12.1 Variable Frequency-Response Analysis

In previous chapters we investigated the response of *RLC* networks to sinusoidal inputs. In particular, we considered 60-Hz sinusoidal inputs. In this chapter we allow the frequency of excitation to become a variable and evaluate network performance as a function of frequency. To begin, let us consider the effect of varying frequency on elements with which we are already quite familiar—the resistor, inductor, and capacitor. The frequency-domain impedance of the resistor shown in **Fig. 12.1a** is

$$\mathbf{Z}_R = R = R\underline{/0^\circ}$$

The magnitude and phase are constant and independent of frequency. Sketches of the magnitude and phase of $\mathbf{Z}_R$ are shown in **Figs. 12.1b** and **c**. Obviously, this is a very simple situation.

For the inductor in **Fig. 12.2a**, the frequency-domain impedance $\mathbf{Z}_L$ is

$$\mathbf{Z}_L = j\omega L = \omega L\underline{/90^\circ}$$

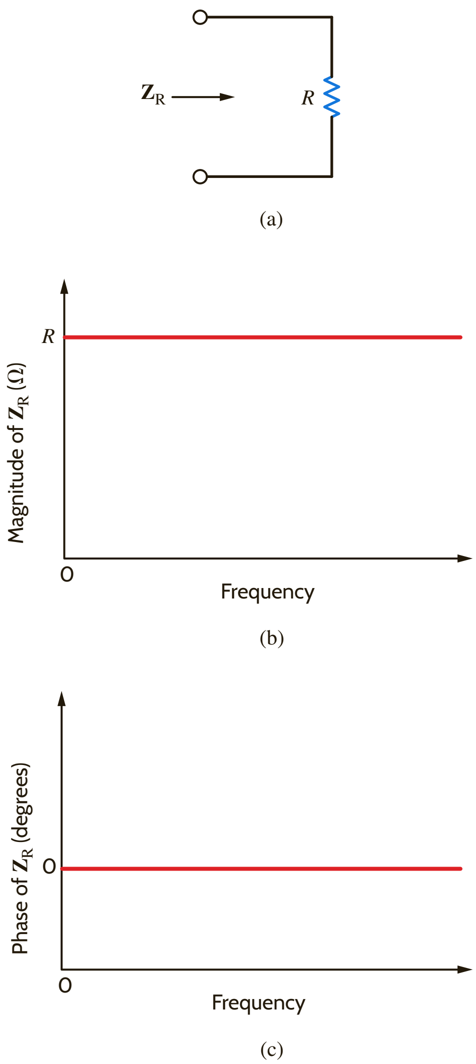


Figure 12.1

Frequency-independent impedance of a resistor.

