10.3 The Ideal Transformer

Consider the situation illustrated in Fig. 10.17, showing two coils of wire wound around a single closed magnetic core. Assume a core flux ϕ, which links all the turns of both coils. In the ideal case, we also neglect wire resistance. Let us now examine the coupling equations under the condition that the same flux goes through each winding, and so

$$v_1(t) = N_1 \frac{d\phi}{dt}$$

and

$$v_2(t) = N_2 \frac{d\phi}{dt}$$

and therefore,

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \frac{\frac{d\phi}{dt}}{\frac{d\phi}{dt}} = \frac{N_1}{N_2} \qquad \textbf{10.24}$$

Ampère's law requires that

$$\oint H \cdot dl = i_{\text{enclosed}} = N_1 i_1 + N_2 i_2 \qquad \textbf{10.25}$$

where H is the magnetic field intensity and the integral is over the closed path traveled by the flux around the transformer core. If $H = 0$, which is the case for an ideal magnetic core with infinite permeability, then

$$N_1 i_1 + N_2 i_2 = 0 \qquad \textbf{10.26}$$

or

$$\frac{i_1}{i_2} = -\frac{N_2}{N_1} \qquad \textbf{10.27}$$

Note that if we divide Eq. (10.26) by N_1 and multiply it by v_1, we obtain

$$v_1 i_1 + \frac{N_2}{N_1} v_1 i_2 = 0$$

However, since $v_1 = (N_1/N_2)v_2$,

$$v_1 i_1 + v_2 i_2 = 0$$

and hence the total power into the device is zero, which means that an ideal transformer is lossless.

The symbol we employ for the ideal transformer is shown in Fig. 10.18a, and the corresponding equations are

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}$$
$$N_1 i_1 + N_2 i_2 = 0 \qquad \textbf{10.28}$$

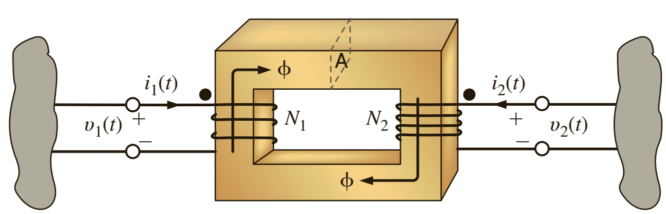

Figure 10.17 Transformer employing a magnetic core.

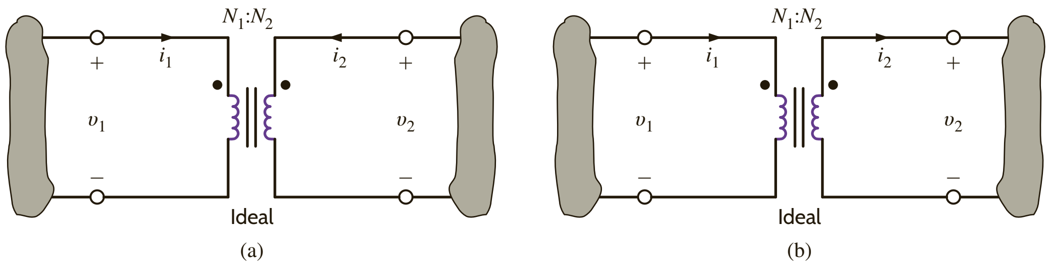

Figure 10.18

Symbol for an ideal transformer: (a) primary and secondary currents into the dots; (b) primary current into, and secondary current out of, the dots.

The normal power flow through a transformer occurs from an input current (i_1) on the primary to an output current (i_2) on the secondary. This situation is shown in **Fig. 10.18b**, and the corresponding equations are

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}$$
$$N_1 i_1 = N_2 i_2 \qquad \textbf{10.29}$$

Note that although the voltage, current, and impedance levels change through a transformer, the power levels do not. The vertical lines between the coils, shown in the figures, represent the magnetic core. Although practical transformers do not use dots per se, they use markings specified by the National Electrical Manufacturers Association (NEMA) that are conceptually equivalent to the dots.

Thus, our model for the ideal transformer is specified by the circuit in Fig. 10.18a and the corresponding Eq. (10.28), or alternatively, by the circuit in Fig. 10.18b, together with Eq. (10.29). Therefore, it is important to note carefully that our model specifies the equations as well as the relationship among the voltages, currents, and the position of the dots. In other words, the equations are valid only for the corresponding circuit diagram. Thus, in a direct analogy to our discussion of the mutual inductance equations and their corresponding circuit, if we change the direction of the current or voltage or the position of the dots, we must make a corresponding change in the equations. The following material will clarify this critical issue.

