12.2 Sinusoidal Frequency Analysis

Although in specific cases a network operates at only one frequency (e.g., a power system network), in general we are interested in the behavior of a network as a function of frequency. In a sinusoidal steady-state analysis, the network function can be expressed as

$$\mathbf{H}(j\omega) = M(\omega)e^{j\phi(\omega)} \tag{12.4}$$

where $M(\omega) = |\mathbf{H}(j\omega)|$ and $\phi(\omega)$ is the phase. A plot of these two functions, which are commonly called the *magnitude* and *phase characteristics*, displays the manner in which the response varies with the input frequency ω. We will now illustrate the manner in which to perform a frequency-domain analysis by simply evaluating the function at various frequencies within the range of interest.

FREQUENCY RESPONSE USING A BODE PLOT If the network characteristics are plotted on a semilog scale (that is, a linear scale for the ordinate and a logarithmic scale for the abscissa), they are known as *Bode plots* (named after Hendrik W. Bode). This graph is a powerful tool in both the analysis and design of frequency-dependent systems and networks such as filters, tuners, and amplifiers. In using the graph, we plot $20 \log_{10} M(\omega)$ versus $\log_{10}(\omega)$ instead of $M(\omega)$ versus ω. The advantage of this technique is that rather than plotting the characteristic point by point, we can employ straight-line approximations to obtain the characteristic very efficiently. The ordinate for the magnitude plot is the decibel (dB). This unit was originally employed to measure the ratio of powers; that is,

$$\text{number of dB} = 10 \log_{10} \frac{P_2}{P_1} \tag{12.5}$$

If the powers are absorbed by two equal resistors, then

$$\begin{aligned} \text{number of dB} &= 10 \log_{10} \frac{|\mathbf{V}_2|^2/R}{|\mathbf{V}_1|^2/R} = 10 \log_{10} \frac{|\mathbf{I}_2|^2 R}{|\mathbf{I}_1|^2 R} \\ &= 20 \log_{10} \frac{|\mathbf{V}_2|}{|\mathbf{V}_1|} = 20 \log_{10} \frac{|\mathbf{I}_2|}{|\mathbf{I}_1|} \end{aligned} \tag{12.6}$$

The term dB has become so popular that it is now used for voltage and current ratios, as illustrated in Eq. (12.6), without regard to the impedance employed in each case.

In the sinusoidal steady-state case, $\mathbf{H}(j\omega)$ in Eq. (12.3) can be expressed in general as

$$\mathbf{H}(j\omega) = \frac{K_0(j\omega)^{\pm N}(1 + j\omega\tau_1)[1 + 2\zeta_3(j\omega\tau_3) + (j\omega\tau_3)^2]\cdots}{(1 + j\omega\tau_a)[1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2]\cdots} \tag{12.7}$$

Note that this equation contains the following typical factors:

1. A frequency-independent factor $K_0 > 0$
2. Poles or zeros at the origin of the form $j\omega$; that is, $(j\omega)^{+N}$ for zeros and $(j\omega)^{-N}$ for poles
3. Poles or zeros of the form $(1 + j\omega\tau)$
4. Quadratic poles or zeros of the form $1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2$

Taking the logarithm of the magnitude of the function $\mathbf{H}(j\omega)$ in Eq. (12.7) yields

$$\begin{aligned} 20 \log_{10}|\mathbf{H}(j\omega)| = {} & 20 \log_{10} K_0 \pm 20N \log_{10} |j\omega| \\ & + 20 \log_{10} |1 + j\omega\tau_1| \\ & + 20 \log_{10} |1 + 2\zeta_3(j\omega\tau_3) + (j\omega\tau_3)^2| \\ & + \cdots - 20 \log_{10}|1 + j\omega\tau_a| \\ & - 20 \log_{10} |1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2| \cdots \end{aligned} \tag{12.8}$$

Note that we have used the fact that the log of the product of two or more terms is equal to the sum of the logs of the individual terms, the log of the quotient of two terms is equal to the difference of the logs of the individual terms, and $\log_{10} A^n = n \log_{10} A$.

The phase angle for $\mathbf{H}(j\omega)$ is

$$\underline{/\mathbf{H}(j\omega)} = 0 \pm N(90°) + \tan^{-1}\omega\tau_1 + \tan^{-1}\left(\frac{2\zeta_3\omega\tau_3}{1 - \omega^2\tau_3^2}\right) + \cdots - \tan^{-1}\omega\tau_a - \tan^{-1}\left(\frac{2\zeta_b\omega\tau_b}{1 - \omega^2\tau_b^2}\right)\cdots \qquad \textbf{12.9}$$

As Eqs. (12.8) and (12.9) indicate, we will simply plot each factor individually on a common graph and then sum them algebraically to obtain the total characteristic. Let us examine some of the individual terms and illustrate an efficient manner in which to plot them on the Bode diagram.

