DANMARKS TEKNISKE UNIVERSITET

PROJECT ASSIGNMENT

31383 - Robotics

TEAM 2

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Introduction

The purpose of this project assignment is to practice the topics learned in the Robotics course 31383 on a realistic robotic application.

The project considers the Alto Robot from the Ramsta Robotics Company. This robot is applied in automatic water jet cleaning of livestock buildings and containers. The Alto Robot is an electrically driven spherical robot with four degrees of freedom.

The project assignment treats tasks with answering the problems by simulation, dynamics, direct kinematics, inverse kinematics, singularities, trajectory planning as well as control.

Table of contributions:

Percentage contribution of the team members for each task, as well as for the whole assignment, are listed below.

	Lukas	Joel	Martin	Kays
Problem 1	50	50	0	0
Problem 2	60	40	0	0
Problem 3	10	0	90	0
Problem 4	15	15	70	0
Problem 5	10	10	80	0
Problem 6	95	0	0	5
Problem 7	0	100	0	0
Problem 8	35	0	65	0
Problem 9	0	100	0	0
Problem 10	80	0	20	0
Problem 11	5	0	95	0
Problem 12	90	10	0	0
Problem 13	0	100	0	0
Problem 14	100	0	0	0
Problem 15	10	0	90	0
Problem 16	50	20	0	30
Problem 17	60	30	0	10
Problem 18	60	30	0	10
Problem 19	0	100	0	0
Problem 20	20	80	0	0
Problem 21	20	80	0	0
SUM	770	765	510	55
PERCENTAGE	36.6%	35.5%	24.3%	3.6%

Contents

Problem 1 - Forward Kinematics

Find the direct kinematic transformation T_4^0 for the robot manipulator.

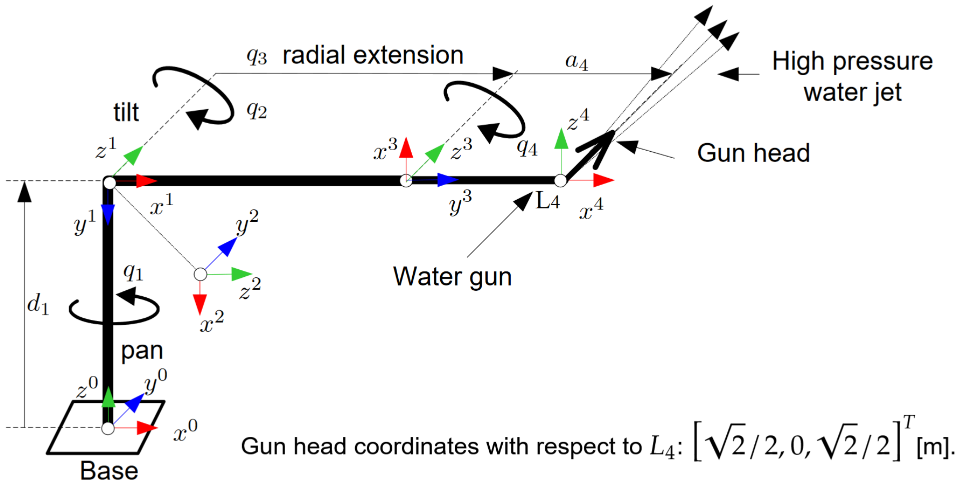


Figure 1: The RRPR-robot used in the project assignment

We fill the Denavit-Hartenberg parameters using the Figure 1.
a_i = distance between $z_{i-1}$and z_i along x_i
α_i = angle from z_{i-1} to z_i around x_i
d_i = distance from o_{i-1} to intersection of x_i and z_{i-1} along z_{i-1}
θ_i = angle from x_{i-1} to x_i around z_{i-1}

Joint	type	a	α	d	θ	initial	min	max
1	R	0	$-\pi/2$	1.5m	q_1	0	$-\pi$	π
2	R	0	$\pi/2$	0	q_2	$\pi/2$	$\pi/6$	$3\pi/4$
3	P	0	$\pi/2$	q_3	π	1.35m	1.35m	3.00m
4	R	1.02m	$\pi/2$	0	q_4	$\pi/2$	$-\pi/2$	$5\pi/4$

Table 1: Denavit-Hartenberg parameters corresponding to the robot

Using Denavit-Hartenberg convention

$$A_i = T_i^{i-1} \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & r_i\cos\theta_i \\ \sin\theta i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & r_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The solution is following:

$$A_1 = T_1^0 = \begin{bmatrix} \cos(q_1) & 0 & -\sin(q_1) & 0 \\ \sin(q_1) & 0 & \cos(q_1) & 0 \\ 0 & -1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = T_2^1 = \begin{bmatrix} \cos(q_2) & 0 & \sin(q_2) & 0 \\ \sin(q_2) & 0 & -\cos(q_2) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = T_3^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = T_4^3 = \begin{bmatrix} \cos(q_4) & 0 & \sin(q_4) & \frac{51\ \cos(q_4)}{50} \\ \sin(q_4) & 0 & -\cos(q_4) & \frac{51\ \sin(q_4)}{50} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the equation $T_4^0 = A_1 \cdot A_2 \cdot A_3 \cdot A_4$ we get the result.

$$T_4^0 = \begin{bmatrix} -cos(\theta_2+\theta_4)cos(\theta_1) & -sin(\theta_1) & -sin(\theta_2+\theta_4)cos(\theta_1) & -\frac{cos(\theta_1)}{50}(51cos(\theta_2+\theta_4)-50q_3sin(\theta_2)) \\ -cos(\theta_2+\theta_4)sin(\theta_1) & cos(\theta_1) & -sin(\theta_2+\theta_4)sin(\theta_1) & -\frac{sin(\theta_1)}{50}(51cos(\theta_2+\theta_4)-50q_3sin(\theta_2)) \\ sin(\theta_2+\theta_4) & 0 & -cos(\theta_2+\theta_4) & \frac{51}{50}sin(\theta_2+\theta_4)+q_3cos(\theta_2)+\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

Problem 2 - Inverse kinematic

Determine the inverse kinematic transformation $q = [q_1, q_2, q_3, q_4]^T = f([x_4]^0, [p_4]^0)$

The inverse kinematic was calculated using forward kinematics matrix $T_4^0(1)$ and the knowledge of $\phi = atan2\left(\frac{sin(\phi)}{cos(\phi)}\right)$. For T_4^0 we use the following convention to express matrix elements.

$$T_4^0 = \begin{bmatrix} x_1 & ? & ? & p_1 \\ x_2 & ? & ? & p_2 \\ x_3 & ? & ? & p_3 \\ x_4 & ? & ? & p_4 \end{bmatrix}$$

From the graphing of the atan function, we can see that the only valid values are inside the interval $(-\pi/2, \pi/2)$. Therefore we can't use $atan(x)$ for the II and III quadrants. For this reason, Matlab implemented atan2, which stands for the four-quadrant inverse tangent, and it works as described below.

$$\phi = atan2(y,x) = \begin{cases} atan(y/x) & \text{if } x > 0 \\ atan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq c \\ atan(y/x) - \pi & \text{if } x < 0 \text{ and } y < c \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

To compute the joint orientation q_i for each DOF of the robot arm, we must find the way how to express q_1, q_2, q_3 and q_4. From Denavit-Hartenberg table 1 we can see that $q_1 = \theta_1$, $q_2 = \theta_2$, $q_3 = d_3$ and $q_4 = \theta_4$. Our goal is to formulate equations to get the angle θ_i as a result. It can be done using the following approach.

$$atan2(y,x) \approx atan\left(\frac{y}{x}\right)$$
$$atan2\left(\frac{sin(\theta_i)}{cos(\theta_i)}\right) = atan2(tan(\theta_i)) = \theta_i$$

Keeping this in mind, now our goal is to find the right input of atan2 y and x, which will result into $\frac{sin(\theta_i)}{cos(\theta_i)}$, because this fraction is equal to $tan(\theta_i)$ and $atan2(tan(\theta_i)) = \theta_i$.

$$\begin{aligned} q_1 &= \theta_1 = atan2(p_2, p_1) \\ q_2 &= \theta_2 = atan2\left(\frac{p_1 - \frac{51}{50}x_1}{cos(q_1)}, p_3 - \frac{3}{2} - \frac{51}{50}x_3)\right) \\ q_3 &= \frac{p_3 - \frac{51}{50}x_3 - \frac{3}{2}}{cos(q_2)} \\ q_4 &= \theta_4 = atan2\left(x_3, \frac{-x_1}{cos(q_1)}\right) - q_2 \end{aligned} \quad (2)$$

Problem 3 - Joint coordinates

Find the corresponding joint coordinates for each knot-point

The path is defined by five frames $T^0_{p_1}, T^0_{p2}, T^0_{p3}, T^0_{p4}$ and T^0_{p5} through which the manipulator is supposed to move the tool.

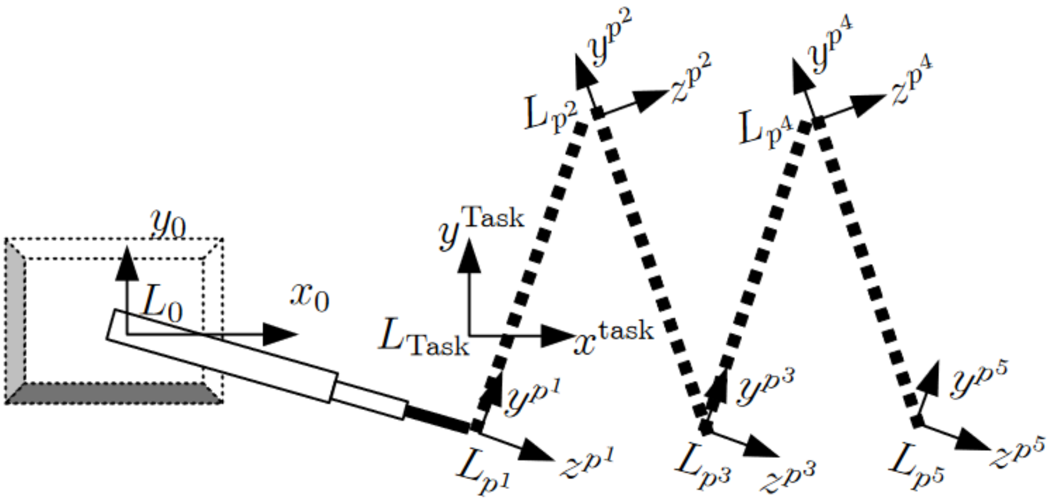


Figure 2: Work scene for the point to point movement

It is necessary to determine all knots in the base frame. Therefore the matrix T^0_{Task} is provided. The right column can be easily obtained by using the [x,y,z] coordinates where we want to move the robot. However, the left column is provided in this example, resulting in an exact solution. If the left column is not provided, inverse kinematics can result in multiple solutions, and choosing the best one is very advanced problematic, which is not included in this course.

$$T^0_{\text{Task}} = \begin{bmatrix} 1 & 0 & 0 & 1.4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{\text{Task}}_{p1} = \begin{bmatrix} 0 & & & 0 \\ 0 & ? & ? & -0.5 \\ -1 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{\text{Task}}_{p2} = \begin{bmatrix} 0 & & & 0.35 \\ 0 & ? & ? & 0.5 \\ -1 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{\text{Task}}_{p3} = \begin{bmatrix} 0 & & & 0.7 \\ 0 & ? & ? & -0.5 \\ -1 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{\text{Task}}_{p4} = \begin{bmatrix} 0 & & & 1.05 \\ 0 & ? & ? & 0.5 \\ -1 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{\text{Task}}_{p5} = \begin{bmatrix} 0 & & & 1.4 \\ 0 & ? & ? & -0.5 \\ -1 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With the knowledge of T^0_{Task} and $T^{Task}_{p_j}$ for each point, we can transform each of the knot-points into the base frame by simply using:

$$T_{Trans} = T^{p_j}_0 = T^0_{Task} \cdot T^{Task}_{p_j}$$

From these matrices, we will denote p_i and x_i the values from the last and first columns respectively.

$$q = [q_1, q_2, q_3, q_4]^T = f(([x_{p_j}]^0.[p_{p_j}]^0)),\ j = 1, 2, 3, 4, 5$$

where $([x_{p_j}]^0.[p_{p_j}]^0)$ are first and last column, respectively of $T^0_{p_j}$, j=1,2,3,4,5

This gives us the following matrices and values, including the unknown variables named b_i:

$$
\begin{aligned}
T_0^{p_1} &= \begin{bmatrix} 0 & b_1 & b_2 & \frac{7}{5} \\ 0 & b_3 & b_4 & -\frac{1}{2} \\ -1 & b_5 & b_6 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; which\ gives\ us: \; x_1 = 0, \; x_3 = -1, \; p_1 = \frac{7}{5}, \; p_2 = -\frac{1}{2}, \; p_3 = \frac{1}{5} \\
T_0^{p_2} &= \begin{bmatrix} 0 & b_1 & b_2 & \frac{7}{4} \\ 0 & b_3 & b_4 & \frac{1}{2} \\ -1 & b_5 & b_6 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; which\ gives\ us: \; x_1 = 0, \; x_3 = -1, \; p_1 = \frac{7}{4}, \; p_2 = \frac{1}{2}, \; p_3 = \frac{1}{5} \\
T_0^{p_3} &= \begin{bmatrix} 0 & b_1 & b_2 & \frac{21}{10} \\ 0 & b_3 & b_4 & -\frac{1}{2} \\ -1 & b_5 & b_6 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; which\ gives\ us: \; x_1 = 0, \; x_3 = -1, \; p_1 = \frac{21}{10}, \; p_2 = -\frac{1}{2}, \; p_3 = \frac{1}{5} \\
T_0^{p_4} &= \begin{bmatrix} 0 & b_1 & b_2 & \frac{49}{20} \\ 0 & b_3 & b_4 & \frac{1}{2} \\ -1 & b_5 & b_6 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; which\ gives\ us: \; x_1 = 0, \; x_3 = -1, \; p_1 = \frac{49}{20}, \; p_2 = \frac{1}{2}, \; p_3 = \frac{1}{5} \\
T_0^{p_5} &= \begin{bmatrix} 0 & b_1 & b_2 & \frac{14}{5} \\ 0 & b_3 & b_4 & -\frac{1}{2} \\ -1 & b_5 & b_6 & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; which\ gives\ us: \; x_1 = 0, \; x_3 = -1, \; p_1 = \frac{14}{5}, \; p_2 = -\frac{1}{2}, \; p_3 = \frac{1}{5}
\end{aligned} \tag{3}
$$

Now that we have these values, we can use them in equations from problem 2 (2). This finally gives us the joint coordinates for each knot-point:

$$
\begin{aligned}
q^{p_1} &= \begin{bmatrix} q_1^{p_1} \\ q_2^{p_1} \\ q_3^{p_1} \\ q_4^{p_1} \end{bmatrix} = \begin{bmatrix} -0.3430 \\ 1.7570 \\ 1.5127 \\ 2.9554 \end{bmatrix} \\
q^{p_2} &= \begin{bmatrix} q_1^{p_2} \\ q_2^{p_2} \\ q_3^{p_2} \\ q_4^{p_2} \end{bmatrix} = \begin{bmatrix} 0.2783 \\ 1.7234 \\ 1.8414 \\ 2.9889 \end{bmatrix} \\
q^{p_3} &= \begin{bmatrix} q_1^{p_3} \\ q_2^{p_3} \\ q_3^{p_3} \\ q_4^{p_3} \end{bmatrix} = \begin{bmatrix} -0.2337 \\ 1.6998 \\ 2.1768 \\ 3.0126 \end{bmatrix} \\
q^{p_4} &= \begin{bmatrix} q_1^{p_4} \\ q_2^{p_4} \\ q_3^{p_4} \\ q_4^{p_4} \end{bmatrix} = \begin{bmatrix} 0.2013 \\ 1.6823 \\ 2.5161 \\ 3.0301 \end{bmatrix} \\
q^{p_5} &= \begin{bmatrix} q_1^{p_5} \\ q_2^{p_5} \\ q_3^{p_5} \\ q_4^{p_5} \end{bmatrix} = \begin{bmatrix} -0.1767 \\ 1.6689 \\ 2.8580 \\ 3.0435 \end{bmatrix}
\end{aligned} \tag{4}
$$

Problem 4 - Quintic interpolation

Determine the coefficients a_{ij} where i ∈ 1,2,...,16 and j ∈ 0,1,2,3,4,5. The time is t = 0 seconds at the beginning of each segment.