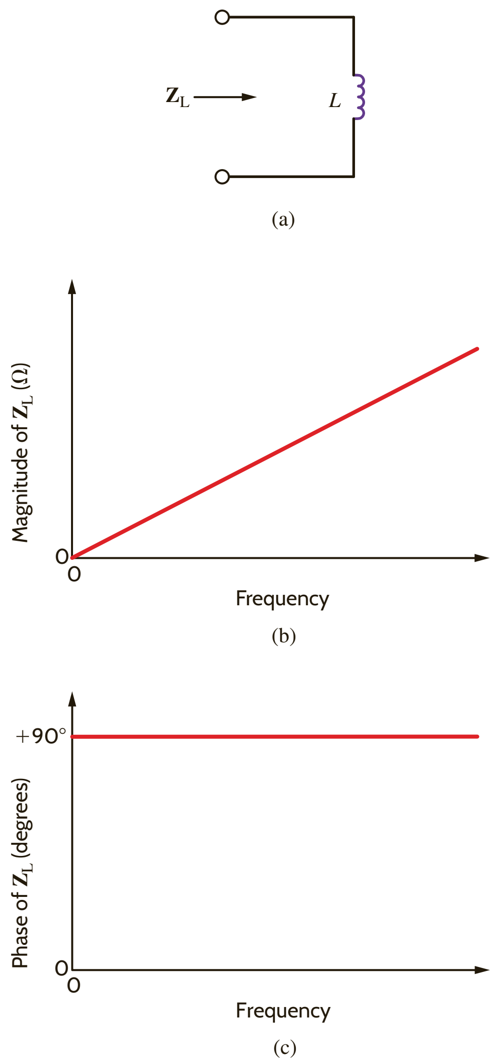
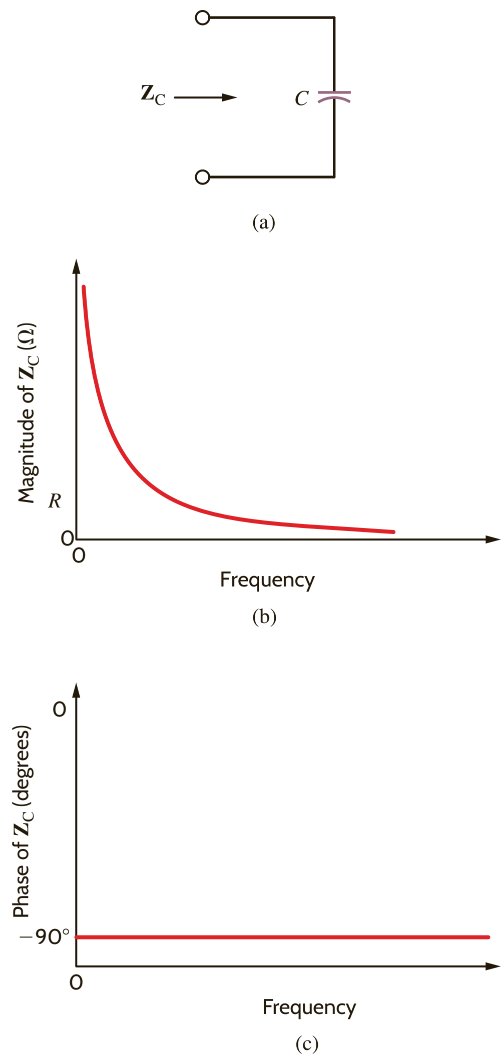


Figure 12.2

Frequency-dependent impedance of an inductor.

Figure 12.3
Frequency-dependent impedance of a capacitor.



The phase is constant at 90°, but the magnitude of $\mathbf{Z}_L$ is directly proportional to frequency. Figs. 12.2b and c show sketches of the magnitude and phase of $\mathbf{Z}_L$ versus frequency. Note that at low frequencies the inductor's impedance is quite small. In fact, at dc, $\mathbf{Z}_L$ is zero, and the inductor appears as a short circuit. Conversely, as frequency increases, the impedance also increases.

Next consider the capacitor of Fig. 12.3a. The impedance is

$$\mathbf{Z}_C = \frac{1}{j\omega C} = \frac{1}{\omega C}\underline{/-90^\circ}$$

Once again the phase of the impedance is constant, but now the magnitude is inversely proportional to frequency, as shown in Figs. 12.3b and c. Note that the impedance approaches infinity, or an open circuit, as ω approaches zero and $\mathbf{Z}_C$ approaches zero as ω approaches infinity.

Now let us investigate a more complex circuit: the *RLC* series network in Fig. 12.4a. The equivalent impedance is

