ORTOGONALIZACION GRAM-SCHMIDT

Dados $\{v_1, \dots, v_n\}$ conjunto l.i. de V un $\mathbb{K}$-e.v
Buscamos $\{q_1, \dots, q_n\}$ b.o.n tal que $\langle v_1, \dots, v_j \rangle = \langle q_1, \dots, q_j \rangle$
para $j = 1, \dots, n$

$$u_1 = v_1 \longrightarrow q_1 = u_1 / \|u_1\|_2$$

$$u_2 = v_2 - P_{u_1}(v_2) \longrightarrow q_2 = u_2 / \|u_2\|_2$$

$$u_3 = v_3 - P_{\langle u_1, u_2 \rangle}(v_3) \longrightarrow q_3 = u_3 / \|u_3\|_2$$

$$\vdots$$

$$u_k = v_k - P_{\langle u_1, \dots, u_{k-1} \rangle}(v_k) \longrightarrow q_k = u_k / \|u_k\|_2$$

$\{u_1, \dots, u_n\}$ base ortogonal
$\{q_1, \dots, q_n\}$ b.o.n.

Ejemplo: ortogonalizamos la base

$$B=\left\{ \begin{array}{ccccc} v_1 & & v_2 & & v_3 \\ \begin{pmatrix}1\\0\\1\end{pmatrix} & , & \begin{pmatrix}1\\0\\0\end{pmatrix} & , & \begin{pmatrix}2\\1\\0\end{pmatrix} \end{array} \right\}$$

1) $u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

2) $u_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}1\\0\\0\end{pmatrix}$

3) $u_3 = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{\langle u_1, u_2\rangle}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_2}\begin{pmatrix}2\\1\\0\end{pmatrix}$

Ejemplo: ortogonalizamos la base $\left\{ \overset{v_1}{\begin{pmatrix}1\\0\\1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}2\\1\\0\end{pmatrix}} \right\}$

1) $u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

2) $u_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \dfrac{(1\ 0\ 1)\begin{pmatrix}1\\0\\0\end{pmatrix}}{(1\ 0\ 1)\begin{pmatrix}1\\0\\1\end{pmatrix}} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} =$

$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \dfrac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1/2\\0\\-1/2\end{pmatrix}$

3) $u_3 = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{\langle u_1, u_2\rangle}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_2}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \dfrac{(1\ 0\ 1)\begin{pmatrix}2\\1\\0\end{pmatrix}}{\|(1\ 0\ 1)\|_2^2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \dfrac{(\frac{1}{2}\ 0\ -\frac{1}{2})\begin{pmatrix}2\\1\\0\end{pmatrix}}{\|(1/2\ 0\ -1/2)\|_2^2} \cdot \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix}$

$= \begin{pmatrix}2\\1\\0\end{pmatrix} - \dfrac{2}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \dfrac{1}{1/2}\begin{pmatrix}1/2\\0\\-1/2\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Ejemplo: ortogonalizamos la base $\left\{ \overset{v_1}{\begin{pmatrix}1\\0\\1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}2\\1\\0\end{pmatrix}} \right\}$

1) $u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ $\qquad q_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \frac{1}{\|u_1\|_2} = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

2) $u_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{(1\ 0\ 1)\begin{pmatrix}1\\0\\0\end{pmatrix}}{(1\ 0\ 1)\begin{pmatrix}1\\0\\1\end{pmatrix}} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} =$

$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1/2\\0\\-1/2\end{pmatrix}$ $\qquad q_2 = \begin{pmatrix}1/2\\0\\-1/2\end{pmatrix} \frac{1}{\|u_2\|_2} = \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}$

3) $u_3 = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{\langle u_1, u_2\rangle}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_1}\begin{pmatrix}2\\1\\0\end{pmatrix} - P_{u_2}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{(1\ 0\ 1)\begin{pmatrix}2\\1\\0\end{pmatrix}}{\|(1\ 0\ 1)\|_2^2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{(\frac{1}{2}\ 0\ -\frac{1}{2})\begin{pmatrix}2\\1\\0\end{pmatrix}}{\|(1/2\ 0\ -1/2)\|_2^2} \cdot \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix}$