The (inverse) transformed knot-points found in problem 3 are to be connected by means of 5.order polynomials. In the knot-points the joint speed and acceleration are 0 rad/s and 0 rad/s^2. Each segment has a duration time of 2 seconds for each run trough. The polynomials are defined by:

Segment 1 : $\{1\} \rightarrow \{2\}, 0 \leq t \leq 2$

$$\begin{aligned}
q_1(t) &= f1(t) = a15 \cdot t^5 + a14 \cdot t^4 + a13 \cdot t^3 + a12 \cdot t^2 + a11 \cdot t + a10 \\
q_2(t) &= f2(t) = a25 \cdot t^5 + a24 \cdot t^4 + a23 \cdot t^3 + a22 \cdot t^2 + a21 \cdot t + a20 \\
q_3(t) &= f3(t) = a35 \cdot t^5 + a34 \cdot t^4 + a33 \cdot t^3 + a32 \cdot t^2 + a31 \cdot t + a30 \\
q_4(t) &= f4(t) = a45 \cdot t^5 + a44 \cdot t^4 + a43 \cdot t^3 + a42 \cdot t^2 + a41 \cdot t + a40
\end{aligned}$$

Segment 2 : $\{2\} \rightarrow \{3\}, 0 \leq t \leq 2$

$$\begin{aligned}
q_1(t) &= f5(t) = a55 \cdot t^5 + a54 \cdot t^4 + a53 \cdot t^3 + a52 \cdot t^2 + a51 \cdot t + a50 \\
q_2(t) &= f6(t) = a65 \cdot t^5 + a64 \cdot t^4 + a63 \cdot t^3 + a62 \cdot t^2 + a61 \cdot t + a60 \\
q_3(t) &= f7(t) = a75 \cdot t^5 + a74 \cdot t^4 + a73 \cdot t^3 + a72 \cdot t^2 + a71 \cdot t + a70 \\
q_4(t) &= f8(t) = a85 \cdot t^5 + a84 \cdot t^4 + a83 \cdot t^3 + a82 \cdot t^2 + a81 \cdot t + a80
\end{aligned}$$

Segment 3 : $\{3\} \rightarrow \{4\}, 0 \leq t \leq 2$

$$\begin{aligned}
q_1(t) &= f9(t) = a95 \cdot t^5 + a94 \cdot t^4 + a93 \cdot t^3 + a92 \cdot t^2 + a91 \cdot t + a90 \\
q_2(t) &= f10(t) = a105 \cdot t^5 + a104 \cdot t^4 + a103 \cdot t^3 + a102 \cdot t^2 + a101 \cdot t + a100 \\
q_3(t) &= f11(t) = a115 \cdot t^5 + a114 \cdot t^4 + a113 \cdot t^3 + a112 \cdot t^2 + a111 \cdot t + a110 \\
q_4(t) &= f12(t) = a125 \cdot t^5 + a124 \cdot t^4 + a123 \cdot t^3 + a122 \cdot t^2 + a121 \cdot t + a120
\end{aligned}$$

Segment 4 : $\{4\} \rightarrow \{5\}, 0 \leq t \leq 2$

$$\begin{aligned}
q_1(t) &= f13(t) = a135 \cdot t^5 + a134 \cdot t^4 + a133 \cdot t^3 + a132 \cdot t^2 + a131 \cdot t + a130 \\
q_2(t) &= f14(t) = a145 \cdot t^5 + a144 \cdot t^4 + a143 \cdot t^3 + a142 \cdot t^2 + a141 \cdot t + a140 \\
q_3(t) &= f15(t) = a155 \cdot t^5 + a154 \cdot t^4 + a153 \cdot t^3 + a152 \cdot t^2 + a151 \cdot t + a150 \\
q_4(t) &= f16(t) = a165 \cdot t^5 + a164 \cdot t^4 + a163 \cdot t^3 + a162 \cdot t^2 + a161 \cdot t + a160
\end{aligned}$$

We need to solve the system of equations that satisfies Ax=b:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will use Matlab "\" operator that represents matrix left division to solve for x. In our case, matrix A is represented by matrix T_{time}. For quintic interpolation seen in the project assignment, we can derive a matrix T_{time} as: [1]

$$T_{time} = \begin{bmatrix} 1 & t_{in} & t_{in}^2 & t_{in}^3 & t_{in}^4 & t_{in}^5 \\ 0 & 1 & 2t_{in} & 3t_{in}^2 & 4t_{in}^3 & 5t_{in}^4 \\ 0 & 0 & 2 & 6t_{in} & 12t_{in}^2 & 20t_{in}^3 \\ 1 & t_A & t_A^2 & t_A^3 & t_A^4 & t_A^5 \\ 0 & 1 & 2t_A & 3t_A^2 & 4t_A^3 & 5t_A^4 \\ 0 & 0 & 2 & 6t_A & 12t_A^2 & 20t_A^3 \end{bmatrix}$$

Time t_{in}, t_A are in and out times for each segment. Since the duration of one segment cleaning is always 2 seconds, t_{in}=0, t_A=2 for all case scenarios. This greatly simplifies T_{time}:

$$T_{time} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 0 & 1 & 4 & 12 & 32 & 80 \\ 0 & 0 & 2 & 12 & 48 & 160 \end{bmatrix}$$

Furthermore, unknown values x are represented as sixteen vectors $a \in R^6$ (4 segments, 4 joints for each segments, resulting in 16 vectors). Each vector a is represented by a row in matrix A, since it is easier to implement functions with better readability.

$$A = \begin{bmatrix} a_{15} & a_{14} & a_{13} & a_{12} & a_{11} & a_{10} \\ a_{25} & a_{24} & a_{23} & a_{22} & a_{21} & a_{20} \\ a_{35} & a_{34} & a_{33} & a_{32} & a_{31} & a_{30} \\ a_{45} & a_{44} & a_{43} & a_{42} & a_{41} & a_{40} \\ a_{55} & a_{54} & a_{53} & a_{52} & a_{51} & a_{50} \\ a_{65} & a_{64} & a_{63} & a_{62} & a_{61} & a_{60} \\ a_{75} & a_{74} & a_{73} & a_{72} & a_{71} & a_{70} \\ a_{85} & a_{84} & a_{83} & a_{82} & a_{81} & a_{80} \\ a_{95} & a_{94} & a_{93} & a_{92} & a_{91} & a_{90} \\ a_{105} & a_{104} & a_{103} & a_{102} & a_{101} & a_{100} \\ a_{115} & a_{114} & a_{113} & a_{112} & a_{111} & a_{110} \\ a_{125} & a_{124} & a_{123} & a_{122} & a_{121} & a_{120} \\ a_{135} & a_{134} & a_{133} & a_{132} & a_{131} & a_{130} \\ a_{145} & a_{144} & a_{143} & a_{142} & a_{141} & a_{140} \\ a_{155} & a_{154} & a_{153} & a_{152} & a_{151} & a_{150} \\ a_{165} & a_{164} & a_{163} & a_{162} & a_{161} & a_{160} \end{bmatrix}$$

The right part of equation Ax=b is a vector of joint angles q, its velocities v and accelerations a, where q_in is at the beginning of the segment, and q_A is at the end of each segment. Since the cleaning head is desired to stop at the end of each segment, in and out velocities and accelerations will be equal to zero.

$$Q = \begin{bmatrix} q_{in} \\ v_{in} \\ a_{in} \\ q_A \\ v_A \\ a_A \end{bmatrix} = \begin{bmatrix} q_{in} \\ 0 \\ 0 \\ q_A \\ 0 \\ 0 \end{bmatrix}$$

Summed up, instead of notation Ax=b, we will use:

$$T_{time} \cdot a = Q$$

The matrix T_{time} is known, and q values that we will use are obtained in problem 3 (4).

```
function [A] = problem4(A, segment, Q_in, Q_A)
    Time_matrix = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 2 0 0 0; 1 2 4 8 16 32; 0 1 4 12 32 80; 0 0 2 12 48 160];
    % we need to solve Ax=B -> x = A\B. In our case its Time_matrix*a=Q ->
    % -> a=Time_matrix\Q
    % Q = right side of equation = [Q_in;v_in;a_in;Q_A;v_A;a_A] but
    % velocity -> acceleration is zero

    for col = 1:4
        Q = [Q_in(col); 0; 0; Q_A(col); 0; 0];
        A(segment*4-4+col,:) = Time_matrix \ Q;
    end
end
```

Figure 3: Matlab function for problem 4

Use of this function is seen above. The unknown vector a is calculated as followings and put into the matrix A for easier implemention in Matlab.

$$T_{Time} \cdot a = Q$$
$$a = T_{Time} \setminus q$$

Finally, the A matrix of unknown vectors is:

$$A = \begin{pmatrix} 0.116 & -0.582 & 0.777 & 0 & 0 & -0.343 \\ -0.006 & 0.031 & -0.042 & 0 & 0 & 1.757 \\ 0.062 & -0.308 & 0.411 & 0 & 0 & 1.513 \\ 0.006 & -0.031 & 0.042 & 0 & 0 & 2.955 \\ -0.096 & 0.48 & -0.64 & 0 & 0 & 0.278 \\ -0.004 & 0.022 & -0.03 & 0 & 0 & 1.723 \\ 0.063 & -0.314 & 0.419 & 0 & 0 & 1.841 \\ 0.004 & -0.022 & 0.03 & 0 & 0 & 2.989 \\ 0.082 & -0.408 & 0.544 & 0 & 0 & -0.234 \\ -0.003 & 0.016 & -0.022 & 0 & 0 & 1.7 \\ 0.064 & -0.318 & 0.424 & 0 & 0 & 2.177 \\ 0.003 & -0.016 & 0.022 & 0 & 0 & 3.013 \\ -0.071 & 0.354 & -0.473 & 0 & 0 & 0.201 \\ -0.003 & 0.013 & -0.017 & 0 & 0 & 1.682 \\ 0.064 & -0.321 & 0.427 & 0 & 0 & 2.516 \\ 0.003 & -0.013 & 0.017 & 0 & 0 & 3.03 \end{pmatrix}$$

Problem 5 - Cartesian straight line path

Determine the coefficients a_{ij} where i $\in$ 1,2,...,8 and j $\in$ 0,1,2,3,4,5 when the robot manipulator is now supposed to be moved along straight lines between the knot-points (Cartesian path control).

For the Cartesian path planner, we use 5. order polynomials similar to the ones we used for the joint path planner in Problem 4. These polynomials are defined by:

Segment 1 : $\{1\} \rightarrow \{2\}, 0 \leq t \leq 2$

$$p_X(t) = f1(t) = a15 \cdot t^5 + a14 \cdot t^4 + a13 \cdot t^3 + a12 \cdot t^2 + a11 \cdot t + a10$$
$$p_Y(t) = f2(t) = a25 \cdot t^5 + a24 \cdot t^4 + a23 \cdot t^3 + a22 \cdot t^2 + a21 \cdot t + a20$$

Segment 2 : $\{2\} \rightarrow \{3\}, 0 \leq t \leq 2$

$$p_X(t) = f3(t) = a35 \cdot t^5 + a34 \cdot t^4 + a33 \cdot t^3 + a32 \cdot t^2 + a31 \cdot t + a30$$
$$p_Y(t) = f4(t) = a45 \cdot t^5 + a44 \cdot t^4 + a43 \cdot t^3 + a42 \cdot t^2 + a41 \cdot t + a40$$

Segment 3 : $\{3\} \rightarrow \{4\}, 0 \leq t \leq 2$

$$p_X(t) = f5(t) = a55 \cdot t^5 + a54 \cdot t^4 + a53 \cdot t^3 + a52 \cdot t^2 + a51 \cdot t + a50$$
$$p_Y(t) = f6(t) = a65 \cdot t^5 + a64 \cdot t^4 + a63 \cdot t^3 + a62 \cdot t^2 + a61 \cdot t + a60$$

Segment 4 : $\{4\} \rightarrow \{5\}, 0 \leq t \leq 2$

$$p_X(t) = f7(t) = a75 \cdot t^5 + a74 \cdot t^4 + a73 \cdot t^3 + a72 \cdot t^2 + a71 \cdot t + a70$$
$$p_Y(t) = f8(t) = a85 \cdot t^5 + a84 \cdot t^4 + a83 \cdot t^3 + a82 \cdot t^2 + a81 \cdot t + a80$$

Where p_x, p_y are the x, y coordinates of the five knot-points in Fig. 1 with respect to the robot base frame.

In this problem, we don't need computed q values. Instead, we need coordinates of end-effector in the known points. The Z coordinate is always the same, so we need only x, y. We have exactly that computed by $T_0^{p_i}$ in problem 3 (3). We know that the last column of this homogeneous transformation matrix is equal to translation, and the first and second row in the last column is equal to x and y.

Then, we will compute $T_{Time} * a = p_x$ and $T_{Time} * a = p_y$ for each segment similarly as last time. For this computation we will use matrix A_c that denotes matrix of unknowns for Cartesian space.

$$A_c = \begin{bmatrix} a_{15} & a_{14} & a_{13} & a_{12} & a_{11} & a_{10} \\ a_{25} & a_{24} & a_{23} & a_{22} & a_{21} & a_{20} \\ a_{35} & a_{34} & a_{33} & a_{32} & a_{31} & a_{30} \\ a_{45} & a_{44} & a_{43} & a_{42} & a_{41} & a_{40} \\ a_{55} & a_{54} & a_{53} & a_{52} & a_{51} & a_{50} \\ a_{65} & a_{64} & a_{63} & a_{62} & a_{61} & a_{60} \\ a_{75} & a_{74} & a_{73} & a_{72} & a_{71} & a_{70} \\ a_{85} & a_{84} & a_{83} & a_{82} & a_{81} & a_{80} \end{bmatrix}$$

Using the function

```
function [A_c] = problem5(A_c, segment, P_in, P_A)
    Time_matrix = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 2 0 0 0; 1 2 4 8 16 32; 0 1 4 12 32 80; 0 0 2 12 48 160];

    for col = 1:2
        P = [P_in(col); 0; 0; P_A(col); 0; 0];
        A_c(segment*2-2+col,:) = Time_matrix \ P;
    end
end
```

Figure 4: Matlab function for problem 5

We receive the final result equal to

$$A_c = \begin{pmatrix} 0.066 & -0.328 & 0.438 & 0 & 0 & 1.4 \\ 0.188 & -0.938 & 1.25 & 0 & 0 & -0.5 \\ 0.066 & -0.328 & 0.438 & 0 & 0 & 1.75 \\ -0.188 & 0.938 & -1.25 & 0 & 0 & 0.5 \\ 0.066 & -0.328 & 0.438 & 0 & 0 & 2.1 \\ 0.188 & -0.938 & 1.25 & 0 & 0 & -0.5 \\ 0.066 & -0.328 & 0.438 & 0 & 0 & 2.45 \\ -0.188 & 0.938 & -1.25 & 0 & 0 & 0.5 \end{pmatrix}$$

Problem 6 - Jacobian

Determine the Jacobian matrix of the manipulator. Use the Jacobian matrix to determine whether there are any singularities along the path in Cartesian space found in Problem 5.

Each column in Jacobian represents one joint. Therefore with 4 joints, the Jacobian will consist of 4 columns.

For prismatic joint, the Jacobian is calculated as follows:

$$J_p = \begin{bmatrix} z_{i-1} \\ o_{3\times 1} \end{bmatrix}$$

For revolute joint, the formula is below:

$$J_r = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}$$

where
z_{i-1} is a vector $[r_{13}\, r_{23}\, r_{33}]^T$ from matrix T_{i-1} (5)
o_{i-1} is a vector $[o_x\, o_y\, o_z]^T$ from matrix T^0_{i-1} (5)
$0_{3\times 1}$ is a vector of zeros $[0\,0\,0]^T$
n represents the number of joints ($n = 4$ in our case)

$$T_{i-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

The Jacobian matrix will be in the form:

$$J = \begin{bmatrix} z_0 \times (o_4 - o_0) & z_1 \times (o_4 - o_1) & z_2 & z_3 \times (o_4 - o_3) \\ z_0 & z_1 & 0_{3\times 1} & z_3 \end{bmatrix} \tag{6}$$

To compute Jacobian homogeneous following transformation matrices are needed. Matrices $A_1,\ A_2,\ A_3,\ A_4$ from Problem 1 are used \eqref{eqn:matrix_forward} in the calculation.

$$T^0_1 = A_1 = \begin{bmatrix} \cos(q_1) & 0 & -\sin(q_1) & 0 \\ \sin(q_1) & 0 & \cos(q_1) & 0 \\ 0 & -1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 \cdot A_2 = \begin{bmatrix} \cos(q_1)\cos(q_2) & -\sin(q_1) & \cos(q_1)\sin(q_2) & 0 \\ \cos(q_2)\sin(q_1) & \cos(q_1) & \sin(q_1)\sin(q_2) & 0 \\ -\sin(q_2) & 0 & \cos(q_2) & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 \cdot A_2 \cdot A_3 = \begin{bmatrix} -\cos(q_1)\cos(q_2) & \cos(q_1)\sin(q_2) & -\sin(q_1) & q_3\cos(q_1)\sin(q_2) \\ -\cos(q_2)\sin(q_1) & \sin(q_1)\sin(q_2) & \cos(q_1) & q_3\sin(q_1)\sin(q_2) \\ \sin(q_2) & \cos(q_2) & 0 & q_3\cos(q_2) + \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = A_1 \cdot A_2 \cdot A_3 \cdot A_4 = \begin{bmatrix} -cos(\theta_2+\theta_4)cos(\theta_1) & -sin(\theta_1) & -sin(\theta_2+\theta_4)cos(\theta_1) & -\frac{cos(\theta_1)}{50}(51cos(\theta_2+\theta_4)-50q_3sin(\theta_2)) \\ -cos(\theta_2+\theta_4)sin(\theta_1) & cos(\theta_1) & -sin(\theta_2+\theta_4)sin(\theta_1) & -\frac{sin(\theta_1)}{50}(51cos(\theta_2+\theta_4)-50q_3sin(\theta_2)) \\ sin(\theta_2+\theta_4) & 0 & -cos(\theta_2+\theta_4) & \frac{51}{50}sin(\theta_2+\theta_4)+q_3cos(\theta_2)+\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From matrices above we can easily find the vectors z_{i-1} and o_{i-1}

$$z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, z_1 = \begin{pmatrix} -\sin(q_1) \\ \cos(q_1) \\ 0 \end{pmatrix}, z_2 = \begin{pmatrix} \cos(q_1)\ \sin(q_2) \\ \sin(q_1)\ \sin(q_2) \\ \cos(q_2) \end{pmatrix}, z_3 = \begin{pmatrix} -\sin(q_1) \\ \cos(q_1) \\ 0 \end{pmatrix}, z_4 = \begin{pmatrix} \cos(q_1)\ \sin(q_2) \\ \sin(q_1)\ \sin(q_2) \\ \cos(q_2) \end{pmatrix}$$

$$o_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, o_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix}, o_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix}, o_3 = \begin{pmatrix} q_3\ \cos(q_1)\ \sin(q_2) \\ q_3\ \sin(q_1)\ \sin(q_2) \\ q_3\ \cos(q_2) + \frac{3}{2} \end{pmatrix} o_4 = \begin{pmatrix} -\frac{\cos(q_1)\ (51\ \cos(q_2+q_4)-50\ q_3\ \sin(q_2))}{50} \\ -\frac{\sin(q_1)\ (51\ \cos(q_2+q_4)-50\ q_3\ \sin(q_2))}{50} \\ \frac{51\ \sin(q_2+q_4)}{50} + q_3\ \cos(q_2) + \frac{3}{2} \end{pmatrix}$$

