13.1 Definition

The Laplace transform of a function $f(t)$ is defined by the equation

$$\mathscr{L}[f(t)] = \mathbf{F}(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \textbf{13.1}$$

where s is the complex frequency

$$s = \sigma + j\omega \qquad \textbf{13.2}$$

and the function $f(t)$ is assumed to possess the property that

$$f(t) = 0 \qquad \text{for } t < 0$$

Note that the Laplace transform is unilateral ($0 \leq t < \infty$), in contrast to the Fourier transform (see Chapter 15), which is bilateral ($-\infty < t < \infty$). In our analysis of circuits using the Laplace transform, we will focus our attention on the time interval $t \geq 0$. It is the initial conditions that account for the operation of the circuit prior to $t = 0$; therefore, our analyses will describe the circuit operation for $t \geq 0$.

For a function $f(t)$ to possess a Laplace transform, it must satisfy the condition

$$\int_0^\infty e^{-\sigma t}\left|f(t)\right|\,dt < \infty \qquad \textbf{13.3}$$

for some real value of σ. Because of the convergence factor $e^{-\sigma t}$, a number of important functions have Laplace transforms, even though Fourier transforms for these functions do not exist. All of the inputs we will apply to circuits possess Laplace transforms. Functions that do not have Laplace transforms $\left(\text{e.g., } e^{t^2}\right)$ are of no interest to us in circuit analysis.

The inverse Laplace transform, which is analogous to the inverse Fourier transform, is defined by the relationship

$$\mathscr{L}^{-1}[\mathbf{F}(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \mathbf{F}(s)e^{st}\,ds \qquad \textbf{13.4}$$

where σ_1 is real and $\sigma_1 > \sigma$ in Eq. (13.3).

Since evaluation of this integral is based on complex variable theory, we will avoid its use. How, then, will we be able to convert our solution in the complex frequency domain back to the time domain? The Laplace transform has a uniqueness property: for a given $f(t)$, there is a unique $\mathbf{F}(s)$. In other words, two different functions $f_1(t)$ and $f_2(t)$ cannot have the same $\mathbf{F}(s)$. Our procedure then will be to use Eq. (13.1) to determine the Laplace transform for a number of functions common to electric circuits and store them in a table of transform pairs. We will use a partial fraction expansion to break our complex frequency-domain solution into a group of terms for which we can utilize our table of transform pairs to identify a time function corresponding to each term.

13.2 Two Important Singularity Functions

Two singularity functions are very important in circuit analysis: (1) the unit step function, $u(t)$, discussed in Chapter 7, and (2) the unit impulse or delta function, $\delta(t)$. They are called *singularity functions* because they are either not finite or they do not possess finite derivatives everywhere. They are mathematical models for signals that we employ in circuit analysis.

The *unit step function* $u(t)$ shown in **Fig. 13.1a** was defined in Section 7.2 as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Recall that the physical analogy of this function, as illustrated earlier, corresponds to closing a switch at $t = 0$ and connecting a voltage source of 1 V or a current source of 1 A to a given circuit. The following example illustrates the calculation of the Laplace transform for unit step functions.

