EXAMPLE 13.7

Let us employ the Laplace transform to solve the equation

$$\frac{dy(t)}{dt} + 2y(t) + \int_0^t y(\lambda)e^{-2(t-\lambda)}d\lambda = 10u(t) \qquad y(0) = 0$$

SOLUTION

Applying property numbers 6 and 10, we obtain

$$s\mathbf{Y}(s) + 2\mathbf{Y}(s) + \frac{\mathbf{Y}(s)}{s+2} = \frac{10}{s}$$

$$\mathbf{Y}(s)\left(s + 2 + \frac{1}{s+2}\right) = \frac{10}{s}$$

$$\mathbf{Y}(s) = \frac{10(s+2)}{s(s^2 + 4s + 5)}$$

This is the solution of the linear constant-coefficient integrodifferential equation in the s-domain. However, we want the solution $y(t)$ in the time domain. $y(t)$ is obtained by performing the inverse transform, which is the topic of the next section, and the solution $y(t)$ is derived in Example 13.9.

LEARNING ASSESSMENTS

E13.3 Find $\mathbf{F}(s)$ if $f(t) = \frac{1}{2}(t - 4e^{-2t})$.

ANSWER: $\mathbf{F}(s) = \frac{1}{2s^2} - \frac{2}{s+2}$.

E13.4 If $f(t) = te^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$, determine $\mathbf{F}(s)$ using the time-shifting theorem.

ANSWER: $\mathbf{F}(s) = \frac{e^{-s}}{(s+1)^2}$.

E13.5 Find $\mathbf{F}(s)$ if $f(t) = e^{-4t}(t - e^{-t})$. Use property number 2.

ANSWER: $\mathbf{F}(s) = \frac{1}{(s+4)^2} - \frac{1}{s+5}$.

13.5 Performing the Inverse Transform

As we begin our discussion of this topic, let us outline the procedure we will use in applying the Laplace transform to circuit analysis. First, we will transform the problem from the time domain to the complex frequency domain (that is, s-domain). Next, we will solve the circuit equations algebraically in the complex frequency domain. Finally, we will transform the solution from the s-domain back to the time domain. It is this latter operation that we discuss now.

The algebraic solution of the circuit equations in the complex frequency domain results in a rational function of s of the form

$$\mathbf{F}(s) = \frac{\mathbf{P}(s)}{\mathbf{Q}(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad \textbf{13.8}$$

The roots of the polynomial $\mathbf{P}(s)$ (i.e., $-z_1, -z_2 \cdots -z_m$) are called the *zeros* of the function $\mathbf{F}(s)$ because at these values of s, $\mathbf{F}(s) = 0$. Similarly, the roots of the polynomial $\mathbf{Q}(s)$ (i.e., $-p_1, -p_2 \cdot\cdot\cdot -p_n$) are called *poles* of $\mathbf{F}(s)$, since at these values of s, $\mathbf{F}(s)$ becomes infinite.

If $\mathbf{F}(s)$ is a proper rational function of s, then $n > m$. However, if this is not the case, we simply divide $\mathbf{P}(s)$ by $\mathbf{Q}(s)$ to obtain a quotient and a remainder; that is,

$$\frac{\mathbf{P}(s)}{\mathbf{Q}(s)} = C_{m-n}s^{m-n} + \cdots + C_2 s^2 + C_1 s + C_0 + \frac{\mathbf{P}_1(s)}{\mathbf{Q}(s)} \qquad \textbf{13.9}$$

Now $\mathbf{P}_1(s)/\mathbf{Q}(s)$ is a proper rational function of s. Let us examine the possible forms of the roots of $\mathbf{Q}(s)$:

1. If the roots are simple, $\mathbf{P}_1(s)/\mathbf{Q}(s)$ can be expressed in partial fraction form as

$$\frac{\mathbf{P}_1(s)}{\mathbf{Q}(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n} \qquad \textbf{13.10}$$

2. If $\mathbf{Q}(s)$ has simple complex roots, they will appear in complex-conjugate pairs, and the partial fraction expansion of $\mathbf{P}_1(s)/\mathbf{Q}(s)$ for each pair of complex-conjugate roots will be of the form

$$\frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{K_1}{s+\alpha-j\beta} + \frac{K_1^*}{s+\alpha+j\beta} + \cdots \qquad \textbf{13.11}$$

where $\mathbf{Q}(s) = \mathbf{Q}_1(s)(s + a - j\beta)(s + \alpha + j\beta)$ and K_1^* in the complex conjugate of K_1.