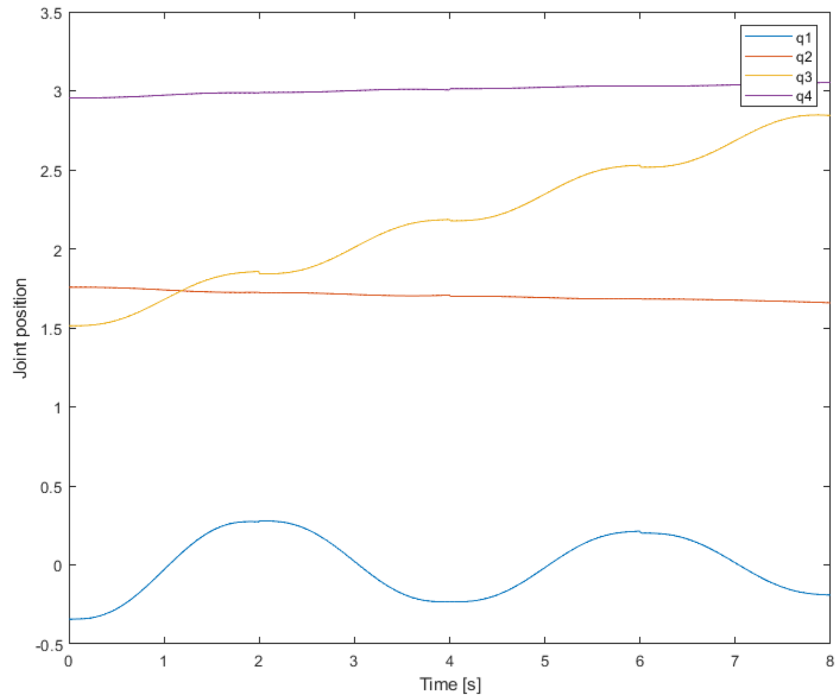
By substituting these vectors into Jacobian (**??**) and knowing that $a_4 = 1.02$, we receive the final result

$$J = \begin{pmatrix} \sin(q_1)\,\sigma_2 & \cos(q_1)\,\sigma_1 & \cos(q_1)\sin(q_2) & a_4\sin(q_2+q_4)\cos(q_1) \\ -\cos(q_1)\,\sigma_2 & \sin(q_1)\,\sigma_1 & \sin(q_1)\sin(q_2) & a_4\sin(q_2+q_4)\sin(q_1) \\ 0 & \sigma_2 & \cos(q_2) & \sigma_3 \\ 0 & -\sin(q_1) & 0 & -\sin(q_1) \\ 0 & \cos(q_1) & 0 & \cos(q_1) \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

where
$\sigma_1 = a_4\sin(q_2+q_4) + q_3\cos(q_2)$
$\sigma_2 = \sigma_3 - q_3\sin(q_2)$
$\sigma_3 = a_4\cos(q_2+q_4)$

The path in Problem 5 does not contain any singularities. To calculate the singularities we must ensure that the rank of Jacobian is the same as the maximum value of Jacobian along the path. ($q^s = [q_1^s, q_2^s, \ldots, q_n^s]$ for $\text{rank}[J(q^s)] < \max_q\{\text{rank}[J(q)]\}$).

To find the $rank(J)$ the angles of each joint along the straight Cartesian line are needed. We can easily calculate these joint angles using problem 4, which represents joints movement for the end-effector path specified in problem 5. The graph of each joint value can be represented as below.



Now we can easily calculate $rank(J)$ for every joint configuration along the path which is every time equal to 4 that is also the maximum Jacobian value. Therefore by definition there are no singularities.:

Problem 7 - Moments of Inertia

Find the following moments of inertia $I_1, I_{1yy}, I_2, I_{2zz}, I_3, I_{3yy}$ and I_4, respectively, when

$L_1 = 0.67[m], m_1 = 4.9[\text{ kg}], r_1 = 0.04[\text{ m}]$	Link 1 is a cylinder with radius R_1, length L_1, and mass m_1
$L_2 = 1.7[m], m_2 = 8.1[\text{ kg}], b_2 = 0.22[\text{ m}]$	Link 2 is a box with sides $b_2 \times b_2$, length L_2, and a mass of m_2
$L_3 = 1.65[m], m_3 = 4.9[\text{ kg}], b_3 = 0.18[\text{ m}]$	Link 2 is a box with sides $b_3 \times b_3$, length L_3, and a mass of m_3
$a_4 = 0.98[m], m_4 = 2.2[\text{ kg}], (a_4 = \text{length})$	Link 4 is an infinitely thin rod with mass m_4 and length a_4

(for use in Problem 9 : $\Delta_2 = 0.34[\text{ m}]$)

The following assumptions can be done in order to determine the center of gravity and the mass moments of inertia of each of the links. The matrix of inertia D_i in center of mass for each link i is defined by:

$$\bar{D_1} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_{1yy} & 0 \\ 0 & 0 & I_1 \end{bmatrix} \bar{D_2} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{2zz} \end{bmatrix} \bar{D_3} = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_{3yy} & 0 \\ 0 & 0 & I_3 \end{bmatrix} \bar{D_4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{bmatrix} \tag{8}$$

where

$$I_1 = I_{1xx} = I_{1zz}, \qquad I_2 = I_{2xx} = I_{2yy}, \qquad I_3 = I_{3xx} = I_{3zz}, \qquad I_4 = I_{4yy} = I_{4zz}$$

To complete the following matrices, we use the values given in the exercise and a few formulas for the inertia of a cylinder, a box and a thin rod.

Inertia of cylinder about central diameter: $I_c = \frac{m \cdot r^2}{4} + \frac{m \cdot L^2}{12}$

Inertia of cylinder about cylinder axis: $I_c = \frac{m \cdot r^2}{2}$

Inertia of a box through center: $I_b = \frac{m \cdot (a^2 + b^2)}{12}$

Inertia pf a thin rod thought center perpendicular to length: $I_r = \frac{m \cdot L^2}{12}$

This gives us the following equations:

$$
\begin{aligned}
I_1 &= \frac{1}{12} m_1 (3r_1^2 + L_1^2) = \frac{1}{12}\, 4,9(3 \cdot 0,04^2 + 0,67^2) = 0,18526\, kg\, m^2 \\
I_{1yy} &= \frac{1}{2} m_1 r_1^2 = \frac{1}{2}\, 4,9 \cdot 0,04^2 = 0,00392\, kg\, m^2 \\
I_2 &= \frac{1}{12} m_2 (b_2^2 + L_2^2) = \frac{1}{12}\, 8,1(0,22^2 + 1,7^2) = 1,98342\, kg\, m^2 \\
I_{2zz} &= \frac{1}{12} m_2 (b_2^2 + b_2^2) = \frac{1}{6}\, 8,1 \cdot 0,22^2 = 0,06534\, kg\, m^2 \\
I_3 &= \frac{1}{12} m_3 (b_3^2 + L_3^2) = \frac{1}{12}\, 4,9(0,18^2 + 1,65^2) = 1,12492\, kg\, m^2 \\
I_{3yy} &= \frac{1}{12} m_3 (b_3^2 + b_3^2) = \frac{1}{6}\, 4,9 \cdot 0,18^2 = 0,02646\, kg\, m^2 \\
I_4 &= \frac{1}{12} m_4\, a_4^2 = \frac{1}{12}\, 2,2 \cdot 0,98^2 = 0,17607\, kg\, m^2
\end{aligned}
$$

Finally, we can insert them in the matrices (8) to find:

$$
\begin{aligned}
\bar{D_1} &= \begin{bmatrix} 0,18526 & 0 & 0 \\ 0 & 0,00392 & 0 \\ 0 & 0 & 0,18526 \end{bmatrix} kg\, m^2 \\
\bar{D_2} &= \begin{bmatrix} 1,98342 & 0 & 0 \\ 0 & 1,98342 & 0 \\ 0 & 0 & 0,06534 \end{bmatrix} kg\, m^2 \\
\bar{D_3} &= \begin{bmatrix} 1,12492 & 0 & 0 \\ 0 & 0,02646 & 0 \\ 0 & 0 & 1,12492 \end{bmatrix} kg\, m^2 \\
\bar{D_4} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0,17607 & 0 \\ 0 & 0 & 0,17607 \end{bmatrix} kg\, m^2
\end{aligned}
$$

Problem 8 - Simulink actuator model

Derive a SIMULINK block diagram for closed-loop motor angle control

The actuator model is given by.

$$
u_i \cdot k_{T,i} - \frac{1}{n_i} \tau_i = J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i} \dot{q}_{M,i}, \quad i = 1, 2, 3, 4
$$

where
u_i : control voltage [V]
$k_{T,i}$: torque constant [Nm/V]
τ_i : load torque (on the load side of the gear)

n_i : gear ratio
$J_{M,i}$: mass moment of inertia of the motor
$f_{M,i}$: viscous friction coefficient for motor
$q_{M,i}$: motor angle [rad]
$\dot{q}_{M,i}$: angular velocity [rad/s]
$\ddot{q}_{M,i}$: angular acceleration [rad/s^2]

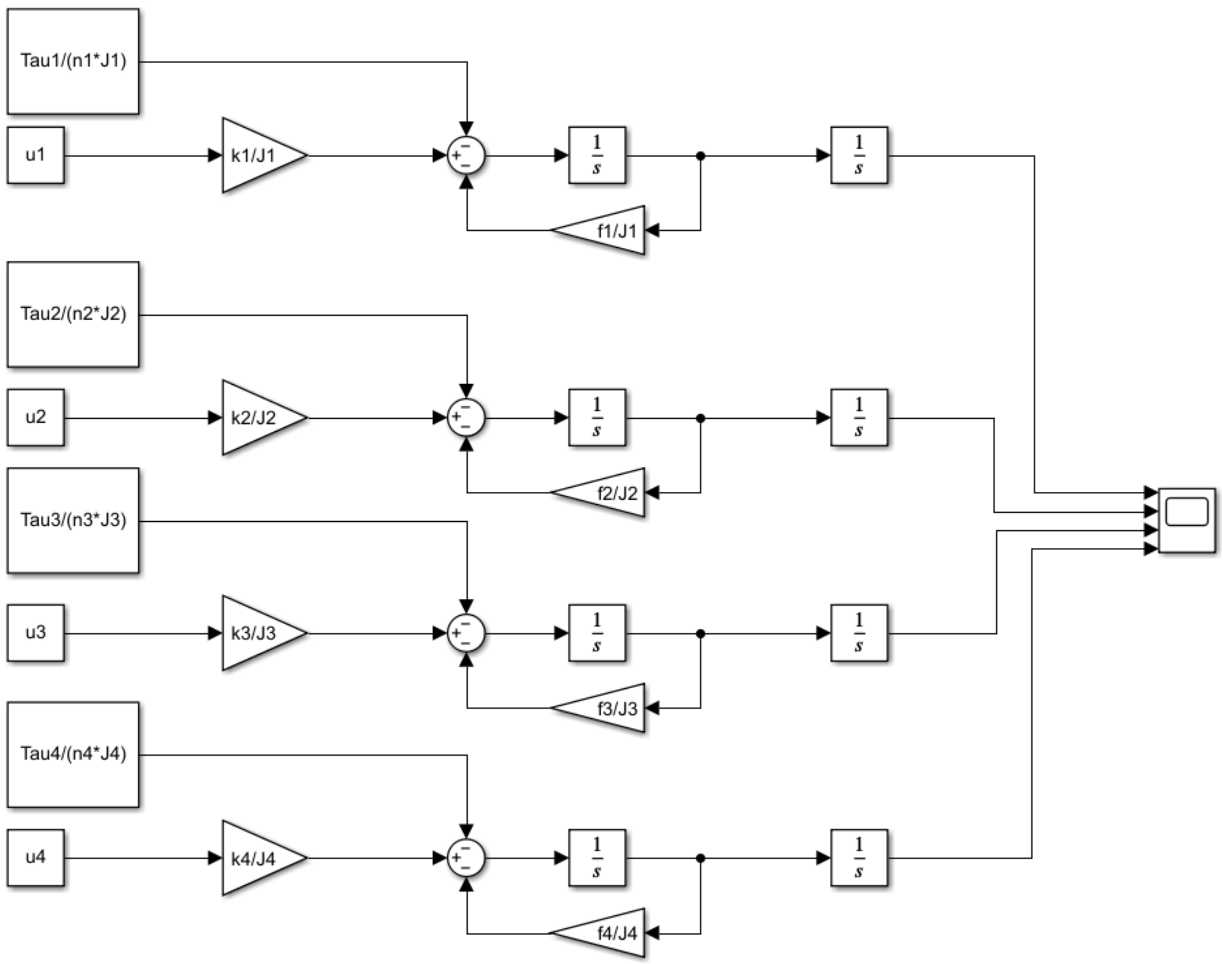
In this exercise, we consider all variables as symbolic (in the following exercise we will give them a numerical value). We assume that there are no losses and backlashes in the gears (losses are not considered in this assignment at all). The masses of all the motors are also neglected.

The only known value are the gear ratios $n_i = 53$, since MAXON DC-servo motors of type RE are mounter in each of the joints of the manipulator for control of the joints $q = [\theta_1, \theta_2, d_3, \theta_4]^T$ (the prismatic joint 3 achieves the gear reduction ratio through a ball lead screw with steep-ness n = 53 [rad/m])

To create the final SIMULINK block diagram, we move the highest derivate to the left side of the equation.

$$\ddot{q}_{M,i} = -\frac{f_{M,i}\dot{q}_{M,i}}{J_{M,i}} + \frac{u_i \cdot k_{T,i}}{J_{M,i}} - \frac{\tau_i}{J_{M,i} \cdot n_i}, \quad i = 1, 2, 3, 4$$

We know that the input to the SIMULINK model is the control voltage u_i and that we need two integrals $\frac{1}{s}$ to receive motor angle q as the output of the model. All other values are input or gain constants. The diagram is created for each joint since each joint has a different angle and thus different constant and input values. Finally, the derived block diagram is below.



Problem 9 - Effective moments of inertia

Determine the effective moments of inertia as seen from the motor axes when we know that

$$J_{\text{eff},i} = \frac{1}{n_i^2} \cdot sup_q(D_{ii}(q)) + J_{M,i},\ i = 1,2,3,4. \tag{9}$$

where:
$D(q)$ = the robot mass/intertia tensor
J_M = mass moment of inertia $J_M = 1320 \times 10^{-7} kg \cdot m^2$
$q = [\theta_1\, \theta_2\, \theta_3\, \theta_4]^T$
$\sup_q(Dii(q))$ = largest numerical value of each diagonal element $(Dii(q))$ in the $D(q)$
i = index for link i
$n_i = 53$ is the gear ration

In order to find the effective moments of inertia we need to first define and find D_{ii}, the other values needed are already known.

The matrix D(q) is the manipulator inertia tensor and is defined as:

$$\begin{bmatrix} D_{11} & 0 & 0 & 0 \\ 0 & K_3 + 2f_1(q_3) + f_2(q_3)sin(q_4) & f(q_4) & 2K_1 + \frac{1}{2}f_2(q_3)sin(q_4) \\ 0 & f(q_4) & K_4 & f(q_4) \\ 0 & 2K_1 + \frac{1}{2}f_2(q_3)sin(q_4) & f(q_4) & 2K_1 \end{bmatrix}$$

The following diagonal functions D_{ii} can be easily extracted:

$$\begin{aligned}
D_{11} &= K_5 + f_1(q_3) - (K_2 + f_1(q_3))cos(2q_2) + K_1 cos(2(q_2 + q_4)) - f_2(q_3)cos(q_2 + q_4)sin(q_2) \\
D_{22} &= K_3 + 2f_1(q_3) + f_2(q_3)sin(q_4) \\
D_{33} &= K_4 \\
D_{44} &= 2K_1
\end{aligned}$$

There are still a lot of unknown values and functions: we will have to find these before optimizing D_{ii}.

