Earlier, in Eq. (13.8), we defined $\mathbf{F}(s)$ as the ratio of two polynomials. Let's suppose that $m = n$ in this equation. In this case, only $C_0$ is nonzero in Eq. (13.9). Recall that we perform a partial fraction expansion on $\mathbf{P}_1(s)/\mathbf{Q}(s)$ and use our table of Laplace transform pairs to determine the corresponding time function for each term in the expansion. What do we do with this constant $C_0$? Looking at our table of transform pairs in Table 13.1, we note that the Laplace transform of the unit impulse function is a constant. As a result, our circuit response would contain a unit impulse function. Earlier we noted that unit impulse functions don't exist in physical systems; therefore, $m < n$ for physical systems.

## 13.6 Convolution Integral

Convolution is a very important concept and has wide application in circuit and systems analysis. We first illustrate the connection that exists between the convolution integral and the Laplace transform. We then indicate the manner in which the convolution integral is applied in circuit analysis.

Property number 10 in Table 13.2 states the following.

If

$$f(t) = f_1(t) \otimes f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda)\, d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda)\, d\lambda \qquad \textbf{13.25}$$

and

$$\mathcal{L}[f(t)] = \mathbf{F}(s),\ \mathcal{L}[f_1(t)] = \mathbf{F}_1(s) \qquad \text{and} \qquad \mathcal{L}[f_2(t)] = \mathbf{F}_2(s)$$

then

$$\mathbf{F}(s) = \mathbf{F}_1(s)\mathbf{F}_2(s) \qquad \textbf{13.26}$$

Our demonstration begins with the definition

$$\mathcal{L}[f(t)] = \int_0^\infty \left[\int_0^t f_1(t-\lambda) f_2(\lambda)\, d\lambda\right] e^{-st}\, dt$$

We now force the function into the proper format by introducing into the integral within the brackets the unit step function $u(t - \lambda)$. We can do this because

$$u(t-\lambda) = \begin{cases} 1 & \text{for } \lambda < t \\ 0 & \text{for } \lambda > t \end{cases} \qquad \textbf{13.27}$$

The first condition in Eq. (13.27) ensures that the insertion of the unit step function has no impact within the limits of integration. The second condition in Eq. (13.27) allows us to change the upper limit of integration from $t$ to $\infty$. Therefore,

$$\mathcal{L}[f(t)] = \int_0^\infty \left[\int_0^\infty f_1(t-\lambda) u(t-\lambda) f_2(\lambda)\, d\lambda\right] e^{-st}\, dt$$

which can be written as

$$\mathcal{L}[f(t)] = \int_0^\infty f_2(\lambda) \left[\int_0^\infty f_1(t-\lambda) u(t-\lambda) e^{-st}\, dt\right] d\lambda$$

Note that the integral within the brackets is the time-shifting theorem illustrated in Eq. (13.6). Hence, the equation can be written as

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^\infty f_2(\lambda) \mathbf{F}_1(s) e^{-s\lambda}\, d\lambda \\ &= \mathbf{F}_1(s) \int_0^\infty f_2(\lambda) e^{-s\lambda}\, d\lambda \\ &= \mathbf{F}_1(s)\mathbf{F}_2(s) \end{aligned}$$

Note that convolution in the time domain corresponds to multiplication in the frequency domain.

Let us now illustrate the use of this property in the evaluation of an inverse Laplace transform.

## EXAMPLE 13.11

The transfer function for a network is given by the expression

$$\mathbf{H}(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_s(s)} = \frac{10}{s+5}$$

The input is a unit step function $\mathbf{V}_S(s) = \frac{1}{s}$. Let us use convolution to determine the output voltage $v_o(t)$.

**SOLUTION**

Since $\mathbf{H}(s) = \frac{10}{(s+5)}$, $h(t) = 10e^{-5t}$ and therefore

$$\begin{aligned} v_o(t) &= \int_0^t 10u(\lambda)e^{-5(t-\lambda)}\,d\lambda \\ &= 10e^{-5t}\int_0^t e^{5\lambda}\,d\lambda \\ &= \frac{10e^{-5t}}{5}[e^{5t} - 1] \\ &= 2[1 - e^{-5t}]u(t) \text{ V} \end{aligned}$$

For comparison, let us determine $v_o(t)$ from $\mathbf{H}(s)$ and $\mathbf{V}_s(s)$ using the partial fraction expansion method. $\mathbf{V}_o(s)$ can be written as

$$\begin{aligned} \mathbf{V}_o(s) &= \mathbf{H}(s)\mathbf{V}_s(s) \\ &= \frac{10}{s(s+5)} = \frac{K_0}{s} + \frac{K_1}{s+5} \end{aligned}$$

Evaluating the constants, we obtain $K_0 = 2$ and $K_1 = -2$. Therefore,

$$\mathbf{V}_o(s) = \frac{2}{s} - \frac{2}{s+5}$$

and hence

$$v_o(t) = 2[1 - e^{-5t}]u(t) \text{ V}$$

Although we can employ convolution to derive an inverse Laplace transform, the example, though quite simple, illustrates that this is a very poor approach. If the function $\mathbf{F}(s)$ is very complicated, the mathematics can become unwieldy. Convolution is, however, a very powerful and useful tool. For example, if we know the impulse response of a network, we can use convolution to determine the network's response to an input that may be available only as an experimental curve obtained in the laboratory. Thus, convolution permits us to obtain the network response to inputs that cannot be written as analytical functions but can be simulated on a digital computer. In addition, we can use convolution to model a circuit, which is completely unknown to us, and use this model to determine the circuit's response to some input signal.

