8.1 Sinusoids

Let us begin our discussion of sinusoidal functions by considering the sine wave

$$x(t) = X_M \sin \omega t \tag{8.1}$$

where $x(t)$ could represent either $v(t)$ or $i(t)$. X_M is the *amplitude, maximum value*, or peak value; ω is the *radian* or *angular frequency*; and ωt is the *argument* of the sine function. A plot of the function in Eq. (8.1) as a function of its argument is shown in **Fig. 8.1a**. Obviously, the function repeats itself every 2π radians. This condition is described mathematically as $x(\omega t + 2\pi) = x(\omega t)$, or in general for period T, as

$$x[\omega(t + T)] = x(\omega t) \tag{8.2}$$

meaning that the function has the same value at time $t + T$ as it does at time t.

HINT
The relationship between frequency and period

The waveform can also be plotted as a function of time, as shown in **Fig. 8.1b**. Note that this function goes through one period every T seconds. In other words, in 1 second it goes through $1/T$ periods or cycles. The number of cycles per second, called Hertz, is the frequency f, where

$$f = \frac{1}{T} \tag{8.3}$$

HINT
The relationship between frequency, period, and radian frequency

Now since $\omega T = 2\pi$, as shown in Fig. 8.1a, we find that

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{8.4}$$

which is, of course, the general relationship among period in seconds, frequency in Hertz, and radian frequency.

Now that we have discussed some of the basic properties of a sine wave, let us consider the following general expression for a sinusoidal function:

$$x(t) = X_M \sin(\omega t + \theta) \tag{8.5}$$

HINT
Phase lag defined

In this case, $(\omega t + \theta)$ is the argument of the sine function, and θ is called the *phase angle*. A plot of this function is shown in **Fig. 8.2**, together with the original function in Eq. (8.1) for comparison. Because of the presence of the phase angle, any point on the waveform $X_M \sin(\omega t + \theta)$ occurs θ radians earlier in time than the corresponding point on the waveform $X_M \sin \omega t$. Therefore, we say that $X_M \sin \omega t$ *lags* $X_M \sin(\omega t + \theta)$ by θ radians. In the more general situation, if

HINT
In phase and out of phase defined

$$x_1(t) = X_{M_1} \sin(\omega t + \theta)$$

and

$$x_2(t) = X_{M_2} \sin(\omega t + \phi)$$

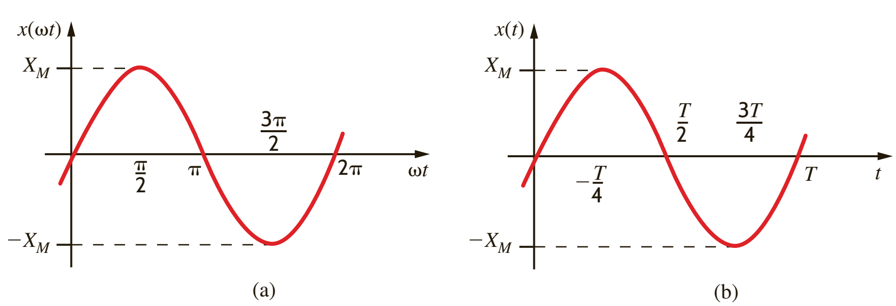

Figure 8.1
Plots of a sine wave as a function of both ωt and t.

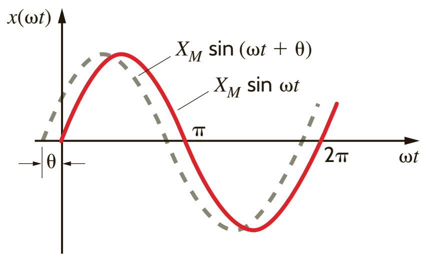

Figure 8.2

Graphical illustration of $X_M \sin(\omega t + \theta)$ leading $X_M \sin \omega t$ by θ radians.

then $x_1(t)$ *leads* $x_2(t)$ by $\theta - \phi$ radians and $x_2(t)$ lags $x_1(t)$ by $\theta - \phi$ radians. If $\theta = \phi$, the waveforms are identical and the functions are said to be *in phase*. If $\theta \neq \phi$, the functions are *out of phase*.

HINT Phase lead graphically illustrated

The phase angle is normally expressed in degrees rather than radians. Therefore, at this point we will simply state that we will use the two forms interchangeably; that is,

$$x(t) = X_M \sin\left(\omega t + \frac{\pi}{2}\right) = X_M \sin(\omega t + 90°) \qquad \textbf{8.6}$$

HINT An important note about phase angles

Rigorously speaking, since ωt is in radians, the phase angle should be as well. However, it is common practice and convenient to use degrees for phase; therefore, that will be our practice in this text.

In addition, it should be noted that adding to the argument integer multiples of either 2π radians or 360° does not change the original function. This can easily be shown mathematically but is visibly evident when examining the waveform, as shown in Fig. 8.2.