The K_i values and f_i functions can easily be calculated using values from Problem 7 - Moments of Inertia and formulas from annex B:

$$\begin{aligned}
K_0 &= m_2(\frac{1}{2}L_2 - \Delta_2)^2 + m_3(\frac{1}{2}L_3)^2 = 5.4419\, kg\, m^2 \\
K_1 &= \frac{1}{2}[I_4 + \frac{1}{4}m_4 a_4^2] = 0.3521\, kg\, m^2 \\
K_2 &= \frac{1}{2}[I_2 - I_{2zz} + I_3 - I_{3yy} + K_0] = 4.2292\, kg\, m^2 \\
K_3 &= I_2 + I_3 + K_0 + 2K_1 = 9.2545\, kg\, m^2 \\
K_4 &= m_3 + m_4 = 7.1\, kg \\
K_5 &= \frac{1}{2}[2I_{1yy} + I_{2zz} + I_{3yy} + K_3] = 4.6771\, kg\, m^2 \\
f_1(q_3) &= \frac{1}{2}K_4 q_3^2 - \frac{1}{2}m_3 L_3 q_3 \\
f_2(q_3) &= m_4 a_4 q_3 \\
f(q_4) &= \frac{1}{2}a_4 m_4 cos(q_4)
\end{aligned}$$

Finally, we can use the right values for functions $D_{ii}(q)$ and optimise them: the effective moments of inertia need the highest value possible for $D_{ii}(q)$ in the adequate range of q_j. These ranges are:

	Min value	Max value	Unit
q_1	$-\pi$	π	rad
q_2	$\frac{\pi}{6}$	$\frac{3\pi}{4}$	rad
q_3	1.35	3.00	m
q_4	$\frac{-\pi}{2}$	$\frac{5\pi}{4}$	rad

Table 2: Range of each joint

Once optimised, we get the following values:

$$\begin{aligned}
&sup_q(D_{11}(q)) = 55.3714\, kg\, m^2 && \forall q_1,\ q_2 = \frac{\pi}{2},\ q_3 = 3,\ q_4 = \frac{\pi}{2} \\
&sup_q(D_{22}(q)) = 67.495\, kg\, m^2 && \forall q_1, \forall q_2,\ q_3 = 3,\ q_4 = \frac{\pi}{2} \\
&sup_q(D_{33}(q)) = K_4 = 7.1\, kg && \forall q_i \quad i = 1,2,3,4 \\
&sup_q(D_{44}(q)) = 2K_1 = 0.7043\, kg\, m^2 && \forall q_i \quad i = 1,2,3,4
\end{aligned}$$

We can finally find the effective moments of inertia using (9):

$$
\begin{aligned}
J_{\text{eff},1} &= \frac{1}{n_1^2} \cdot sup_q(D_{11}(q)) + J_{M,1} = 0.0198\,kg\,m^2 \\
J_{\text{eff},2} &= \frac{1}{n_2^2} \cdot sup_q(D_{22}(q)) + J_{M,2} = 0.0242\,kg\,m^2 \\
J_{\text{eff},3} &= \frac{1}{n_3^2} \cdot sup_q(D_{33}(q)) + J_{M,3} = 0.0027\,kg\,m^2 \\
J_{\text{eff},4} &= \frac{1}{n_4^2} \cdot sup_q(D_{44}(q)) + J_{M,4} = 0.00038\,kg\,m^2
\end{aligned}
$$

Problem 10 - Joint values in Laplace domain

Express the angle position in the Laplace domain using the actuator model from problem 8, control from problem 9, and PD control law below

To express angle positions in Laplace domain we need the transfer functions (all $f(s)$ are polynomials):

$$q_i(s) = \frac{1}{N_i(s)} \left[F_i(s)q_i^r(s) - G_i(s)T_{L,i}(s)\right] \tag{10}$$

where:
q_i^r is input (motor angle in radians)
$q_i t$ is output (motor angle in radians)
N_i, F_i, G_i are transfer functinos
$u_i$is control voltage
$k_{T,i}$ is torque constant
τ_i is load torque derived from euler-lagrange equations (written as τ_l in a book)*
T_L is load torque $T_L = \frac{\tau_i}{n_i}$ (considered as disturbance D)
n_i is gear ratio equal to 53
$J_{M,i}$ is mass moment of inertia of the motor
$J_{\text{eff},i}$ effective moment of inertia
$f_{M,i}$ is viscous friction coefficient for motor
D_{ii} = each diagonal element (Dii(q)) in the D(q)
$D(q)$ = the robot mass/intertia tensor
*more information are at chapter 7.2 in [2]

From the actuator model seen in Problem 8, we know that:

$$u_i \cdot k_{T,i} - \frac{1}{n_i}\tau_i = J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i} \tag{11}$$

We also know from PD control law that

$$u_i = n_i \cdot (K_{p,i} \cdot (q_i^r - q_i) + K_{D,i} \cdot (\dot{q}_i^r - \dot{q}_i)) \tag{12}$$

Let's plug (12) into (11) to get:

$$J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i} = n_i k_{T,i} \left(K_{p,i} \cdot (q_i^r - q_i) + K_{D,i} \cdot (\dot{q}_i^r - \dot{q}_i)\right) - \frac{1}{n_i}\tau_i$$

We can now further simplify and isolate values as outputs on the left side and inputs and constants on the right side of equation:

$$J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i} = n_i K_{p,i} k_{T,i} q_i^r - n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} \dot{q}_i^r - n_i K_{D,i} k_{T,i} \dot{q}_i - \frac{1}{n_i}\tau_i$$
$$J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i} + n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} \dot{q}_i = n_i K_{p,i} k_{T,i} q_i^r + n_i K_{D,i} k_{T,i} \dot{q}_i^r - \frac{1}{n_i}\tau_i$$

We also know from Problem 9 and Appendix C we know that

$$J_{\text{eff}} = \frac{sup_q(D_{ii}(q_i))}{n_i^2} + J_{M,i} \rightarrow J_{M,i} = J_{\text{eff}} - \frac{sup_q(D_{ii}(q_i))}{n_i^2}$$
$$\hat{D}(q_i)\ddot{q} = \tau - T_L$$
$$sup_q(D_{ii}(q_i)) = \hat{D}(q_i) = \frac{\tau_i - T_{L,i}}{\ddot{q}_i}$$
$$\rightarrow J_{M,i} = J_{\text{eff},i} - \frac{\tau_\text{i} - T_{L,i}}{n_i^2 \ddot{q}_i} \tag{13}$$
$$\rightarrow \tau_i \;=\; J_{\text{eff},i} n_i^2 + T_{L,i} - J_{M,i}$$

where $\hat{D}(q) = \text{diag}\left(I_{1\max}, \ldots, I_{m\max}\right)$ is a constant diagonal matrix determined by $I_{i\max} = \sup_q\left(d_{ii}\right)$

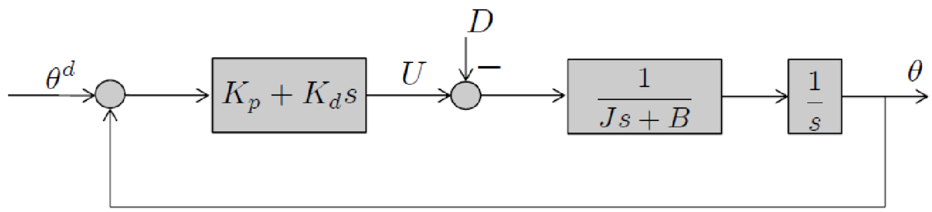
$$\left(J_{\text{eff,i}} - \frac{\tau_\text{i} - T_{L,i}}{n_i^2 \ddot{q}_s}\right) \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i} + n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} \dot{q}_i = n_i K_{p,i} k_{T,i} q_i^r + n_i K_{D,i} k_{T,i} \dot{q}_i^r - \frac{1}{n_i}\tau_i$$
$$\left(J_{\text{eff,i}}\ddot{q}_{M,i} - \frac{J_{\text{eff},i} \cdot n_i^2 + T_{L,i} - J_{M,i} - T_{L,i}}{n_i^2 \ddot{q}_s} \cdot \ddot{q}_{M,i}\right) + f_{M,i}\dot{q}_{M,i} + n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} \dot{q}_i =$$
$$= n_i K_{p,i} k_{T,i} q_i^r + n_i K_{D,i} k_{T,i} \dot{q}_i^r - \frac{1}{n_i}\left(\ J_{\text{eff},i} n_i^2 + T_{L,i} - J_{M,i}\right)$$
$$\left(J_{\text{eff,i}}\ddot{q}_{M,i} - \frac{J_{\text{eff},i} \cdot n_i^2 - J_{M,i}}{n_i^2 \ddot{q}_s} \cdot \ddot{q}_{M,i}\right) + f_{M,i}\dot{q}_{M,i} + n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} \dot{q}_i =$$
$$= n_i K_{p,i} k_{T,i} q_i^r + n_i K_{D,i} k_{T,i} \dot{q}_i^r - J_{\text{eff},i} n_i - \frac{T_{L,i}}{n_i} + \frac{J_{M,i}}{n_i}$$

The $q_{M,i}$ from Problem 8 does not consider the gear ratio, therefore during Laplace Transform we need to multiply this joint angle by the gear ratio n_i.While applying Laplace transform we know that at each segment the robot arm end-effector will stop and accelerate again into another segment. Thus, the initial values are zero. For sake of simple notation (s) elements will not be written. We can plug that into Laplaced equation above and get:

$$\left(J_{\text{eff,i}} n_i q_i s^2 - \frac{J_{\text{eff},i} \cdot n_i^2 - J_{M,i}}{n_i^2 q_i s^2} \cdot n_i q_i s^2\right) + f_{M,i} n_i q_i s + n_i K_{p,i} k_{T,i} q_i + n_i K_{D,i} k_{T,i} q_i s =$$
$$= n_i K_{p,i} k_{T,i} q_i^r + n_i K_{D,i} k_{T,i} q_i^r s - J_{\text{eff},i} n_i - \frac{T_{L,i}}{n_i} + \frac{J_{M,i}}{n_i}$$
$$J_{\text{eff,i}} n_i^3 q_i s^2 - J_{\text{eff},i} n_i^3 + J_{M,i} n_i + f_{M,i} n_i^3 q_i s + n_i^3 K_{p,i} k_{T,i} q_i + n_i^3 K_{D,i} k_{T,i} q_i s =$$
$$= n_i^3 K_{p,i} k_{T,i} q_i^r + n_i^3 K_{D,i} k_{T,i} q_i^r s - J_{\text{eff},i} n_i^3 - T_{L,i} n_i + J_{M,i} n_i$$
$$J_{\text{eff,i}} n_i^3 q_i s^2 + f_{M,i} n_i^3 q_i s + n_i^3 K_{p,i} k_{T,i} q_i + n_i^3 K_{D,i} k_{T,i} q_i s =$$
$$= n_i^3 K_{p,i} k_{T,i} q_i^r + n_i^3 K_{D,i} k_{T,i} q_i^r s - J_{\text{eff},i} n_i^3 - T_{L,i} n_i + J_{M,i} n_i - J_{M,i} n_i + J_{\text{eff},i} n_i^3$$
$$q_i\left(J_{\text{eff,i}} n_i^3 s^2 + f_{M,i} n_i^3 s + n_i^3 K_{p,i} k_{T,i} + n_i^3 K_{D,i} k_{T,i} s\right) = q_i^r\left(n_i^3 K_{p,i} k_{T,i} + n_i^3 K_{D,i} k_{T,i} s\right) - T_{L,i} n_i$$

$$q_i = \frac{(K_{p,i} \ \cdot k_{T,i} + K_{D,i} s \cdot k_{T,i}) \cdot q_i^r(s) - \frac{T_{L,i}(s)}{n_i^2}}{J_{\mathrm{eff,i}} s^2 + (f_{M,i} + K_{D,i} \cdot k_{T,i}) s + K_{p,i} \cdot k_{T,i}} \tag{14}$$

That result is corresponding to the following block diagram from the lecture



$$\left[(\theta^d - \theta)(K_p + K_D s) - D(s)\right] \frac{1}{Js+B} \frac{1}{s} = \theta$$

$$\downarrow$$

$$\theta = \frac{(K_p + K_D s)\theta^d(s) - D(s)}{Js^2 + (B + K_D)s + K_p}$$

Using the constant mentioned in the task
$f_{\mathrm{eff},i} = f_{M,i} + \frac{f_{L,i}}{N_i^2} = f_{M,i} = 2.4 \cdot 10^{-5} \left[\frac{\mathrm{Nm}}{\mathrm{rad/s}}\right]$
$f_L = 0$ (viscous friction coefficient)
$f_{\mathrm{eff},i} = B_{\mathrm{eff},i} \quad f_{M,i} = B$
$k_T = 0.17 \left[\frac{\mathrm{Nm}}{\mathrm{V}}\right]$
$n_i = 53$

We receive the following equation:

$$q_i = \frac{(0.17 K_{p,i} + 0.17 K_{D,i} s) \cdot q_i^r(s) - \frac{T_{L,i}(s)}{2809}}{J_{\mathrm{eff,i}} s^2 + (2.4 \cdot 10^{-5} + 0.17 K_{D,i})\, s + 0.17 K_{p,i}}$$

We have to express the result in format $q_i(s) = \frac{1}{N_i(s)} \left[F_i(s) q_i^r(s) - G_i(s) T_{L,i}(s)\right]$, where variables F, G, N are Transfer functions. Index i represents each joint and q_i express the angle positions in the Laplace domain by the transfer function, and n_i is constant value equal to gear ration (in out case n=53). T_L can be considered as a disturbance and is considered independent of q when used in simple stability analysis.

By comparing (14) with (10) to get N(s), F(s) and G(s) we receive that

$$N_i(s) = J_{\mathrm{eff,i}} s^2 + (f_{M,i} + K_{D,i} \cdot k_{T,i}) s + K_{p,i} \cdot k_{T,i}$$

$$F_i(s) = K_{p,i} \ \cdot k_{T,i} + K_{D,i} s \cdot k_{T,i}$$

$$G_i(s) = \frac{1}{n_i^2}$$

Problem 11 - PD Control Parameters

Show that the PD control parameters can be expressed as follows

$$K_{P,i} = \frac{\omega_{n,i}^2 \cdot J_{\mathrm{eff},i}}{k_{T,i}} \quad K_{D,i} = \frac{2\zeta_i}{\omega_{n,i}} \cdot K_{P,i} - \frac{f_{\mathrm{eff},i}}{k_{T,i}}, \quad i = 1, 2, 3, 4$$

where ω_n and ζ are the natural frequency and the damping of the closed-loop system. Be aware that a PD controller can not be applied in practice. (for more info search LEAD controller)

We will use the knowledge of the second-order system's general transfer function with a correlation of damping coefficient ζ and natural frequency ω_n :

$$s^2 + 2\omega_{n,i}\zeta_i s + \omega_{n,i}^2$$

Therefore, we will be comparing the denominators with our transfer function $q_i(s)$ from Problem 10.

As for comparing the first terms, we need to get rid of J_{eff} term in $q_i(s)$ denominator by expanding the whole fraction by $1/J_{\text{eff}}$. Then we will need to take the gearing ratio into consideration and therefore multiply by n_i term on several occasions. Other than that, we will work with only the denominator of $q_i(s)$ from now on to get desired answers:

$$J_{\text{eff}}s^2 + \left(f_{M,i} + K_{D,i}k_{T,i}\right)s + K_{p,i}k_{T,i} \rightarrow s^2 + \frac{f_{M,i}}{J_{\text{eff}}} + \frac{k_{T,i}}{J_{\text{eff}}}K_{D,i} + \frac{k_{T,i}}{J_{\text{eff}}}K_{p,i}$$

Comparing the first term, we already have desired $s^2 = s^2$ and we can continue on. Comparing the third term to get K_p :

$$\omega_{n,i}^2 = K_{p,i}k_{T,i} \cdot \frac{1}{J_{eff,i}}$$

$$K_{p,i} = \frac{\omega_{n,i}^2 \cdot J_{eff,i}}{k_{T,i}}$$

Which is what we wanted.

Second terms will now be compared, with an expansion of $\omega_{n,i}$ to get desired $K_{p,i}$ inside the notation:

$$2\omega_{n,i}\zeta_i = \frac{1}{J_{\text{eff },i}}\left(f_{\text{eff },i} + K_{D,i}k_{T,i}\right)$$

$$K_{D,i} = \frac{2\omega_{n,i}\zeta_2 J_{\text{eff },i} - f_{\text{eff },i}}{k_{T,i}}$$

$$K_{D,i} = \frac{2\zeta_i}{\omega_{n,i}} \cdot \frac{\omega_{n,i}^2 J_{\text{eff },i}}{k_{T,i}} - \frac{f_{\text{eff },i}}{k_{T,i}}$$

We now can plug the $K_{p,i}$ term to get the desired expression of $K_{D,i}$:

$$K_{D,i} = \frac{2\zeta_i}{\omega_{n,i}} \cdot K_{p,i} - \frac{f_{eff,i}}{k_{T,i}}$$

Problem 12 - PD Controller in Simulink

Derive a model in SIMULINK for each joint servo axis with $q_i^r(s)$ as input and $q_i(s)$ as output. Use the closed loop control described above. Simulate the closed loop model response of a reference step at the size below.Plot q_i^r, q_1, q_2, q_3 and q_4 as a function of time t, on the same plot, denoted A_1 when $\zeta_i = 1$ and A_1 when $\zeta_i = 1$. Comment the results.

$T_{L,i} = 0$
$K_{P,i} = \frac{\omega_{n,i}^2 \cdot J_{\text{eff},i}}{k_{T,i}}$
$K_{D,i} = \frac{2\zeta_i}{\omega_{n,i}} \cdot K_{P,i} - \frac{f_{\text{eff},i}}{k_{T,i}}$ $\omega_{n,i} = 15\left[\frac{\text{rad}}{\text{s}}\right]$
$\zeta_i = 1$
$q_i^r = 0.34[\text{rad}]$ for $i = 1,2,4$
$q_3^r = 0.34[\text{ m}]$ for $i = 3$

$k_T = 0.17 \frac{Nm}{V}$
$f_{\mathrm{eff},i} = 2.410^{-5} \left[\frac{\mathrm{Nm}}{\mathrm{rad/s}}\right]$

From Problem 9 - Effective moments of inertia we know the effective moment of inertia for each joint.
$J_{\mathrm{eff},1} = 0.0198\, kg\, m^2$
$J_{\mathrm{eff},2} = 0.0242\, kg\, m^2$
$J_{\mathrm{eff},3} = 0.0027\, kg\, m^2$
$J_{\mathrm{eff},4} = 0.00038\, kg\, m^2$

To build the block diagram we will use the PD controller system from the course.

