TOEPLITZ RANDOM ENCODING MR IMAGING USING COMPRESSED SENSING

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ABSTRACT

Compressed Sensing (CS), as a new framework for data acquisition and signal recovery, has been applied to accelerate conventional magnetic resonance imaging (MRI) with Fourier encoding. However, Fourier encoding is not universal and weakly spreads out the energy of most natural images. This limits the achievable reduction factors. In this paper, we propose a Toeplitz random encoding method that is universal and spreads out the image energy more evenly. The MR physical feasibility of the proposed encoding method is verified by Bloch simulation, and the superior performance of the proposed method is demonstrated in simulation results.

Index Terms— Compressed Sensing, magnetic resonance imaging, Fourier encoding, Toeplitz random encoding

1. INTRODUCTION

In conventional MRI, the data is collected in the spatial frequency domain, which is often called *k*-space and is related to the image of the object through a Fourier transform relationship [1]. A well-known problem with MRI is its long imaging time, which is usually proportional to the number of acquired data. The minimum number of data samples is determined by the Shannon-Nyquist sampling theory.

Recently, Compressed Sensing (CS) has emerged as an alternative sampling theory which goes against this conventional wisdom [2-3]. This theory allows sparse or compressible signals to be sampled at a rate that is close to their intrinsic information rate and well below their Nyquist rate, and still allows the signal to be reconstructed exactly from very few incoherent measurements by a non-linear procedure. Several methods have applied CS to conventional MRI for fast imaging [4-6]. These methods take advantage of the fact that conventional MRI meets the two conditions of CS: MR images are sparse or compressible after certain transformations and the Fourier encoding is incoherent with some sparse transformations. Therefore, the MR images can be reconstructed from far fewer samples than what determined by the Nyquist rate in conventional MRI.

Because Fourier encoding is not universal, the incoherent condition is only weakly satisfied for some sparse transforms. For example, the coarse scale of wavelet transform has its energy concentrated rather than spread out in the Fourier domain, which suggests the incoherence condition is barely satisfied [7]. Attempts have been made to use random encoding in replace of the Fourier encoding along the phase encoding direction [8, 9].

In this paper, we propose a pseudo 2D Toeplitz random encoding scheme. Specifically, the encoding matrix has a Toeplitz structure with random independent elements. With this encoding scheme, the signal energy is more spread out in the sampling domain, and thus the incoherent condition is better satisfied. The experimental results show that the reconstruction quality using the proposed encoding scheme is superior to that using Fourier encoding with the same reduction factor.

2. BACKGROUND

The central problem in compressed sensing (CS) is the recovery of a signal $\mathbf{x}$ with size n from its linear measurements $\mathbf{y}$ with size m:

$$\mathbf{y} = \mathbf{\Phi x} \tag{1}$$

where m is assumed to be much smaller than n. This underdetermined system of equations makes the recovery of the original vector $\mathbf{x}$ impossible without further assumptions. However, if $\mathbf{x}$ has a sparse representation under some sparse transformation $\mathbf{\Psi}$, one can actually recover $\mathbf{x}$ from $\mathbf{y}$ by solving a convex program

$$\text{minimize } \|\mathbf{\Psi x}\|_1 \quad \text{s.t.} \quad \mathbf{\Phi x} = \mathbf{y} \tag{2}$$

under the condition that the encoding matrix $\mathbf{\Phi}$ is incoherent with $\mathbf{\Psi}$ [10].

This feature is very desirable in MRI for a significant reduction in imaging time because it is directly related to the number of data acquired. SparseMRI is one of the methods to apply CS to conventional MRI with Cartesian sampling [4]. Considering the practical limitations, the method fully samples the readout and randomly undersamples the phase-encoding lines using a variable-density sampling scheme with denser sampling near the center of the *k*-space. The desired MR image is reconstructed by solving

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$$\arg\min_{\mathbf{x}}\{\|\mathbf{b}-\mathbf{F}_u\mathbf{f}\|_2^2+\lambda_1\|\mathbf{Wf}\|_1+\lambda_2\,\mathrm{TV}(\mathbf{f})\} \tag{3}$$

where $\mathbf{b}$ is the measured *k*-space data, $\mathbf{F}_u$ is the random subset of the rows of the Fourier encoding matrix, $\mathbf{W}$ is the sparsifying transform matrix, and $\mathrm{TV}(\cdot)$ is total variation.