$= \begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{1/2}\begin{pmatrix}1/2\\0\\-1/2\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\qquad q_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ B = \Big\{ & \begin{pmatrix}1\\0\\1\end{pmatrix}, & \begin{pmatrix}1\\0\\0\end{pmatrix}, & \begin{pmatrix}2\\1\\0\end{pmatrix} \Big\} \end{array} \longrightarrow \{q_1, q_2, q_3\} = \left\{ \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \text{ b.o.n}$$

Sabemos que $\langle v_1 \rangle = \langle q_1 \rangle$, $\langle v_1, v_2 \rangle = \langle q_1, q_2 \rangle$, $\langle v_1, v_2, v_3 \rangle = \langle q_1, q_2, q_3 \rangle$

$$B = \left\{ \overset{v_1}{\begin{pmatrix}1\\0\\1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}2\\1\\0\end{pmatrix}} \right\} \longrightarrow \{q_1, q_2, q_3\} = \left\{ \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \text{ b.o.n}$$

Sabemos que $\langle v_1 \rangle = \langle q_1 \rangle$, $\langle v_1, v_2 \rangle = \langle q_1, q_2 \rangle$, $\langle v_1, v_2, v_3 \rangle = \langle q_1, q_2, q_3 \rangle$

$$\left\{ \begin{aligned}
v_1 &= \begin{pmatrix}1\\0\\1\end{pmatrix} = \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} = \sqrt{2}\, q_1 \\
v_2 &= \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix} = \frac{\sqrt{2}}{2} q_1 + \frac{\sqrt{2}}{2} q_2 \\
v_3 &= \begin{pmatrix}2\\1\\0\end{pmatrix} = \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} + \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = \underbrace{\sqrt{2}}_{= q_1^t v_3}\, q_1 + \underbrace{\sqrt{2}}_{= q_2^t v_3}\, q_2 + \underbrace{q_3}_{q_3^t v_3}
\end{aligned} \right.$$

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ B = \Bigg\{ & \begin{pmatrix}1\\0\\1\end{pmatrix}, & \begin{pmatrix}1\\0\\0\end{pmatrix}, & \begin{pmatrix}2\\1\\0\end{pmatrix} \Bigg\} \end{array} \longrightarrow \{q_1, q_2, q_3\} = \left\{ \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \text{ s.o.n}$$

Sabemos que $\langle v_1 \rangle = \langle q_1 \rangle$, $\langle v_1, v_2 \rangle = \langle q_1, q_2 \rangle$, $\langle v_1, v_2, v_3 \rangle = \langle q_1, q_2, q_3 \rangle$

$$\begin{cases} v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} = \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} = \sqrt{2}\, q_1 \\[2ex] v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix} = \frac{\sqrt{2}}{2} q_1 + \frac{\sqrt{2}}{2} q_2 \\[2ex] v_3 = \begin{pmatrix}2\\1\\0\end{pmatrix} = \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} + \sqrt{2}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\, q_1 + \sqrt{2}\, q_2 + q_3 \end{cases}$$

$$\left(v_1 \middle| v_2 \middle| v_3 \right) = \begin{pmatrix}1 & 1 & 2\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix} = \underbrace{\left(q_1 \middle| q_2 \middle| q_3 \right)}_{\substack{\text{matriz}\\ \text{"ortogonal"}}} \underbrace{\begin{pmatrix}\sqrt{2} & \sqrt{2}/2 & \sqrt{2}\\0 & \sqrt{2}/2 & \sqrt{2}\\0 & 0 & 1\end{pmatrix}}_{\substack{\text{matriz}\\ \text{triangular superior.}}} = QR \rightarrow \underline{\text{factorización } QR}$$

MATRICES ORTOGONALES / UNITARIAS

Una matriz $Q \in \mathbb{R}^{n \times n}$ se dice **ortogonal** si $Q^{-1} = Q^t$

Son equivalentes:

1. $Q^{-1} = Q^t$
2. Las columnas de Q forman una b.o.n
3. Las filas de Q forman una b.o.n
4. $\|Qx\|_2 = \|x\|_2 \quad \forall x \in \mathbb{R}^n$

En $\mathbb{C}^{n \times n}$ las llamamos matrices **unitarias**: $Q^{-1} = Q^*$

Algunas propiedades de matrices unitarias.