Consider now the circuit shown in **Fig. 10.19**. If we compare this circuit to that shown in Fig. 10.18b, we find that the direction of both the currents and voltages is the same. Hence, the equations for the network are

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1}{N_2}$$

and

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_2}{N_1}$$

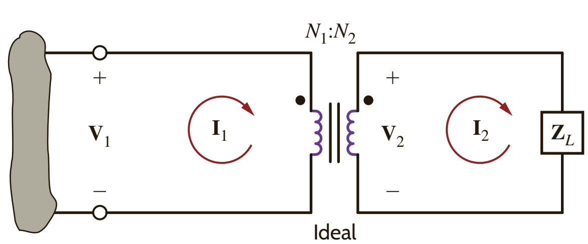

Figure 10.19

Ideal transformer circuit used to illustrate input impedance.

These equations can be written as

$$\mathbf{V}_1 = \frac{N_1}{N_2}\mathbf{V}_2 \qquad \mathbf{10.30}$$

$$\mathbf{I}_1 = \frac{N_2}{N_1}\mathbf{I}_2$$

Also note that

$$\mathbf{Z}_L = \frac{\mathbf{V}_2}{\mathbf{I}_2}$$

and therefore, the input impedance is

$$\mathbf{Z}_1 = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \left(\frac{N_1}{N_2}\right)^2 \mathbf{Z}_L \qquad \mathbf{10.31}$$

where $\mathbf{Z}_L$ is reflected into the primary side by the turns ratio.

If we now define the turns ratio as

$$n = \frac{N_2}{N_1} \qquad \mathbf{10.32}$$

then the defining equations for the *ideal transformer* in this configuration are

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{n}$$

$$\mathbf{I}_1 = n\mathbf{I}_2 \qquad \mathbf{10.33}$$

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_L}{n^2}$$

Care must be exercised in using these equations because the signs on the voltages and currents are dependent on the assigned references and their relationship to the dots.

EXAMPLE 10.8

Given the circuit shown in Fig. 10.20, we wish to determine all indicated voltages and currents.

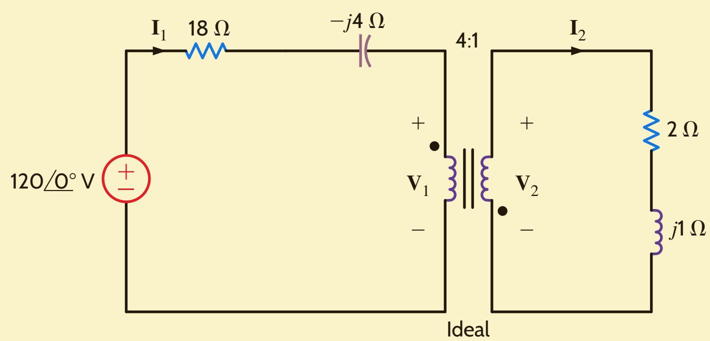

Figure 10.20
Ideal transformer circuit.

SOLUTION

Because of the relationships between the dots and the currents and voltages, the transformer equations are

$$\mathbf{V}_1 = -\frac{\mathbf{V}_2}{n} \quad \text{and} \quad \mathbf{I}_1 = -n\mathbf{I}_2$$

where $n = 1/4$. The reflected impedance at the input to the transformer is

$$\mathbf{Z}_1 = 4^2\mathbf{Z}_L = 16(2 + j1) = 32 + j16\ \Omega$$

Therefore, the current in the source is

$$\mathbf{I}_1 = \frac{120\underline{/0^\circ}}{18 - j4 + 32 + j16} = 2.33\underline{/-13.5^\circ}\text{A}$$

The voltage across the input to the transformer is then

$$\begin{aligned}\mathbf{V}_1 &= \mathbf{I}_1\mathbf{Z}_1\\ &= (2.33\underline{/-13.5^\circ})(32 + j16)\\ &= 83.49\underline{/13.07^\circ}\text{ V}\end{aligned}$$

Hence, $\mathbf{V}_2$ is

$$\begin{aligned}\mathbf{V}_2 &= -n\mathbf{V}_1\\ &= -\frac{1}{4}(83.49\underline{/13.07^\circ})\\ &= 20.87\underline{/193.07^\circ}\text{ V}\end{aligned}$$

The current $\mathbf{I}_2$ is

$$\begin{aligned}\mathbf{I}_2 &= -\frac{\mathbf{I}_1}{n}\\ &= -4(2.33\underline{/-13.5^\circ})\\ &= 9.33\underline{/166.50^\circ}\text{A}\end{aligned}$$

LEARNING ASSESSMENTS

E10.9 Compute the current $\mathbf{I}_1$ in the network in Fig. E10.9.