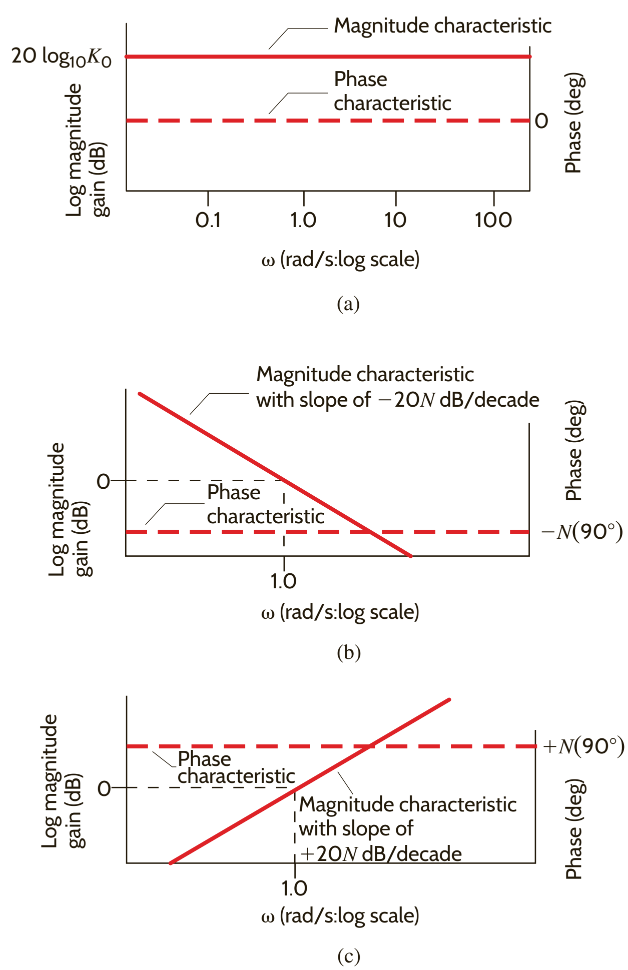
CONSTANT TERM The term $20 \log_{10} K_0$ represents a constant magnitude with zero phase shift, as shown in Fig. 12.10a.

POLES OR ZEROS AT THE ORIGIN Poles or zeros at the origin are of the form $(j\omega)^{\pm N}$, where + is used for a zero and—is used for a pole. The magnitude of this function is $\pm 20N \log_{10}\omega$, which is a straight line on semilog paper with a slope of $\pm 20N$ dB/decade; that is, the value will change by $20N$ each time the frequency is multiplied by 10, and the phase of this function is a constant $\pm N(90°)$. The magnitude and phase characteristics for poles and zeros at the origin are shown in Figs. 12.10b and c, respectively.

SIMPLE POLE OR ZERO Linear approximations can be employed when a simple pole or zero of the form $(1 + j\omega\tau)$ is present in the network function. For $\omega\tau \ll 1$, $(1 + j\omega\tau) \approx 1$ and, therefore, $20 \log_{10}|(1 + j\omega\tau)| = 20 \log_{10} 1 = 0$ dB. Similarly, if $\omega\tau \gg 1$, then $(1 + j\omega\tau) \approx j\omega\tau$,

Figure 12.10

Magnitude and phase characteristics for a constant term and poles and zeros at the origin.



