The s-domain representation of this element is shown in Fig. 14.2c.

Using the passive sign convention, we find that the voltage–current relationships for the coupled inductors shown in Fig. 14.2d are

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \qquad \textbf{14.12}$$
$$v_2(t) = L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt}$$

The relationships in the s-domain are then

$$\mathbf{V}_1(s) = L_1 s\mathbf{I}_1(s) - L_1 i_1(0) + Ms\mathbf{I}_2(s) - Mi_2(0) \qquad \textbf{14.13}$$
$$\mathbf{V}_2(s) = L_2 s\mathbf{I}_2(s) - L_2 i_2(0) + Ms\mathbf{I}_1(s) - Mi_1(0)$$

Independent and dependent voltage and current sources can also be represented by their transforms; that is,

$$\mathbf{V}_1(s) = \mathscr{L}[v_1(t)] \qquad \textbf{14.14}$$
$$\mathbf{I}_2(s) = \mathscr{L}[i_2(t)]$$

and if $v_1(t) = Ai_2(t)$, which represents a current-controlled voltage source, then

$$\mathbf{V}_1(s) = A\mathbf{I}_2(s) \qquad \textbf{14.15}$$

Note carefully the direction of the current sources and the polarity of the voltage sources in the transformed network that result from the initial conditions. If the polarity of the initial voltage or direction of the initial current is reversed, the sources in the transformed circuit that results from the initial condition are also reversed.

PROBLEM-SOLVING STRATEGY

s-DOMAIN CIRCUITS

STEP 1. Solve for initial capacitor voltages and inductor currents. This may require the analysis of a circuit valid for $t < 0$ drawn with all capacitors replaced by open circuits and all inductors replaced by short circuits.

STEP 2. Draw an s-domain circuit by substituting an s-domain representation for all circuit elements. Be sure to include initial conditions for capacitors and inductors if nonzero.

STEP 3. Use the circuit analysis techniques presented in this textbook to solve for the appropriate voltages and/or currents. The voltages and/or currents will be described by a ratio of polynomials in s.

STEP 4. Perform an inverse Laplace transform to convert the voltages and/or currents back to the time domain.

14.3 Analysis Techniques

Now that we have the s-domain representation for the circuit elements, we are in a position to analyze networks using a transformed circuit.

EXAMPLE 14.1

Given the network in Fig. 14.3a, let us draw the s-domain equivalent circuit and find the output voltage in both the s and time domains.

SOLUTION

The s-domain network is shown in Fig. 14.3b. We can write the output voltage as

$$\mathbf{V}_o(s) = \left[R // \frac{1}{sC}\right]\mathbf{I}_S(s)$$

or

$$\mathbf{V}_o(s) = \left[\frac{1/C}{s + (1/RC)}\right]\mathbf{I}_S(s)$$

Given the element values, $\mathbf{V}_o(s)$ becomes

$$\mathbf{V}_o(s) = \left(\frac{40{,}000}{s+4}\right)\left(\frac{0.003}{s+1}\right) = \frac{120}{(s+4)(s+1)}$$

Expanding $\mathbf{V}_o(s)$ into partial fractions yields

$$\mathbf{V}_o(s) = \frac{120}{(s+4)(s+1)} = \frac{40}{s+1} - \frac{40}{s+4}$$

Performing the inverse Laplace transform yields the time-domain representation

$$v_o(t) = 40[e^{-t} - e^{-4t}]u(t)\text{ V}$$

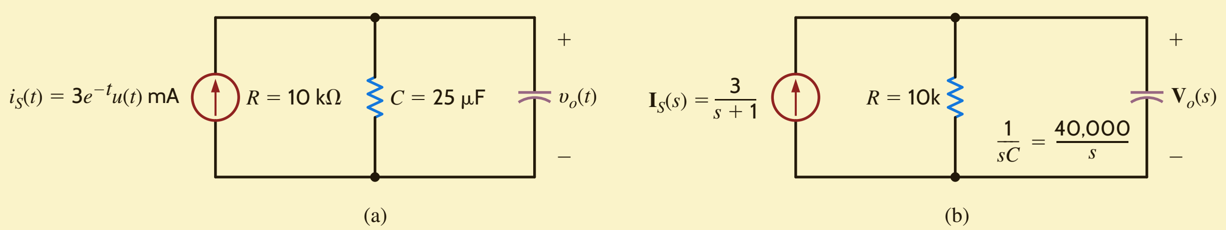


Figure 14.3
Time-domain and s-domain representations of an RC parallel network.

Now that we have demonstrated the use of the Laplace transform in the solution of a simple circuit, let us consider the more general case. Note that in Fig. 14.2 we have shown two models for the capacitor and inductor when initial conditions are present. Let us now consider an example in which we will illustrate the use of these models in deriving both the node and loop equations for the circuit.

EXAMPLE 14.2

Given the circuits in Figs. 14.4a and b, we wish to write the mesh equations in the s-domain for the network in Fig. 14.4a and the node equations in the s-domain for the network in Fig. 14.4b.