Sea $Q \in \mathbb{K}^{n \times n}$ unitaria.

- El producto de matrices unitarias es unitaria.
- $\|Q\|_2 = 1$
- $\mathrm{cond}_2(Q) = 1$
- $|\det(Q)| = 1$

PROYECTORES

Consideremos la siguiente transformación lineal

$P: \mathbb{R}^2 \to \mathbb{R}^2$

$P(1,1) = (1,1)$
$P(2,1) = (1,1)$

Analizar la t.l. y hallar $[P]_E$

PROYECTORES

Def: Una t.l. $P: V \to V$ se dice **proyector** si $\underbrace{P \circ P}_{P^2} = P$

Proposición: $P: V \to V$ es proyector $\iff P(v) = v \ \forall v \in \operatorname{Im}(P)$

Se cumple:

1. Si $P: V \to V$ es proyector $\Rightarrow \operatorname{Nu}(P) \cap \operatorname{Im}(P) = \{0\}$
2. Si $P: V \to V$ es proyector $\Rightarrow v - P(v) \in \operatorname{Nu}(P) \ \forall v \in V$
3. Si $P: V \to V$ es proyector $\Rightarrow \operatorname{Nu}(P) \oplus \operatorname{Im}(P) = V$

Def: Un proyector P se dice ortogonal si

$$Nu(P) \perp Im(P)$$

P es un proyector ortogonal si y sólo si $[P]^t = [P]$

Ejemplos

$B = \{(1,1,0), (0,1,1), (1,1,1)\}$

Definir P proyector tal que

$Im P = \langle (1\,1\,0), (0\,1\,1) \rangle$

$Nu P = \langle (1\,1\,1) \rangle$

Construir proyector P tal que

$Im P = \langle (1,2,3), (0,1,0) \rangle$

¿Cómo lo definiría si además P debe ser ortogonal?

QR vía reflexiones de Householder.

Def Una matriz $H \in \mathbb{R}^{n \times n}$ se dice matriz de Householder si $\exists\, u \in \mathbb{R}^n$, $\|u\|_2 = 1$ tal que

$$H = I - 2uu^t$$

Obs.: si $\|u\|_2 \neq 1$ también podemos definirla

como $$H = I - 2\frac{uu^t}{\underbrace{u^t u}_{\|u\|_2^2}} = I - 2\frac{u}{\|u\|_2}\frac{u^t}{\|u\|_2}$$

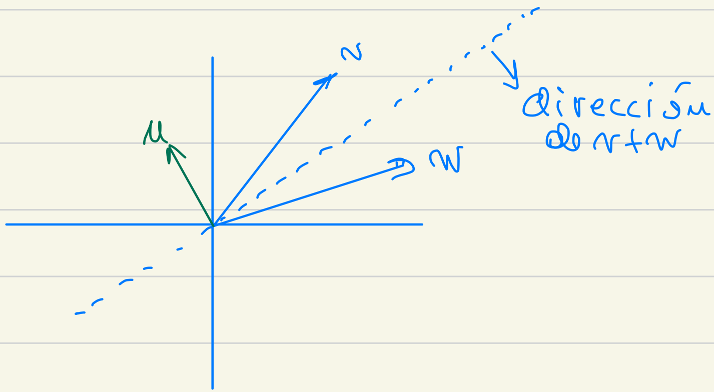
Propiedad: H es ortogonal y simétrica.