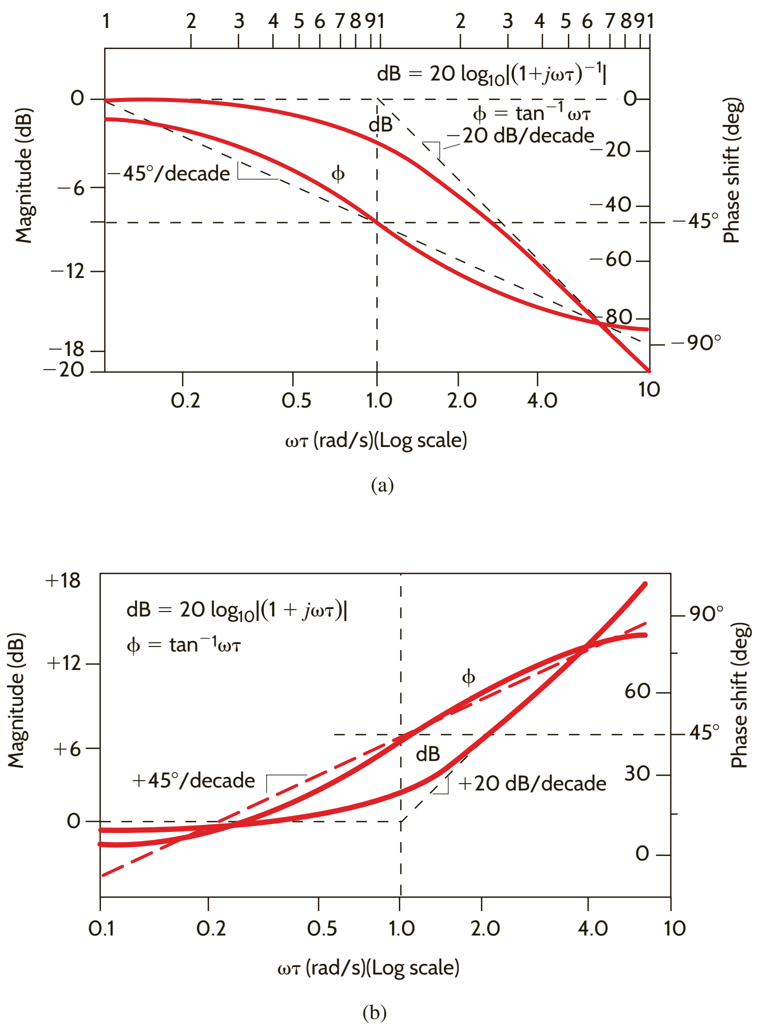


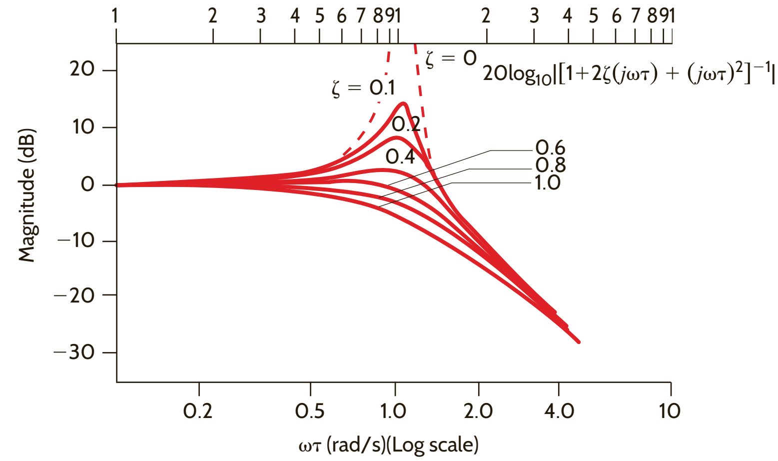
Figure 12.11 Magnitude and phase plot (a) for a simple pole and (b) for a simple zero.

and hence $20 \log_{10}|(1 + j\omega\tau)| \approx 20 \log_{10}\omega\tau$. Therefore, for $\omega\tau \ll 1$ the response is 0 dB and for $\omega\tau \gg 1$ the response has a slope that is the same as that of a simple pole or zero at the origin. The intersection of these two asymptotes, one for $\omega\tau \ll 1$ and one for $\omega\tau \gg 1$, is the point where $\omega\tau = 1$ or $\omega = 1/\tau$, which is called the *break frequency*. At this break frequency, where $\omega = 1/\tau$, $20 \log_{10}|(1 + j1)| = 20 \log_{10}(2)^{1/2} = 3$ dB. Therefore, the actual curve deviates from the asymptotes by 3 dB at the break frequency. It can be shown that at one-half and twice the break frequency, the deviations are 1 dB. The phase angle associated with a simple pole or zero is $\phi = \tan^{-1} \omega\tau$, which is a simple arctangent curve. Therefore, the phase shift is 45° at the break frequency and 26.6° and 63.4° at one-half and twice the break frequency, respectively. The actual magnitude curve for a pole of this form is shown in **Fig. 12.11a**. For a zero the magnitude curve and the asymptote for $\omega\tau \gg 1$ have a positive slope, and the phase curve extends from 0° to +90°, as shown in **Fig. 12.11b**. If multiple poles or zeros of the form $(1 + j\omega\tau)^N$ are present, then the slope of the high-frequency asymptote is multiplied by N, the deviation between the actual curve and the asymptote at the break frequency is $3N$ dB, and the phase curve extends from 0 to $N(90°)$ and is $N(45°)$ at the break frequency.