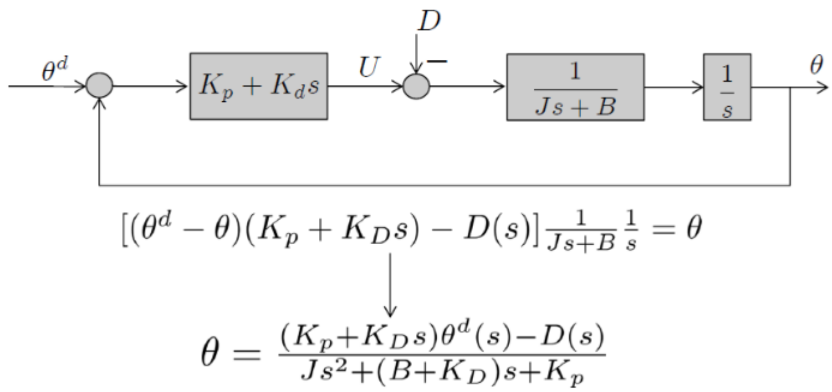


Figure 5: Second-Order System with PD Control

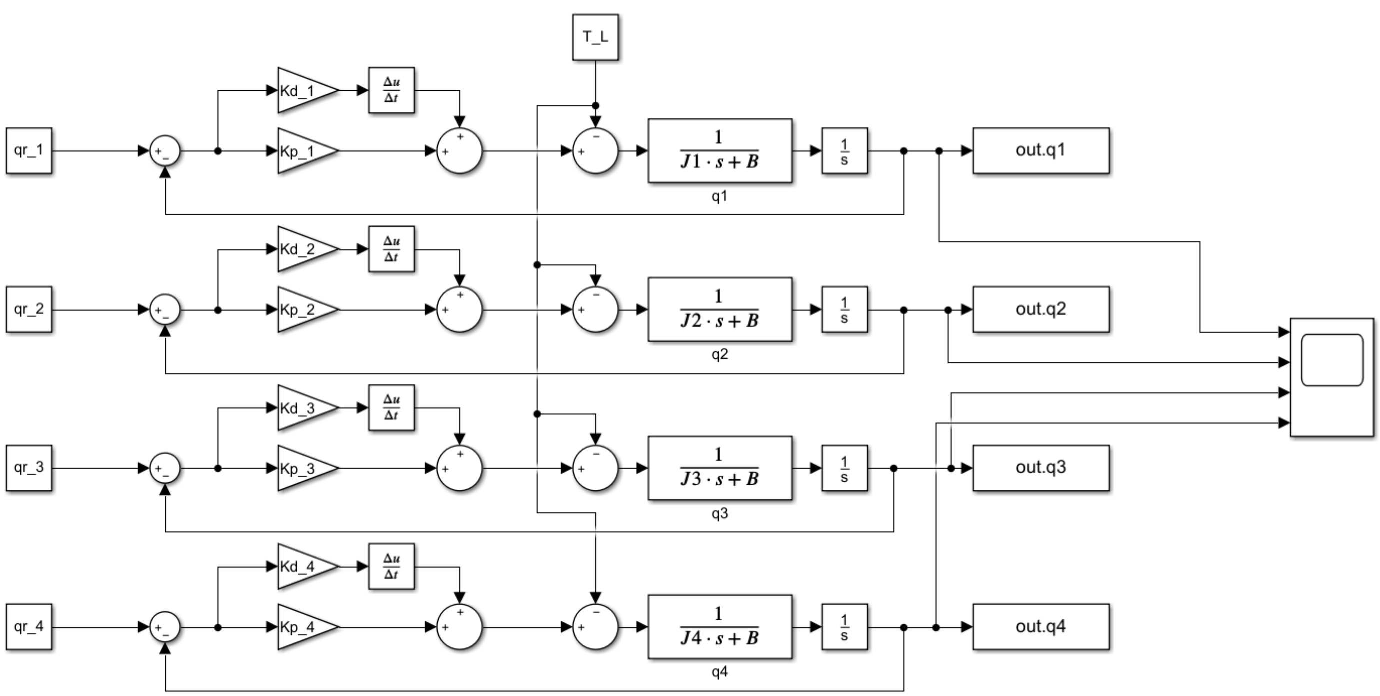
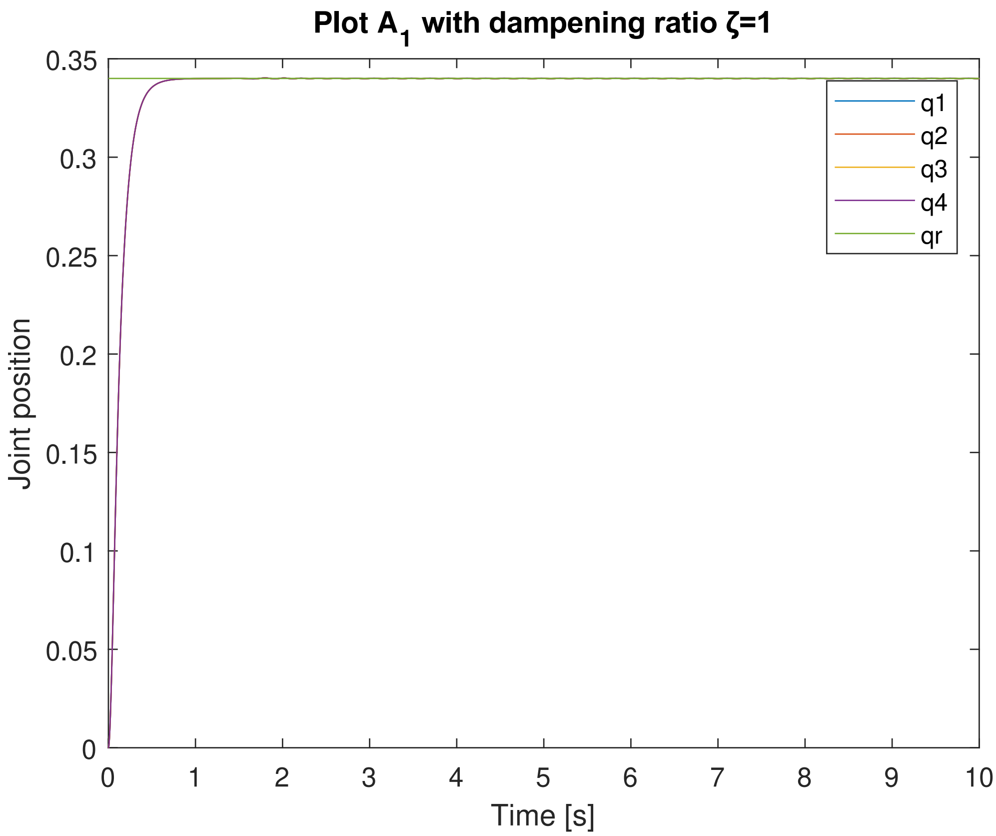


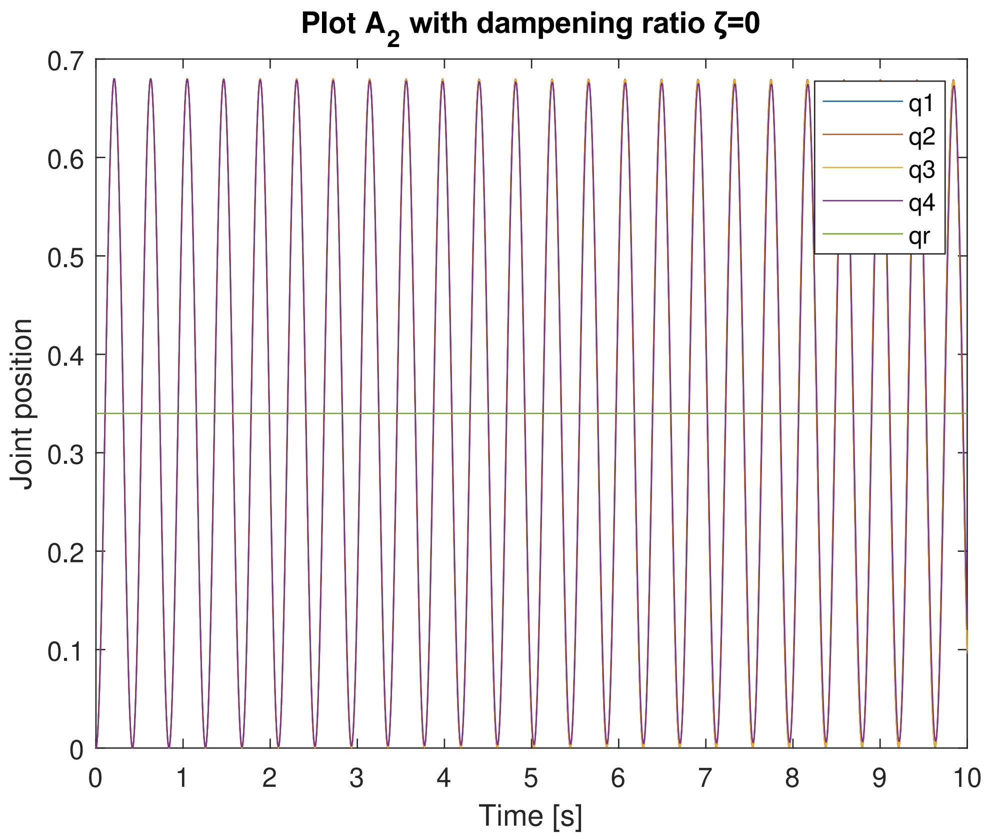
Figure 6: Simulink model for PD Control

Now we have all we need to calculate $K_{D,i}$ and $K_{P,i}$ for each joint. Firstly, we use dampening ratio $\zeta = 1$ and plot the graph.
$K_{P,1} = 26.2059$ and $K_{D,1} = 3.4940$
$K_{P,2} = 32.0294$ and $K_{D,2} = 4.2704$
$K_{P,3} = 3.5735$ and $K_{D,3} = 0.4763$
$K_{P,4} = 0.5029$ and $K_{D,4} = 0.0669$



Using damping ratio $\zeta = 1$ is common in robotics applications as it is critically damped. It provides the fastest non-oscillatory response - the steady-state is reached with no overshoot and no steady-state error. In our example, ω determines the speed of response.

Secondly, we reproduce the same approach with dampening ratio $\zeta = 0$ and receive the following values. $K_{P,1} = 26.2059$ and $K_{D,1} = -0.00014$
$K_{P,1} = 32.0294$ and $K_{D,1} = -0.00014$
$K_{P,1} = 3.5735$ and $K_{D,1} = -0.00014$
$K_{P,1} = 0.5029$ and $K_{D,1} = -0.00014$



As we can see in the graph, with damping ratio $\zeta = 0$ the joint values are not damped, resulting in oscillation around the input value. For this reason, in the next problems the damping ratio $\zeta = 1$ will be used.

Problem 13 - Load torques and force

The masses of the manipulator are now introduced to the model by means of a load torque T_L. Please determine the joint values when the torque/force load T_L is maximum for each joint (use the $h(q)$ matrix in Annex B), and give the relevant configuration of the robot arm. Add these constant load torques T_L to the simulation model. Simulate a response of a reference step input, $q_i^r = 0.35$, damping ratio $\zeta_i = 1.0$ and natural frequency $\omega_{n,i} = 15$. On the same plot (A3), show q_i^r, q_1, q_2, q_3 and q_4 as a function of time. Make comments on the results. Determine the steady-state error (calculate and compare).

Similarly to Problem 9 - Effective moments of inertia, we have functions that we have to optimise while taking into account the minimum and maximum values of each joint. This time, we want to find the maximal torque T_L of each joint, found with the gravity load vector found in annex B:

$$h(q) = g \begin{bmatrix} 0 \\ \frac{1}{2}m_4 a_4 cos(q_2 + q_4) + [m_2(\Delta_2 - \frac{1}{2}L_2) + m_3(\frac{1}{2}L_3 - q_3) - m_4 q_3]sin(q_2) \\ (m_3 + m_4)cos(q_2) \\ \frac{1}{2}a_4 m_4 cos(q_2 + q_4) \end{bmatrix}$$

All the values used can be found in Problem 7 - Moments of Inertia. Now we only need to optimize each component of h(q) and find its maximal values between the values from the table2.

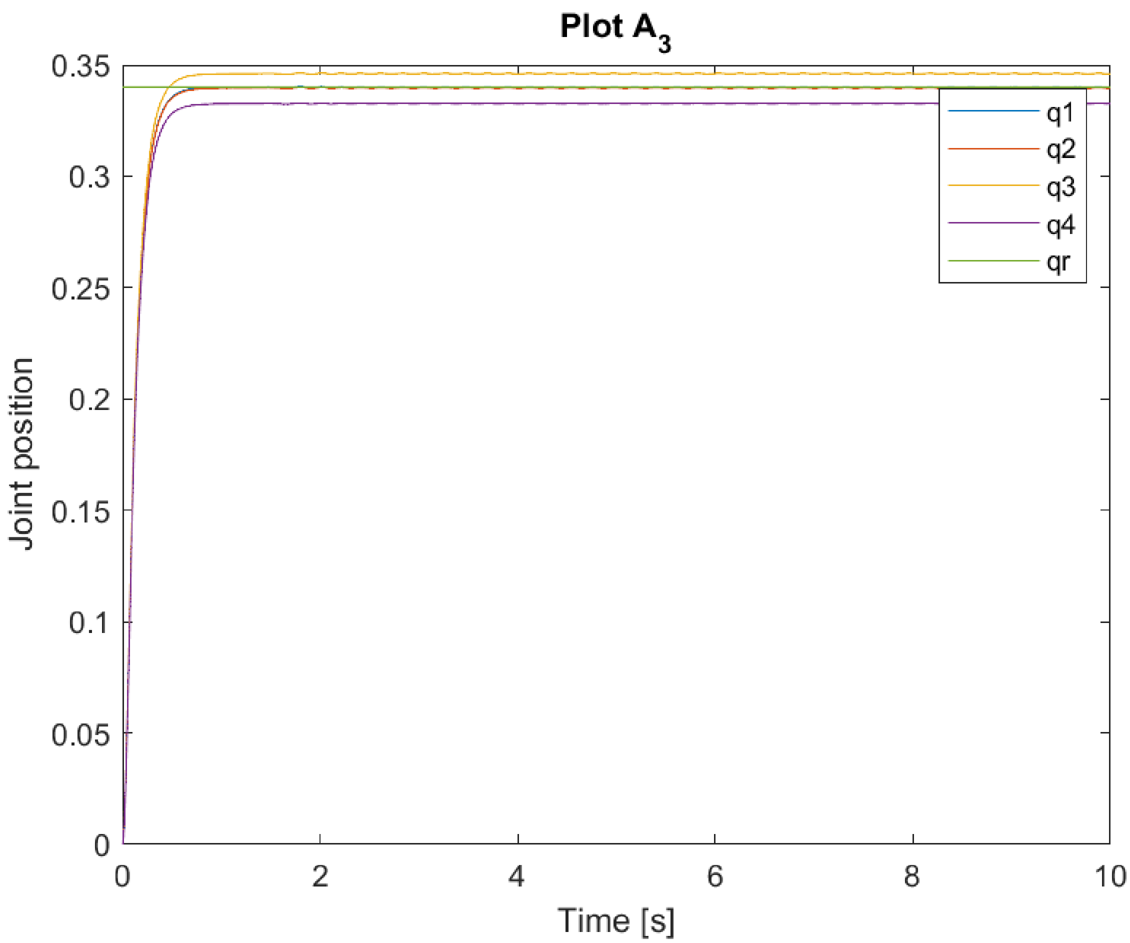
$$T_{L1} = 0\ Nm \qquad \forall q_i,\ i = 1,2,3,4$$

$$T_{L2} = -36.8734\ Nm \qquad \forall q_1,\ q_2 = \frac{\pi}{6},\ q_3 = 1.35,\ q_4 = -\frac{\pi}{6}$$

$$T_{L3} = 60.3195\ N \qquad q_2 = \frac{\pi}{6},\ \forall q_i,\ i = 1,3,4$$

$$T_{L4} = -10.5752\ Nm \qquad q_1 = 0,\ q_2 = \frac{\pi}{2}\ q_4 = 2,\ q_3 = \frac{\pi}{2}$$

Finally, by plotting the simulation and adding these load torques and force, we get:



We can see that adding these additional values will have a non-negligible effect on our system. Their differences can be found as :

$$\Delta q_1 = 0.34 - 0.3399 = 6.04 \cdot 10^{-5}\ rad$$

$$\Delta q_2 = 0.34 - 0.3403 = -3.49 \cdot 10^{-4}\ rad$$

$$\Delta q_3 = 0.34 - 0.3339 = 0.0061\ m$$

$$\Delta q_4 = 0.34 - 0.3474 = -0.0074\ rad$$

Problem 14 - PID Controller Parameters

For each robot-axis, introduce a PID controller of type 2

This problem is similar to Problem 10 - Joint values in Laplace domain. We use the previous actuator model (11).

$$u_i \cdot k_{T,i} - \frac{1}{n_i}\tau_i = J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i}$$

Now instead of the PD control law as in Problem 10, the PID control law is introduced.

$$u_i = n_i \cdot \left[K_{p,i} \cdot (q_i^r - q_i) + K_{D,i} \cdot (\dot{q}_i^r - \dot{q}_i) + K_{I,i} \int (q_i^r - q_i)\, dt\right], \quad i = 1, 2, 3, 4$$

Combining the two equations above, we receive the following:

$$n_i \cdot \left[K_{p,i} \cdot (q_i^r - q_i) + K_{D,i} \cdot (\dot{q}_i^r - \dot{q}_i) + K_{I,i} \int (q_i^r - q_i)\, dt\right] \cdot k_{T,i} - \frac{1}{n_i}\tau_i = J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i}$$

From Probelm 10 we know that:

$$\begin{aligned} \tau_i &= J_{\text{eff},i}n_i^2 + T_{L,i} - J_{M,i} \\ J_{M,i} &= J_{\text{eff,i}} - \frac{\tau_\text{i} - T_{L,i}}{n_i^2 \ddot{q}_s} \end{aligned}$$

Plugging these two equation into the one above we get:

$$n_i \cdot \left(K_{p,i} \cdot (q_i^r - q_i) + K_{D,i} \cdot (\dot{q}_i^r - \dot{q}_i) + K_{L,i} \int (q_i^r - q_i)\, dt\right) \cdot k_{T,i} - \frac{J_{\text{eff},i}n_i^2 + T_{L,i} - J_{M,i}}{n_i} =$$

$$= \left(J_{\text{eff},i} - \frac{J_{\text{eff},i}n_i^2 + T_{L,i} - J_{M,i} - T_{L,i}}{n_i^2 \ddot{q}_i}\right) \cdot \ddot{q}_{M,i} + f_{M,i}\dot{q}_{M,i}$$

