Università degli Studi di Milano

Bachelor of Science in Artificial Intelligence

Coding assignment of Mathematics for Imaging and Signal Processing

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Project Report: Generalized Tikhonov Regularization & Error Analysis

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Abstract

The project explores the problem of image blurring and deconvolution. The script consists of implementations of various types of image blur in the Fourier domain (gaussian, linear motion and out-of-focus), as well as different variants of the deblurring solution (spectral window and Tikhonov regularization, with L2, H1 and H2 penalties). Qualitative comparison is made between the different approaches to reconstruction, w.r.t. the smoothness of edges and the presence of artifacts. The tradeoff between bias and variance of the solution is explored.

1 Script overview

1.1 The forward problem

The code assumes the input image to be square. The transformation to grayscale is implemented by setting the intensity value Y for each pixel as a linear combination of it's red, green and blue values according to:

$$Y = 0.299R + 0.587G + 114B \tag{1}$$

For each blur type, the blurring kernel in the Fourier domain is computed as:

- Gaussian: $\hat{K} = \exp\left(-\frac{k_x^2+k_y^2}{2\sigma^2}\right)$
- Linear motion: $\hat{K} = \exp\left(-iL\frac{\pi k_x}{N}\right) sinc(Lk_x/N)$
- Out-of-focus: $\hat{K} = 2 \cdot \frac{J_1(R \cdot \frac{2\pi|k|}{N})}{R \cdot \frac{2\pi|k|}{N}}$

Parameters σ, L and R can be freely adjusted.

k_x, k_y are integers in the range $[-N/2, N/2-1]$, which corresponds to a sampling grid of frequencies $(\omega_x, \omega_y) = \frac{2\pi}{N}(k_x, k_y)$ where N is the side length of the image. Using k in place of ω introduces to $\hat{K}$ only a scaling factor that is absorbed by the parameter value.

The standard deviation of the noise distribution is computed based on the signal-to-noise ratio according to:

$$\sigma_{noise} = 10^{-SNR_{dB}/20} \cdot std(g) \tag{2}$$

where g is the blurred image.

1.2 Reconstruction filters

For Tikhonov regularization, the following reconstruction kernel has been constructed:

$$\hat{R}_\mu = \frac{\hat{K}^*}{|\hat{K}|^2 + \mu P} \tag{3}$$

Where μ is the regularization parameter and the polynomial P is defined based on the penalty type:

- L2: $P = \mathbf{1}$
- H1: $P = k_x^2 + k_y^2$
- H2: $P = (k_x^2 + k_y^2)^2$

The spectral window kernel is defined as:

$$\hat{R}_\Omega = \frac{\hat{W}_\Omega}{\hat{K}}, \quad \hat{W}_\Omega(k_x, k_y) = \begin{cases} 1 & \text{if } \sqrt{k_x^2 + k_y^2} < \Omega \\ 0 & \text{if } \sqrt{k_x^2 + k_y^2} \geq \Omega \end{cases} \tag{4}$$

where Ω is an adjustable parameter determining the cutoff threshold.

In the naive solution, the reconstruction kernel is defined as:

$$\hat{R}_{naive} = \begin{cases} \frac{1}{\hat{K}} & \text{if } \hat{K} > 10^{-5} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The threshold has been included to make computation possible.

2 Comparison of reconstructed images

2.1 Changing the regularization penalty

The necessity of regularization is exhibited in fig. 1. The naive solution is completely overtaken by noise. The hard spectral cut-off constitutes a solution similar to the naive reconstruction. It avoids completely flooding the image with noise by excluding high frequencies from the reconstruction process. As a result, it is less accurate at preserving details than the smooth tikhonov filters.