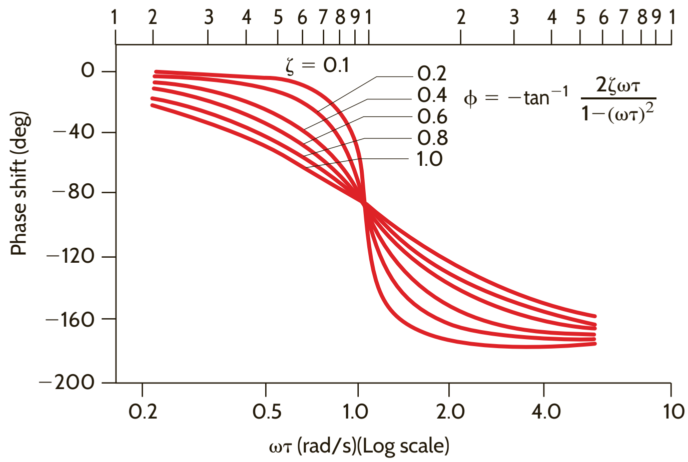
QUADRATIC POLES OR ZEROS Quadratic poles or zeros are of the form $1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2$. This term is a function not only of ω but also of the dimensionless term ζ, which is called the *damping ratio*. If $\zeta > 1$ or $\zeta = 1$, the roots are real and unequal or real and equal, respectively, and these two cases have already been addressed. If $\zeta < 1$, the roots are complex conjugates, and it is this case that we will examine now. Following the preceding argument for a simple pole or zero, the log magnitude of the quadratic factor is 0 dB for $\omega\tau \ll 1$. For $\omega\tau \gg 1$,

$$20 \log_{10}\left|1 - (\omega\tau)^2 + 2j\zeta(\omega\tau)\right| \approx 20 \log_{10}\left|(\omega\tau)^2\right| = 40 \log_{10}|\omega\tau|$$

Figure 12.12
Magnitude and phase characteristics for quadratic poles.



(a)



(b)

and therefore, for $\omega\tau \gg 1$, the slope of the log magnitude curve is +40 dB/decade for a quadratic zero and −40 dB/decade for a quadratic pole. Between the two extremes, $\omega\tau \ll 1$ and $\omega\tau \gg 1$, the behavior of the function is dependent on the damping ratio ζ. **Fig. 12.12a** illustrates the manner in which the log magnitude curve for a quadratic *pole* changes as a function of the damping ratio. The phase shift for the quadratic factor is $\tan^{-1}2\zeta\omega\tau/[1-(\omega\tau)^2]$. The phase plot for quadratic *poles* is shown in **Fig. 12.12b**. Note that in this case the phase changes from 0° at frequencies for which $\omega\tau \ll 1$ to −180° at frequencies for which $\omega\tau \gg 1$. For quadratic zeros the magnitude and phase curves are inverted; that is, the log magnitude curve has a slope of +40 dB/decade for $\omega\tau \gg 1$, and the phase curve is 0° for $\omega\tau \ll 1$ and +180° for $\omega\tau \gg 1$.

EXAMPLE 12.3

We want to generate the magnitude and phase plots for the transfer function

$$\mathbf{G}_v(j\omega) = \frac{10(0.1j\omega + 1)}{(j\omega + 1)(0.02j\omega + 1)}$$

SOLUTION

Note that this function is in standard form, since every term is of the form $(j\omega\tau + 1)$. To determine the composite magnitude and phase characteristics, we will plot the individual asymptotic terms and then add them as specified in Eqs. (12.8) and (12.9). Let us consider the magnitude plot first. Since $K_0 = 10$, $20 \log_{10} 10 = 20$ dB, which is a constant independent of frequency, as shown in **Fig. 12.13a**. The zero of the transfer function contributes a term of the form $+20 \log_{10}|1 + 0.1j\omega|$, which is 0 dB for $0.1\omega \ll 1$, has a slope of +20 dB/decade