ANSWER:
$\mathbf{I}_1 = 3.07\underline{/39.81^\circ}$ A.

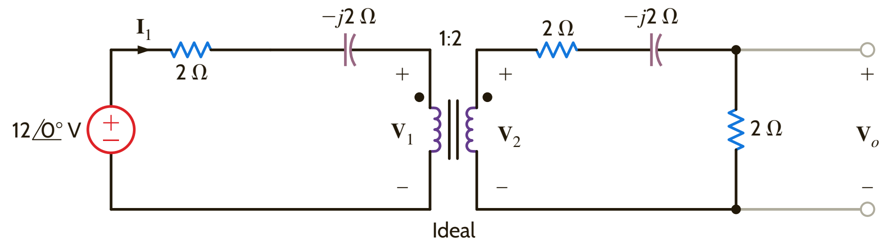

Figure E10.9

E10.10 Find $\mathbf{V}_o$ in the network in Fig. E10.9.

ANSWER:
$\mathbf{V}_o = 3.07\underline{/39.81^\circ}$ V.

E10.11 Determine $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{V}_1$, and $\mathbf{V}_2$ in Fig. E10.11.

ANSWER:
$\mathbf{I}_1 = 2.12\underline{/-25.6^\circ}$ A;
$\mathbf{I}_2 = 8.49\underline{/154.42^\circ}$ A;
$\mathbf{V}_1 = 64.16\underline{/44.1^\circ}$ V;
$\mathbf{V}_2 = 16.04\underline{/-135.9^\circ}$ V.

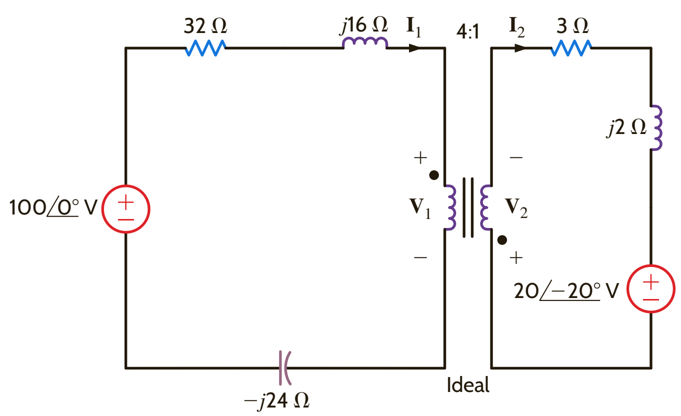

Figure E10.11

E10.12 Determine $\mathbf{V}_o$ in Fig. E10.12.

ANSWER:
$\mathbf{V}_o = 24.95\underline{/-62.65^\circ}$ V.

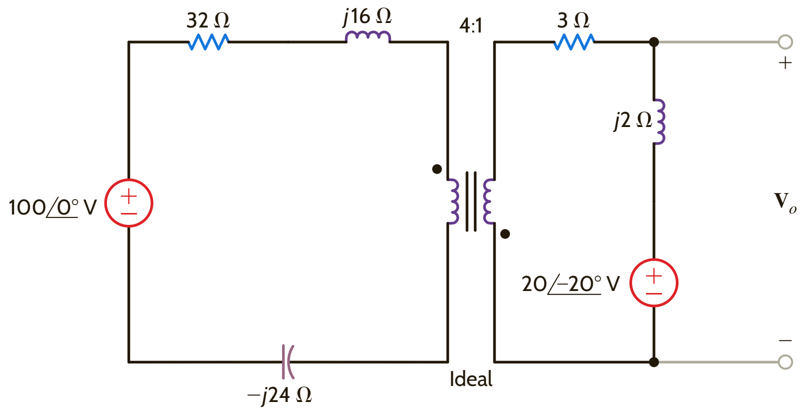

Figure E10.12

E10.13 Determine $\mathbf{V}_o$ in Fig. E10.13.

ANSWER:
$\mathbf{V}_o = 93.68\underline{/-83^\circ}$ V.