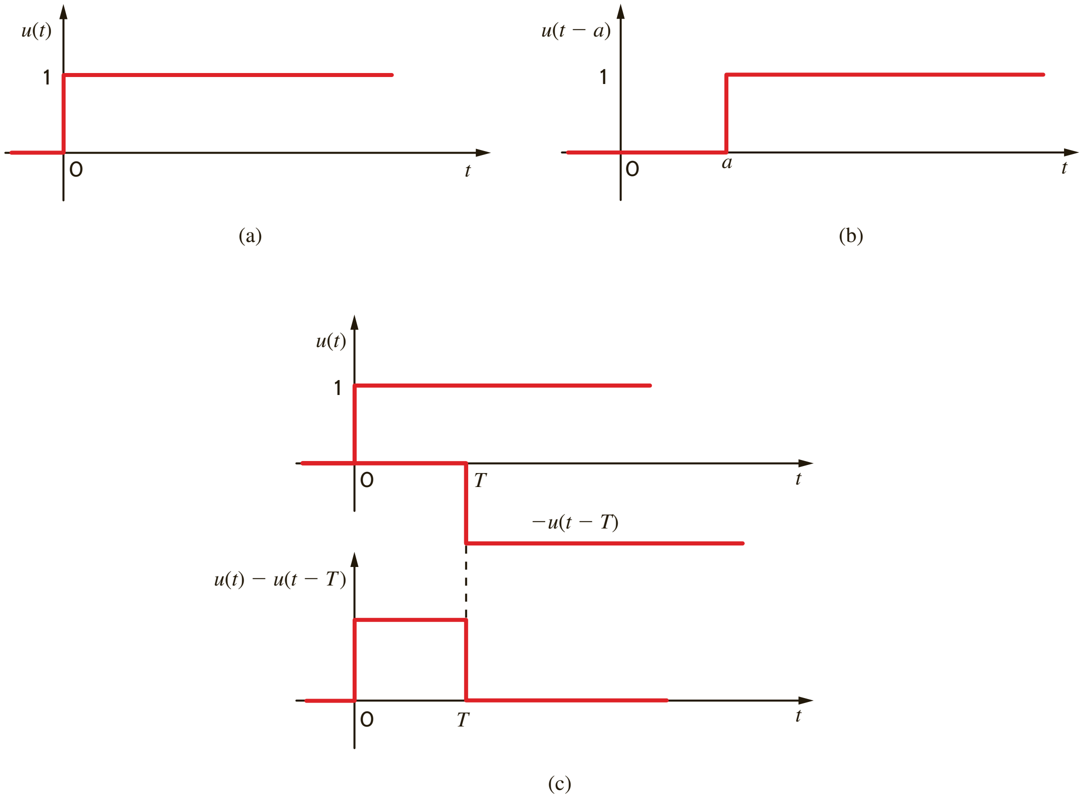

Figure 13.1

Representations of the unit step function.

EXAMPLE 13.1

Let us determine the Laplace transform for the waveforms in Fig. 13.1.

SOLUTION

The Laplace transform for the unit step function in Fig. 13.1a is

$$\begin{aligned}\mathbf{F}(s) &= \int_0^\infty u(t)e^{-st}\,dt \\ &= \int_0^\infty 1e^{-st}\,dt \\ &= -\frac{1}{s}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s} \qquad \sigma > 0\end{aligned}$$

Therefore,

$$\mathscr{L}[u(t)] = \mathbf{F}(s) = \frac{1}{s}$$

The Laplace transform of the time-shifted unit step function shown in Fig. 13.1b is

$$\mathbf{F}(s) = \int_0^\infty u(t-a)e^{-st}\,dt$$

Note that

$$u(t-a) = \begin{cases} 1 & a < t < \infty \\ 0 & t < a \end{cases}$$

Therefore,

$$\mathbf{F}(s) = \int_a^\infty e^{-st}\,dt$$

$$= \frac{e^{-as}}{s} \qquad \sigma > 0$$

Finally, the Laplace transform of the pulse shown in Fig. 13.1c is

$$\mathbf{F}(s) = \int_0^\infty [u(t) - u(t-T)]e^{-st}\,dt$$

$$= \frac{1 - e^{-Ts}}{s} \qquad \sigma > 0$$

The unit impulse function can be represented in the limit by the rectangular pulse shown in Fig. 13.2a as $a \to 0$. The function is defined by the following:

$$\delta(t - t_0) = 0 \qquad t \neq t_0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

The unit impulse is zero except at $t = t_0$, where it is undefined, but it has unit area (sometimes referred to as *strength*). We represent the unit impulse function on a plot as shown in Fig. 13.2b.

An important property of the unit impulse function is what is often called the *sampling property*, which is exhibited by the following integral:

$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)\,dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1,\ t_0 > t_2 \end{cases}$$

for a finite t_0 and any $f(t)$ continuous at t_0. Note that the unit impulse function simply samples the value of $f(t)$ at $t = t_0$.

Now that we have defined the unit impulse function, let's consider the following question: why introduce the unit impulse function? We certainly cannot produce a voltage or current signal with zero width and infinite height in a physical system. For engineers, the unit impulse function is a convenient mathematical function that can be utilized to model a physical process. For example, a lightning stroke is a short-duration event. If we were analyz-

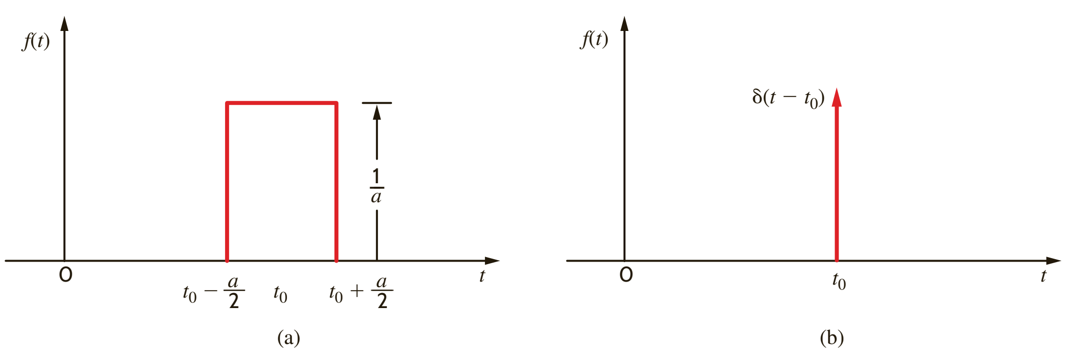

Figure 13.2

Representations of the unit impulse.