$$\mathbf{Z}_{eq} = R + j\omega L + \frac{1}{j\omega C}$$

or

$$\mathbf{Z}_{eq} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C}$$

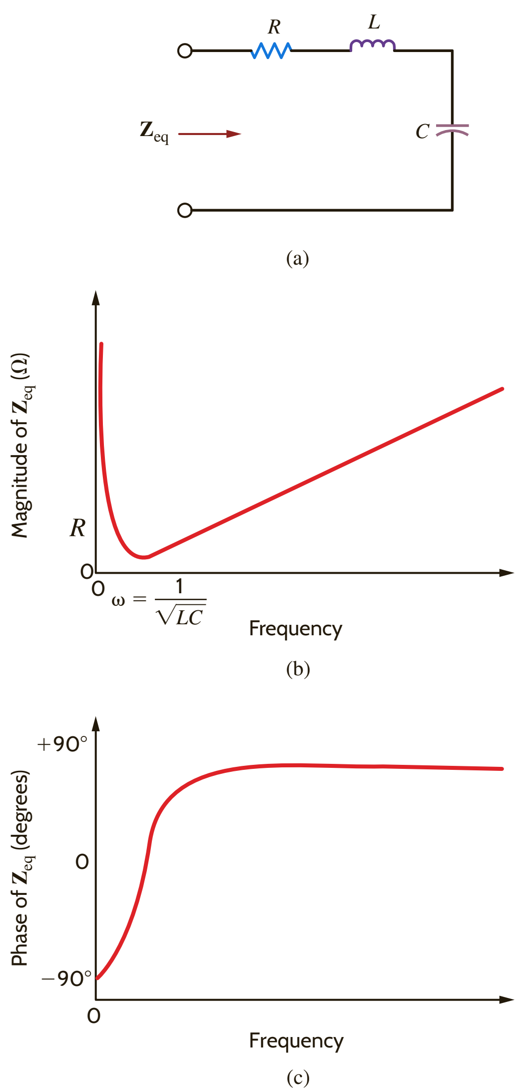


Figure 12.4

Frequency-dependent impedance of an *RLC* series network.

Sketches of the magnitude and phase of this function are shown in **Figs. 12.4b** and **c**.

Note that at very low frequencies, the capacitor appears as an open circuit and, therefore, the impedance is very large in this range. At high frequencies, the capacitor has very little effect and the impedance is dominated by the inductor, whose impedance keeps rising with frequency.

As the circuits become more complicated, the equations become more cumbersome. In an attempt to simplify them, let us make the substitution $j\omega = s$. (This substitution has a more important meaning, which we will describe in later chapters.) With this substitution, the expression for $\mathbf{Z}_{eq}$ becomes

$$\mathbf{Z}_{eq} = \frac{s^2LC + sRC + 1}{sC}$$

If we review the four circuits we investigated thus far, we will find that in every case the impedance is the ratio of two polynomials in s and is of the general form

$$\mathbf{Z}(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad \textbf{12.1}$$

where $N(s)$ and $D(s)$ are polynomials of order m and n, respectively. An extremely important aspect of Eq. (12.1) is that it holds not only for impedances but also for all voltages, currents, admittances, and gains in the network. The only restriction is that the values of all circuit elements (resistors, capacitors, inductors, and dependent sources) must be real numbers.

Let us now demonstrate the manner in which the voltage across an element in a series *RLC* network varies with frequency.

EXAMPLE 12.1

Consider the network in Fig. 12.5a. We wish to determine the variation of the output voltage as a function of frequency over the range from 0 to 1 kHz.

SOLUTION

Using voltage division, we can express the output as

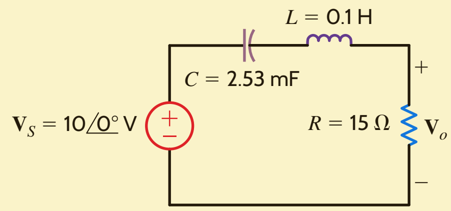
$$\mathbf{V}_o = \left(\frac{R}{R + j\omega L + \dfrac{1}{j\omega C}}\right)\mathbf{V}_S$$