Theoretically, CS with Fourier encoding should work well in some imaging scenarios, such as angiograph [4] and dynamic imaging [11], but may not be an optimal choice in other imaging scenarios due to two shortcomings of Fourier encoding. First, Fourier encoding is not universal. It is only maximum incoherent with the canonical basis or the fine scales of a wavelet transform. Therefore, the incoherence requirement of CS may not be best guaranteed when the MR images are sparse in some other transform domains. Second, for most natural images, the energy is not spread out in *k*-space but is typically concentrated in the center of *k*-space. For example, wavelet basis functions are much spread out in Fourier domain at fine scales but show very strong localization at coarse scales [7]. Therefore, Fourier encoding may not be the best choice for CS reconstruction of most MR images which are sparse with wavelet transform.

3. PROPOSED METHOD

3.1. Theory

Our objective is to design a practical, non-Fourier encoding scheme for MRI that is superior to Fourier encoding in the CS framework. The most desirable encoding schemes should (a) be universal (i.e., incoherent with any fixed sparsifying basis), (b) spread out the energy of the sparsifying basis function evenly in the sampling domain, and (c) be able to be implemented practically in MRI. Several non-Fourier matrices (e.g., IID Gaussian [12-15]) have been demonstrated in the signal processing community to satisfy requirements (a) and (b). Among them, Toeplitz-family matrices [13-15] are of significant interest because of the existence of fast algorithms and the requirement of small storage. For example, Romberg et al. proposed a Toeplitz CS matrix for circular convolutions [15]. In this setting, the signal $\mathbf{x}$ with size n is circularly convolved with a random sequence $\mathbf{h}=[h_1,h_2,\cdots h_n]\in\mathbb{R}^n$. This convolution can be written as matrix operation $\mathbf{Hx}$, where

$$\mathbf{H}=\begin{pmatrix} h_1 & h_2 & \cdots & h_n \\ h_n & h_1 & h_2 & h_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ h_2 & h_3 & \cdots & h_1 \end{pmatrix}=n^{-1/2}\mathbf{F}_1^*\sum\mathbf{F}_1 \tag{4}$$

where $\mathbf{F_1}$ is the 1D Fourier matrix, and $\sum=diag([\sigma_1,\sigma_2,\cdots\sigma_n])$ is a diagonal matrix whose non-zero entries are the 1D Fourier transform of $\mathbf{h}$, σ_i are unit magnitude complex numbers with random phases that are generated as follows:

- $i=1:\sigma_i=\pm1$ with equal probability (5)
- $2\le i<n/2+1:\sigma_i=e^{j\theta_i}$ where θ_i is uniformly at random on $[0,2\pi]$
- $i=n/2+1:\sigma_{n/2+1}=\pm1$ with equal probability
- $n/2+2\le i<n:\sigma_i=\sigma^*_{n-i+2}$

where $j=\sqrt{-1}$, superscript "*" means complex conjugate. The conjugate property of σ_i comes from the conjugate property of the 1D Fourier transform.

From Eq. (4), it is easily seen that the multiplication of $\mathbf{H}$ with $\mathbf{x}$ can be efficiently calculated by a 1D Fourier transform followed by a randomization of the phase (unit magnitude), and then an inverse 1D Fourier transform. It has been proven that this orthogonal, Toeplitz-structured $\mathbf{H}$ is incoherent with any fixed sparsifying transforms [14].

3.2. Practical encoding scheme

The above Toeplitz random encoding matrix is desirable, but difficult to implement in practical MR imaging; it requires a long RF pulse whose length is the same as the image size. In order to make it more practical, we modify the encoding matrix to be a pseudo 2D Toeplitz random encoding matrix

$$\mathbf{H}_2=n^{-1/2}\mathbf{F}_2^*\textstyle\sum_2\mathbf{F}_2 \tag{6}$$

where $\mathbf{F}_2$ is 2D Fourier matrix, $\sum_2=diag([\eta_1,\eta_2,\cdots\eta_n])$ is a diagonal matrix whose non-zero entries are designed as follows:

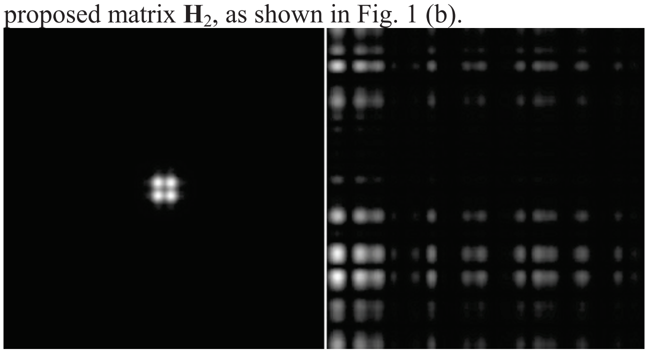
- Generate two 1D random vectors $\mathbf{r}=[r_1,r_2,\cdots r_{\sqrt{n}}]$ and $\mathbf{s}=[s_1,s_2,\cdots s_{\sqrt{n}}]$ respectively following the properties in Eq. (5).
- $\boldsymbol{\eta}=[\eta_1,\eta_2,\cdots\eta_n]$ is the Kronecker product of $\mathbf{r}$ and $\mathbf{s}$

