13.8 Solving Differential Equations with Laplace Transforms

As a prelude to Chapter 14, in which we will employ the power and versatility of the Laplace transform in a wide variety of circuit analysis problems, we will now demonstrate how the techniques outlined in this chapter can be used in the solution of a circuit problem via the differential equation that describes the network.

EXAMPLE 13.14

Consider the network shown in Fig. 13.5a. Assume that the network is in steady state prior to $t = 0$. Let us find the current $i(t)$ for $t > 0$.

SOLUTION

In steady state prior to $t = 0$, the network is as shown in Fig. 13.5b, since the inductor acts like a short circuit to dc and the capacitor acts like an open circuit to dc. From Fig. 13.5b we note that $i(0) = 4$ A and $v_C(0) = 4$ V. For $t > 0$, the KVL equation for the network is

$$12u(t) = 2i(t) + 1\frac{di(t)}{dt} + \frac{1}{0.1}\int_0^t i(x)\,dx + v_C(0)$$

Using the results of Example 13.1 and properties 7 and 10, the transformed expression becomes

$$\frac{12}{s} = 2\mathbf{I}(s) + s\mathbf{I}(s) - i(0) + \frac{10}{s}\mathbf{I}(s) + \frac{v_C(0)}{s}$$

Using the initial conditions, we find that the equation becomes

$$\frac{12}{s} = \mathbf{I}(s)\left(2 + s + \frac{10}{s}\right) - 4 + \frac{4}{s}$$

or

$$\mathbf{I}(s) = \frac{4(s+2)}{s^2 + 2s + 10} = \frac{4(s+2)}{(s + 1 - j3)(s + 1 + j3)}$$

and then

$$K_1 = \left.\frac{4(s+2)}{s + 1 + j3}\right|_{s = -1 + j3}$$
$$= 2.11\underline{/-18.4^\circ}$$

Therefore,

$$i(t) = 2(2.11)e^{-t}\cos(3t - 18.4^\circ)u(t) \text{ A}$$

Note that this expression satisfies the initial condition $i(0) = 4$ A.

In the introduction to this chapter, we stated that the Laplace transform would yield both the natural and forced responses for a circuit. Our solution to this problem contains only one term. Is it the forced response or the natural response? Remember that the forced response always has the same form as the forcing function or source. The source for this problem is a dc voltage source, so the forced response should be a constant. In fact, the forced response is zero for our circuit, and the natural response is the damped cosine function. Does a zero forced response make sense? Yes! If we look at our circuit, the capacitor is going to charge up to the source voltage. Once the capacitor voltage reaches the source voltage, the current will become zero.

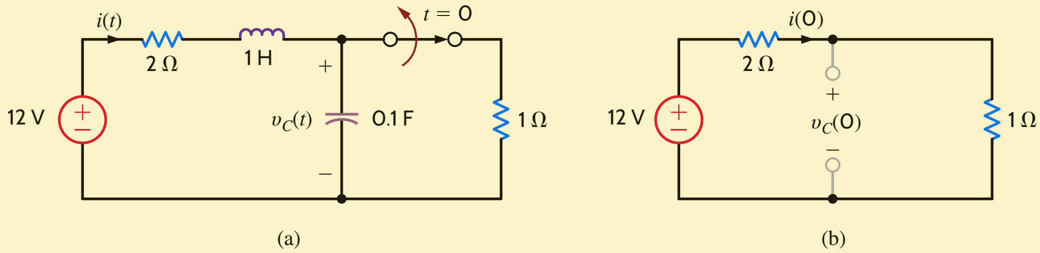

Figure 13.5 Circuits used in Example 13.14.

LEARNING ASSESSMENTS

E13.14 Find the initial and final values of the function $f(t)$ if $\mathbf{F}(s) = \mathscr{L}[f(t)]$ is given by the expression

$$\mathbf{F}(s) = \frac{(s+1)^2}{s(s+2)(s^2+2s+2)}$$

ANSWER:
$f(0) = 0$ and $f(\infty) = \frac{1}{4}$.

E13.15 Find the initial and final values of the time function $f(t)$ if $\mathbf{F}(s) = \dfrac{8s^2 - 20s + 500}{s(s^2 + 4s + 50)}$.

ANSWER:
$f(0) = 8$; $f(\infty) = 10$.

E13.16 Use the Laplace transform to find $y(t)$ if

$$\frac{dy}{dt} + 4y(t) + 4\int_0^t y(x)dx = 10u(t),\ y(0) = 10$$

ANSWER:
$y(t) = \left(10e^{-2t} - 10te^{-2t}\right)u(t)$.

PROBLEM-SOLVING STRATEGY

THE LAPLACE TRANSFORM AND TRANSIENT CIRCUITS

STEP 1. Assume that the circuit has reached steady state before a switch is moved. Draw the circuit valid for $t = 0^-$ replacing capacitors with open circuits and inductors with short circuits. Solve for the initial conditions: voltages across capacitors and currents flowing through inductors. Remember that

$v_C(0-) = v_C(0+) = v_C(0)$ and $i_L(0-) = i_L(0+) = i_L(0)$.

STEP 2. Draw the circuit valid for $t > 0$. Use circuit analysis techniques to determine the differential or integrodifferential equation that describes the behavior of the circuit.

STEP 3. Convert this differential/integrodifferential equation to an algebraic equation using the Laplace transform.

STEP 4. Solve this algebraic equation for the variable of interest. Your result will be a ratio of polynomials in the complex variable s.

STEP 5. Perform an inverse Laplace transform to solve for the circuit response in the time domain.

LEARNING ASSESSMENTS

E13.17 Assuming that the network in Fig. E13.17 is in steady state prior to $t = 0$, find $i(t)$ for $t > 0$.

ANSWER:
$i(t) = (3 - e^{-2t})u(t)$ A.

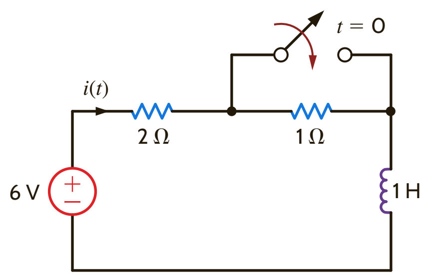

Figure E13.17