SOLUTION

The transformed circuit for the network in Fig. 14.4a is shown in Fig. 14.4c. The mesh equations for this network are

$$\left(R_1 + \frac{1}{sC_1} + \frac{1}{sC_2} + sL_1\right)\mathbf{I}_1(s) - \left(\frac{1}{sC_2} + sL_1\right)\mathbf{I}_2(s)$$
$$= \mathbf{V}_A(s) - \frac{v_1(0)}{s} + \frac{v_2(0)}{s} - L_1 i_1(0)$$

$$-\left(\frac{1}{sC_2} + sL_1\right)\mathbf{I}_1(s) + \left(\frac{1}{sC_2} + sL_1 + sL_2 + R_2\right)\mathbf{I}_2(s)$$
$$= L_1 i_1(0) - \frac{v_2(0)}{s} - L_2 i_2(0) + \mathbf{V}_B(s)$$

The transformed circuit for the network in Fig. 14.4b is shown in **Fig. 14.4d**. The node equations for this network are

$$\left(G_1 + \frac{1}{sL_1} + sC_1 + \frac{1}{sL_2}\right)\mathbf{V}_1(s) - \left(\frac{1}{sL_2} + sC_1\right)\mathbf{V}_2(s)$$
$$= \mathbf{I}_A(s) - \frac{i_1(0)}{s} + \frac{i_2(0)}{s} - C_1 v_1(0)$$
$$-\left(\frac{1}{sL_2} + sC_1\right)\mathbf{V}_1(s) + \left(\frac{1}{sL_2} + sC_1 + G_2 + sC_2\right)\mathbf{V}_2(s)$$
$$= C_1 v_1(0) - \frac{i_2(0)}{s} - C_2 v_2(0) - \mathbf{I}_B(s)$$

HINT

Note that the equations employ the same convention used in dc analysis.

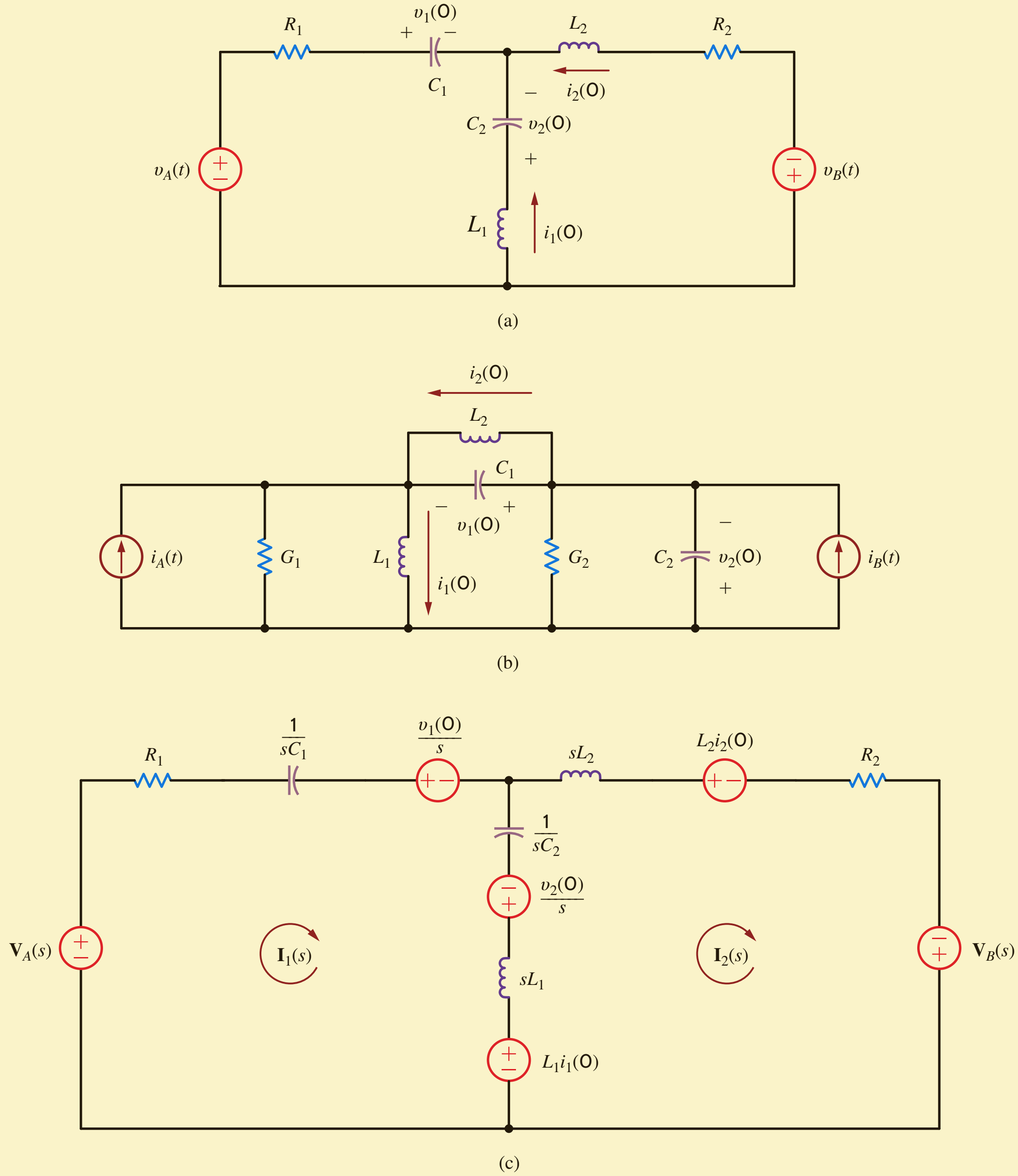
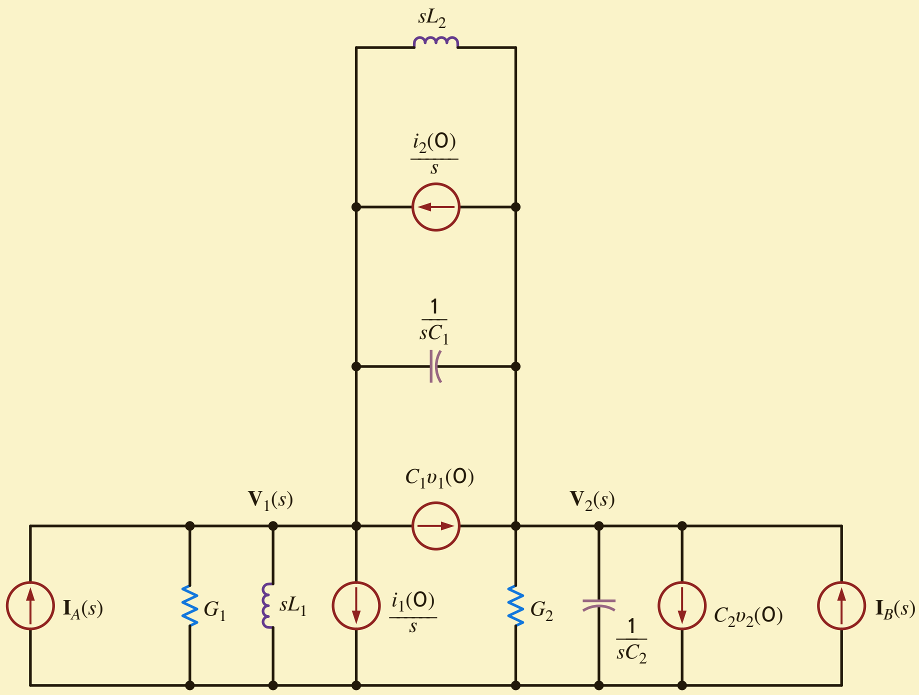