or, equivalently,

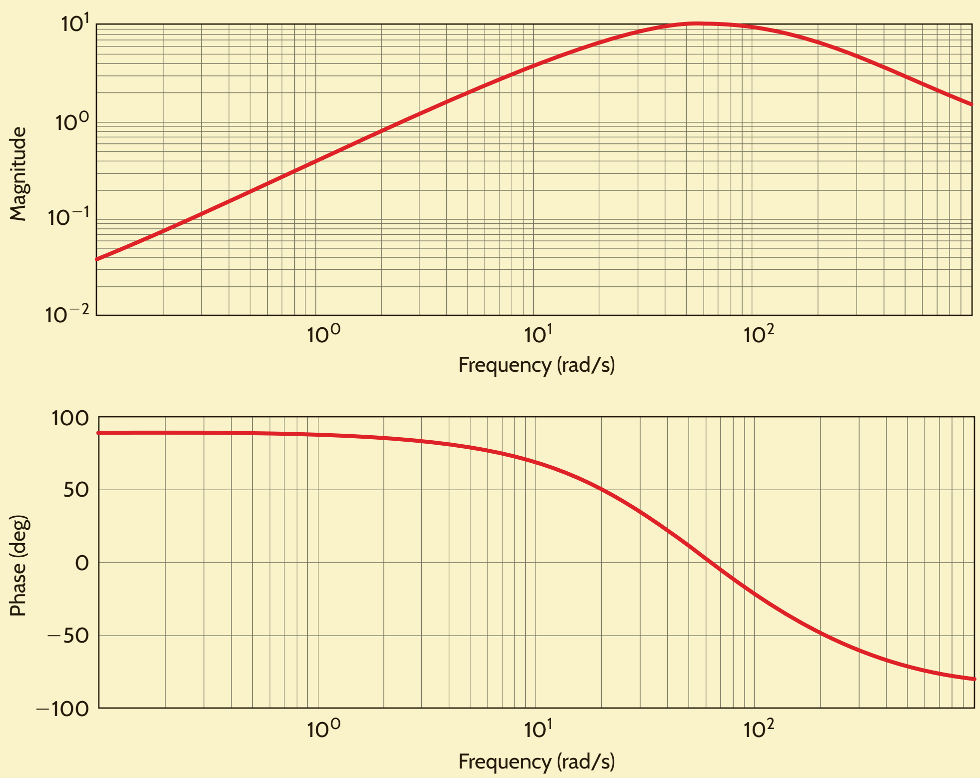
$$\mathbf{V}_o = \left(\frac{j\omega CR}{(j\omega)^2 LC + j\omega CR + 1}\right)\mathbf{V}_S$$

Using the element values, we find that the equation becomes

$$\mathbf{V}_o = \left(\frac{(j\omega)(37.95 \times 10^{-3})}{(j\omega)^2(2.53 \times 10^{-4}) + j\omega(37.95 \times 10^{-3}) + 1}\right)10\underline{/0^\circ}$$



(a)



(b)

Figure 12.5 (a) Network and (b) its frequency-response simulation.

The resultant magnitude and phase characteristics are semilog plots in which the frequency is displayed on the log axis. The plots for the function $\mathbf{V}_o$ are shown in Fig. 12.5b.

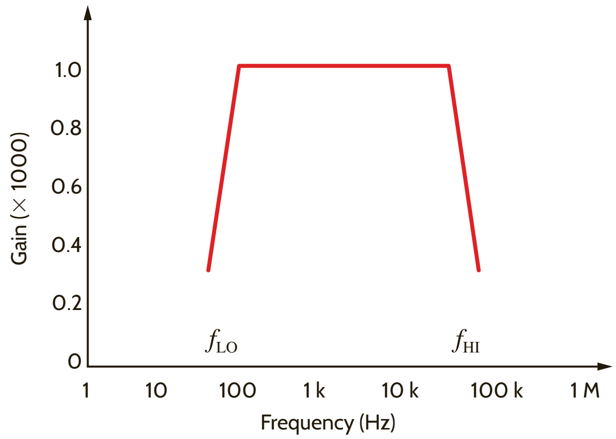


Figure 12.6

Amplifier frequency-response requirements.

