time  $t_1$ , we let the current  $i_2(t)$  increase from zero to some value  $I_2$  at time  $t_2$  while holding  $i_1(t)$  constant at  $I_1$ . The energy delivered through the right-side terminals is

$$\int_{t_1}^{t_2} v_2(t)i_2(t) dt = \int_0^{t_2} L_2 i_2(t) di_2(t) = \frac{1}{2} L_2 I_2^2$$

However, during the interval  $t_1$  to  $t_2$  the voltage  $v_1(t)$  is

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

Since  $i_1(t)$  is a constant  $I_1$ , the energy delivered through the left-side terminals is

$$\begin{aligned} \int_{t_1}^{t_2} v_1(t)i_1(t) dt &= \int_{t_1}^{t_2} M \frac{di_2(t)}{dt} I_1 dt = MI_1 \int_0^{t_2} di_2(t) \\ &= MI_1 I_2 \end{aligned}$$

10.2

## Energy Analysis

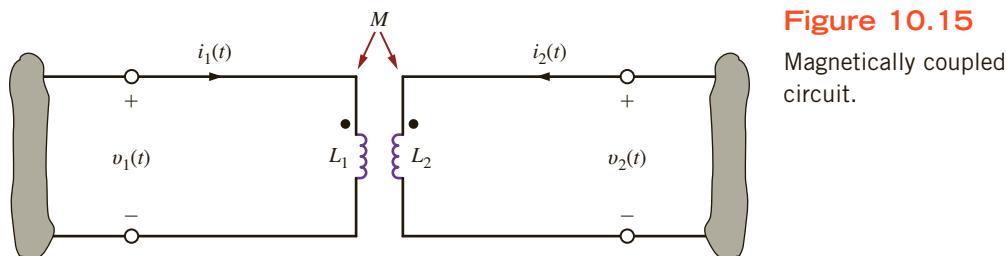


Figure 10.15

Magnetically coupled circuit.

Therefore, the total energy stored in the network for  $t > t_2$  is

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \quad 10.18$$

We could, of course, repeat our entire experiment with either the dot on  $L_1$  or  $L_2$ , but not both, reversed, and in this case the sign on the mutual inductance term would be negative, producing

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2$$

It is very important for the reader to realize that in our derivation of the preceding equation, by means of the experiment, the values  $I_1$  and  $I_2$  could have been any values at *any time*; therefore, the energy stored in the magnetically coupled inductors at any instant of time is given by the expression

$$w(t) = \frac{1}{2} L_1 [i_1(t)]^2 + \frac{1}{2} L_2 [i_2(t)]^2 \pm M i_1(t) i_2(t) \quad 10.19$$

The two coupled inductors represent a passive network, and therefore, the energy stored within this network must be nonnegative for any values of the inductances and currents.

The equation for the instantaneous energy stored in the magnetic circuit can be written as

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2$$

Adding and subtracting the term  $1/2(M^2/L_2)i_1^2$  and rearranging the equation yields

$$w(t) = \frac{1}{2} \left( L_1 - \frac{M^2}{L_2} \right) i_1^2 + \frac{1}{2} L_2 \left( i_2 + \frac{M}{L_2} i_1 \right)^2$$

From this expression, we recognize that the instantaneous energy stored will be nonnegative if

$$M \leq \sqrt{L_1 L_2} \quad 10.20$$

Note that this equation specifies an upper limit on the value of the mutual inductance.

We define the coefficient of coupling between the two inductors  $L_1$  and  $L_2$  as

$$k = \frac{M}{\sqrt{L_1 L_2}} \quad 10.21$$

and we note from Eq. (10.20) that its range of values is

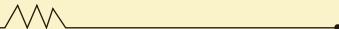
$$0 \leq k \leq 1 \quad 10.22$$

This coefficient is an indication of how much flux in one coil is linked with the other coil; that is, if all the flux in one coil reaches the other coil, then we have 100% coupling and  $k = 1$ . For large values of  $k$  (i.e.,  $k > 0.5$ ), the inductors are said to be tightly coupled, and for small values of  $k$  (i.e.,  $k \leq 0.5$ ), the coils are said to be loosely coupled. If there is no coupling,  $k = 0$ . The previous equations indicate that the value for the mutual inductance is confined to the range

$$0 \leq M \leq \sqrt{L_1 L_2} \quad 10.23$$

and that the upper limit is the geometric mean of the inductances  $L_1$  and  $L_2$ .