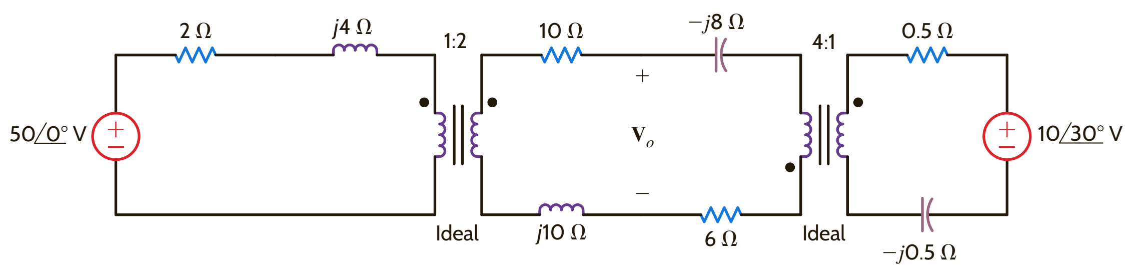

Figure E10.13

E10.14 If $\mathbf{V}_o = 10\underline{/30^\circ}$V in Fig. E10.14, find $\mathbf{V}_S$.

ANSWER:
$\mathbf{V}_S = 32.34\underline{/-125.3^\circ}$ V.

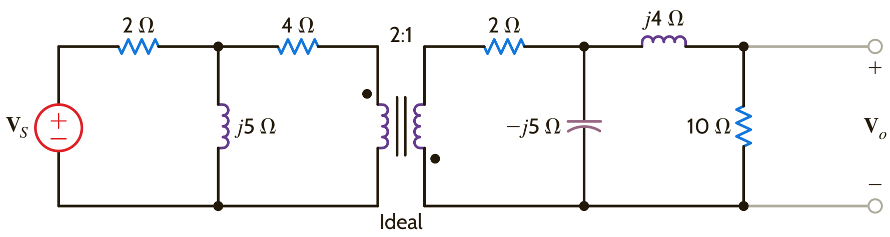

Figure E10.14

EXAMPLE 10.9

The local transformer in Fig. 10.21 provides the last voltage stepdown in a power distribution system. A common sight on utility poles in residential areas, it is a single-phase transformer that typically has a 13.8-kV rms line to neutral on its primary coil, and a center tap secondary coil provides both 120 V rms and 240 V rms to service several residences. A typical example of this transformer, often referred to as a *pole pig*, is shown in Fig. 10.22.

Let us find the turns ratio necessary to produce the 240-V rms secondary voltage. Assuming that the transformer provides 200-A rms service to each of 10 houses, let us determine the minimum power rating for the transformer and the maximum current in the primary.

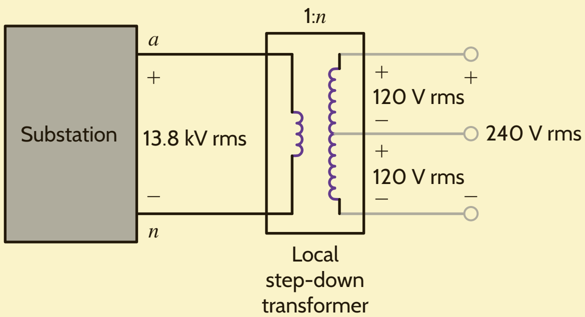

Figure 10.21
Local transformer subcircuit with center tap.

SOLUTION

The turns ratio is given by

$$n = \frac{V_2}{V_1} = \frac{240}{13{,}800} = \frac{1}{57.5}$$

If I_H is the maximum current per household, then the maximum primary current is

$$I_1 = nI_2 = n(10I_H) = 34.78 \text{ A rms}$$

The maximum power delivered to the primary is then

$$S_1 = V_1 I_1 = (13{,}800)(34.78) = 480 \text{ kVA}$$

Therefore, the transformer must have a power rating of at least 480 kVA.

Figure 10.22
A residential utility transformer (©tomba/iStockphoto).