We can simplify the equation as

$$k_{T,i}n_iK_{p,i}q_i^r - k_{T,i}n_iK_{p,i}q_i + k_{T,i}n_iK_{D,i}\dot{q}_i^r - k_{T,i}n_iK_{D,i}\dot{q}_i + k_{T,i}n_iK_{I,i}\int (q_i^r - q_i)\, dt - J_{\text{eff},i}n_i - \frac{T_{L,i}}{n_i} + \frac{J_{M,i}}{n_i} =$$

$$= J_{\text{eff,i}}\ddot{q}_{M,i} - \frac{J_{\text{eff},i}n_i^2\ddot{q}_{M,i}}{n_i^2\ddot{q}_i} + \frac{J_{M,i}\ddot{q}_{M,i}}{n_i^2\ddot{q}_i} + f_{M,i}\dot{q}_{M,i}$$

Now we use the Laplace transform. The $q_{M,i}$ from Problem 8 does not consider the gear ratio, therefore during Laplace Transform we need to multiply this joint angle by the gear ratio n_i.

$$k_{T,i}n_iK_{p,i}q_i^r - k_{T,i}n_iK_{p,i}q_i + k_{T,i}n_iK_{D,i}q_i^r s - k_{T,i}n_iK_{D,i}q_i s + k_{T,i}n_iK_{I,i}\frac{1}{s}q_i^r - k_{T,i}n_iK_{I,i}\frac{1}{s}q_i - J_{\text{eff},i}n_i - \frac{T_{L,i}}{n_i} + \frac{J_{M,i}}{n_i} =$$

$$= J_{\text{eff,i}}q_i s^2 n_i - \frac{J_{\text{eff},i}n_i^3 s^2 q_i}{n_i^2 q_i s^2} + \frac{J_{M,i}n_i q_i s^2}{n_i^2 q_i s^2} + f_{M,i}n_i q_i s$$

Now we need to separate the output value q_i to one side of the equation.

$$k_{T,i}n_i^2K_{p,i}q_i^r - k_{T,i}n_i^2K_{p,i}q_i + k_{T,i}n_i^2K_{D,i}q_i^r s - k_{T,i}n_i^2K_{D,i}q_i s + k_{T,i}n_i^2K_{I,i}\frac{1}{s}q_i^r - k_{T,i}n_i^2K_{I,i}\frac{1}{s}q_i - J_{\text{eff},i}n_i^2 - T_{L,i} + J_{M,i} =$$

$$= J_{\text{eff,i}}q_i s^2 n_i^2 - J_{\text{eff},i}n_i^2 + J_{M,i} + f_{M,i}n_i^2 q_i s$$

$$k_{T,i}n_i^2K_{p,i}q_i^r s + k_{T,i}n_i^2K_{D,i}q_i^r s^2 + k_{T,i}n_i^2K_{I,i}q_i^r - T_{L,i}s =$$

$$= f_{M,i}n_i^2 q_i s^2 + k_{T,i}n_i^2K_{p,i}q_i s + k_{T,i}n_i^2K_{D,i}q_i s^2 + k_{T,i}n_i^2K_{I,i}q_i + J_{\text{eff,i}}q_i s^3 n_i^2$$

$$q_i^r\left(k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{D,i}s^2 + k_{T,i}n_i^2K_{I,i}\right) - T_{L,i}s =$$
$$= q_i\left(f_{M,i}n_i^2s^2 + k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{D,i}s^2 + k_{T,i}n_i^2K_{I,i} + J_{\text{eff,i}}s^3n_i^2\right)$$

Now we need to express the result in the format:

$$q_i(s) = \frac{1}{N_i(s)}\left[F_i(s)q_i^r(s) - G_i(s)T_{L,i}(s)\right] \tag{15}$$

Therefore we adjust the equation

$$q_i = \frac{q_i^r\left(k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{D,i}s^2 + k_{T,i}n_i^2K_{I,i}\right) - T_{L,i}s}{J_{\text{eff,i}}s^3n_i^2 + f_{M,i}n_i^2s^2 + k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{D,i}s^2 + k_{T,i}n_i^2K_{I,i}}$$

$$q_i = \frac{\left(k_{T,i}n_i^2K_{D,i}s^2 + k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{I,i}\right)q_i^r - T_{L,i}s}{J_{\text{eff,i}}s^3n_i^2 + \left(f_{M,i}n_i^2 + k_{T,i}n_i^2K_{D,i}\right)s^2 + k_{T,i}n_i^2K_{p,i}s + k_{T,i}n_i^2K_{I,i}}$$

To receive the final result

$$q_i = \frac{\left(k_{T,i}K_{D,i}s^2 + k_{T,i}K_{p,i}s + k_{T,i}K_{I,i}\right)q_i^r - \frac{T_{L,i}s}{n_i^2}}{J_{\text{eff,i}}s^3 + (f_{M,i} + k_{T,i}K_{D,i})s^2 + k_{T,i}K_{p,i}s + k_{T,i}K_{I,i}}$$

Using the same constant as in Problem 12
$f_{\text{eff},i} = f_{M,i} + \frac{f_{L,i}}{N_i^2} = f_{M,i} = 2.4 \cdot 10^{-5}\left[\frac{\text{Nm}}{\text{rad/s}}\right]$
$f_L = 0$ (viscous friction coefficient)
$f_{\text{eff},i} = B_{\text{eff},i}$ $\quad f_{M,i} = B$
$g = 9.8\left[\frac{\text{m}}{\text{s}^2}\right]$
$k_T = 0.17\left[\frac{\text{Nm}}{\text{V}}\right]$
$n_i = 53$

We get the result

$$q_i = \frac{\left(0.17 \cdot K_{D,i}s^2 + 0.17 \cdot K_{p,i}s + 0.17 \cdot K_{I,i}\right)q_i^r - \frac{T_{L,i}s}{2809}}{J_{\text{eff,i}}s^3 + (2.4 \cdot 10^{-5} + 0.17 \cdot K_{D,i})\,s^2 + 0.17 \cdot K_{p,i}s + 0.17 \cdot K_{I,i}}$$

Problem 15 - PID in Simulink

Find $K_{P,i}, K_{D,i}, K_{I,i}$ expressed by the natural frequency ω, damping ζ and the time constant τ' of the closed loop system $\left(s^2 + 2\zeta_i\omega_{n,i}s + \omega_{n,i}^2\right)(1 + \tau_i's)$. Put $\tau_i' = |J_{\textbf{eff}\ ,i}|$ (in seconds) and plot $q_i^r, q_1, q_2, q_3, q_4$ as a function of time.

We will compare the characteristic polynomial P of the transfer function from Problem 14 with polynomial Q from the Assignment:

$$P = s^3 \cdot J_{\text{eff},i} + s^2 \cdot (f_{M,i} + k_{T,i} \cdot n_i \cdot K_{D,i}) + s \cdot k_{T,i} \cdot n_i \cdot K_{p,i} + k_{T,i} \cdot n_i \cdot K_{I,i}$$
$$Q = \left(s^2 + 2\zeta_i\omega_{n,i}s + \omega_{n,i}^2\right)(1 + \tau_i's) = s^3\tau_i' + s^2\left(1 + 2\zeta_i\omega_{n,i}\tau_i'\right) + s\left(2\zeta_i\omega_{n,i} + \omega_{n,i}^2\tau_i'\right) + \omega_{n,i}^2$$

Third-order term:

$$s^3 \cdot J_{\text{eff},i} = s^3\tau_i'$$
$$J_{\text{eff}\ ,i} = \tau_i'$$

Second-order term:

$$s^2 \cdot (f_{M,i} + k_{T,i} \cdot K_{D,i}) = s^2 \left(1 + 2\zeta_i \omega_{n,i} \tau_i'\right)$$

$$K_{D,i} = \frac{1 + 2\zeta_i \omega_{n,i} \tau_i' - f_{M,i}}{k_{T,i}}$$

First-order term:

$$s k_{T,i} n_i K_{p,i} = s \left(2\zeta_i \omega_{n,i} + \omega_{n,i}^2 \tau_i'\right)$$

$$K_{p,i} = \frac{2\zeta_i \omega_{n,i} + \omega_{n,i}^2 \tau_i'}{k_{T,i}}$$

Linear term:

$$k_{T,i} n_i K_{I,i} = \omega_{n,i}^2$$

$$K_{I,i} = \frac{\omega_{n,i}^2}{k_{T,i}}$$

Now we will determine PID gains for $i = 1, 2, 3, 4$ with parameters from Problem 14 and as follows:

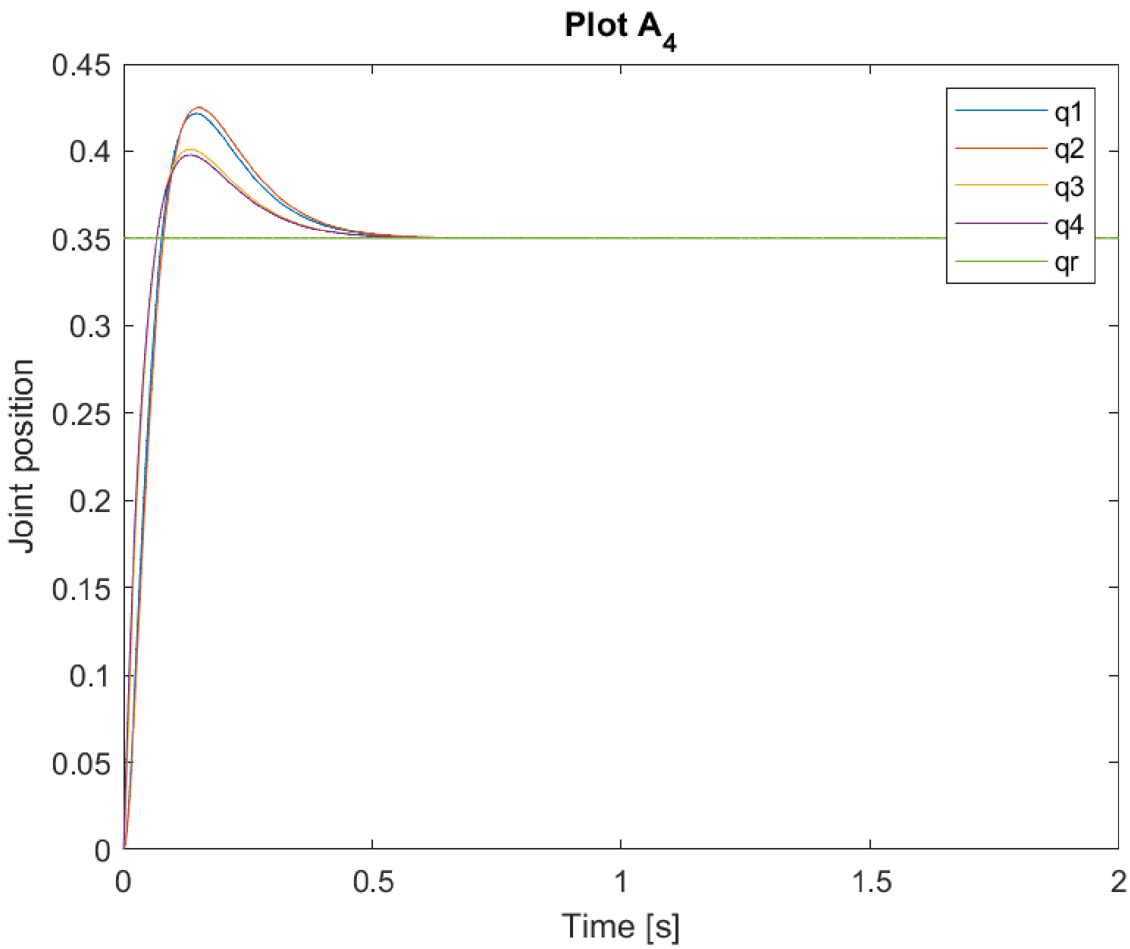
$$\omega_{n,i} = 15 \left[\frac{\text{rad}}{\text{s}}\right]$$

$$\zeta_i = 1$$

$$q_i^r = 0.35[\text{rad}] \text{ or } [\text{m}]$$

Therefore we have:

$$\begin{array}{cccc} K_{D,1} = 9.3763 & K_{D,2} = 10.1528 & K_{D,3} = 6.35869 & K_{D,4} = 5.94927 \\ K_{p,1} = 202.6765 & K_{p,2} = 208.5 & K_{p,3} = 180.0441 & K_{p,4} = 176.9735 \\ K_{I,1} = 1323.5294 & K_{I,2} = 1323.5294 & K_{I,3} = 1323.5294 & K_{I,4} = 1323.5294 \end{array}$$



As anticipated, the I term in our controller suppresses the steady-state error to 0. However, an overshoot is introduced due to the integration of the initial error, which is, of course, big enough to affect future behavior, resulting in an overshoot.

Problem 16 - Prefilter

Add the following pre-filter to the simulation model. Simulate a reference step response, and plot the results on the same plot (A5) as a function of time. Make comments on the result. Draw a symbolic block diagram (no numeric values) for the closed-loop system for link 1 including the control-object, the PID-controller, and the pre-filter (plot A6).

The pre-filter is made using the following transfer function:

$$F_{P,i}(s) = \frac{1}{\frac{K_{D,i}}{K_{I,i}}s^2 + \frac{K_{P,i}}{K_{I,i}}s + 1}, \quad i = 1, 2, 3, 4$$

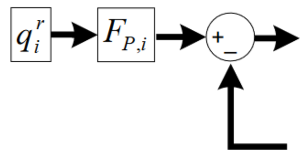
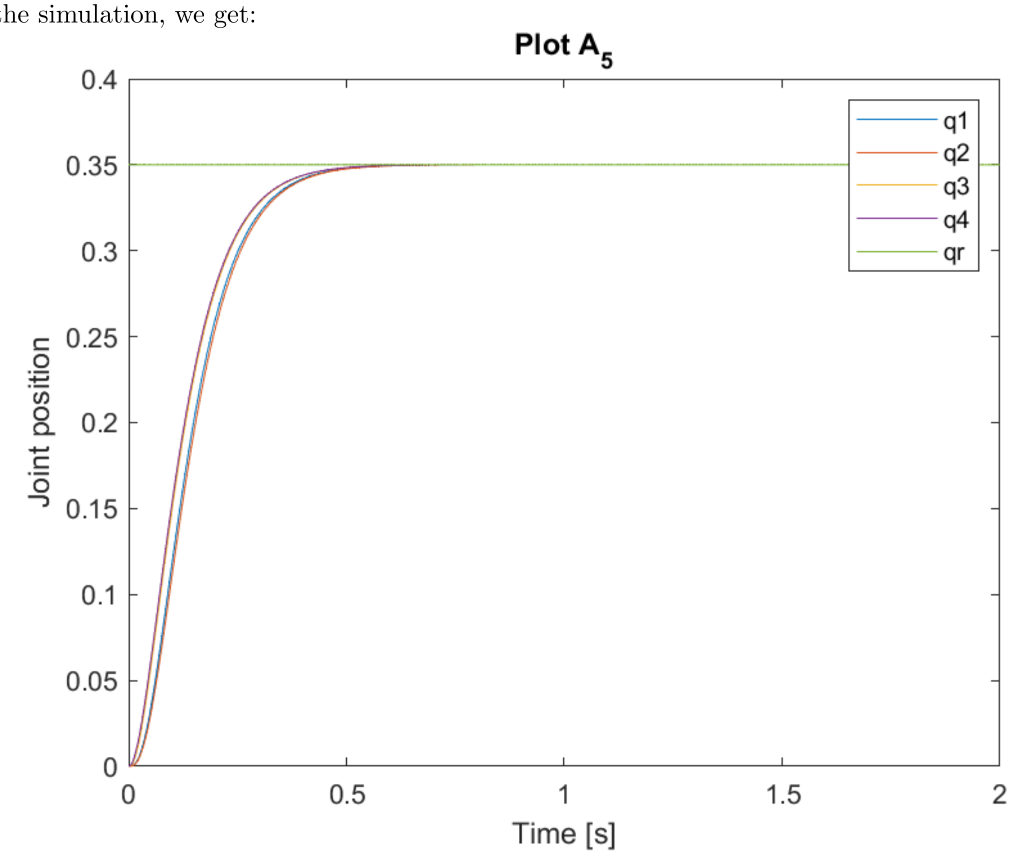


Figure 2: Prefilter

From this formula and the values from Problem 15, we get the following prefilters:

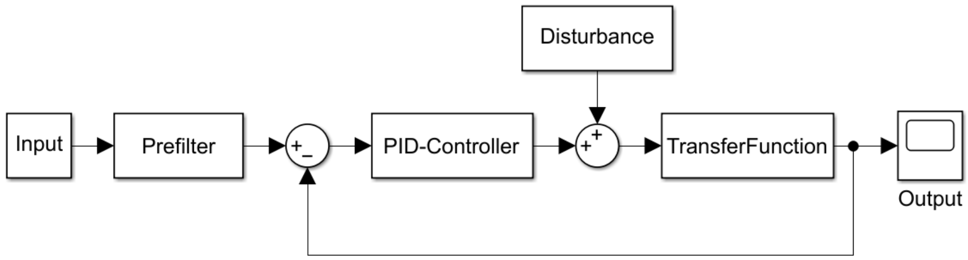
$$F_{P,1}(s) = \frac{1}{\frac{K_{D,1}}{K_{I,1}}s^2 + \frac{K_{p,1}}{K_{I,1}}s + 1} = \frac{1}{\frac{9.3763}{1323.5294}s^2 + \frac{202.6765}{1323.5294}s + 1} = \frac{1}{0.007s^2 + 0.1531s + 1}$$
$$F_{P,2}(s) = \frac{1}{\frac{K_{D,2}}{K_{I,2}}s^2 + \frac{K_{p,2}}{K_{I,2}}s + 1} = \frac{1}{\frac{10.1528}{1323.5294}s^2 + \frac{208.5}{1323.5294}s + 1} = \frac{1}{0.0077s^2 + 0.1575s + 1}$$
$$F_{P,3}(s) = \frac{1}{\frac{K_{D,3}}{K_{I,3}}s^2 + \frac{K_{p,3}}{K_{I,3}}s + 1} = \frac{1}{\frac{6.3587}{1323.5294}s^2 + \frac{180.0441}{1323.5294}s + 1} = \frac{1}{0.0048s^2 + 0.136s + 1}$$
$$F_{P,4}(s) = \frac{1}{\frac{K_{D,4}}{K_{I,4}}s^2 + \frac{K_{p,4}}{K_{I,4}}s + 1} = \frac{1}{\frac{5.9493}{1323.5294}s^2 + \frac{176.9735}{1323.5294}s + 1} = \frac{1}{0.0045s^2 + 0.1337s + 1}$$

Once added to the simulation, we get:



We see that adding a prefilter is a very effective way of preventing overshoots. There is no steady error, no ripples, and no overshoot. However, the rise time is bigger, but on the other hand, the settling time is very similar to both graphs with and without pre-filter.