Figure 14.4

Circuits used in Example 14.2.



(d)

Figure 14.4
(continued)

Example 14.2 attempts to illustrate the manner in which to employ the two s-domain representations of the inductor and capacitor circuit elements when initial conditions are present. In the following examples, we illustrate the use of a number of analysis techniques in obtaining the complete response of a transformed network. The circuits analyzed have been specifically chosen to demonstrate the application of the Laplace transform to circuits with a variety of passive and active elements.

EXAMPLE 14.3

Let us examine the network in Fig. 14.5a. We wish to determine the output voltage $v_o(t)$.

SOLUTION

As a review of the analysis techniques presented earlier in this text, we will solve this problem using nodal analysis, mesh analysis, superposition, source exchange, Thévenin's theorem, and Norton's theorem.

The transformed network is shown in Fig. 14.5b. In our employment of nodal analysis, rather than writing KCL equations at the nodes labeled $\mathbf{V}_1(s)$ and $\mathbf{V}_o(s)$, we will use only the former node and use voltage division to find the latter.

KCL at the node labeled $\mathbf{V}_1(s)$ is

$$-\frac{4}{s} + \frac{\mathbf{V}_1(s) - \frac{12}{s}}{s} + \frac{\mathbf{V}_1(s)}{\frac{1}{s} + 2} = 0$$