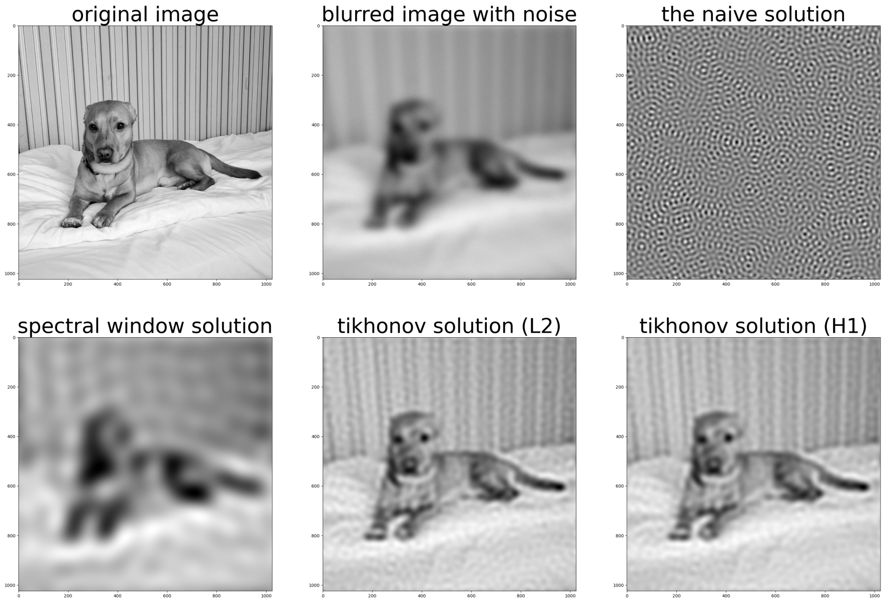
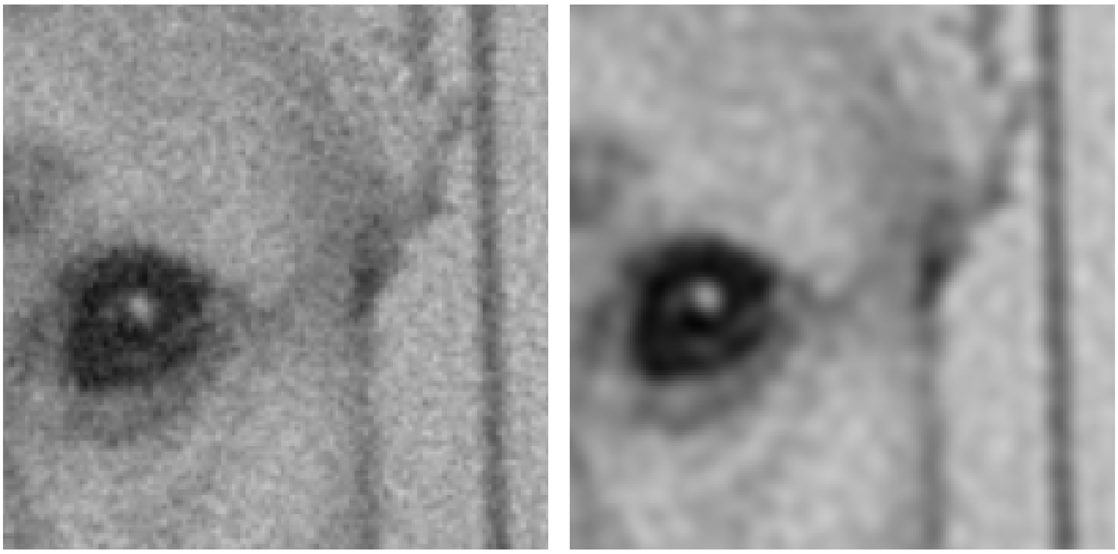


Figure 1: Reconstruction comparison. The original image (1), image with strong gaussian blur and low SNR (2), naive reconstruction (3), spectral window solution (4), tikhonov regularization with L2 (5) and H1 (6) penalty

The L2 regularization suppresses the absolute value of the reconstructed image, while H2 strongly suppresses the signal's derivative. As a result, H2 reconstruction smoothes out any edges present in the image. L2 regularization does not counter the sudden jumps in intensity caused by the addition of noise, making the reconstructed image appear grainy, as shown in fig. 2 .

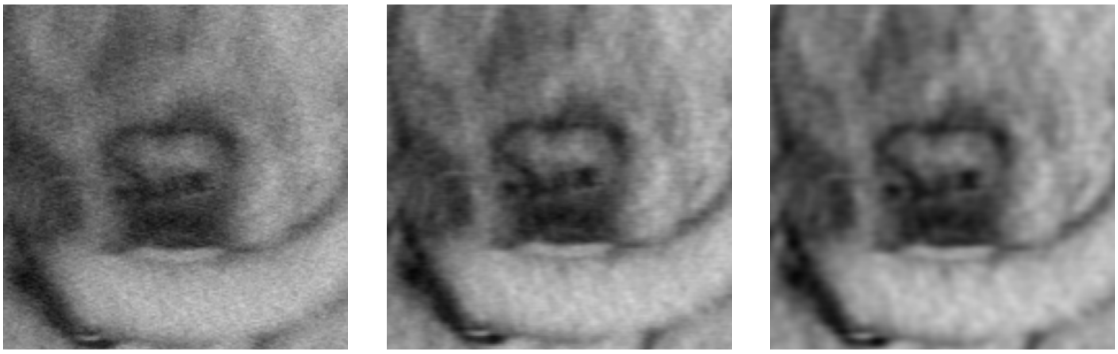
The smoother H1 and H2 penalties cause the assignment of a single value to groups of pixels that have similar values. As a result, the reconstruction has regions of constant intensity, giving the image an appearance of higher contrast. In comparison L2 appears less saturated, due to a lack of uniformly black or white regions.

For this image, Tikhonov with H2 penalty was the best performing regularization technique, over a range of blur types, strengths, and noise ratios.
Natural images are mainly composed of regions of uniform colors or smooth gradients caused by shade. This is in agreement with the assumptions of the H2 penalty, which rewards such regions. The error caused by the insufficient sharpness of the edges is outweighed by the correctly reconstructed surfaces.



(a) L2 penalty (b) H2 penalty

Figure 2: Comparison betweeen an L2 and H2 reconstruction.



(a) L2 penalty (b) H1 penalty (c) H2 penalty

Figure 3: Comparison betweeen an L2, H1 and H2 reconstruction.

