Network Science

Lecture 18: Sep 15, 2009 Lecturer: Madhav Marathe

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Congestion Game

18.1 Traffic at Equilibrium

We first analyze a sample network with traffic flowing from A to B in Figure 18.1. Each edge is labeled with the required travel time in minutes when there are x cars on it. Suppose we have 4000 cars in total. With drivers choose different strategies, the total travel time would be different. For instance, when all 4000 cars travel from A to B via C, the total travel time for each car is 85 minutes. If 2000 cars choose to go via C while the other 2000 cars choose to go via D, the total travel time for each car is 65 minutes; this is actually what will happen at equilibrium of the congestion game.

Now we analyze the congestion game with respect to Nash Equilibrium. 4000 cars in the system means 4000 individual players. This game does have a Nash Equilibrium, which is obtained by equal balance among the cars, i.e., 2000 cars travel on $A \to C \to D$ and the other 2000 on $A \to D \to B$. That is because with an even balance between the two routes, no driver has an incentive to switch over to the other route. And if there is any unbalanced traffic going on, the route with more traffic will result in longer travel time, giving the drivers on the slow route incentives to switch to the faster and less congested route.

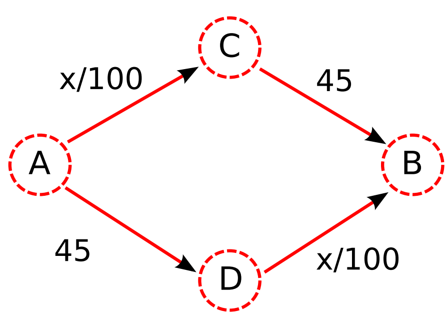

Figure 18.1: A highway network.

18.2 Braess's Paradox

Braess's Paradox is, briefly, "additional capacity/resources may sometimes degrade performance at equilibrium." This is counter-intuitive, since one would expect that adding capacity to a system

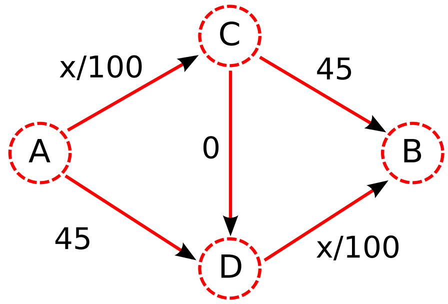

Figure 18.2: A highway network with added resources.

would improve overall performance, yet there are real-life examples where this is exactly the opposite case. We will look at the following example and find that it is actually not so "paradoxical."

In Figure 18.2, we show a traffic network identical to the one in Figure 18.1, but with an additional highway added from C to D. And let us assume that the travel time is far shorter than the other routes and is almost 0 regardless of the amount of traffic through it. Although the traffic system has been "upgraded", the travel time at equilibrium will be increased to 80 minutes now, "surprisingly". That is because at equilibrium, every driver gains by switching route to $A \to C \to D \to B$, regardless of the current traffic pattern.

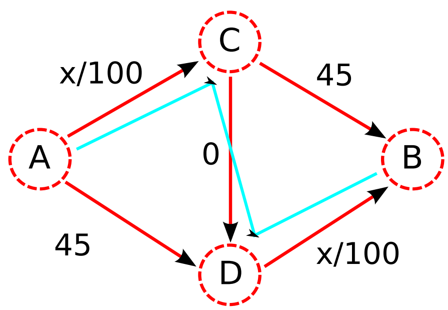

Figure 18.3: Route (in blue) chosen by all at equilibrium.

18.3 The Social Cost of Traffic at Equilibrium

Definition 1 *Travel-time Function* $T_e(x) = a_e x + b_e$, gives the time it takes all drivers to cross the edge e when there are x drivers using it.

Definition 2 *Traffic Pattern*, is simply a choice of a path by each driver.