3. If $\mathbf{Q}(s)$ has a root of multiplicity r, the partial fraction expansion for each such root will be of the form

$$\frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+p_1)^r} = \frac{K_{11}}{(s+p_1)} + \frac{K_{12}}{(s+p_1)^2} + \cdots + \frac{K_{1r}}{(s+p_1)^r} + \cdots \qquad \textbf{13.12}$$

The importance of these partial fraction expansions stems from the fact that once the function $\mathbf{F}(s)$ is expressed in this form, the individual inverse Laplace transforms can be obtained from known and tabulated transform pairs. The sum of these inverse Laplace transforms then yields the desired time function, $f(t) = \mathscr{L}^{-1}[\mathbf{F}(s)]$.

SIMPLE POLES Let us assume that all the poles of $\mathbf{F}(s)$ are simple, so that the partial fraction expansion of $\mathbf{F}(s)$ is of the form

$$\mathbf{F}(s) = \frac{\mathbf{P}(s)}{\mathbf{Q}(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n} \qquad \textbf{13.13}$$

Then the constant K_i can be computed by multiplying both sides of this equation by $(s + p_i)$ and evaluating the equation at $s = -p_i$; that is,

$$\left.\frac{(s+p_i)\mathbf{P}(s)}{\mathbf{Q}(s)}\right|_{s=-p_i} = 0 + \cdots + 0 + K_i + 0 + \cdots + 0 \qquad i = 1, 2, \ldots, n \qquad \textbf{13.14}$$

Once all of the K_i terms are known, the time function $f(t) = \mathscr{L}^{-1}[\mathbf{F}(s)]$ can be obtained using the Laplace transform pair:

$$\mathscr{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at} \qquad \textbf{13.15}$$

EXAMPLE 13.8

Given that

$$\mathbf{F}(s) = \frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)}$$

let us find the function $f(t) = \mathscr{L}^{-1}[\mathbf{F}(s)]$.

SOLUTION Expressing $\mathbf{F}(s)$ in a partial fraction expansion, we obtain

$$\frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

To determine K_0, we multiply both sides of the equation by s to obtain the equation

$$\frac{12(s+1)(s+3)}{(s+2)(s+4)(s+5)} = K_0 + \frac{K_1 s}{s+2} + \frac{K_2 s}{s+4} + \frac{K_3 s}{s+5}$$

Evaluating the equation at $s = 0$ yields

$$\frac{(12)(1)(3)}{(2)(4)(5)} = K_0 + 0 + 0 + 0$$

or

$$K_0 = \frac{36}{40}$$

Similarly,

$$(s+2)\mathbf{F}(s)\Big|_{s=-2} = \frac{12(s+1)(s+3)}{s(s+4)(s+5)}\Big|_{s=-2} = K_1$$

or

$$K_1 = 1$$

Using the same approach, we find that $K_2 = \frac{36}{8}$ and $K_3 = -\frac{32}{5}$. Hence, $\mathbf{F}(s)$ can be written as

$$\mathbf{F}(s) = \frac{36/40}{s} + \frac{1}{s+2} + \frac{36/8}{s+4} - \frac{32/5}{s+5}$$

Then $f(t) = \mathscr{L}^{-1}[\mathbf{F}(s)]$ is

$$f(t) = \left(\frac{36}{40} + 1e^{-2t} + \frac{36}{8}e^{-4t} - \frac{32}{5}e^{-5t}\right)u(t)$$

LEARNING ASSESSMENTS

E13.6 Find $f(t)$ if $\mathbf{F}(s) = 10(s+6)/(s+1)(s+3)$.

ANSWER:
$f(t) = (25e^{-t} - 15e^{-3t})u(t)$.

E13.7 If $\mathbf{F}(s) = 12(s+2)/s(s+1)$, find $f(t)$.

ANSWER:
$f(t) = (24 - 12e^{-t})u(t)$.

E13.8 Given $\mathbf{F}(s) = \dfrac{s^2 + 5s + 1}{s(s+1)(s+4)}$, find $f(t)$.

ANSWER:
$f(t) = (0.25 + e^{-t} - 0.25e^{-4t})u(t)$.

