Multivariable Calculus

And Why We Care

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Chapter 1

Multivariable Precalculus

Learning everything we will need about math in multiple dimensions, before we apply calculus.

1.1 Geometry in $\mathbb{R}^3$

Welcome to the world of 3D geometry! In this section, we will explore how to measure distances in three-dimensional space and describe spheres. While we'll cover the essential formulas and definitions, our focus will be on understanding why these ideas matter and seeing how they come up in real-life situations. Let's start by looking at a familiar scenario— something you might encounter every day—and then connect it to the underlying math.

Remark 1.1.1 Motivation: Navigating in 3D. Imagine you're operating a drone that can move in any direction above the ground. You want to figure out how far the drone is from its takeoff spot, so you can plan the best path to fly back safely. Or picture a submarine exploring the ocean depths: it might need to maintain a certain distance from the seafloor. These situations both revolve around figuring out the distance between two points in three-dimensional space.

Insight 1.1.2 What is $\mathbb{R}^3$?$Or\mathbb{R}^n$? We will see this notation alot. What does it mean? You might recall that the xy-plane is the same as $\mathbb{R}^2$ -- which basically means two real number lines slapped on each other at 90 degrees. Well, $\mathbb{R}^3$ is just doing that again, with another number line that is perpendicular to both. (Imagine the xy-plane is your desk, so the only thing 90 degrees to that would be straight up out of the desk, or toward the floor).

To extend this idea to $\mathbb{R}^n$ we just need a bit of imagination. The same thing is happening - slapping on another "number line" that is perpendicular to all the others... we just can't draw it anymore. n can be any natural number, and this will work out. This n is the number of "dimensions", which is just the number of things we care to measure. For our purposes, 3 dimensions will typically be enough. But just nearly everything we do will work in higher dimensions.

1.1.1 The Distance Formula in 3D

1.1.1.1 Coordinates of Points in 3D

When we work in three dimensions, we label each point with three coordinates (x, y, z). Think of x as the left–right direction, y as the forward–back direction, and z as the up–down direction.

- A point P is located at (x, y, z).
- Another point Q is located at (x, y, z).

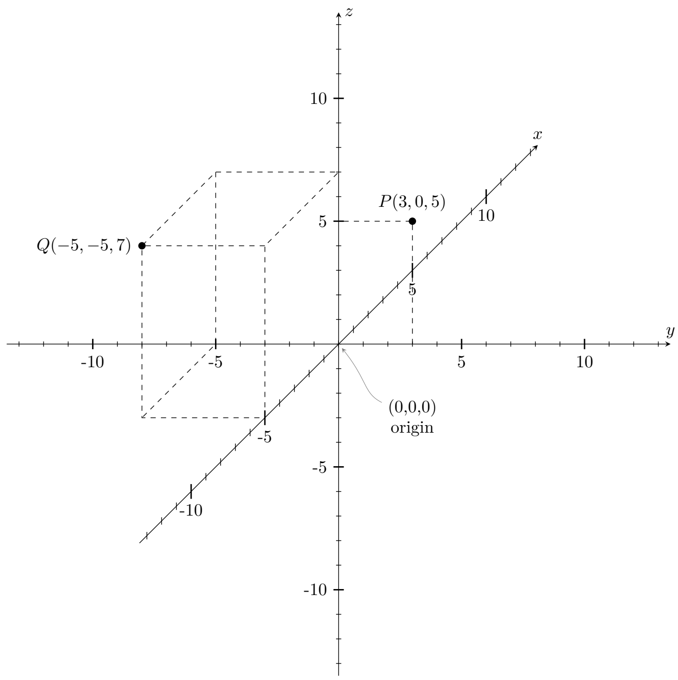

Figure 1.1.3 A figure generated with TikZ in LaTeX

1.1.1.2 The Distance Formula

If you need the distance d between P and Q, you can use a direct extension of the Pythagorean theorem from two dimensions to three dimensions. The formula is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This formula captures the combined difference in each of the three directions. Example (Drone Navigation):

If your drone starts at $(2, -1, 5)$ and ends up at $(5, 3, 7)$, the distance traveled is

$$\sqrt{(5-2)^2 + (3-(-1))^2 + (7-5)^2} = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{9 + 16 + 4} = \sqrt{29}.$$

So the drone is $\sqrt{29}$ units away from its starting point.