Solving for $\mathbf{V}_1(s)$ we obtain

$$\mathbf{V}_1(s) = \frac{4(s+3)(2s+1)}{s(s^2+2s+1)}$$

Now employing voltage division,

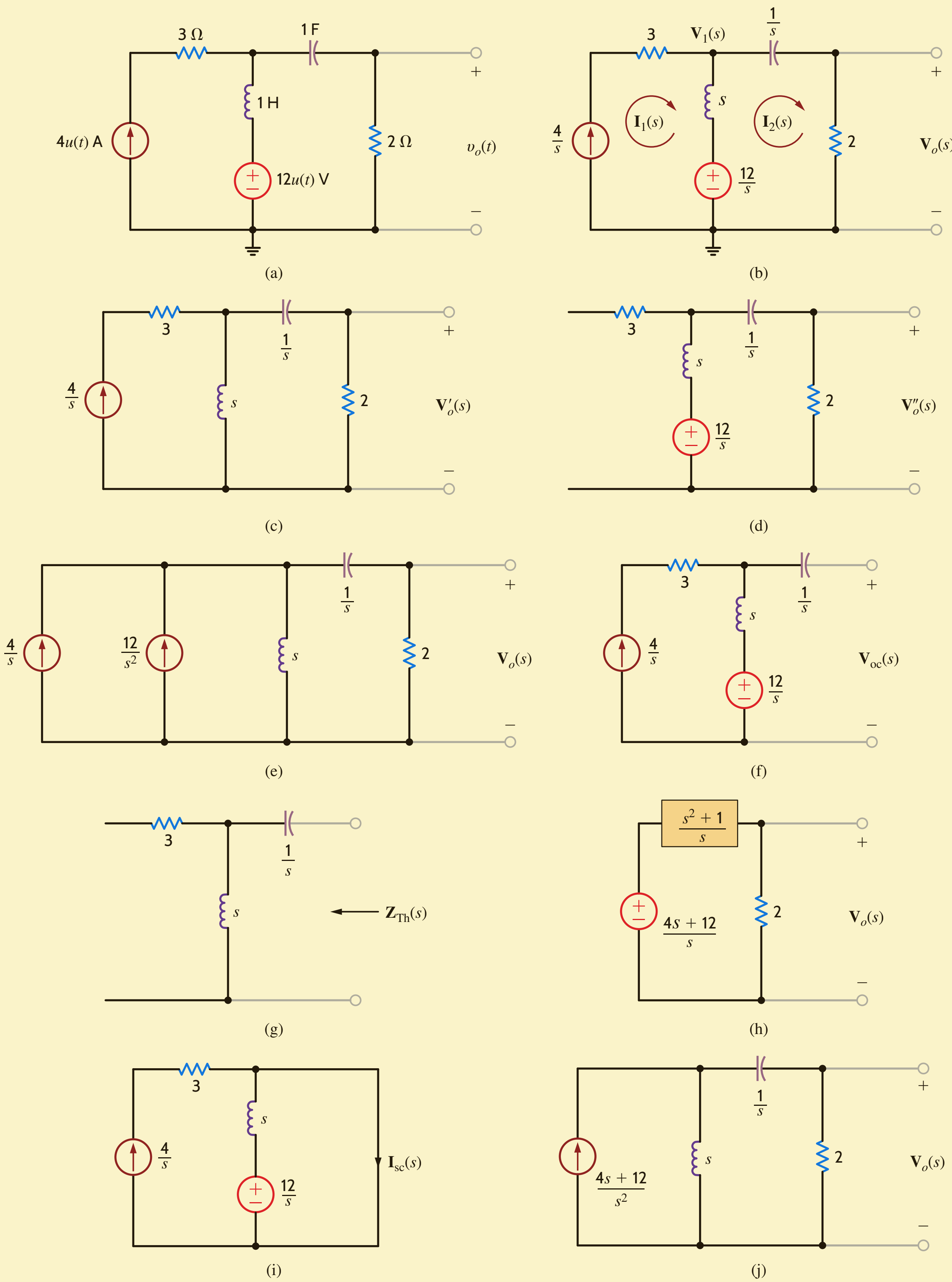


Figure 14.5

Circuits used in Example 14.3.

$$\mathbf{V}_o(s) = \mathbf{V}_1(s)\left[\frac{2}{\frac{1}{s} + 2}\right] = \mathbf{V}_1(s)\left(\frac{2s}{2s+1}\right)$$

$$= \frac{8(s+3)}{(s+1)^2}$$

In our mesh analysis we note that the current $\mathbf{I}_1(s)$ goes through the current source, and therefore KVL for the right-hand loop is

$$\frac{12}{s} - [\mathbf{I}_2(s) - \mathbf{I}_1(s)]s - \frac{\mathbf{I}_2(s)}{s} - 2\mathbf{I}_2(s) = 0$$

However, $\mathbf{I}_1(s) = 4/s$, and hence

$$\mathbf{I}_2(s) = \frac{4(s+3)}{(s+1)^2}$$

Therefore,

$$\mathbf{V}_o(s) = \frac{8(s+3)}{(s+1)^2}$$

The 3-Ω resistor never enters our equations. Furthermore, it will not enter our other analyses either. Why?

In using superposition, we first consider the current source acting alone as shown in Fig. 14.5c. Applying current division, we obtain

$$\mathbf{V}_o'(s) = \left[\frac{\frac{4}{s}(s)}{s + \frac{1}{s} + 2}\right](2)$$

$$= \frac{8s}{s^2 + 2s + 1}$$

With the voltage source acting alone, as shown in Fig. 14.5d, we obtain

$$\mathbf{V}_o''(s) = \left[\frac{\frac{12}{s}}{s + \frac{1}{s} + 2}\right](2)$$

$$= \frac{24}{s^2 + 2s + 1}$$

Hence,

$$\mathbf{V}_o(s) = \mathbf{V}_o'(s) + \mathbf{V}_o''(s)$$

$$= \frac{8(s+3)}{(s+1)^2}$$

In applying source exchange, we transform the voltage source and series inductor into a current source with the inductor in parallel as shown in Fig. 14.5e. Adding the current sources and applying current division yields

$$\mathbf{V}_o(s) = \left(\frac{12}{s^2} + \frac{4}{s}\right)\left[\frac{s}{s + \frac{1}{s} + 2}\right](2)$$

$$= \frac{\left(\frac{12}{s} + 4\right)(2)}{s + \frac{1}{s} + 2}$$

$$\mathbf{V}_o(s) = \frac{8(s+3)}{(s+1)^2}$$

To apply Thévenin's theorem, we first find the open-circuit voltage shown in Fig. 14.5f. $\mathbf{V}_{\mathrm{oc}}(s)$ is then

$$\mathbf{V}_{\mathrm{oc}}(s) = \left(\frac{4}{s}\right)(s) + \frac{12}{s}$$
$$= \frac{4s + 12}{s}$$

The Thévenin equivalent impedance derived from Fig. 14.5g is

$$\mathbf{Z}_{\mathrm{Th}}(s) = \frac{1}{s} + s$$
$$= \frac{s^2 + 1}{s}$$

Now, connecting the Thévenin equivalent circuit to the load produces the circuit shown in Fig. 14.5h. Then, applying voltage division, we obtain

$$\mathbf{V}_o(s) = \frac{4s + 12}{s}\left[\frac{2}{\dfrac{s^2 + 1}{s} + 2}\right]$$
$$= \frac{8(s + 3)}{(s + 1)^2}$$

In applying Norton's theorem, for simplicity we break the network to the right of the first mesh. In this case, the short-circuit current is obtained from the circuit in Fig. 14.5i; that is,

$$\mathbf{I}_{\mathrm{sc}}(s) = \frac{\dfrac{12}{s}}{s} + \frac{4}{s}$$
$$= \frac{4s + 12}{s^2}$$

The Thévenin equivalent impedance in this application of Norton's theorem is $\mathbf{Z}_{\mathrm{Th}}(s) = s$. Connecting the Norton equivalent circuit to the remainder of the original network yields the circuit in Fig. 14.5j. Then

$$\mathbf{V}_o(s) = \frac{4s + 12}{s^2}\left[\frac{s}{s + \dfrac{1}{s} + 2}\right](2)$$
$$= \frac{8(s + 3)}{(s + 1)^2}$$

Finally, $\mathbf{V}_o(s)$ can now be transformed to $v_o(t)$. $\mathbf{V}_o(s)$ can be written as

$$\mathbf{V}_o(s) = \frac{8(s + 3)}{(s + 1)^2} = \frac{K_{11}}{(s + 1)^2} + \frac{K_{12}}{s + 1}$$

Evaluating the constants, we obtain

$$8(s + 3)\big|_{S = -1} = K_{11}$$
$$16 = K_{11}$$

and

$$\frac{d}{ds}[8(s + 3)]\bigg|_{S = -1} = K_{12}$$
$$8 = K_{12}$$

Therefore,

$$v_o(t) = (16te^{-t} + 8e^{-t})u(t) \text{ V}$$

EXAMPLE 14.4

Consider the network shown in Fig. 14.6a. We wish to determine the output voltage $v_o(t)$.