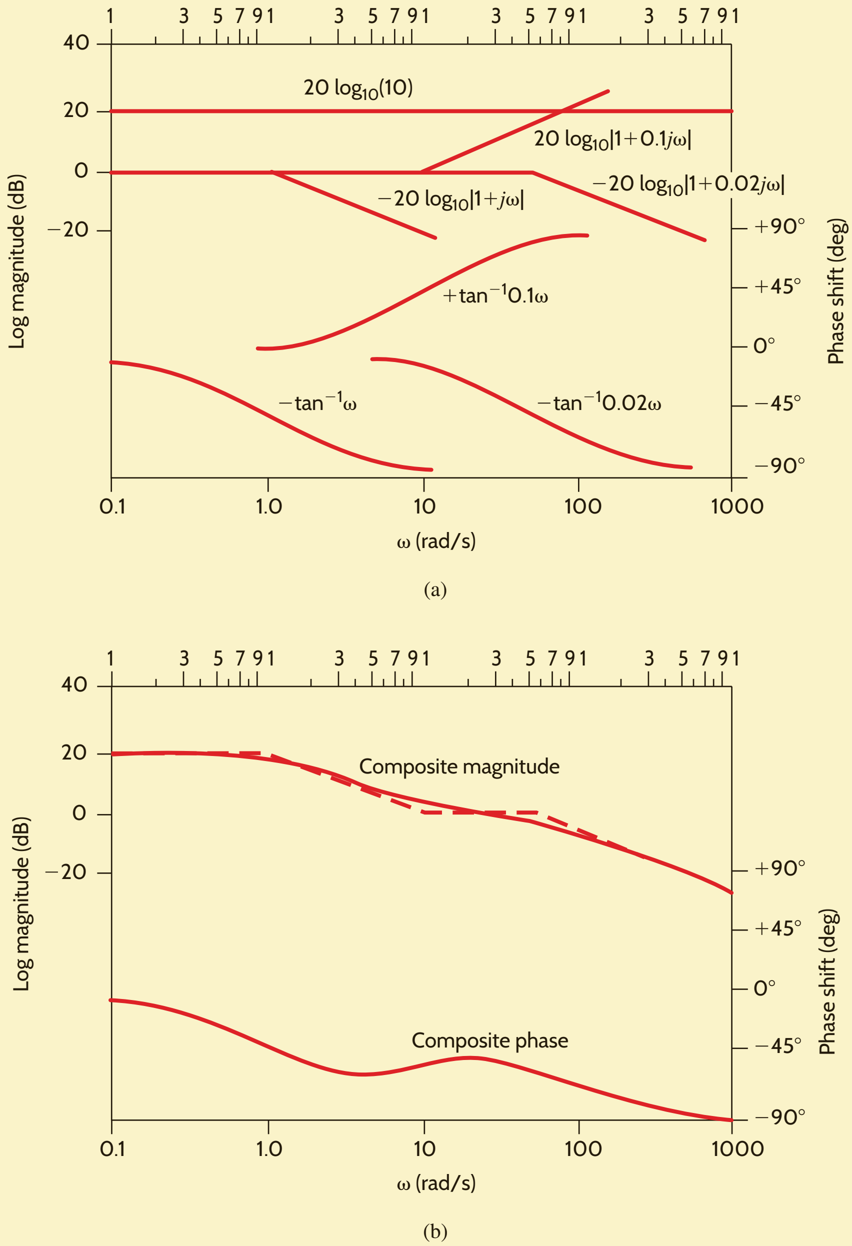


Figure 12.13 (a) Magnitude and phase components for the poles and zeros of the transfer function in Example 12.3; (b) Bode plot for the transfer function in Example 12.3.

for $0.1\omega \gg 1$, and has a break frequency at $\omega = 10$ rad/s. The poles have break frequencies at $\omega = 1$ and $\omega = 50$ rad/s. The pole with a break frequency at $\omega = 1$ rad/s contributes a term of the form $-20 \log_{10}|1 + j\omega|$, which is 0 dB for $\omega \ll 1$ and has a slope of -20 dB/decade for $\omega \gg 1$. A similar argument can be made for the pole that has a break frequency at $\omega = 50$ rad/s. These factors are all plotted individually in Fig. 12.13a.

Consider now the individual phase curves. The term K_0 is not a function of ω and does not contribute to the phase of the transfer function. The phase curve for the zero is $+\tan^{-1} 0.1\omega$, which is an arctangent curve that extends from 0° for $0.1\omega \ll 1$ to $+90°$ for $0.1\omega \gg 1$ and has a phase of $+45°$ at the break frequency. The phase curves for the two poles are $-\tan^{-1}\omega$ and $-\tan^{-1} 0.02\omega$. The term $-\tan^{-1}\omega$ is 0° for $\omega \ll 1$, $-90°$ for $\omega \gg 1$, and $-45°$ at the break frequency $\omega = 1$ rad/s. The phase curve for the remaining pole is plotted in a similar fashion. All the individual phase curves are shown in Fig. 12.13a.

As specified in Eqs. (12.8) and (12.9), the composite magnitude and phase of the transfer function are obtained simply by adding the individual terms. The composite curves are plotted in Fig. 12.13b. Note that the actual magnitude curve (solid line) differs from the straight-line approximation (dashed line) by 3 dB at the break frequencies and 1 dB at one-half and twice the break frequencies.

EXAMPLE 12.4

Let us draw the Bode plot for the following transfer function:

$$\mathbf{G}_v(j\omega) = \frac{25(j\omega + 1)}{(j\omega)^2(0.1j\omega + 1)}$$

SOLUTION

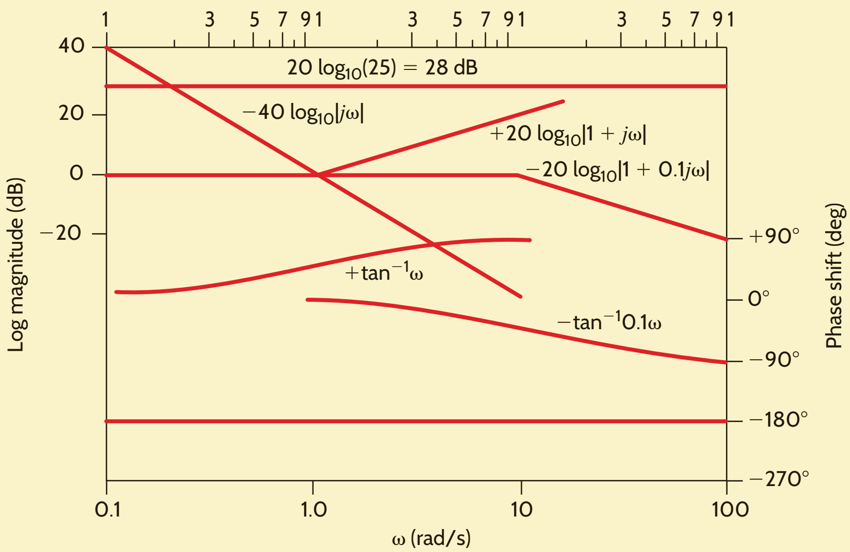
Once again all the individual terms for both magnitude and phase are plotted in **Fig. 12.14a**. The straight line with a slope of -40 dB/decade is generated by the double pole at the origin. This line is a plot of $-40 \log_{10} \omega$ versus ω and therefore passes through 0 dB at $\omega = 1$ rad/s. The phase for the double pole is a constant $-180°$ for all frequencies. The remainder of the terms are plotted as illustrated in Example 12.3.

The composite plots are shown in **Fig. 12.14b**. Once again they are obtained simply by adding the individual terms in Fig. 12.14a. Note that for frequencies for which $\omega \ll 1$, the slope of the magnitude curve is -40 dB/decade. At $\omega = 1$ rad/s, which is the break frequency of the zero, the magnitude curve changes slope to -20 dB/decade. At $\omega = 10$ rad/s, which is the break frequency of the pole, the slope of the magnitude curve changes back to -40 dB/decade.

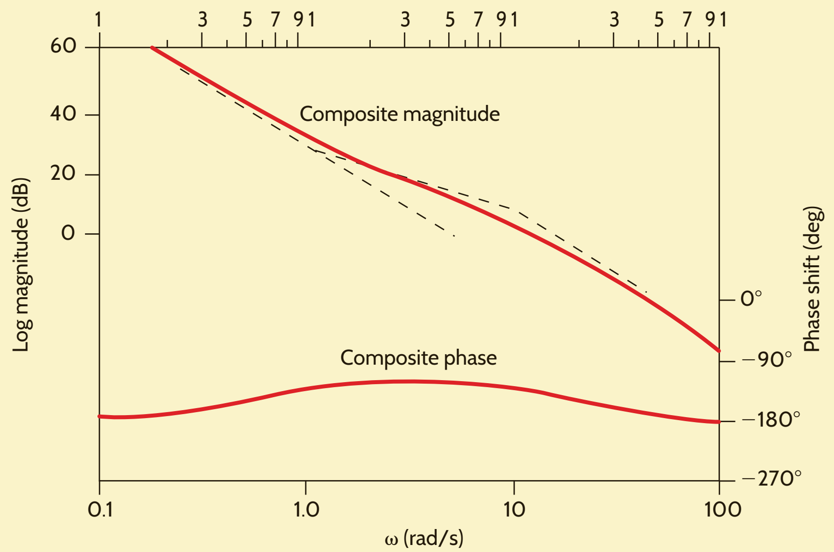
The composite phase curve starts at $-180°$ due to the double pole at the origin. Since the first break frequency encountered is a zero, the phase curve shifts toward $-90°$. However, before the composite phase reaches $-90°$, the pole with break frequency $\omega = 10$ rad/s begins to shift the composite curve back toward $-180°$.

Figure 12.14

(a) Magnitude and phase components for the poles and zeros of the transfer function in Example 12.4;
(b) Bode plot for the transfer function in Example 12.4.



(a)



(b)

Example 12.4 illustrates the manner in which to plot directly terms of the form $K_0/(j\omega)^N$. For terms of this form, the initial slope of $-20N$ dB/decade will intersect the 0-dB axis at a frequency of $(K_0)^{1/N}$ rad/s; that is, $-20\log_{10}|K_0/(j\omega)^N| = 0$ dB implies that $K_0/(j\omega)^N = 1$ and, therefore, $\omega = (K_0)^{1/N}$ rad/s. Note that the projected slope of the magnitude curve in Example 12.4 intersects the 0-dB axis at $\omega = (25)^{1/2} = 5$ rad/s.

Similarly, it can be shown that for terms of the form $K_0(j\omega)^N$, the initial slope of $+20N$ dB/decade will intersect the 0-dB axis at a frequency of $\omega = (1/K_0)^{1/N}$ rad/s; that is, $+20\log_{10}|K_0/(j\omega)^N| = 0$ dB implies that $K_0/(j\omega)^N = 1$ and, therefore, $\omega = (1/K_0)^{1/N}$ rad/s.

By applying the concepts we have just demonstrated, we can normally plot the log magnitude characteristic of a transfer function directly in one step.