Another technique for simplifying the analysis of circuits containing an ideal transformer involves the use of either Thévenin's or Norton's theorem to obtain an equivalent circuit that replaces the transformer and either the primary or secondary circuit. This technique usually requires more effort, however, than the approach presented thus far. Let us demonstrate this approach by employing Thévenin's theorem to derive an equivalent circuit for the transformer and primary circuit of the network shown in **Fig. 10.23a**. The equations for the transformer in view of the direction of the currents and voltages and the position of the dots are

$$\mathbf{I}_1 = n\mathbf{I}_2$$

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{n}$$

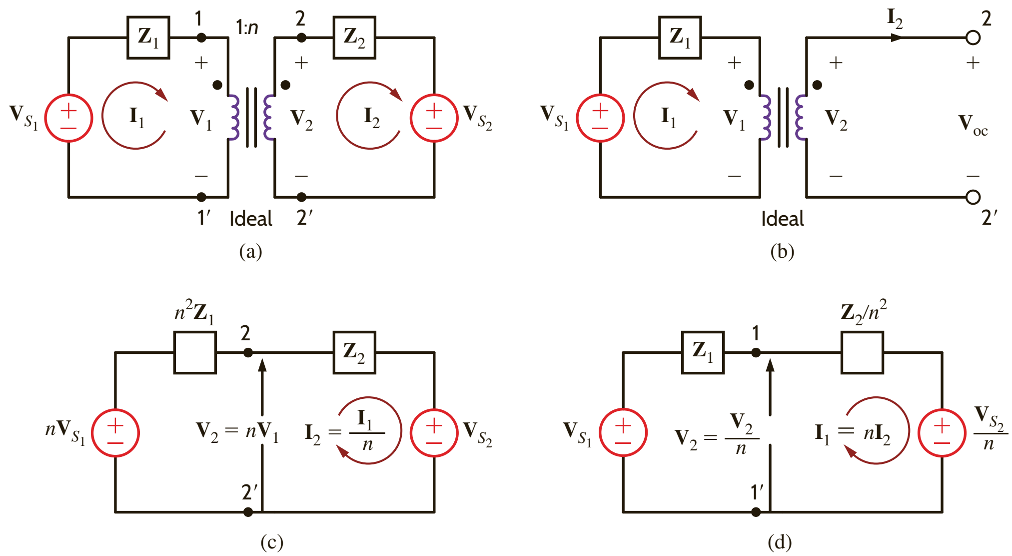

Figure 10.23

Circuit containing an ideal transformer and some of its equivalent networks.

Forming a Thévenin equivalent at the secondary terminals $2 = 2'$, as shown in **Fig. 10.23b**, we note that $\mathbf{I}_2 = 0$ and therefore $\mathbf{I}_1 = 0$. Hence,

$$\mathbf{V}_{\text{oc}} = \mathbf{V}_2 = n\mathbf{V}_1 = n\mathbf{V}_{S_1}$$

The Thévenin equivalent impedance obtained by looking into the open-circuit terminals with $\mathbf{V}_{S_1}$ replaced by a short circuit is $\mathbf{Z}_1$, which when reflected into the secondary by the turns ratio is

$$\mathbf{Z}_{\text{Th}} = n^2\mathbf{Z}_1$$

Therefore, one of the resulting equivalent circuits for the network in Fig. 10.23a is as shown in **Fig. 10.23c**. In a similar manner, we can show that replacing the transformer and its secondary circuit by an equivalent circuit results in the network shown in **Fig. 10.23d**.

It can be shown in general that when developing an equivalent circuit for the transformer and its primary circuit, each primary voltage is multiplied by n, each primary current is divided by n, and each primary impedance is multiplied by n^2. Similarly, when developing an equivalent circuit for the transformer and its secondary circuit, each secondary voltage is divided by n, each secondary current is multiplied by n, and each secondary impedance is divided by n^2. Powers are the same, whether calculated on the primary or secondary side.

Recall from our previous analysis that if either dot on the transformer is reversed, then n is replaced by $-n$ in the equivalent circuits. In addition, note that the development of these equivalent circuits is predicated on the assumption that removing the transformer will divide the network into two parts; that is, there are no connections between the primary and secondary other than through the transformer. If any external connections exist, the equivalent circuit technique cannot in general be used. Finally, we point out that if the primary or secondary circuits are more complicated than those shown in Fig. 10.23a, Thévenin's theorem may be applied to reduce the network to that shown in Fig. 10.23a. Also, we can simply reflect the complicated circuit component by component from one side of the transformer to the other.

EXAMPLE 10.10

Given the circuit in **Fig. 10.24a**, we wish to draw the two networks obtained by replacing the transformer and the primary, and the transformer and the secondary, with equivalent circuits.