COMPLEX-CONJUGATE POLES Let us assume that $\mathbf{F}(s)$ has one pair of complex-conjugate poles. The partial fraction expansion of $\mathbf{F}(s)$ can then be written as

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{K_1}{s+\alpha-j\beta} + \frac{K_1^*}{s+\alpha+j\beta} + \cdots \quad \textbf{13.16}$$

The constant K_1 can then be determined using the procedure employed for simple poles; that is,

$$(s+\alpha-j\beta)\mathbf{F}(s)\Big|_{s=-\alpha+j\beta} = K_1 \quad \textbf{13.17}$$

In this case K_1 is in general a complex number that can be expressed as $|K_1| \underline{/\theta}$. Then $K_1^* = |K_1| \underline{/-\theta}$. Hence, the partial fraction expansion can be expressed in the form

$$\mathbf{F}(s) = \frac{|K_1| \underline{/\theta}}{s + \alpha - j\beta} + \frac{|K_1| \underline{/-\theta}}{s + \alpha - j\beta} + \cdots \qquad \textbf{13.18}$$

$$= \frac{|K_1| e^{j\theta}}{s + \alpha - j\beta} + \frac{|K_1| e^{-j\theta}}{s + \alpha + j\beta} + \cdots$$

HINT

Recall that

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

The corresponding time function is then of the form

$$f(t) = \mathscr{L}^{-1}[\mathbf{F}(s)] = |K_1| e^{j\theta} e^{-(\alpha - j\beta)t} + |K_1| e^{-j\theta} e^{-(\alpha + j\beta)t} + \cdots$$

$$= |K_1| e^{-\alpha t}\left[e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}\right] + \cdots \qquad \textbf{13.19}$$

$$= 2|K_1| e^{-\alpha t} \cos(\beta t + \theta) + \cdots$$

EXAMPLE 13.9

Let us determine the time function $y(t)$ for the function

$$\mathbf{Y}(s) = \frac{10(s + 2)}{s\left(s^2 + 4s + 5\right)}$$

SOLUTION

Expressing the function in a partial fraction expansion, we obtain

$$\frac{10(s + 2)}{s(s + 2 - j1)(s + 2 + j1)}$$

$$= \frac{K_0}{s} + \frac{K_1}{s + 2 - j1} + \frac{K_1^*}{s + 2 - j1}$$

$$\left.\frac{10(s + 2)}{s^2 + 4s + 5}\right|_{s = 0} = K_0$$

$$4 = K_0$$

In a similar manner,

$$\left.\frac{10(s + 2)}{s(s + 2 + j1)}\right|_{s = -2 + j1} = K_1$$

$$2.236 \underline{/-153.43^\circ} = K_1$$

Therefore,

$$2.236 \underline{/153.43^\circ} = K_1^*$$

The partial fraction expansion of $\mathbf{Y}(s)$ is then

$$\mathbf{Y}(s) = \frac{4}{s} + \frac{2.236 \underline{/-153.43^\circ}}{s + 2 - j1} + \frac{2.236 \underline{/153.43^\circ}}{s + 2 + j1}$$

and therefore,

$$y(t) = [4 + 4.472 e^{-2t} \cos(t - 153.43^\circ)]\, u(t)$$

LEARNING ASSESSMENTS

E13.9 Determine $f(t)$ if $\mathbf{F}(s) = s/(s^2 + 4s + 8)$.

ANSWER: $f(t) = 1.41 e^{-2t} \cos(2t + 45^\circ) u(t)$.

E13.10 Given $\mathbf{F}(s) = \dfrac{4(s + 3)}{(s + 1)(s^2 + 2s + 5)}$, find $f(t)$.

ANSWER: $f(t) = (2e^{-t} + 2\sqrt{2}\, e^{-t} \cos(2t - 135^\circ)) u(t)$.

MULTIPLE POLES Let us suppose that $\mathbf{F}(s)$ has a pole of multiplicity r. Then $\mathbf{F}(s)$ can be written in a partial fraction expansion of the form

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s + p_1)^r}$$

$$= \frac{K_{11}}{s + p_1} + \frac{K_{12}}{(s + p_1)^2} + \cdots + \frac{K_{1r}}{(s + p_1)^r} + \cdots \qquad \textbf{13.20}$$

Employing the approach for a simple pole, we can evaluate K_{1_r} as

$$(s + p_1)^r \mathbf{F}(s)\Big|_{s = -p_1} = K_{1_r} \qquad \textbf{13.21}$$

To evaluate $K_{1_{r-1}}$ we multiply $\mathbf{F}(s)$ by $(s + p_1)^r$ as we did to determine K_{1_r}; however, prior to evaluating the equation at $s = -p_1$, we take the derivative with respect to s. The proof that this will yield $K_{1_{r-1}}$ can be obtained by multiplying both sides of Eq. (13.20) by $(s + p_1)^r$ and then taking the derivative with respect to s. Now when we evaluate the equation at $s = -p_1$, the only term remaining on the right side of the equation is $K_{1_{r-1}}$, and therefore,

$$\frac{d}{ds}[(s + p_1)^r \mathbf{F}(s)]\Big|_{s = -p_1} = K_{1_{r-1}} \qquad \textbf{13.22}$$