LEARNING ASSESSMENTS

E12.4 Sketch the magnitude characteristic of the Bode plot, labeling all critical slopes and points for the function

$$\mathbf{G}(j\omega) = \frac{10^4(j\omega + 2)}{(j\omega + 10)(j\omega + 100)}$$

ANSWER:

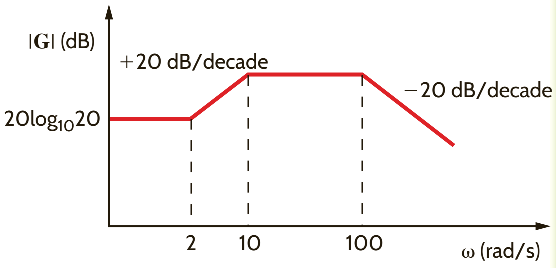


Figure E12.4

E12.5 Sketch the magnitude characteristic of the Bode plot for the transfer function

$$\mathbf{H}(j\omega) = \frac{5(j\omega + 10)}{j\omega(j\omega + 100)}$$

ANSWER:

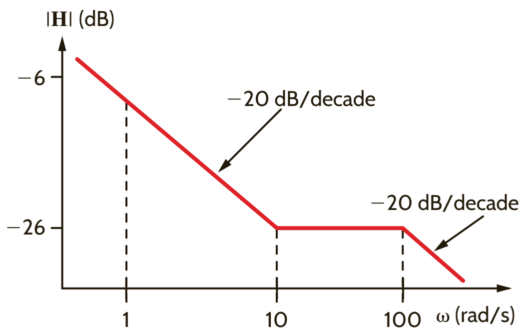


Figure E12.5

E12.6 Sketch the magnitude characteristic of the Bode plot, labeling all critical slopes and points for the function

$$\mathbf{G}(j\omega) = \frac{100(0.02j\omega + 1)}{(j\omega)^2}$$

ANSWER:

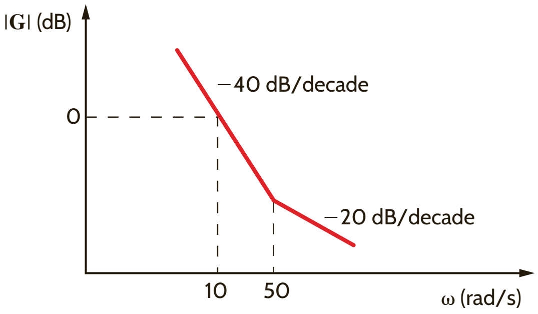


Figure E12.6

E12.7 Sketch the magnitude characteristic of the Bode plot, labeling all critical slopes and points for the function

$$\mathbf{G}(j\omega) = \frac{10j\omega}{(j\omega + 1)(j\omega + 10)}$$

ANSWER:

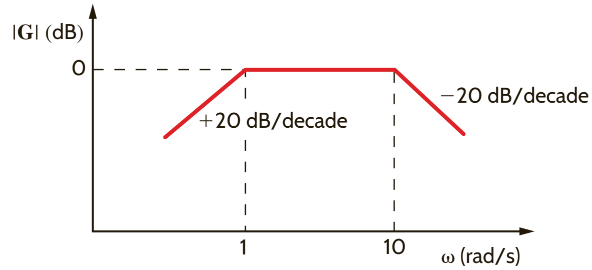


Figure E12.7

EXAMPLE 12.5

We wish to generate the Bode plot for the following transfer function:

$$\mathbf{G}_v(j\omega) = \frac{25j\omega}{(j\omega + 0.5)[(j\omega)^2 + 4j\omega + 100]}$$

SOLUTION

Expressing this function in standard form, we obtain

$$\mathbf{G}_v(j\omega) = \frac{0.5j\omega}{(2j\omega + 1)[(j\omega/10)^2 + j\omega/25 + 1]}$$

The Bode plot is shown in Fig. 12.15. The initial low-frequency slope due to the zero at the origin is +20 dB/decade, and this slope intersects the 0-dB line at $\omega = 1/K_0 = 2$ rad/s. At $\omega = 0.5$ rad/s the slope changes from +20 dB/decade to 0 dB/decade due to the presence of the pole with a break frequency at $\omega = 0.5$ rad/s. The quadratic term has a center frequency of $\omega = 10$ rad/s (i.e., $\tau = 1/10$). Since

$$2\zeta\tau = \frac{1}{25}$$

and

$$\tau = 0.1$$

then

$$\zeta = 0.2$$

Plotting the curve in Fig. 12.12a with a damping ratio of $\zeta = 0.2$ at the center frequency $\omega = 10$ rad/s completes the composite magnitude curve for the transfer function.

The initial low-frequency phase curve is +90° due to the zero at the origin. This curve and the phase curve for the simple pole and the phase curve for the quadratic term, as defined in Fig. 12.12b, are combined to yield the composite phase curve.