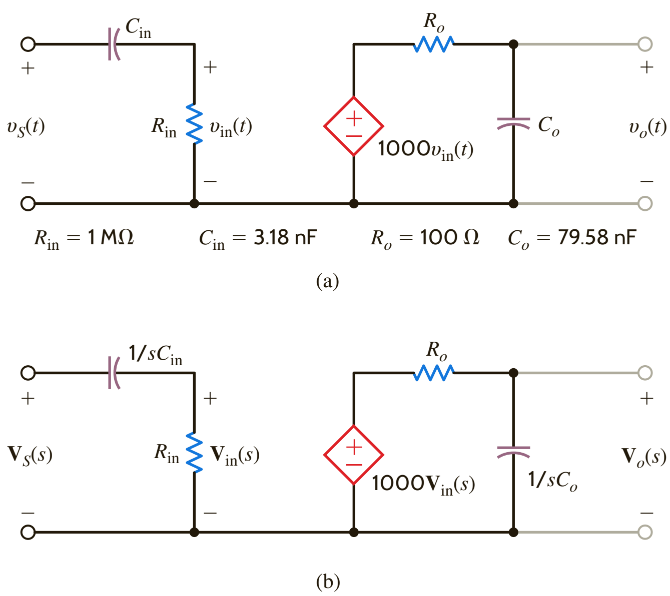


Figure 12.7

Amplifier equivalent network.

In subsequent sections we will illustrate that the use of a semilog plot is a very useful tool in deriving frequency-response information.

As an introductory application of variable frequency-response analysis and characterization, let us consider a stereo amplifier. In particular, we should consider first the frequency range over which the amplifier must perform and then exactly what kind of performance we desire. The frequency range of the amplifier must exceed that of the human ear, which is roughly 50 Hz to 15,000 Hz. Accordingly, typical stereo amplifiers are designed to operate in the frequency range from 50 Hz to 20,000 Hz. Furthermore, we want to preserve the fidelity of the signal as it passes through the amplifier. Thus, the output signal should be an exact duplicate of the input signal times a gain factor. This requires that the gain be independent of frequency over the specified frequency range of 50 Hz to 20,000 Hz. An ideal sketch of this requirement for a gain of 1000 is shown in **Fig. 12.6**, where the midband region is defined as that portion of the plot where the gain is constant and is bounded by two points, which we will refer to as f_{LO} and f_{HI}. Notice once again that the frequency axis is a log axis and, thus, the frequency response is displayed on a semilog plot.

A model for the amplifier described graphically in Fig. 12.6 is shown in **Fig. 12.7a**, with the frequency-domain equivalent circuit in **Fig. 12.7b**.

If the input is a steady-state sinusoid, we can use frequency-domain analysis to find the gain

$$\mathbf{G}_v(j\omega) = \frac{\mathbf{V}_o(j\omega)}{\mathbf{V}_S(j\omega)}$$

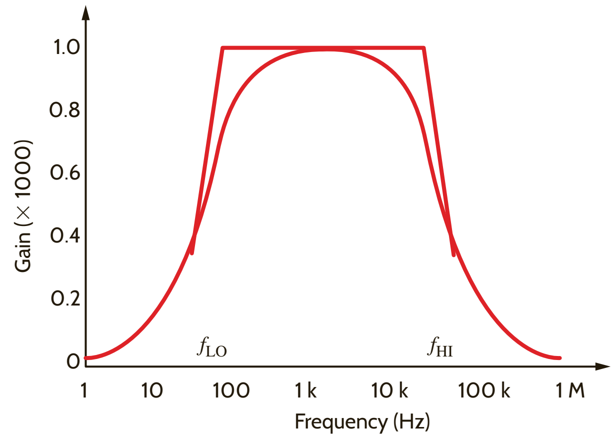


Figure 12.8 Exact and approximate amplifier gain versus frequency plots.

which with the substitution $s = j\omega$ can be expressed as

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_S(s)}$$

Using voltage division, we find that the gain is

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_S(s)} = \frac{\mathbf{V}_{in}(s)}{\mathbf{V}_S(s)}\frac{\mathbf{V}_o(s)}{\mathbf{V}_{in}(s)} = \left[\frac{R_{in}}{R_{in} + 1/sC_{in}}\right](1000)\left[\frac{1/sC_o}{R_o + 1/sC_o}\right]$$

or

$$\mathbf{G}_v(s) = \left[\frac{sC_{in}R_{in}}{1 + sC_{in}R_{in}}\right](1000)\left[\frac{1}{1 + sC_oR_o}\right]$$