Checkpoint 1.1.4 Find the distance from the point (-6,-1,4) to the xy-plane.

1.1.2 Equations of Spheres

A sphere in three-dimensional space is the set of points that are all at the same distance, called the radius, from a fixed point called the center. Spheres are often used in physics (like modeling planets or atoms), in computer graphics (3D animations), and even in global positioning systems (GPS relies on distances from satellite “spheres”).

1.1.3 Standard Form of a Sphere

If a sphere has a center at (h, k, l) and a radius r, any point (x, y, z) on the sphere satisfies:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

This is basically the 3D distance formula set equal to a constant radius r.

Example (GPS Satellites):

GPS satellites broadcast their positions and send signals that travel at the speed of light. The distance between a satellite and your receiver on Earth is like the radius of a sphere. Your receiver calculates its position by finding the intersection of several of these "distance spheres" from different satellites. Wherever they intersect is where you are located on the planet's surface.

1.1.4 Visualizing a Sphere

- Center: (h, k, l)
- Radius: r

If you wanted to picture it in your mind, think of wrapping a basketball around a point in space. The surface of the sphere is the collection of all points that are exactly r units from the center. (Note: it does not contain the points inside -- it is specifically the set of points on the surface)

Let's put these ideas into practice with a quick exercise.

Example 1.1.5 Satellite in Orbit. A certain GPS satellite orbits at a center point $(20000, 0, 0)$ in kilometers (this is not an actual orbit location—just an example). The sphere representing all the points it can reach on Earth at a given instant has a radius of 2000 km. Write an equation for this sphere.

Solution.

1. The center of the sphere: $(h, k, l) = (20000, 0, 0)$.
2. The radius of the sphere: $r = 2000$.
3. Plug into the formula:

   $$(x - 20000)^2 + (y - 0)^2 + (z - 0)^2 = 2000^2.$$

   Or more simply:

   $$(x - 20000)^2 + y^2 + z^2 = 4,000,000.$$

That's it! You've written the equation of the sphere. □

5. Key Takeaways

- Distance in 3D – To find the distance between points (x, y, z) and (x, y, z), remember
  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
- Equation of a Sphere – A sphere with center (h, k, l) and radius r is given by
  $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$
- Real-World Applications – From drone flights to GPS technology, understanding how to calculate distances and describe spheres in 3D helps us solve practical, real-life problems.

Looking Ahead. You'll use distance formulas and the concept of spheres to tackle more advanced topics like level surfaces, gradients, and optimization in 3D. By appreciating the real-world motivations—like finding paths for vehicles, mapping planetary orbits, and designing engineering solutions—these mathematical tools become more intuitive and more powerful.

Keep practicing, and remember to look for ways these ideas show up in everyday life. You'll soon see that the world around us is brimming with three-dimensional geometry in action!

1.1.5 Exercises

Exercise Group. In these exercises, you will practice applying the distance formula in 3D and writing equations for spheres.

1. Use the distance formula to find the distance between the points $(1, 2, 3)$ and $(-2, 4, 5)$.

2. Find the equation of the sphere with center $(2, -1, 3)$ and radius 7.

Hint. Remember the standard form:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

3. A drone is hovering at $(5, 10, 3)$ in 3D space. If its takeoff point was $(0, 0, 0)$, how far has it traveled?

4. If a point (x, y, z) satisfies

$$x^2 + y^2 + z^2 = 16,$$

what geometric surface does it represent?

Colophon

This book was authored in PreTeXt.