SOLUTION

As we begin to attack the problem, we note two things. First, because the source $12u(t)$ is connected between $v_1(t)$ and $v_2(t)$, we have a supernode. Second, if $v_2(t)$ is known, $v_o(t)$ can be easily obtained by voltage division. Hence, we will use nodal analysis in conjunction with voltage division to obtain a solution. Then for purposes of comparison, we will find $v_o(t)$ using Thévenin's theorem.

HINT

Summing the currents leaving the supernode.

The transformed network is shown in Fig. 14.6b. KCL for the supernode is

$$\frac{\mathbf{V}_1(s)}{2} + \mathbf{V}_1(s)\frac{s}{2} - 2\mathbf{I}(s) + \frac{\mathbf{V}_2(s)}{s+1} = 0$$

However,

$$\mathbf{I}(s) = -\frac{\mathbf{V}_1(s)}{2}$$

and

$$\mathbf{V}_1(s) = \mathbf{V}_2(s) - \frac{12}{s}$$

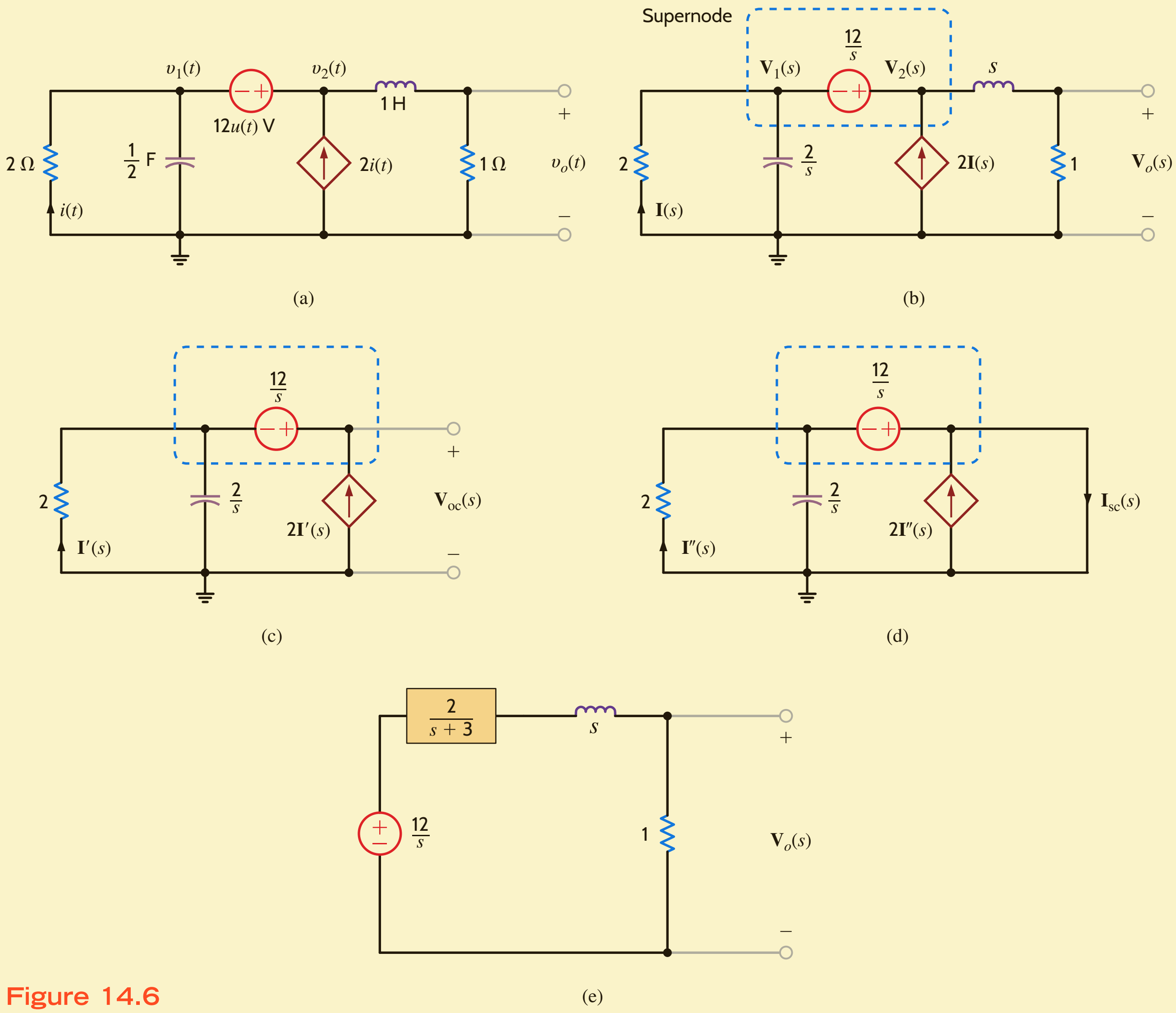


Figure 14.6

Circuits used in Example 14.4.