SOLUTION

Due to the relationship between the assigned currents and voltages and the location of the dots, the network containing an equivalent circuit for the primary and the network containing an equivalent circuit for the secondary are shown in **Figs. 10.24b** and **c**, respectively. The reader should note carefully the polarity of the voltage sources in the equivalent networks.

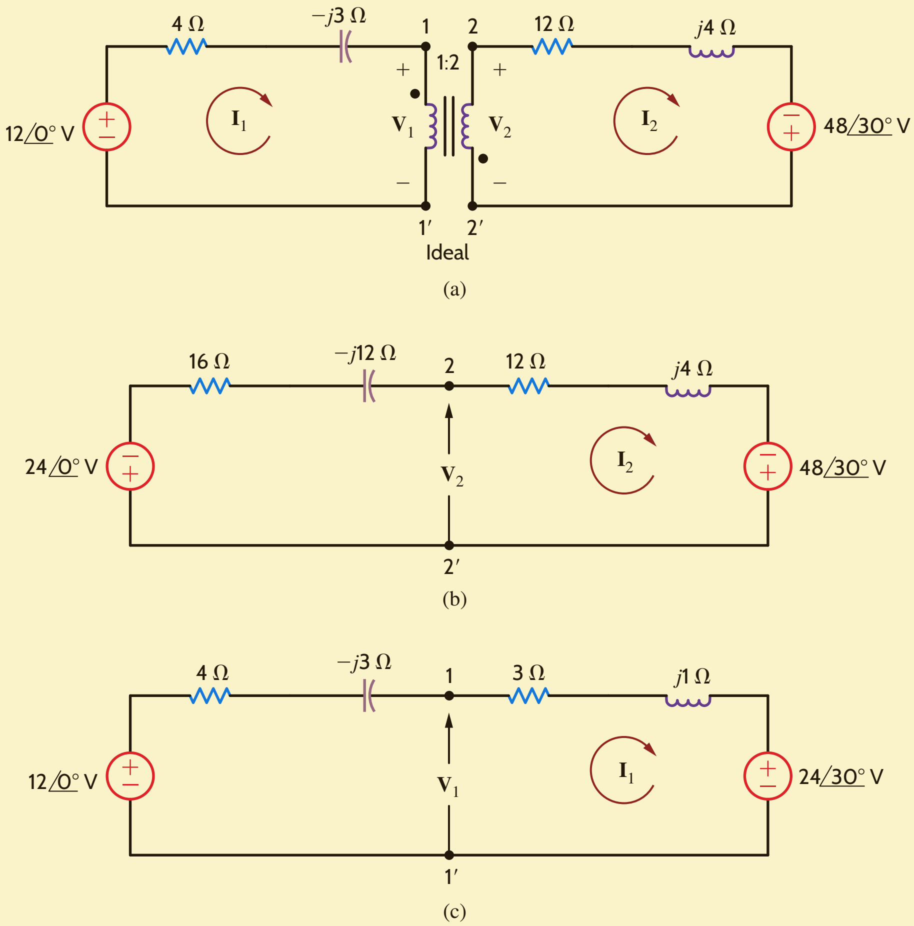

Figure 10.24 Example circuit and two equivalent circuits.

PROBLEM-SOLVING STRATEGY

CIRCUITS CONTAINING IDEAL TRANSFORMERS

STEP 1. Carefully examine the circuit diagram to determine the assigned voltage polarities and current directions in relation to the transformer dots.

- If both voltages are referenced positive at the dotted terminals or undotted terminals, then $v_1/v_2 = N_1/N_2$. If this is not true, then $v_1/v_2 = -N_1/N_2$.
- If one current is defined as entering a dotted terminal and the other current is defined as leaving a dotted terminal, then $N_1 i_1 = N_2 i_2$. If this condition is not satisfied, then $N_1 i_1 = -N_2 i_2$.

STEP 2. If there are no electrical connections between two transformer windings, reflect all circuit elements on one side of the transformer through to the other side, thus eliminating the ideal transformer. Be careful to apply the statements above when reflecting elements through the transformer. Remember that impedances are scaled in magnitude only. Apply circuit analysis techniques to the circuit that results from eliminating all ideal transformers. After this circuit has been analyzed, reflect voltages and currents back through the appropriate ideal transformers to find the answer.

STEP 3. As an alternative approach, use Thévenin's or Norton's theorem to simplify the circuit. Typically, calculation of the equivalent circuit eliminates the ideal transformer. Solve the simplified circuit.