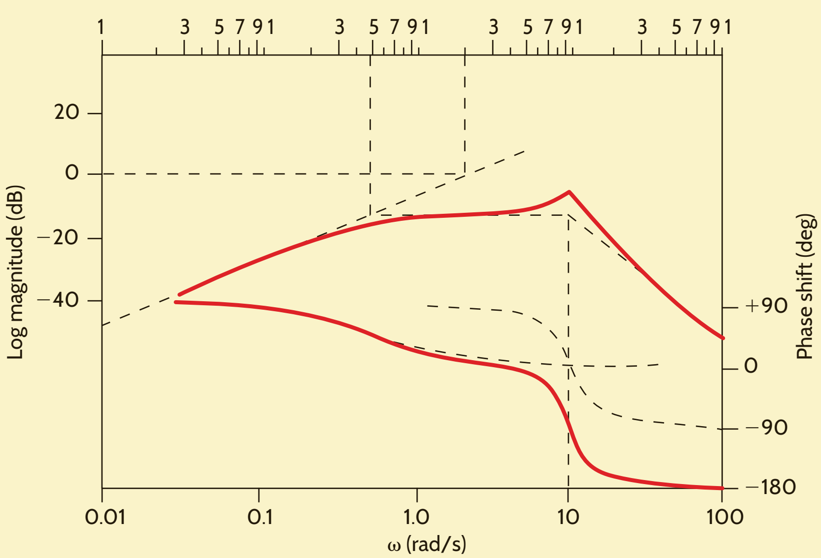


Figure 12.15 Bode plot for the transfer function in Example 12.5.

LEARNING ASSESSMENT

E12.8 Given the following function $\mathbf{G}(j\omega)$, sketch the magnitude characteristic of the Bode plot, labeling all critical slopes and points:

$$\mathbf{G}(j\omega) = \frac{0.2(j\omega + 1)}{j\omega[(j\omega/12)^2 + j\omega/36 + 1]}$$

ANSWER:

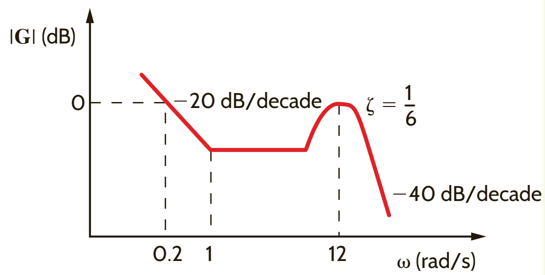


Figure E12.8

DERIVING THE TRANSFER FUNCTION FROM THE BODE PLOT The following example will serve to demonstrate the derivation process.

EXAMPLE 12.6

Given the asymptotic magnitude characteristic shown in Fig. 12.16, we wish to determine the transfer function $\mathbf{G}_v(j\omega)$.

SOLUTION

Since the initial slope is 0 dB/decade, and the level of the characteristic is 20 dB, the factor K_0 can be obtained from the expression

$$20 \text{ dB} = 20 \log_{10} K_0$$

and hence

$$K_0 = 10$$

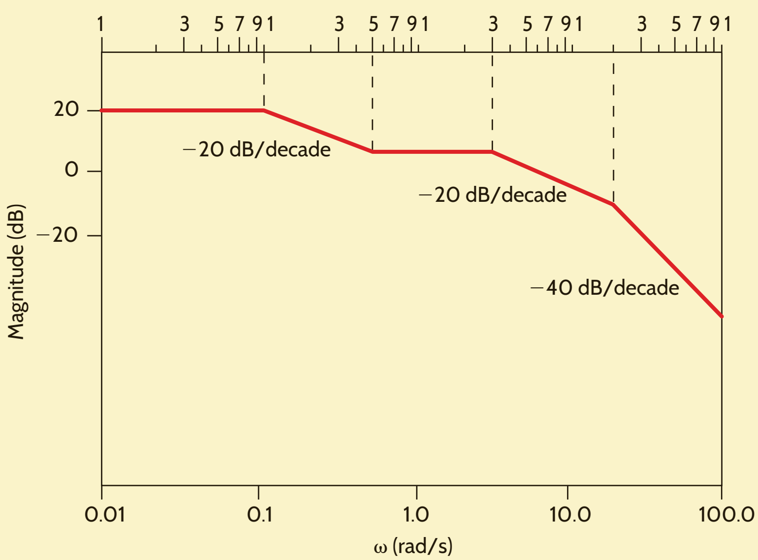


Figure 12.16 Straight-line magnitude plot employed in Example 12.6.

The −20-dB/decade slope starting at $\omega = 0.1$ rad/s indicates that the first pole has a break frequency at $\omega = 0.1$ rad/s and, therefore, one of the factors in the denominator is $(10j\omega + 1)$. The slope changes by +20 dB/decade at $\omega = 0.5$ rad/s, indicating that there is a zero