Substituting the last two equations into the first equation yields

$$\left[\mathbf{V}_2(s) - \frac{12}{s}\right]\frac{s+3}{2} + \frac{\mathbf{V}_2(s)}{s+1} = 0$$

or

$$\mathbf{V}_2(s) = \frac{12(s+1)(s+3)}{s(s^2+4s+5)}$$

Employing a voltage divider, we obtain

$$\begin{aligned}\mathbf{V}_o(s) &= \mathbf{V}_2(s)\frac{1}{s+1}\\ &= \frac{12(s+3)}{s(s^2+4s+5)}\end{aligned}$$

To apply Thévenin's theorem, we break the network to the right of the dependent current source as shown in Fig. 14.6c. KCL for the supernode is

$$\frac{\mathbf{V}_{oc}(s) - \frac{12}{s}}{2} + \frac{\mathbf{V}_{oc}(s) - \frac{12}{s}}{\frac{2}{s}} - 2\mathbf{I}'(s) = 0$$

where

$$\mathbf{I}'(s) = -\left(\frac{\mathbf{V}_{oc}(s) - \frac{12}{s}}{2}\right)$$

Solving these equations for $\mathbf{V}_{oc}(s)$ yields

$$\mathbf{V}_{oc}(s) = \frac{12}{s}$$

The short-circuit current is derived from the network in Fig. 14.6d as

$$\mathbf{I}_{sc}(s) = 2\mathbf{I}''(s) + \frac{\frac{12}{s}}{\frac{(2)\left(\frac{2}{s}\right)}{2+\frac{2}{s}}}$$

where

$$\mathbf{I}''(s) = \frac{\frac{12}{s}}{2}$$

Solving these equations for $\mathbf{I}_{sc}(s)$ yields

$$\mathbf{I}_{sc}(s) = \frac{6(s+3)}{s}$$

The Thévenin equivalent impedance is then

$$\begin{aligned}\mathbf{Z}_{Th}(s) &= \frac{\mathbf{V}_{oc}(s)}{\mathbf{I}_{sc}(s)}\\ &= \frac{\frac{12}{s}}{\frac{6(s+3)}{s}}\\ &= \frac{2}{s+3}\end{aligned}$$

If we now connect the Thévenin equivalent circuit to the remainder of the original network, we obtain the circuit shown in Fig. 14.6e. Using voltage division,

$$\begin{aligned}\mathbf{V}_o(s) &= \frac{1}{\frac{2}{s+3} + s + 1}\left(\frac{12}{s}\right)\\ &= \frac{12(s+3)}{s\left(s^2+4s+5\right)}\end{aligned}$$

or

$$\mathbf{V}_o(s) = \frac{12(s + 3)}{s(s + 2 - j1)(s + 2 + j1)}$$

To obtain the inverse transform, the function is written as

$$\frac{12(s + 3)}{s(s + 2 - j1)(s + 2 + j1)} = \frac{K_0}{s} + \frac{K_1}{s + 2 - j1} + \frac{K_1^*}{s + 2 + j1}$$

Evaluating the constants, we obtain

$$\left.\frac{12(s + 3)}{s^2 + 4s + 5}\right|_{s=0} = K_0$$

$$\frac{36}{5} = K_0$$

and

$$\left.\frac{12(s + 3)}{s(s + 2 + j1)}\right|_{s=-2+j1} = K_0$$

$$3.79\underline{/161.57^\circ} = K_1$$

Therefore,

$$v_o(t) = \left[7.2 + 7.58e^{-2t}\cos(t + 161.57^\circ)\right]u(t)\text{ V}$$

LEARNING ASSESSMENTS

E14.1 Find $i_o(t)$ in the network in Fig. E14.1 using node equations.

ANSWER:
$i_o(t) = 6.53e^{-t/4}\cos\left[\left(\sqrt{15}/4\right)t - 156.72^\circ\right]u(t)$ A.

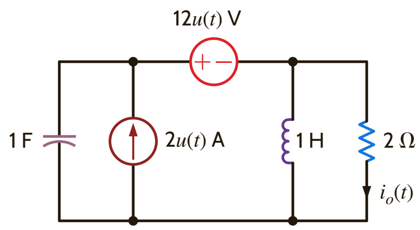


Figure E14.1

E14.2 Find $v_o(t)$ for $t > 0$ in Fig. E14.2 using nodal analysis.

ANSWER:
$v_o(t) = (10.64e^{-0.75t}\cos(0.97t - 19.84^\circ))u(t)$ V.

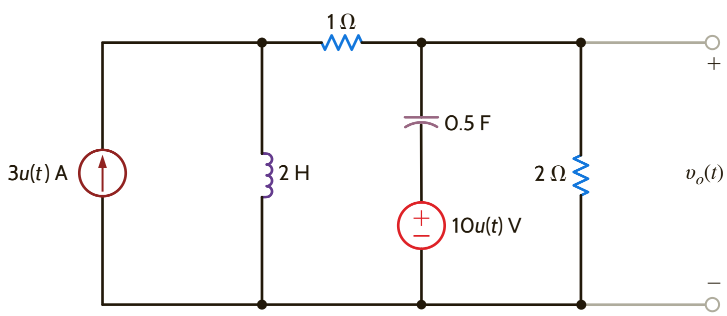


Figure E14.2

E14.3 Find $v_o(t)$ in the network in Fig. E14.3 using loop equations.

ANSWER:
$v_o(t) = (4 - 8.93e^{-3.73t} + 4.93e^{-0.27t})u(t)$ V.