STEP 4. If there are electrical connections between two transformer windings, use nodal analysis or mesh analysis to write equations for the circuits. Solve the equations using the proper relationships between the voltages and currents for the ideal transformer.

EXAMPLE 10.11

Let us determine the output voltage $\mathbf{V}_o$ in the circuit in Fig. 10.25a.

SOLUTION

We begin our attack by forming a Thévenin equivalent for the primary circuit. From Fig. 10.25b we can show that the open-circuit voltage is

$$\begin{aligned}\mathbf{V}_{oc} &= \frac{24\underline{/0^\circ}}{4 - j4}(-j4) - 4\underline{/-90^\circ} \\ &= 12 - j8 = 14.42\underline{/-33.69^\circ}\text{ V}\end{aligned}$$

The Thévenin equivalent impedance looking into the open-circuit terminals with the voltage sources replaced by short circuits is

$$\begin{aligned}\mathbf{Z}_{Th} &= \frac{(4)(-j4)}{4 - j4} + 2 \\ &= 4 - j2\ \Omega\end{aligned}$$

The circuit in Fig. 10.25a thus reduces to that shown in Fig. 10.25c. Forming an equivalent circuit for the transformer and primary results in the network shown in Fig. 10.25d.

Therefore, the voltage $\mathbf{V}_o$ is

$$\begin{aligned}\mathbf{V}_o &= \frac{-28.84\underline{/-33.69^\circ}}{20 - j5}(2) \\ &= 2.80\underline{/160.35^\circ}\text{ V}\end{aligned}$$