## EXAMPLE 13.12

To demonstrate the power of convolution, we will create a model for a "black-box" linear band-pass filter, shown as a block in Fig. 13.3. We have no details about the filter circuitry at all—no circuit diagram, no component list, no component values. As a result, our filter model must be based solely on measurements. Using our knowledge of convolution and the Laplace transform, let us discuss appropriate measurement techniques, the resulting model, and how to employ the model in subsequent simulations.

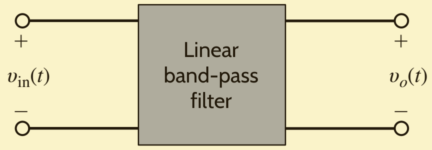


**Figure 13.3** Conceptual diagram for a band-pass filter.

**SOLUTION**

Because the filter is linear, $v_o(t)$ can be written

$$v_o(t) = h(t) \otimes v_{\text{in}}(t) \qquad \textbf{13.28}$$

Thus, the function $h(t)$ will be our model for the filter. To determine $h(t)$, we must input some $v_{\text{in}}(t)$, measure the response, $v_o(t)$, and perform the appropriate mathematics. One obvious option for $v_{\text{in}}(t)$ is the impulse function, $\delta(t)$; then $\mathbf{V}_{\text{in}}(s)$ is 1, and the output is the desired model, $h(t)$:

$$v_o(t) = h(t)$$

Unfortunately, creating an adequate impulse, infinite amplitude, and zero width in the laboratory is nontrivial. It is much easier, and more common, to apply a step function such as $10\,u(t)$. Then $\mathbf{V}_{\text{in}}(s)$ is $10/s$, and the output can be expressed in the $s$-domain as

$$\mathbf{V}_o(s) = \mathbf{H}(s)\left[\frac{10}{s}\right]$$

or

$$\mathbf{H}(s) = \left[\frac{s}{10}\right]\mathbf{V}_o(s)$$

Since multiplication by $s$ is equivalent to the time derivative, we have for $h(t)$

$$h(t) = \left[\frac{1}{10}\right]\frac{dv_o(t)}{dt} \qquad \textbf{13.29}$$

Thus, $h(t)$ can be obtained from the derivative of the filter response to a step input!

In the laboratory, the input $10\,u(t)$ was applied to the filter and the output voltage was measured using a digital oscilloscope. Data points for time and $v_o(t)$ were acquired every 50 μs over the interval 0 to 50 ms; that is, 1,000 data samples. The digital oscilloscope formats the data as a text file, which can be transferred to a personal computer where the data can be processed. [In other words, we can find our derivative in Eq. (13.29), $dv_o(t)/dt$.] The results are shown in Table 13.3. The second and third columns in the table show the elapsed time and the output voltage for the first few data samples. To produce $h(t)$, the derivative was approximated in software using the simple algorithm,

$$\frac{dv_o(t)}{dt} \approx \frac{\Delta V_o}{\Delta t} = \frac{V_o[(n+1)T_S] - V_o[nT_S]}{T_S}$$

where $T_S$ is the sample time, 50 μs, and $n$ is the sample number. Results for $h(t)$ are shown in the fourth column of the table. At this point, $h(t)$ exists as a table of data points and the filter is now modeled.

To test our model, $h(t)$, we let the function $v_{\text{in}}(t)$ contain a combination of dc and sinusoid components such as

$$v_{\text{in}}(t) = \begin{cases} 1\sin[(2\pi)100t] + 1\sin[(2\pi)1234t] + 4 & 0 \le t < 25\text{ ms} \\ 0 & t \ge 25\text{ ms} \end{cases} \qquad \textbf{13.30}$$

How will the filter perform? What will the output voltage look like? To find out, we must convolve $h(t)$ and $v_{\text{in}}(t)$. A data file for $v_{\text{in}}(t)$ can be created by simply evaluating the function in Eq. (13.30) every 50 μs. This convolution can be performed using any convenient computational method.

**TABLE 13.3** The first five data samples of the step response and the evaluation of $h(t)$

| N | TIME(s) | STEP RESPONSE (V) | $h(t)$ |
|---|---|---|---|
| 0 | 0.00E+00 | 0.00E+00 | 3.02E+02 |
| 1 | 5.00E−05 | 1.51E−01 | 8.98E+02 |
| 2 | 1.00E−04 | 6.00E−01 | 9.72E+02 |
| 3 | 1.50E−04 | 1.09E+00 | 9.56E+02 |
| 4 | 2.00E−04 | 1.56E+00 | 9.38E+02 |