A general system and whole Simulink model with a prefilter would look like this:



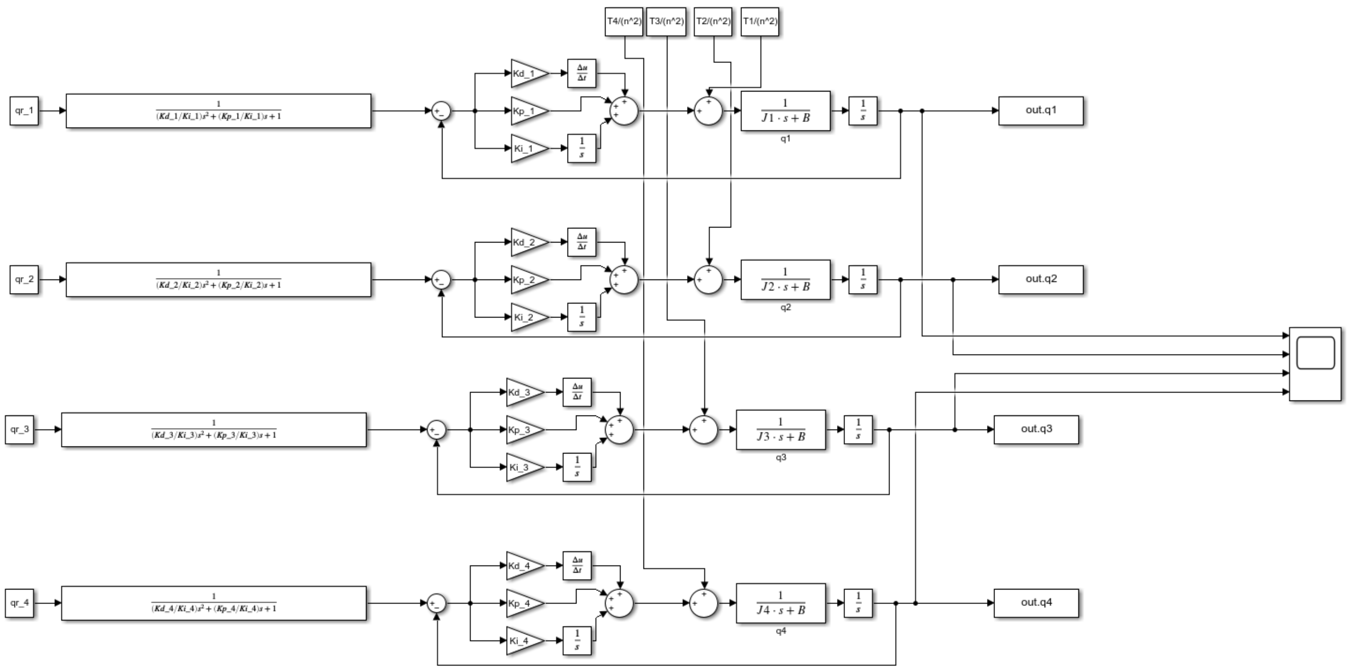
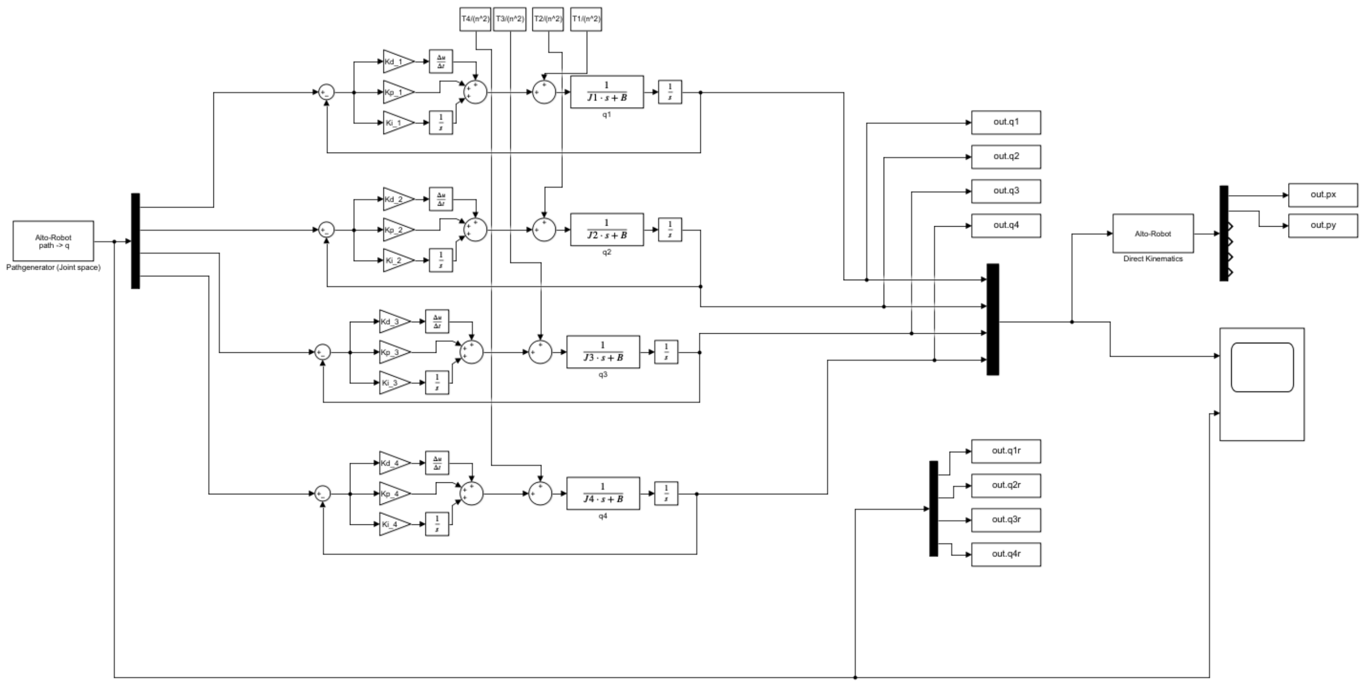


Figure 7: Plot A_6

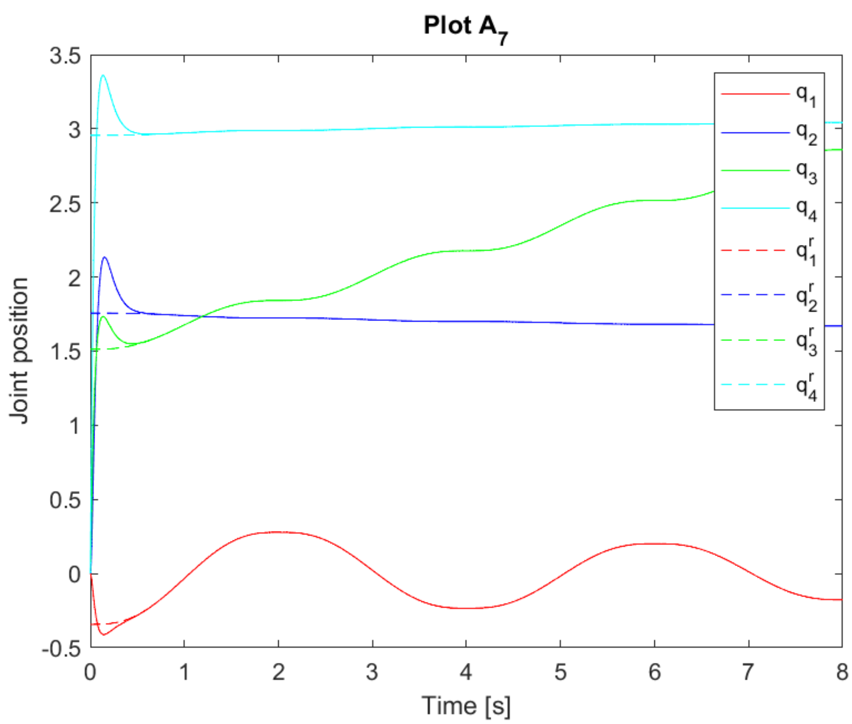
Problem 17 - Direct kinematic in Simulink

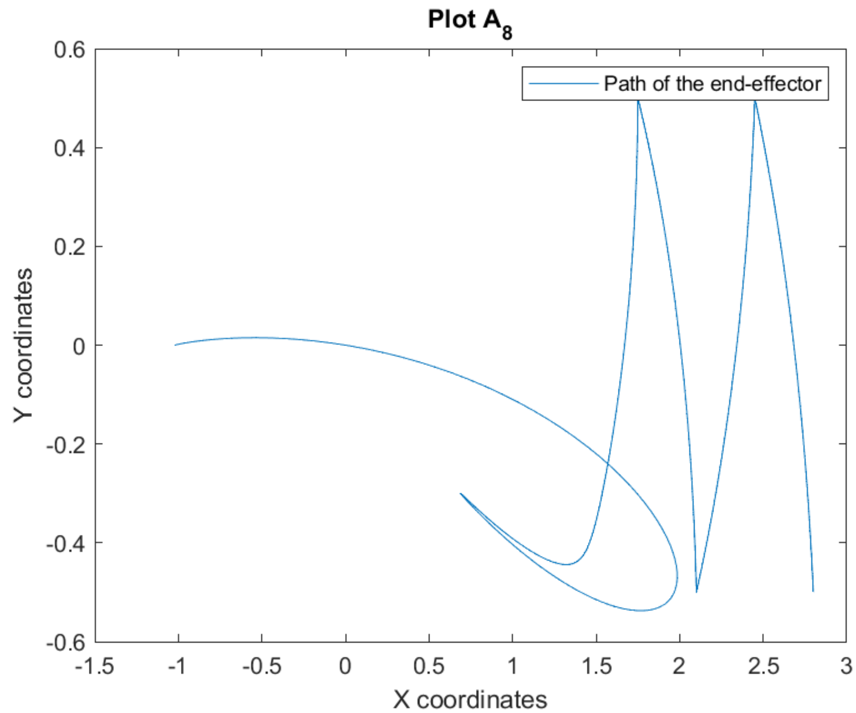
Add the SIMULINK block named Path generator, and the Block Direct kinematics to model. Add the GM-matrix (found in Problem 4) to MATLABs workspace. Simulate the point-to-point movement without the pre-filter (the duration time is 8sec.). Plot q_i^r as a function of time on the same plot A7 and show p_y as a function of p_x on plot A8. Make comments on the results. Animate the inclusion of the block named Show 4link.

The block diagram for this problem was derived using the PID Regulator from Problem 16 - Prefilter without the pre-filter and the quantic interpolation from Problem 4 - Quintic interpolation was used to simulate the 8 seconds of the robot movements.

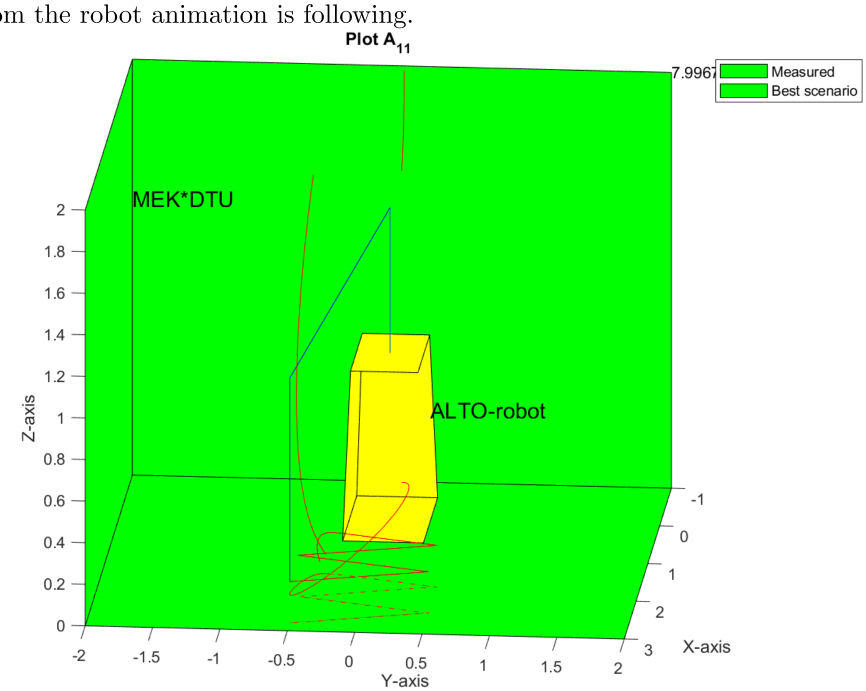


The input is based on Path generator, so this time we simulate the whole robot movement and not only the step response. Since we are working with four joints. Muxes and demuxes are used to seperate the values for each joint. To show the movement on the plot A8, we must use the Direct Kinematics to get the Cartesian coordinates from the joint values. The value throughout the time is shown on the graph A7, and the movement in the Cartesian space could be seen in Plot A8.





Moreover, the snap from the robot animation is following.

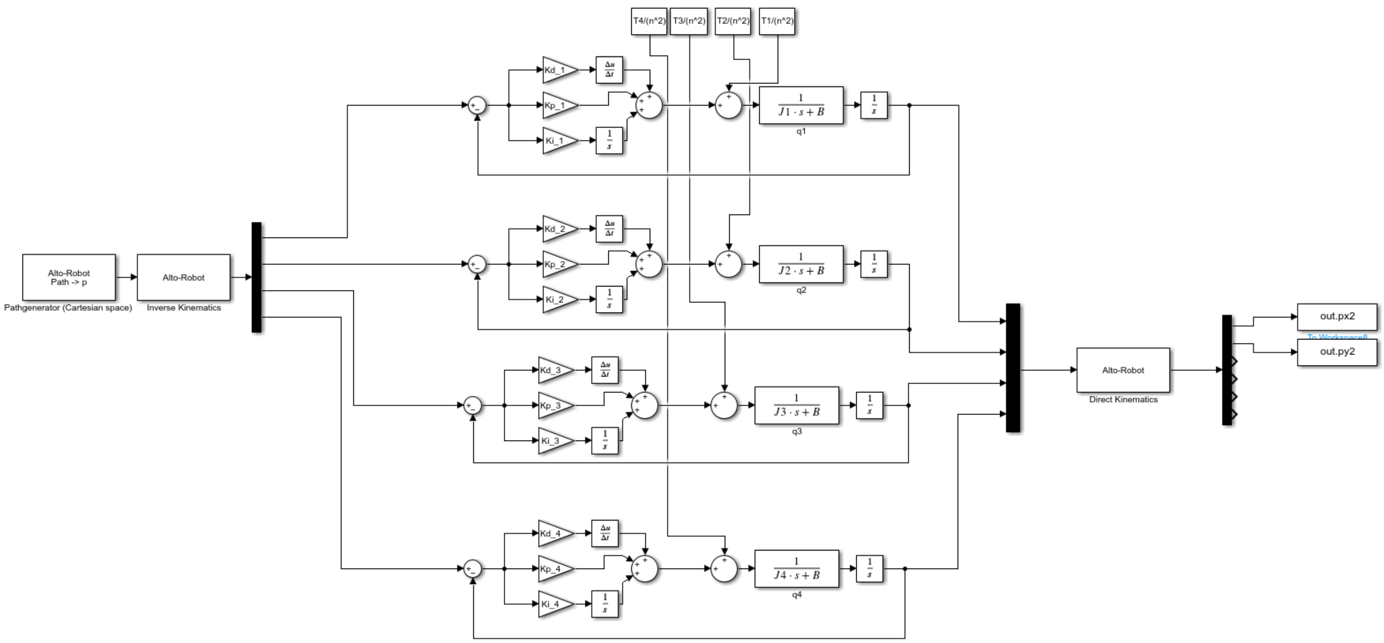


Since the pre-filter is not included in the block diagram, we can see a little overshoot before the robot starts to copy the desired path. This can be solved by starting the robot closer to deired values, so the initial movement from the home coordinates [-1,0] might be achieved in faster.

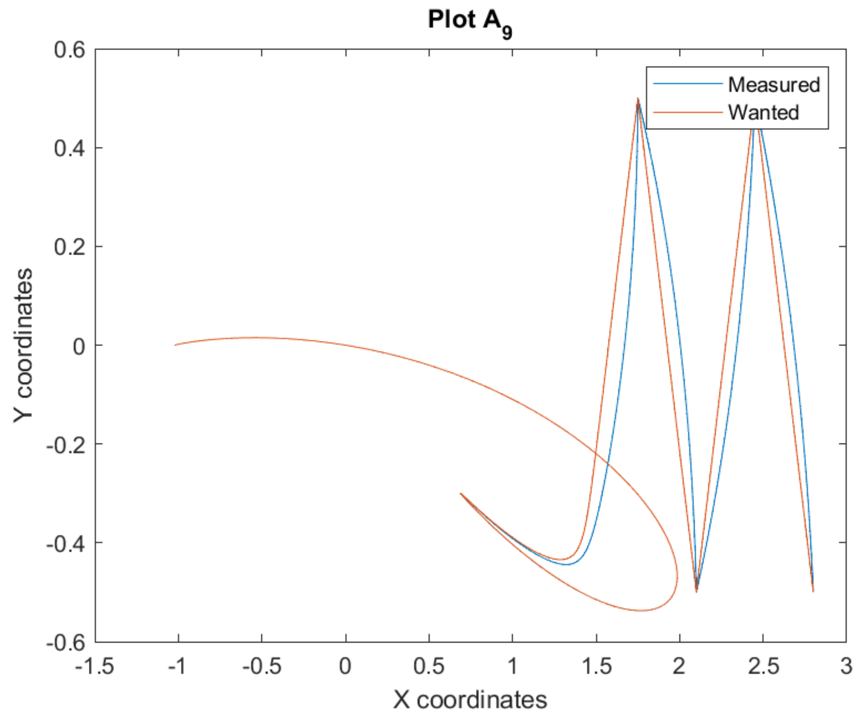
Problem 18 - Inverse kinematic in Simulink

Add to the simulation model the module named Inverse kinematics. Add the matrix found in Problem 5 to MATLABs workspace. Execute the m-file. After that, simulate the total movement in the Cartesian space (duration 8 seconds). Please plot the wanted Cartesian path and the measured Cartesian path (p_y as a function of p_x) on the plot A9. Make comments on the results.