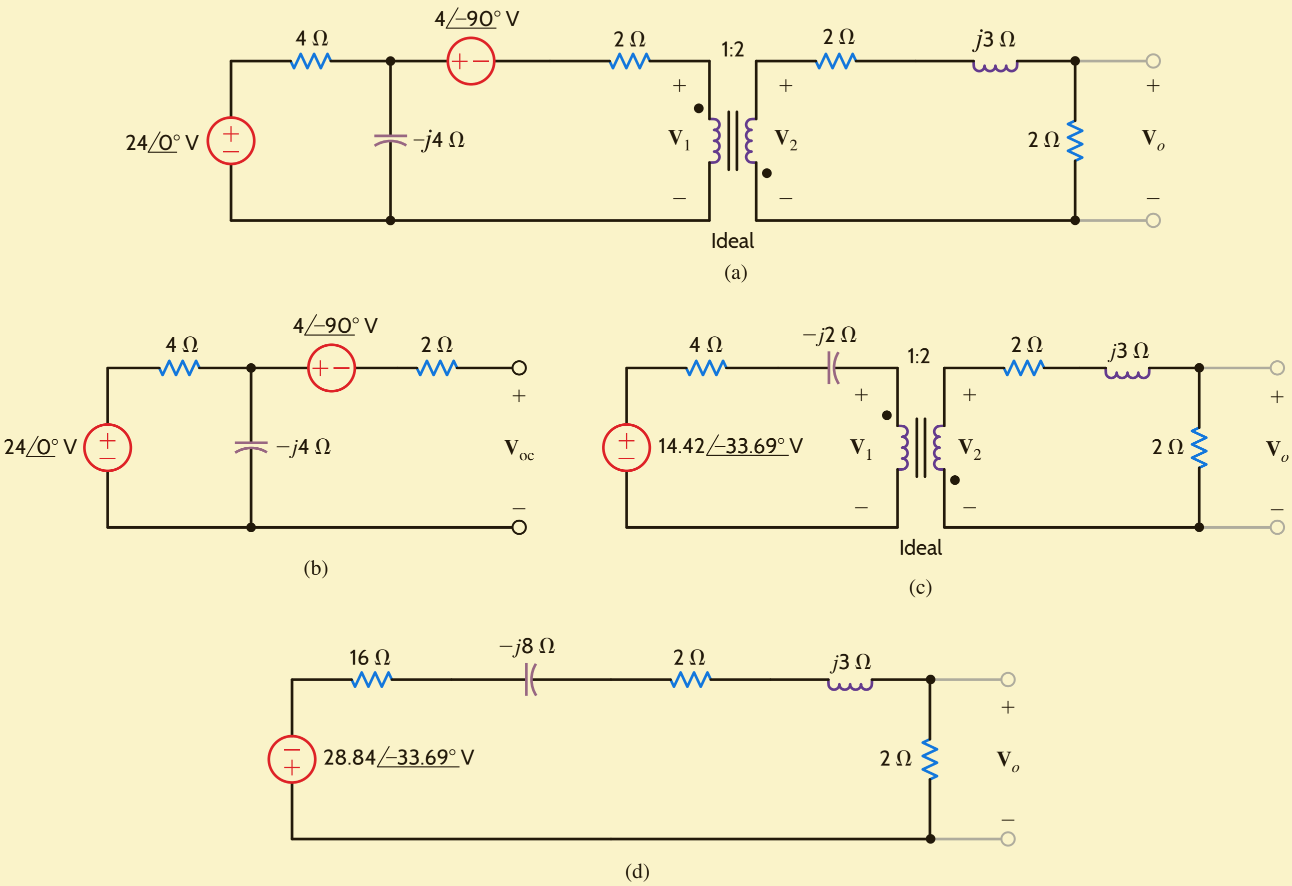

Figure 10.25

Example network and other circuits used to derive an equivalent network.

LEARNING ASSESSMENTS

E10.15 Given the network in Fig. E10.15, form an equivalent circuit for the transformer and secondary, and use the resultant network to compute $\mathbf{I}_1$.

ANSWER:
$\mathbf{I}_1 = 13.12\angle 38.66°$ A.

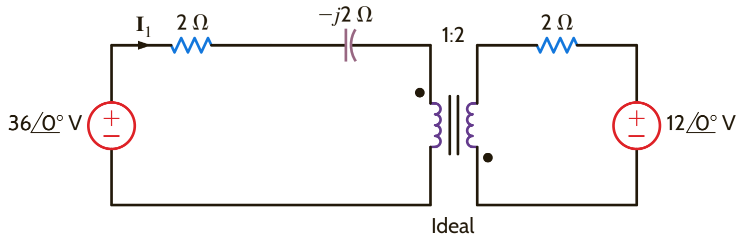

Figure E10.15

E10.16 Given the network in Fig. E10.16, form an equivalent circuit for the transformer and primary, and use the resultant network to find $\mathbf{V}_o$.

ANSWER:
$\mathbf{V}_o = 3.12\angle 38.66°$ V.

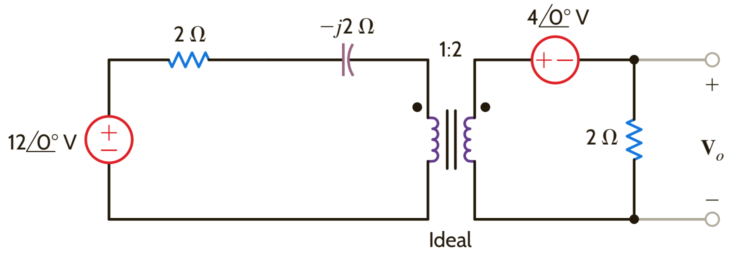

Figure E10.16

EXAMPLE 10.12

Determine $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{V}_1$, and $\mathbf{V}_2$ in the network in Fig. 10.26.

SOLUTION

The nodal equations at nodes 1 and 2 are

$$\frac{10 - \mathbf{V}_1}{2} = \frac{\mathbf{V}_1 - \mathbf{V}_2}{2} + \mathbf{I}_1$$

$$\mathbf{I}_2 + \frac{\mathbf{V}_1 - \mathbf{V}_2}{2} = \frac{\mathbf{V}_2}{2j}$$

The transformer relationships are $\mathbf{V}_2 = 2\mathbf{V}_1$ and $\mathbf{I}_1 = 2\mathbf{I}_2$. The first nodal equation yields $\mathbf{I}_1 = 5$ A and therefore $\mathbf{I}_2 = 2.5$ A. The second nodal equation, together with the constraint equations specified by the transformer, yields $\mathbf{V}_1 = \sqrt{5}\,\angle 63°$ V and $\mathbf{V}_2 = 2\sqrt{5}\,\angle 63°$ V.

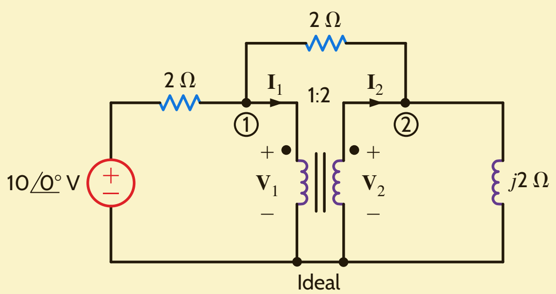

Figure 10.26 Circuit used in Example 10.12.