Definition 3 *Social Cost*, $SC(Z)$, of a given traffic pattern Z, is the sum of the travel times incurred by all drivers when they use this traffic pattern.

A traffic pattern which achieves the minimum possible social cost, is said to be *socially optimal*. Note that the sum of the drivers payoffs is the negativeof the social cost.

Next we will show that that there is always an equilibrium whose social cost is at most twice that of the optimum [1]. That is,

$$SC(Z_{Nash}) \leq 2SC(Z_{OPT}).$$

18.3.1 Best Response Strategy

First we search for an equilibrium by constantly having new travelers perform their best response to the current configuration. We start with any traffic pattern Z, if it reaches equilibrium, we are done. Otherwise, we at least have one traveler t who is deciding on which path to take from source to destination, and we have that $cost(P_{new}(t)) < cost(P_{current}(t))$, we will let t take the new path $P_{new}(t)$. If the procedure ever stops, we have arrived at a equilibrium for the system. Now we will show that the best response strategy must eventually stop at an equilibrium.

Lemma 1 *The potential energy of the system strictly decreases by following the best response strategy.*

Proof: When a driver switches paths, the net change in potential energy is simply the difference between the new travel time minus the old travel time. According to the best response strategy, a driver only changes paths when it causes his travel time to decrease. ■

Since the potential energy cannot decrease forever, it eventually reaches an equilibrium where it would not change any more.

18.3.2 Analysis

Definition 4 *Potential Energy* of a traffic pattern, is defined as

$$Energy(e) = T_e(1) + T_e(2) + \ldots + T_e(x) \text{ ,if edge } e \text{ currently has } x \text{ drivers on it.}$$

And $Energy(e) = 0$ if $x = 0$.

Definition 5 *Total Travel Time* of an edge e with x travelers is

$$TotalTravelTime(e) = xTe(x).$$

Lemma 2 $\frac{1}{2}TotalTravelTime(e) \leq Energy(e) \leq TotalTravelTime(e).$

Proof: Obviously, the following inequality is true.

$$Energy(e) \leq TotalTravelTime(e).$$

Besides, since T_e is a linear function, we can obtain

$$\begin{aligned} Energy(e) &= Te(1) + Te(2) + \ldots + Te(x) \\ &= a_e(1 + 2 + \ + x) + b_e x \\ &= \frac{a_e x(x+1)}{2} + b_e x \\ &\geq \tfrac{1}{2} x(a_e x + b_e) \\ &= \tfrac{1}{2} x T_e(x) \\ &= \tfrac{1}{2} TotalTravelTime(e). \end{aligned}$$

The above two inequalities imply that

$$\frac{1}{2} TotalTravelTime(e) \leq Energy(e) \leq TotalTravelTime(e).$$

■

Let $Energy(Z)$ denote the total potential energy of all edges when drivers follow the traffic pattern Z.

According to the definition of social cost, we have

$$\frac{1}{2} SC(Z) \leq Energy(Z) \leq SC(Z).$$

Let Z_{OPT} be the starting traffic configuration. At optimum, we have

$$\frac{1}{2} SC(Z_{OPT}) \leq Energy(Z_{OPT}) \leq SC(Z_{OPT}).$$

Then, throught the operation of best response strategy, at equilibrium, we have

$$\frac{1}{2} SC(Z_{Nash}) \leq Energy(Z_{Nash}) \leq SC(Z_{Nash}).$$

Because $Energy(Z_{Nash}) \leq Energy(Z_{OPT})$, we can obtain that

$$\frac{1}{2} SC(Z_{Nash}) \leq Energy(Z_{Nash}) \leq Energy(Z_{OPT}) \leq SC(Z_{OPT}).$$

That is, what we wanted to show,

$$SC(Z_{Nash}) \leq 2SC(Z_{OPT}).$$

References

[1] T. Roughgarden and E. Tardos, "How bad is selfish routing?" *Journal of the ACM*, vol. 49, pp. 236–259, 2002.