It can be seen that η_i's are still unit magnitude complex numbers with random phase. The conjugate property of $\mathbf{r}$ and $\mathbf{s}$ corresponds to the conjugate property of 2D Fourier transform along both dimensions. The construction of $\boldsymbol{\eta}$ as a Kronecker product comes from the property that the 2D Fourier transform matrix $\mathbf{F}_2$ can be regarded as the Kronecker product of two 1D Fourier transform matricies $\mathbf{F}_1$ and $\mathbf{F}_1$. Similar to the 1D case, multiplication of $\mathbf{H}_2$ with $\mathbf{x}$ can be efficiently implemented by a 2D Fourier transform followed by a randomization of the phase (unit magnitude), and then an 2D inverse Fourier transform. The resulting $\mathbf{H}_2$ is Toeplitz-block-structured as shown in Eq. (7), where $\mathbf{R}=[R_1,R_2,\cdots R_{\sqrt{n}}]$ and $\mathbf{S}=[S_1,S_2,\cdots S_{\sqrt{n}}]$ are 1D Fourier transform of $\mathbf{r}$ and $\mathbf{s}$ respectively. We can see that each

$\sqrt{n}\times\sqrt{n}$ small block is Toeplitz-structured and $\mathbf{H}_2$ is $\sqrt{n}$ -order Toeplitz structured.

$$\begin{pmatrix}
R_1\bullet\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix} & R_2\bullet\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix} & \cdots & R_{\sqrt{n}}\bullet\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix} \\
R_{\sqrt{n}}\bullet\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix} & R_1\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \cdots & R_1\bullet\begin{bmatrix} S_1 & S_2 & \cdots & S_{\sqrt{n}} \\ S_{\sqrt{n}} & S_1 & S_2 & S_{\sqrt{n}-1} \\ \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_1 \end{bmatrix}
\end{pmatrix} \quad (7)$$

Figure 1 illustrates the motivation of the proposed encoding scheme in replace of Fourier encoding. Figure 1 (a) shows the magnitude spectrum of a 2D Daubechies-8 wavelet basis. The basis is sparse in wavelet domain, but is not spread out in Fourier domain as required by the CS incoherence condition. In contrast, this wavelet basis is much more spread out in the domain encoded by the proposed matrix $\mathbf{H}_2$, as shown in Fig. 1 (b).



(a) (b)

Fig.1 (a) Magnitude spectrum of a 2D Daubechies-8 wavelet basis. (b) Magnitude of the wavelet basis encoded by the proposed scheme.

After undersampling, the original signal $\mathbf{x}$ can be reconstructed from the undersampled data $\mathbf{y}$ by solving

$$\text{minimize } \|\mathbf{\Psi x}\|_1 \text{ s.t. } \mathbf{H}_{2,u}\mathbf{x}=\mathbf{y} \quad (8)$$

where the subscript u means undersampling and $\mathbf{\Psi}$ is a sparse transform matrix.

3.3. Implementation

The pulse sequence designed for the proposed Toeplitz random encoding is shown in Fig. 2. Basically, a random RF pulse is applied to generate a random flip angle along the phase encoding direction. This modulation in image domain corresponds to a convolution in *k*-space along the phase encoding direction. In addition, after the conventional phase encoding and frequency encoding procedure, the acquired

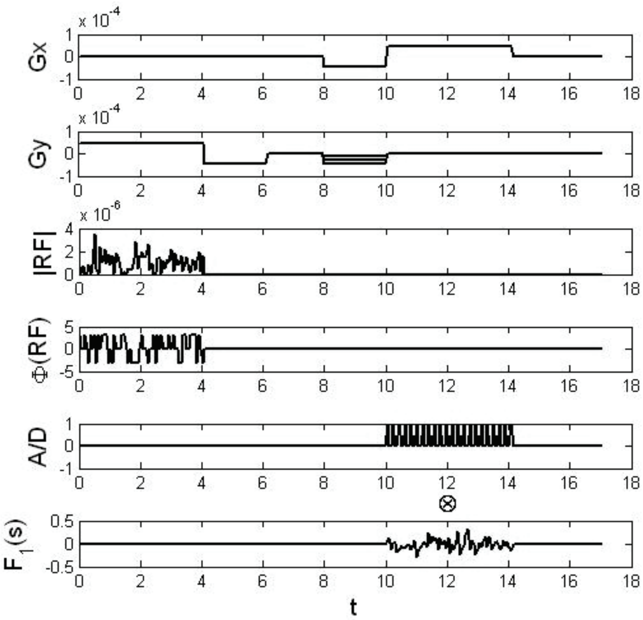


Fig.2 Timing diagram of the designed pulse sequence for the proposed encoding scheme, where $\otimes$ means convolution.