Using the element values in Fig. 12.7a,

$$\mathbf{G}_v(s) = \left[\frac{s}{s + 100\pi}\right](1000)\left[\frac{40{,}000\pi}{s + 40{,}000\pi}\right]$$

where 100π and $40{,}000\pi$ are the radian equivalents of 50 Hz and 20,000 Hz, respectively. Since $s = j\omega$, the network function is indeed complex. An exact plot of $\mathbf{G}_v(s)$ is shown in **Fig. 12.8** superimposed over the sketch of Fig. 12.6. The exact plot exhibits smooth transitions at f_{LO} and f_{HI}; otherwise the plots match fairly well.

Let us examine our expression for $\mathbf{G}_v(s)$ more closely with respect to the plot in Fig. 12.8. Assume that f is well within the midband frequency range; that is,

$$f_{LO} \ll f \ll f_{HI}$$

or

$$100\pi \ll |s| \ll 40{,}000\pi$$

Under these conditions, the network function becomes

$$\mathbf{G}_v(s) \approx \left[\frac{s}{s}\right](1000)\left[\frac{1}{1 + 0}\right]$$

or

$$\mathbf{G}_v(s) = 1000$$

Thus, well within midband, the gain is constant. However, if the frequency of excitation decreases toward f_{LO}, then $|s|$ is comparable to 100π and

$$\mathbf{G}_v(s) \approx \left[\frac{s}{s + 100\pi}\right](1000)$$

Since $R_{in}C_{in} = 1/100\pi$, we see that C_{in} causes the rolloff in gain at low frequencies. Similarly, when the frequency approaches f_{HI}, the gain rolloff is due to C_o.

TABLE 12.1 Network transfer functions

INPUT	OUTPUT	TRANSFER FUNCTION	SYMBOL
Voltage	Voltage	Voltage gain	$\mathbf{G}_v(s)$
Current	Voltage	Transimpedance	$\mathbf{Z}(s)$
Current	Current	Current gain	$\mathbf{G}_i(s)$
Voltage	Current	Transadmittance	$\mathbf{Y}(s)$

Through this amplifier example, we have introduced the concept of frequency-dependent networks and have demonstrated that frequency-dependent network performance is caused by the reactive elements in a network.

NETWORK FUNCTIONS In the previous section, we introduced the term *voltage gain*, $\mathbf{G}_v(s)$. This term is actually only one of several network functions, designated generally as $\mathbf{H}(s)$, which define the ratio of response to input. Since the function describes a reaction due to an excitation at some other point in the circuit, network functions are also called *transfer functions*. Furthermore, transfer functions are not limited to voltage ratios. Since in electrical networks inputs and outputs can be either voltages or currents, there are four possible network functions, as listed in Table 12.1.

There are also *driving point functions*, which are impedances or admittances defined at a single pair of terminals. For example, the input impedance of a network is a driving point function.

EXAMPLE 12.2

We wish to determine the transfer admittance $[\mathbf{I}_2(s)/\mathbf{V}_1(s)]$ and the voltage gain of the network shown in Fig. 12.9.

SOLUTION

The mesh equations for the network are

$$(R_1 + sL)\mathbf{I}_1(s) - sL\mathbf{I}_2(s) = \mathbf{V}_1(s)$$
$$-sL\mathbf{I}_1(s) + \left(R_2 + sL + \frac{1}{sC}\right)\mathbf{I}_2(s) = 0$$
$$\mathbf{V}_2(s) = \mathbf{I}_2(s)R_2$$

Solving the equations for $\mathbf{I}_2(s)$ yields

$$\mathbf{I}_2(s) = \frac{sL\mathbf{V}_1(s)}{(R_1 + sL)(R_2 + sL + 1/sC) - s^2L^2}$$