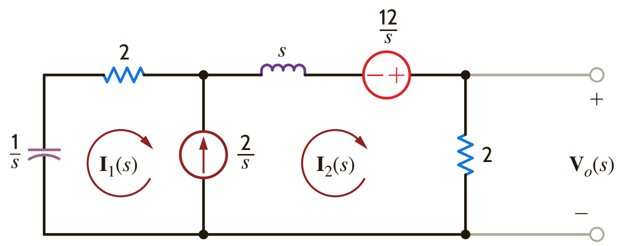


Figure E14.3

E14.4 Find $v_o(t)$ for $t > 0$ in Fig. E14.2 using mesh analysis.

ANSWER:
$v_o(t) = (10.64e^{-0.75t} \cos(0.97t - 19.84°))u(t)$ V.

E14.5 Use Thévenin's theorem to determine $v_o(t)$ for $t > 0$ in Fig. E14.2.

ANSWER:
$v_o(t) = (10.64e^{-0.75t} \cos(0.97t - 19.84°))u(t)$ V.

E14.6 Use Thévenin's theorem to determine $v_o(t)$ for $t > 0$ in Fig. E14.6.

ANSWER:
$v_o(t) = (21.5 + 12.29e^{-1.267t})u(t)$ V.

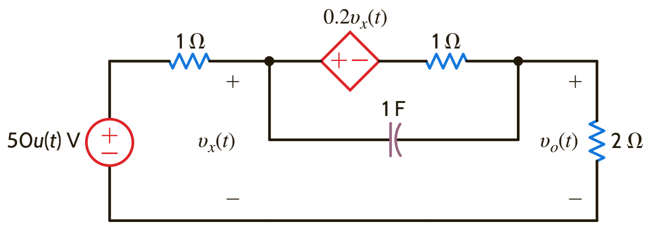


Figure E14.6

We will now illustrate the use of the Laplace transform in the transient analysis of circuits. We will analyze networks such as those considered in Chapter 7. Our approach will first be to determine the initial conditions for the capacitors and inductors in the network, and then we will employ the element models that were specified at the beginning of this chapter together with the circuit analysis techniques to obtain a solution. The following example demonstrates the approach.

EXAMPLE 14.5

Let us determine the output voltage of the network shown in Fig. 14.7a for $t > 0$.

SOLUTION

At $t = 0$, the initial voltage across the capacitor is 1 V and the initial current drawn through the inductor is 1 A. The circuit for $t > 0$ is shown in Fig. 14.7b with the initial conditions. The transformed network is shown in Fig. 14.7c.

The mesh equations for the transformed network are

$$(s + 1)\mathbf{I}_1(s) - s\mathbf{I}_2(s) = \frac{4}{s} + 1$$

$$-s\mathbf{I}_1(s) + \left(s + \frac{2}{s} + 1\right)\mathbf{I}_2(s) = \frac{-1}{s} - 1$$

which can be written in matrix form as

$$\begin{bmatrix} s + 1 & -s \\ -s & \dfrac{s^2 + s + 2}{s} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1(s) \\ \mathbf{I}_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s + 4}{s} \\ \dfrac{-(s + 1)}{s} \end{bmatrix}$$

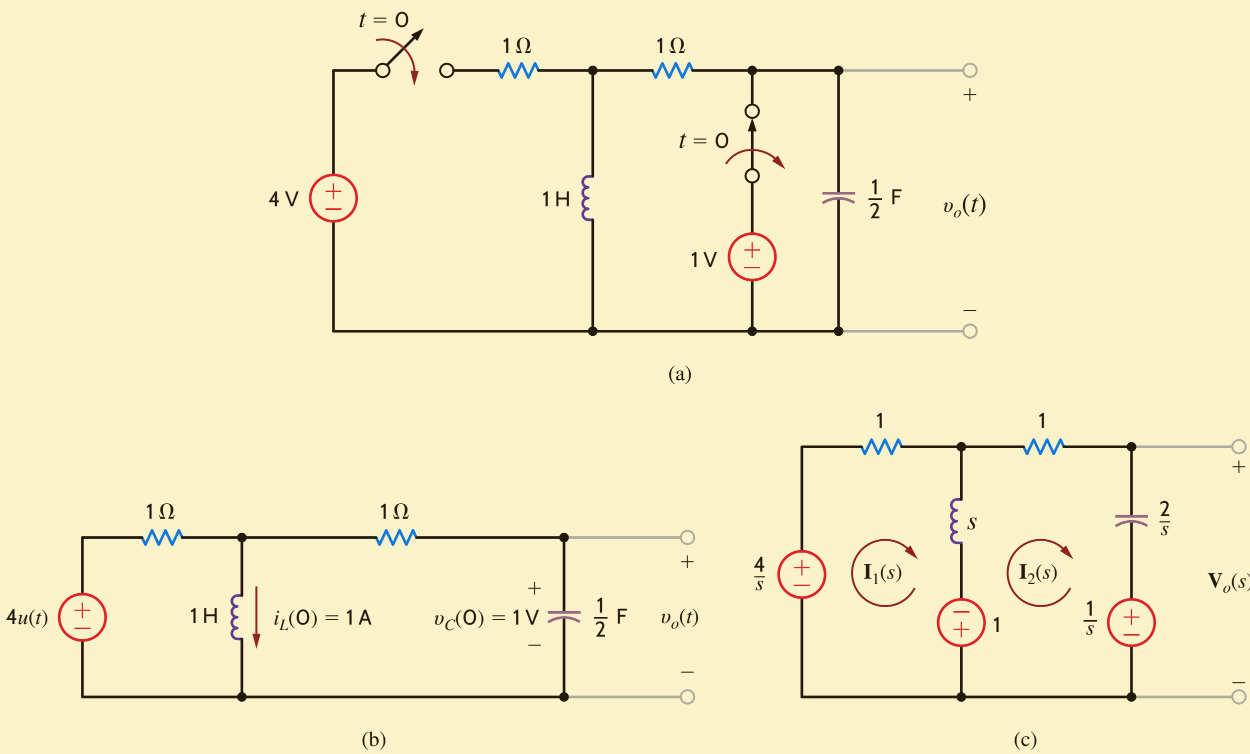


Figure 14.7

Circuits employed in Example 14.5.