## EXAMPLE 10.7



The coupled circuit in Fig. 10.16a has a coefficient of coupling of 1 (i.e.,  $k = 1$ ). We wish to determine the energy stored in the mutually coupled inductors at time  $t = 5$  ms.  $L_1 = 2.653$  mH and  $L_2 = 10.61$  mH.

### SOLUTION

From the data, the mutual inductance is

$$M = \sqrt{L_1 L_2} = 5.31 \text{ mH}$$

The frequency-domain equivalent circuit is shown in Fig. 10.16b, where the impedance values for  $X_{L_1}$ ,  $X_{L_2}$ , and  $X_M$  are 1, 4, and 2, respectively. The mesh equations for the network are then

$$(2 + j1)\mathbf{I}_1 - j2\mathbf{I}_2 = 24/0^\circ$$

$$-j2\mathbf{I}_1 + (4 + j4)\mathbf{I}_2 = 0$$

Solving these equations for the two mesh currents yields

$$\mathbf{I}_1 = 9.41/-11.31^\circ \text{ A} \quad \text{and} \quad \mathbf{I}_2 = 3.33/+33.69^\circ \text{ A}$$

and therefore,

$$i_1(t) = 9.41 \cos(377t - 11.31^\circ) \text{ A}$$

$$i_2(t) = 3.33 \cos(377t + 33.69^\circ) \text{ A}$$

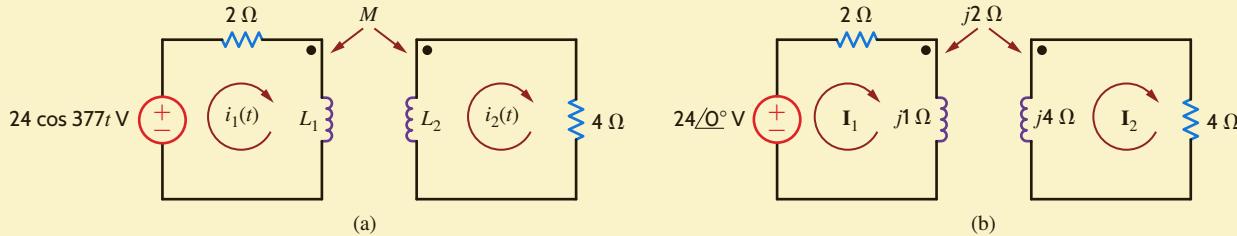
At  $t = 5 \text{ ms}$ ,  $377t = 1.885$  radians or  $108^\circ$ , and therefore,

$$i_1(t = 5 \text{ ms}) = 9.41 \cos(108^\circ - 11.31^\circ) = -1.10 \text{ A}$$

$$i_2(t = 5 \text{ ms}) = 3.33 \cos(108^\circ + 33.69^\circ) = -2.61 \text{ A}$$

Therefore, the energy stored in the coupled inductors at  $t = 5 \text{ ms}$  is

$$\begin{aligned} w(t)|_{t=0.005 \text{ s}} &= \frac{1}{2}(2.653)(10^{-3})(-1.10)^2 + \frac{1}{2}(10.61)(10^{-3})(-2.61)^2 \\ &\quad - (5.31)(10^{-3})(-1.10)(-2.61) \\ &= (1.61)(10^{-3}) + (36.14)(10^{-3}) - (15.25)(10^{-3}) \\ &= 22.5 \text{ mJ} \end{aligned}$$



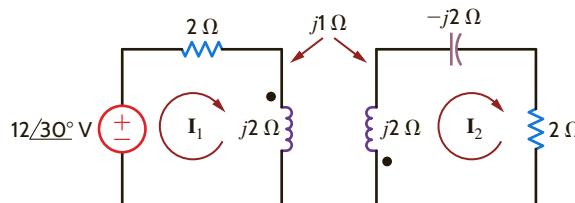
**Figure 10.16**

Example of a magnetically coupled circuit drawn in the time and frequency domains.

## LEARNING ASSESSMENT

**E10.8** The network in Fig. E10.8 operates at 60 Hz. Compute the energy stored in the mutually coupled inductors at time  $t = 10 \text{ ms}$ .

**ANSWER:**  
 $w(10 \text{ ms}) = 39 \text{ mJ}$ .



**Figure E10.8**