$K_{1_{r-2}}$ can be computed in a similar fashion, and in that case the equation is

$$\frac{d^2}{ds^2}[(s + p_1)^r \mathbf{F}(s)]\Big|_{s = -p_1} = (2!)K_{1_{r-2}} \qquad \textbf{13.23}$$

The general expression for this case is

$$K_{1j} = \frac{1}{(r - j)!} \frac{d^{r-j}}{ds^{r-j}}[(s + p_1)^r \mathbf{F}(s)]\Big|_{s = -p_1} \qquad \textbf{13.24}$$

Let us illustrate this procedure with an example.

EXAMPLE 13.10

Given the following function $\mathbf{F}(s)$, let us determine the corresponding time function $f(t) = \mathcal{L}^{-1}[\mathbf{F}(s)]$.

$$\mathbf{F}(s) = \frac{10(s + 3)}{(s + 1)^3(s + 2)}$$

SOLUTION

Expressing $\mathbf{F}(s)$ as a partial fraction expansion, we obtain

$$\mathbf{F}(s) = \frac{10(s + 3)}{(s + 1)^3(s + 2)}$$

$$= \frac{K_{11}}{s + 1} + \frac{K_{12}}{(s + 1)^2} + \frac{K_{13}}{(s + 1)^3} + \frac{K_2}{s + 2}$$

Then

$$(s + 1)^3 \mathbf{F}(s)\Big|_{s = -1} = K_{13}$$

$$20 = K_{13}$$

K_{12} is now determined by the equation

$$\frac{d}{ds}[(s + 1)^3 \mathbf{F}(s)]\Big|_{s = -1} = K_{12}$$

$$\frac{-10}{(s + 2)^2}\Big|_{s = -1} = -10 = K_{12}$$

In a similar fashion, K_{11} is computed from the equation

$$\frac{d^2}{ds^2}\left[(s+1)^3\mathbf{F}(s)\right]\bigg|_{s=-1} = 2K_{11}$$

$$\frac{20}{(s+2)^3}\bigg|_{s=-1} = 20 = 2K_{11}$$

Therefore,

$$10 = K_{11}$$

In addition,

$$(s+2)\mathbf{F}(s)\bigg|_{s=-2} = K_2$$

$$-10 = K_2$$

Hence, $\mathbf{F}(s)$ can be expressed as

$$\mathbf{F}(s) = \frac{10}{s+1} - \frac{10}{(s+1)^2} + \frac{20}{(s+1)^3} - \frac{10}{s+2}$$

Now we employ the transform pair

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{t^n}{n!}e^{-at}$$

and hence,

$$f(t) = (10e^{-t} - 10te^{-t} + 10t^2e^{-t} - 10e^{-2t})u(t)$$

LEARNING ASSESSMENTS

E13.11 Determine $f(t)$ if $\mathbf{F}(s) = s/(s+1)^2$.

ANSWER: $f(t) = (e^{-t} - te^{-t})u(t)$.

E13.12 If $\mathbf{F}(s) = (s+2)/s^2(s+1)$, find $f(t)$.

ANSWER: $f(t) = (-1 + 2t + e^{-t})u(t)$.

E13.13 Given $\mathbf{F}(s) = \dfrac{100}{s^3(s+5)}$, find $f(t)$.

ANSWER: $f(t) = (0.8 - 4t + 10t^2 - 0.8e^{-5t})u(t)$.

Back in Chapter 7 we discussed the characteristic equation for a second-order transient circuit. The polynomial $\mathbf{Q}(s) = 0$ is the characteristic equation for our circuit. The roots of the characteristic equation, also called the poles of $\mathbf{F}(s)$, determine the time response for our circuit. If $\mathbf{Q}(s) = 0$ has simple roots, then the time response will be characterized by decaying exponential functions. Multiple roots produce a time response that contains decaying exponential terms such as e^{-at}, te^{-at}, and t^2e^{-at}. The time response for simple complex-conjugate roots is a sinusoidal function whose amplitude decays exponentially. Note that all of these time responses decay to zero with time. Suppose our circuit response contained a term such as $3e^{2t}$. A quick plot of this function reveals that it increases without bound for $t > 0$. Certainly, if our circuit was characterized by this type of response, we would need eye protection as our circuit destructed before us!