Solving for the currents, we obtain

$$\begin{bmatrix} \mathbf{I}_1(s) \\ \mathbf{I}_2(s) \end{bmatrix} = \begin{bmatrix} s+1 & -s \\ -s & \dfrac{s^2+s+2}{s} \end{bmatrix} \begin{bmatrix} \dfrac{s+4}{s} \\ \dfrac{-(s+1)}{s} \end{bmatrix}$$

$$= \frac{s}{2s^2+3s+2} \begin{bmatrix} \dfrac{s^2+s+2}{s} & s \\ s & s+1 \end{bmatrix} \begin{bmatrix} \dfrac{s+4}{s} \\ \dfrac{-(s+1)}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4s^2+6s+8}{s(2s^2+3s+2)} \\ \dfrac{2s-1}{2s^2+3s+2} \end{bmatrix}$$

The output voltage is then

$$\mathbf{V}_o(s) = \frac{2}{s}\mathbf{I}_2(s) + \frac{1}{s}$$

$$= \frac{2}{s}\left(\frac{2s-1}{2s^2+3s+2}\right) + \frac{1}{s}$$

$$= \frac{s+\frac{7}{2}}{s^2+\frac{3}{2}s+1}$$

This function can be written in a partial fraction expansion as

$$\frac{s+\frac{7}{2}}{s^2+\frac{3}{2}s+1} = \frac{K_1}{s+\frac{3}{4}-j\left(\sqrt{7}/4\right)} + \frac{K_1^*}{s+\frac{3}{4}+j\left(\sqrt{7}/4\right)}$$

Evaluating the constants, we obtain

$$\left.\frac{s+\frac{7}{2}}{s+\frac{3}{4}+j(\sqrt{7}/4)}\right|_{s=-(3/4)+j(\sqrt{7}/4)} = K_1$$

$$2.14\underline{/-76.5^\circ} = K_1$$

Therefore,

$$v_o(t) = \left[4.29e^{-(3/4)t}\cos\left(\frac{\sqrt{7}}{4}t - 76.5^\circ\right)\right]u(t)\,\text{V}$$

LEARNING ASSESSMENTS

E14.7 Solve Learning Assessment E7.3 on page 261 using Laplace transforms.

ANSWER:
$i_1(t) = (1e^{-9t})u(t)$ A.

E14.8 Solve Learning Assessment E7.6 on page 268 using Laplace transforms.

ANSWER:
$v_o(t) = \left(6 - \frac{10}{3}e^{-2t}\right)u(t)$ V.

E14.9 Find $i_0(t)$ for $t > 0$ in Fig. E14.9.

ANSWER:
$i_0(t) = (-2e^{-2t} + e^{-4t})u(t)$ A.

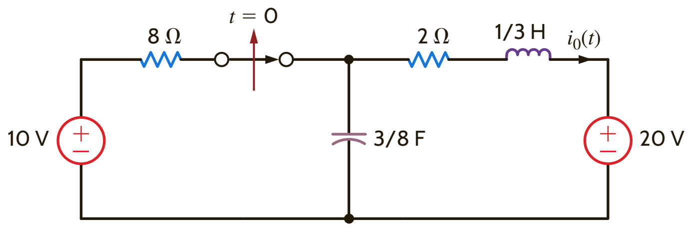


Figure E14.9

E14.10 Find $v_o(t)$ for $t > 0$ in Fig. E14.10.

ANSWER:
$v_o(t) = (-2.93e^{-4.13t} - 9.07e^{-14.54t})u(t)$ V.

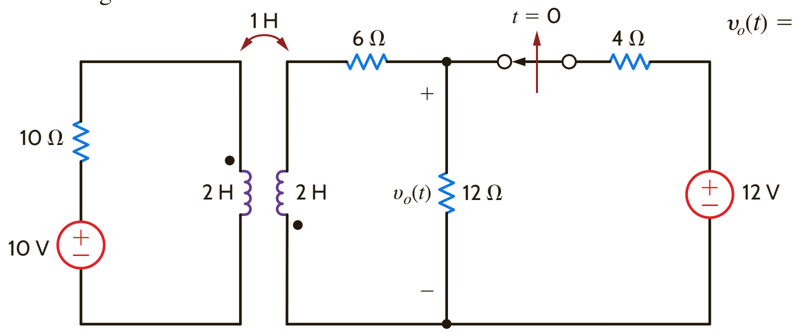


Figure E14.10

E14.11 The input voltage for the circuit in Fig. E14.11 is given in the plot. Determine the output voltage $v_o(t)$.

ANSWER:
$v_o(t) = [(4 - 4e^{-1.5t})u(t) - (4 - 4e^{-1.5(t-1)})u(t-1)]$V.

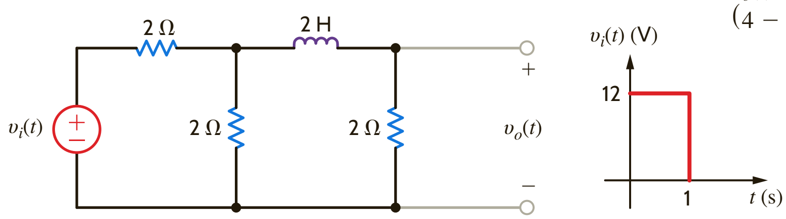


Figure E14.11