To generate the Cartesian path, we used the matrix from Problem 5 - Cartesian straight line path. Again, the block diagram for this problem was derived using the PID Regulator from Problem 16 - Prefilter without the pre-filter. Till now, the input to the PID were 4 joint values for each robot arm. Now, from the path generator in Cartesian space we get 6 values $p_1, p_2, p_3, x_1 x_2 and x_3$. Therefore we first need to use the Inverse kinematics to receive four joint values that can be used in out PID controller.



Simulating the block diagram above, we received the following graph (we plot the values in Matlab workspace, so that everything can be simulate at one plae).



As we can see, the measured path is slightly skewed, as a proper pre-filter is not included. Altough the path is not followed perfectly, the maximum offset is 10cm, which is still acceptable in consideration of the robot's application and design.

Problem 19 - Manipulator dynamics matrix

Show that equation (1) and (2) can be reduced to $\widetilde{D}(q) \cdot \ddot{q} + \widetilde{v}(q, \dot{q}) = u$ where $\widetilde{D}(q)$ is a 4x4 matrix, $\widetilde{v}(q, \dot{q})$ is a 4x1 vector and $u = [u_1 u_2 u_3 u_4]^T$ is a 4x1 vector.

From equation (1), we know the τ_i values which can be implemented in equation (2).

$$u_i \cdot k_{T,i} - \frac{1}{n_i} \cdot (\sum_{j=1}^{4} D_{ij}(q) \cdot \ddot{q}_j + v_i(q, \dot{q})) = J_{M,i} \cdot \ddot{q}_{M_i} + f_{M,i} \cdot \dot{q}_{M,i}$$

From which, with in mind that we have to translate the motor angles to usual angles with $q_{M,i} = n_i \cdot q_i$, we get :

$$\begin{aligned}
u_i \cdot k_{T,i} &= J_{M,i} \cdot \ddot{q}_i \cdot n_i + f_{M,i} \cdot \dot{q}_i \cdot n_i + \frac{1}{n_i} \cdot (\sum_{j=1}^{4} D_{ij}(q) \cdot \ddot{q}_j + v_i(q, \dot{q})) \\
u_i &= \frac{J_{M,i} \cdot n_i}{k_{T,i}} \cdot \ddot{q}_i + \frac{f_{M,i} \cdot n_i}{k_{T,i}} \cdot \dot{q}_i + \frac{\sum_{j=1}^{4} D_{ij}(q) \cdot \ddot{q}_j}{n_i \cdot k_{T,i}} + \frac{v_i(q, \dot{q})}{n_i \cdot k_{T,i}} \\
u_i &= \left(\frac{J_{M,i} \cdot n_i}{k_{T,i}} \cdot \ddot{q}_i + \frac{\sum_{j=1}^{4} D_{ij}(q) \cdot \ddot{q}_j}{n_i \cdot k_{T,i}} \right) + \left(\frac{f_{M,i} \cdot n_i}{k_{T,i}} \cdot \dot{q}_i + \frac{v_i(q, \dot{q})}{n_i \cdot k_{T,i}} \right) \\
u_i &= \widetilde{D}_i(q) \cdot \ddot{q}_i + \widetilde{v}_i(q, \dot{q})
\end{aligned}$$

We write $\widetilde{D}_i(q)$ the 1x4 vector, i-th row of the matrix $\widetilde{D}(q)$.

We can now define the $\widetilde{D}(q)$ matrix:

$$\widetilde{D}_i(q)\cdot\ddot{q}_i = \frac{J_{M,i}\cdot n_i}{k_{T,i}}\cdot\ddot{q}_i + \frac{\sum_{j=1}^{4} D_{ij}(q)\cdot\ddot{q}_j}{n_i\cdot k_{T,i}} = \frac{J_{M,i}\cdot n_i}{k_{T,i}}\cdot\ddot{q}_i + \frac{\sum_{j=1}^{4} D_{ij}(q)\cdot\ddot{q}_j}{n_i\cdot k_{T,i}}$$

$$\widetilde{D}(q)\cdot\ddot{q} = \begin{bmatrix} \frac{J_{M,1}\cdot n_1}{k_{T,1}} & 0 & 0 & 0 \\ 0 & \frac{J_{M,2}\cdot n_2}{k_{T,2}} & 0 & 0 \\ 0 & 0 & \frac{J_{M,3}\cdot n_3}{k_{T,3}} & 0 \\ 0 & 0 & 0 & \frac{J_{M,4}\cdot n_4}{k_{T,4}} \end{bmatrix}\cdot\ddot{q} + \begin{bmatrix} \frac{D_{11}(q)}{n_1\cdot k_{T,1}} & \frac{D_{12}(q)}{n_1\cdot k_{T,1}} & \frac{D_{13}(q)}{n_1\cdot k_{T,1}} & \frac{D_{14}(q)}{n_1\cdot k_{T,1}} \\ \frac{D_{21}(q)}{n_2\cdot k_{T,2}} & \frac{D_{22}(q)}{n_2\cdot k_{T,2}} & \frac{D_{23}(q)}{n_2\cdot k_{T,2}} & \frac{D_{24}(q)}{n_2\cdot k_{T,2}} \\ \frac{D_{31}(q)}{n_3\cdot k_{T,3}} & \frac{D_{32}(q)}{n_3\cdot k_{T,3}} & \frac{D_{33}(q)}{n_3\cdot k_{T,3}} & \frac{D_{34}(q)}{n_3\cdot k_{T,3}} \\ \frac{D_{41}(q)}{n_4\cdot k_{T,4}} & \frac{D_{42}(q)}{n_4\cdot k_{T,4}} & \frac{D_{43}(q)}{n_4\cdot k_{T,4}} & \frac{D_{44}(q)}{n_4\cdot k_{T,4}} \end{bmatrix}\cdot\ddot{q}$$

$$\widetilde{D}(q) = \begin{bmatrix} \frac{J_{M,1}\cdot n_1}{k_{T,1}} + \frac{D_{11}(q)}{n_1\cdot k_{T,1}} & \frac{D_{12}(q)}{n_1\cdot k_{T,1}} & \frac{D_{13}(q)}{n_1\cdot k_{T,1}} & \frac{D_{14}(q)}{n_1\cdot k_{T,1}} \\ \frac{D_{21}(q)}{n_2\cdot k_{T,2}} & \frac{J_{M,2}\cdot n_2}{k_{T,2}} + \frac{D_{22}(q)}{n_2\cdot k_{T,2}} & \frac{D_{23}(q)}{n_2\cdot k_{T,2}} & \frac{D_{24}(q)}{n_2\cdot k_{T,2}} \\ \frac{D_{31}(q)}{n_3\cdot k_{T,3}} & \frac{D_{32}(q)}{n_3\cdot k_{T,3}} & \frac{J_{M,3}\cdot n_3}{k_{T,3}} + \frac{D_{33}(q)}{n_3\cdot k_{T,3}} & \frac{D_{34}(q)}{n_3\cdot k_{T,3}} \\ \frac{D_{41}(q)}{n_4\cdot k_{T,4}} & \frac{D_{42}(q)}{n_4\cdot k_{T,4}} & \frac{D_{43}(q)}{n_4\cdot k_{T,4}} & \frac{J_{M,4}\cdot n_4}{k_{T,4}} + \frac{D_{44}(q)}{n_4\cdot k_{T,4}} \end{bmatrix}$$

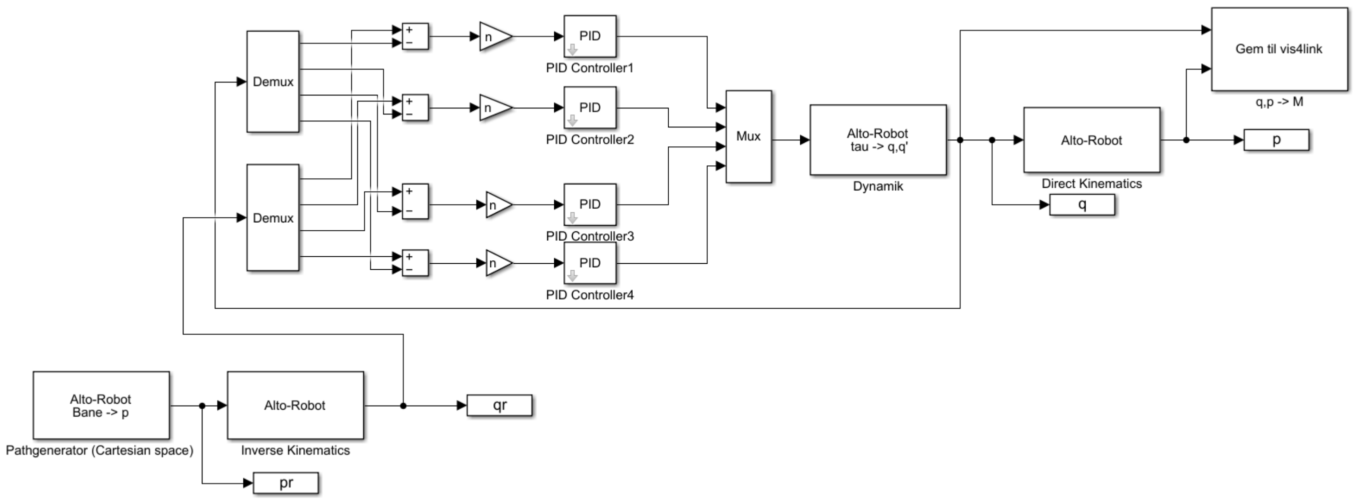
The matrix $\widetilde{v}(q,\dot{q})$ is defined as:

$$\widetilde{v}(q,\dot{q}) = \begin{pmatrix} \frac{f_{M,1}\cdot n_1}{k_{T,1}}\cdot\dot{q}_1 + \frac{v_1(q,\dot{q})}{n_1\cdot k_{T,1}} \\ \frac{f_{M,2}\cdot n_2}{k_{T,2}}\cdot\dot{q}_2 + \frac{v_2(q,\dot{q})}{n_2\cdot k_{T,2}} \\ \frac{f_{M,3}\cdot n_3}{k_{T,3}}\cdot\dot{q}_3 + \frac{v_3(q,\dot{q})}{n_3\cdot k_{T,3}} \\ \frac{f_{M,4}\cdot n_4}{k_{T,4}}\cdot\dot{q}_4 + \frac{v_4(q,\dot{q})}{n_4\cdot k_{T,4}} \end{pmatrix}$$

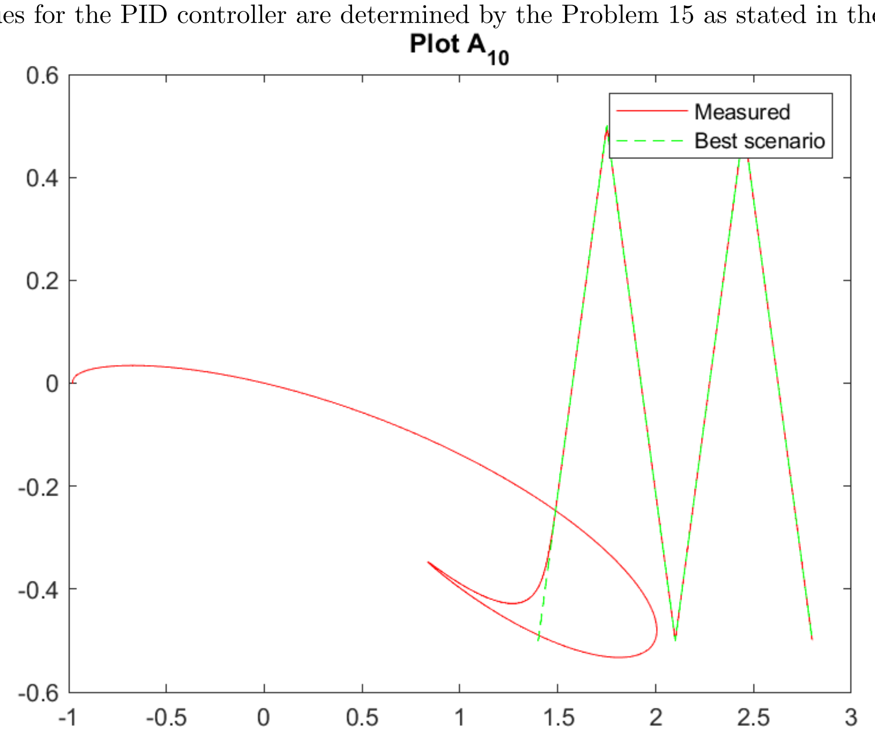
Problem 20 - PID Real scenario

For the last two Problems, consider the "exact" model and a square generator included in provided trc dyn.mdl. In this exact model, the gravity load is dependent of the joint variables as shown in Annex B, and the gravitational load is no longer approximated by the constant "worst-case" load torque TL. Add the control parameters determined in Problem 15 and the GM-matrix used in Problem 18 to MATLAB's workspace. Execute the m-file. Simulate the movements. Plot the wanted Cartesian path and the measured Cartesian path (p_y as a function of p_x) on the same plot (A10). Compare the results with the results obtained in Problem 18.

Running the Simulink below, we achieved the graph using the PID "exact" model, considering the gravitational load dependent on the joint variable, not on the worst case load torque T_L. Nevertheless, friction, air rezistance, and other losses were not considered.



For plotting the following graph, only p values are used, since the joint angles are not considered in the final graph for this problem. The values for the PID controller are determined by the Problem 15 as stated in the assignment.



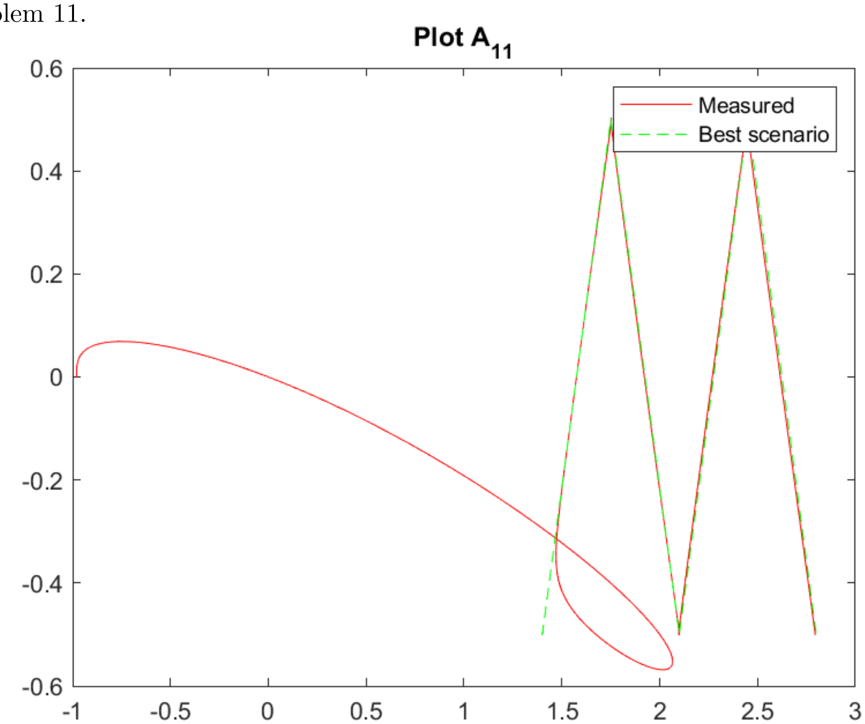
We can see from the graph, that the desired path was followed without any errors. This is the result of not considering the worst-case scenario for load torque T_L for each joint. There is still small delay at the start of the path, but it is because we are starting too far from the starting point. This can be resolved by moving the end-effector closer to the first point of desired path.

Problem 21 - PD Real scenario

Considering the real scenario from Problem 20, add the control parameters determined in Problem 11. Simulate without using the integration control using a PD controller. Plot the wanted Cartesian path and

the measured Cartesian path on plot A11 (p_y as a function of p_x). Compare the results with the results obtained in Problem 20.

We use the same block diagram as before, the only difference is in PID controller. This time, we set the K_I values to zero. Therefore there is a zero impact for the final results from the integrator. K_P and K_D values are defined using the calculations from Problem 11.



There are two differences between the plots A_{20} and A_{21}. Contrary to the PID graph, using the PD controller, the robot arm moves directly on the desired trajectory without any path overshoot. There is however an offset introduced along the path due to an insufficient disturbance rejection ability. By adding and adjusting the I term, a suitable compromise between an offset elimination and wind up overshooting could be accomplished.

References

[1] K. Poulios, *Trajectory Planning*. `https://learn.inside.dtu.dk/d2l/le/content/79600/viewContent/281144/View`, 2021. [Slide 25].

[2] M. W. S. . S. H. . M. Vidyasagar, *Robot Modeling and Control*. `https://www.researchgate.net/profile/Mohamed-Mourad-Lafifi/post/How_to_avoid_singular_configurations/attachment/59d6361b79197b807799389a/AS%3A386996594855942%401469278586939/download/Spong+-+Robot+modeling+and+Control.pdf`, First Edition.