Therefore, the transfer admittance $[\mathbf{I}_2(s)/\mathbf{V}_1(s)]$ is

$$\mathbf{Y}_T(s) = \frac{\mathbf{I}_2(s)}{\mathbf{V}_1(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (L + R_1R_2C)s + R_1}$$

and the voltage gain is

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_2(s)}{\mathbf{V}_1(s)} = \frac{LCR_2s^2}{(R_1 + R_2)LCs^2 + (L + R_1R_2C)s + R_1}$$

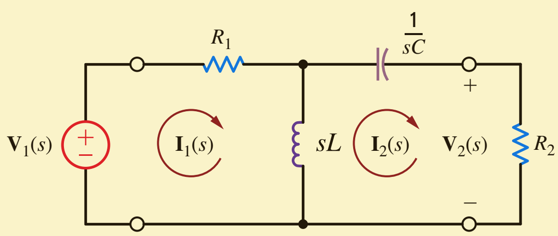


Figure 12.9 Circuit employed in Example 12.2.

POLES AND ZEROS As we have indicated, the network function can be expressed as the ratio of the two polynomials in s. In addition, we note that since the values of our circuit elements, or controlled sources, are real numbers, the coefficients of the two polynomials will be real. Therefore, we will express a network function in the form

$$\mathbf{H}(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad \textbf{12.2}$$

where $N(s)$ is the numerator polynomial of degree m and $D(s)$ is the denominator polynomial of degree n. Equation (12.2) can also be written in the form

$$\mathbf{H}(s) = \frac{K_0(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \qquad \textbf{12.3}$$

where K_0 is a constant, $z_1, \ldots, z_m$ are the roots of $N(s)$, and $p_1, \ldots, p_n$ are the roots of $D(s)$. Note that if $s = z_1$, or $z_2, \ldots, z_m$, then $\mathbf{H}(s)$ becomes zero and hence $z_1, \ldots, z_m$ are called zeros of the transfer function. Similarly, if $s = p_1$, or $p_2, \cdots, p_n$, then $\mathbf{H}(s)$ becomes infinite and, therefore, $p_1, \ldots, p_n$ are called poles of the function. The zeros or poles may actually be complex. However, if they are complex, they must occur in conjugate pairs since the coefficients of the polynomial are real. The representation of the network function specified in Eq. (12.3) is extremely important and is generally employed to represent any linear time-invariant system. The importance of this form lies in the fact that the dynamic properties of a system can be gleaned from an examination of the system poles.

LEARNING ASSESSMENTS

E12.1 Find the driving-point impedance at $\mathbf{V}_S(s)$ in the amplifier shown in Fig. 12.7b.

ANSWER:

$$\mathbf{Z}(s) = R_{\text{in}} + \frac{1}{sC_{\text{in}}} = \left[1 + \left(\frac{100\pi}{s}\right)\right] \text{M}\Omega.$$

E12.2 Find the pole and zero locations in hertz and the value of K_0 for the amplifier network in Fig. 12.7.

ANSWER:

$z_1 = 0$ Hz (dc);
$p_1 = -50$ Hz;
$p_2 = -20{,}000$ Hz;
$K_0 = (4 \times 10^7)\pi$.

E12.3 Determine the voltage transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$ as a function of s in Fig. E12.3.

ANSWER:

$$\frac{s}{R_1C_2\left[s^2 + \dfrac{C_1R_2 + C_2R_2 + C_1R_1}{R_1R_2C_1C_2}s + \dfrac{1}{R_1R_2C_1C_2}\right]}$$

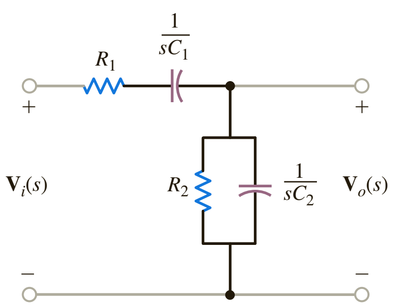


Figure E12.3