Teorema Sean $v, w \in \mathbb{R}^n$ tal que $\|v\|_2 = \|w\|_2$ y sea $u = \dfrac{v-w}{\|v-w\|_2}$ con $H = I - 2uu^t$ la matriz de Householder asociada.

Entonces se cumple: $Hv = w$ y $Hw = v$.

Dem $Hv = \left(I - 2\dfrac{(v-w)(v-w)^t}{\|v-w\|_2^2}\right)v = v - 2\dfrac{(v-w)(v-w)^t v}{\|v-w\|_2^2} =$

$$= v - \frac{\cancel{2}(v-w)\cancel{(v-w)^t v}}{\cancel{2}\,\cancel{(v-w)^t v}} = v - (v-w) = w.$$



$$u = \frac{v-w}{\|v-w\|_2} \qquad \|v\|_2 = \|w\|_2$$

$$H = I - 2uu^t$$

- $v-w \perp v+w$
- H refleja de una dirección ortogonal a u

Efecto esperado: $H(v+w) = v+w$

$Hu = -u$

Hallar factorización QR de $A = \begin{pmatrix} 1/2 & 1 \\ \sqrt{3}/2 & 0 \end{pmatrix}$

triangulando con reflexiones de Householder.

TRIANGULACION CON REFLEXIONES

$$\underset{A}{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}} \xrightarrow{} H_1 A \quad \underset{A_1}{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}} \xrightarrow{} H_2 A_1 \quad \underset{A_2}{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}} \xrightarrow{} H_3 A_2 \quad \overbrace{\underset{R}{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}}}^{\text{triangular superior}}$$

$$H_1 \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \tilde{H}_2 \begin{pmatrix} * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \qquad \tilde{H}_3 \begin{pmatrix} * \\ * \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$H_2 = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \tilde{H}_2 & \\ 0 & & & \end{array}\right) \qquad H_3 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\boxed{\tilde{H}_3}} \\ 0 & 0 & & \end{array}\right)$$

$$H_1 \begin{array}{c} A \\ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \end{array} = \begin{array}{c} A_1 \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \end{array} \qquad H_2 \begin{array}{c} A_1 \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \end{array} = \begin{array}{c} A_2 \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \end{array}$$

$$H_1 A = A_1 \quad \Longrightarrow \quad H_2 (H_1 A) = A_2 \quad \Longrightarrow$$

$$H_3 \begin{array}{c} A_2 \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \end{array} = \begin{array}{c} R \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \end{array}$$

H_1, H_2, H_3 son ortogonales y simétricas $\Longrightarrow H_i^{-1} = H_i$, $i = 1, 2, 3$

$$\Longrightarrow H_3 (H_2 H_1 A) = R \quad \Longrightarrow \quad A = \overbrace{H_1^{-1} H_2^{-1} H_3^{-1}}^{Q} R$$

factorización QR

Ventajas QR via Householder

- si $H = I - 2uu^t \Longrightarrow Hx = \underbrace{x -}_{n \text{ restas}} \overbrace{2u}^{n \text{ productos}} \underbrace{(u^t x)}_{\substack{n \text{ productos} \\ n-1 \text{ sumas}}}$ $\quad$ Hx es $O(n)$

  $\underline{H \in \mathbb{R}^{n \times n}}$ $\quad$ n productos

  $\boxed{\text{NO HACE FALTA CONSTRUIR } H}$

- Solo hace falta guardar u

- Resolver sistemas con QR

$$Ax = b \iff Q\underbrace{Rx}_{y} = b \longrightarrow \begin{cases} Qy = b \xrightarrow{\text{fácil!}} y = Q^t b \\ Rx = y \longrightarrow \text{sistema triangular.} \end{cases}$$

Teorema: Sea $A \in \mathbb{R}^{n \times n}$, no singular. Entonces existen únicas $Q \in \mathbb{R}^{n \times n}$ ortogonal y $R \in \mathbb{R}^{n \times n}$ triangular superior con $r_{ii} > 0$ tal que $A = QR$.