data is further convolved with a 1D random sequence. This post-processing of acquired data does not prolong the acquisition time. As a result, the post-processed data vector $\mathbf{y}$ is related to the *k*-space vector $\mathbf{x}$ of the desired image by

$$\mathbf{y}=\mathbf{H}_2\mathbf{x}\,. \quad (9)$$

where $\mathbf{H}_2$ is exactly of the same form as in Eq. (7). That is, the acquired data with post-processing is the desired image in *k*-space convolved with two separable random sequences in both directions. The two random sequences, vector R and S are the applied random RF pulse and the random sequence for post-processing of the acquired data.

In acquisition, the readout direction is fully sampled and only the phase encoding direction is reduced for improved speed. According to the property that $\mathbf{H}_2$ is incoherent with any sparse transformation, the image can be obtained by taking the Fourier transform of $\mathbf{x}$, which is reconstructed by

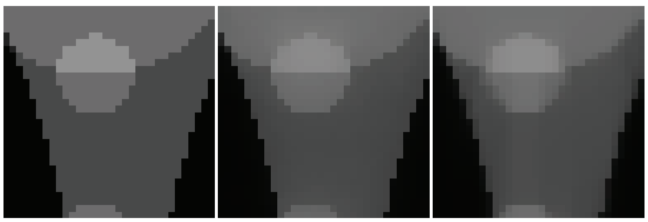
$$\text{minimize } \|\mathbf{\Psi F}_2\mathbf{x}\|_1 \text{ s.t. } \mathbf{H}_{2,u}\mathbf{x}=\mathbf{y}\,. \quad (10)$$

where $\mathbf{\Psi}$ is the sparse transform of the desired image, or equivalently, $\mathbf{\Psi F_2}$ is the sparse transform of the desired *k*-space image $\mathbf{x}$.

4. SIMULATIONS

In this section, we conducted two computer simulations to compare the performance of the proposed random encoding with the conventional Fourier encoding in the CS context. For Fourier encoding, we follow the sampling scheme used in [4], while for Toeplitz random encoding, the phase encodings are randomly undersampled. Non-linear conjugate gradient [4] was used to solve the optimization problem.

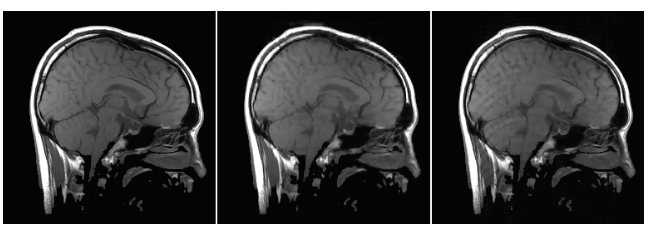
In the first simulation, the Shepp-Logan phantom was used as the original image to compare two different encoding schemes. The simulated full *k*-space data were acquired by the proposed encoding scheme. The reduced data were generated by manually removing some data to simulate a reduction factor of 2. Total variation was used as the sparse representation. The reconstructions using both encoding schemes look almost the same as the original phantom (not shown). However, when zoomed in as shown in Fig. 3, the proposed encoding method is seen to better preserve edges than Fourier encoding.



(a) (b) (c)

Fig.3 (a) Zoomed region of interest of the original Shepp-Logan phantom. (b) Reconstruction from the encoded data using the proposed method. (c) Reconstruction from Fourier encoded data.

Figure 4 shows the reconstruction of a sagittal brain image with wavelet as the sparse transform when the reduction factor is 1.5. Although neither encoding schemes can give the exact reconstruction, the proposed encoding scheme has fewer artifacts and preserves more details than Fourier encoding.



(a) (b) (c)

Fig.4 (a) Original sagittal brain image. (b) Reconstructed image using the proposed encoding. (c) Reconstructed image using the Fourier encoding.

5. CONCLUSION

In this paper, we propose a novel pseudo 2D Toeplitz random encoding scheme which is universal, able to spread out the signal energy in the sampling domain, and also practically implementable according to the MR physics. The simulations show that the proposed encoding scheme outperforms Fourier encoding in preserving details.

6. ACKNOWLEDGMENT

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