Although our discussion has centered on the sine function, we could just as easily have used the cosine function, since the two waveforms differ only by a phase angle; that is,

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \textbf{8.7}$$

$$\sin \omega t = \cos\left(\omega t - \frac{\pi}{2}\right) \qquad \textbf{8.8}$$

HINT Some trigonometric identities that are useful in phase angle calculations

We are often interested in the phase difference between two sinusoidal functions. Three conditions must be satisfied before we can determine the phase difference: (1) the frequency of both sinusoids must be the same, (2) the amplitude of both sinusoids must be positive, and (3) both sinusoids must be written as sine waves or cosine waves. Once in this format, the phase angle between the functions can be computed as outlined previously. Two other trigonometric identities that normally prove useful in phase angle determination are

$$-\cos(\omega t) = \cos(\omega t \pm 180°) \qquad \textbf{8.9}$$

$$-\sin(\omega t) = \sin(\omega t \pm 180°) \qquad \textbf{8.10}$$

Finally, the angle-sum and angle-difference relationships for sines and cosines may be useful in the manipulation of sinusoidal functions. These relations are

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned} \qquad \textbf{8.11}$$

EXAMPLE 8.1

We wish to plot the waveforms for the following functions:

a. $v(t) = 1\cos(\omega t + 45°)$,

b. $v(t) = 1\cos(\omega t + 225°)$, and

c. $v(t) = 1\cos(\omega t - 315°)$.

SOLUTION

Figure 8.3a shows a plot of the function $v(t) = 1\cos \omega t$. **Figure 8.3b** is a plot of the function $v(t) = 1\cos(\omega t + 45°)$. **Figure 8.3c** is a plot of the function $v(t) = 1\cos(\omega t + 225°)$. Note that since

$$v(t) = 1\cos(\omega t + 225°) = 1\cos(\omega t + 45° + 180°)$$

this waveform is 180° out of phase with the waveform in Fig. 8.3b; that is, $\cos(\omega t + 225°) = -\cos(\omega t + 45°)$, and Fig. 8.3c is the negative of Fig. 8.3b. Finally, since the function

$$v(t) = 1\cos(\omega t - 315°) = 1\cos(\omega t - 315° + 360°) = 1\cos(\omega t + 45°)$$

this function is identical to that shown in Fig. 8.3b.

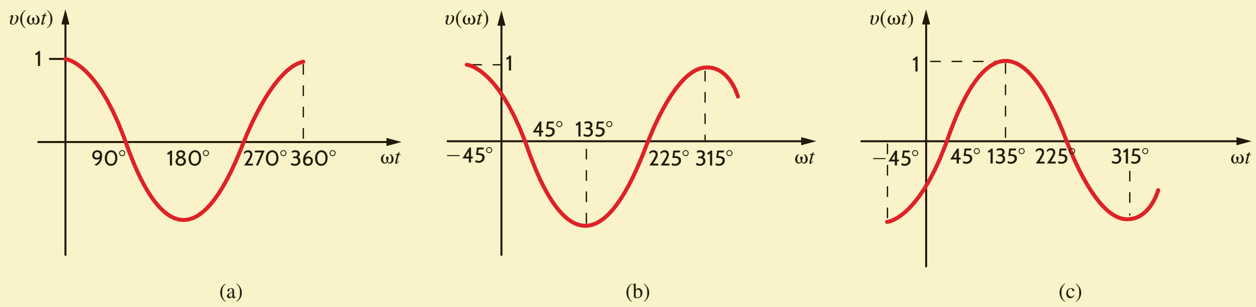

Figure 8.3

Cosine waveforms with various phase angles.

EXAMPLE 8.2

Determine the frequency and the phase angle between the two voltages $v_1(t) = 12\sin(1000t + 60°)$ V and $v_2(t) = -6\cos(1000t + 30°)$ V.

SOLUTION

The frequency in Hertz (Hz) is given by the expression

$$f = \frac{\omega}{2\pi} = \frac{1000}{2\pi} = 159.2 \text{ Hz}$$

Using Eq. (8.9), $v_2(t)$ can be written as

$$v_2(t) = -6\cos(\omega t + 30°) = 6\cos(\omega t + 210°) \text{ V}$$

Then employing Eq. (8.7), we obtain

$$6\sin(\omega t + 300°) \text{ V} = 6\sin(\omega t - 60°) \text{ V}$$

Now that both voltages of the same frequency are expressed as sine waves with positive amplitudes, the phase angle between $v_1(t)$ and $v_2(t)$ is $60° - (-60°) = 120°$; that is, $v_1(t)$ leads $v_2(t)$ by 120° or $v_2(t)$ lags $v_1(t)$ by 120°.

LEARNING ASSESSMENTS

E8.1 Given the voltage $v(t) = 120\cos(314t + \pi/4)$ V, determine the frequency of the voltage in Hertz and the phase angle in degrees.

ANSWER:
$f = 50$ Hz;
$\theta = 45°$.