2.2 High vs. low SNR

The application of blur to an image is a deterministic process. While it can lead to a loss of information, it is highly reversible, with the exception of zeros of the blurring kernel. The need for regularization exists because of the presence of noise – we want to reconstruct only the underlying true image, not the noise. As a result, the value of the optimal regularization parameter μ differs depending on the SNR, as shown in table 1.

Any regularization method performs better in the high SNR scenario, since the range of frequencies uncontaminated by noise is bigger.

	Ω	μ_{L2}	μ_{H1}	μ_{H2}
$SNR = 40dB$	196	10^{-4}	10^{-9}	10^{-13}
$SNR = 20dB$	52	10^{-2}	10^{-7}	10^{-11}

Table 1: Comparison of optimal regularization parameters with low-noise and high-noise input

2.3 Artifacts, Ringing phenomena

Gibbs phenomenon refers to the spikes in the Fourier series of a discontinuous function, which occur around its jump point. The effect is attenuated with the inclusion of more components in the series. In the spectral window solution, the high-frequency components needed to counter this phenomenon are excluded, which results in a clearly observable artifact: along a sharp edge of the image, the reconstruction includes a light band which was not present in the original. This effect is visibly stronger in the spectral window case, than it is when using the very smooth kernel of the H2 penalty.

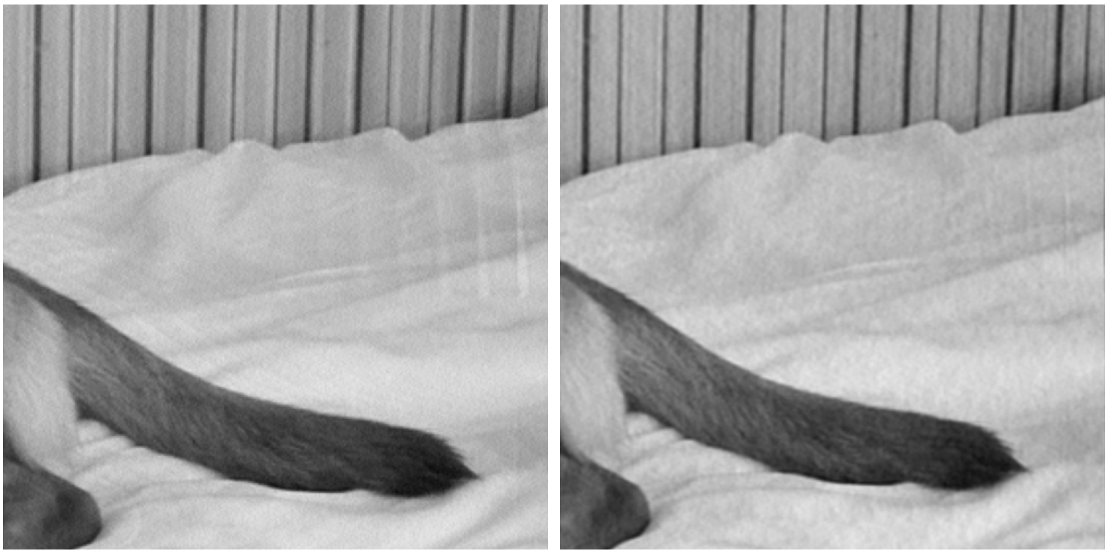
Near transitions between dark and bright areas of the image, the reconstruction filter effectively overcompensates the recovered pixel intensity, leading to repeating brighter ”echos” of the original edge, as visible in fig. 5. H2 penalty rewards regions with uniform values or smooth gradients of intensity, countering this effect more successfuly than L2.

(a) Original image

(b) Spectral window reconstruction

(c) Thikhonov with H2 penalty (best)

Figure 4: Comparison of the presence of ringing artifacts, between the sharp cut-off (b) and smooth filter (c)



(a) Tikhonov, L2 penalty

(b) Tikhonov, H2 penalty

Figure 5: Comparison of the echo (ripple) effect with L2 and H2 regularization

3 Bias-variance tradeoff

While a completely unregularized reconstruction leads to an unacceptable solution, regularization which is too strong can suppress the recovery of any detail, rewarding instead a reconstruction which is completely smooth.
With high values of μ, the constraint imposed on the solution outweighs the accuracy of the reconstruction. To achieve the optimal solution, we must consider a trade-off between the approximation error (caused by the inclusion of the penalty), and random error (caused by the presence of noise).
The bias and variance terms of the reconstruction error have been computed as:

$$B = ||\hat{R} \cdot \hat{g}_{clean} - \hat{f}||^2 \tag{6}$$

$$V = ||\hat{R} \cdot \hat{n}||^2 \tag{7}$$

The total error is then found to be:

$$E = ||\hat{R} \cdot \hat{g}_{noised} - \hat{f}||^2 = B + V \tag{8}$$

since the error term $\hat{n}$ has zero mean.

The optimal solution is found near the intersection of bias and variance curves as a function of μ (regularization strength), as portrayed in fig. 6.

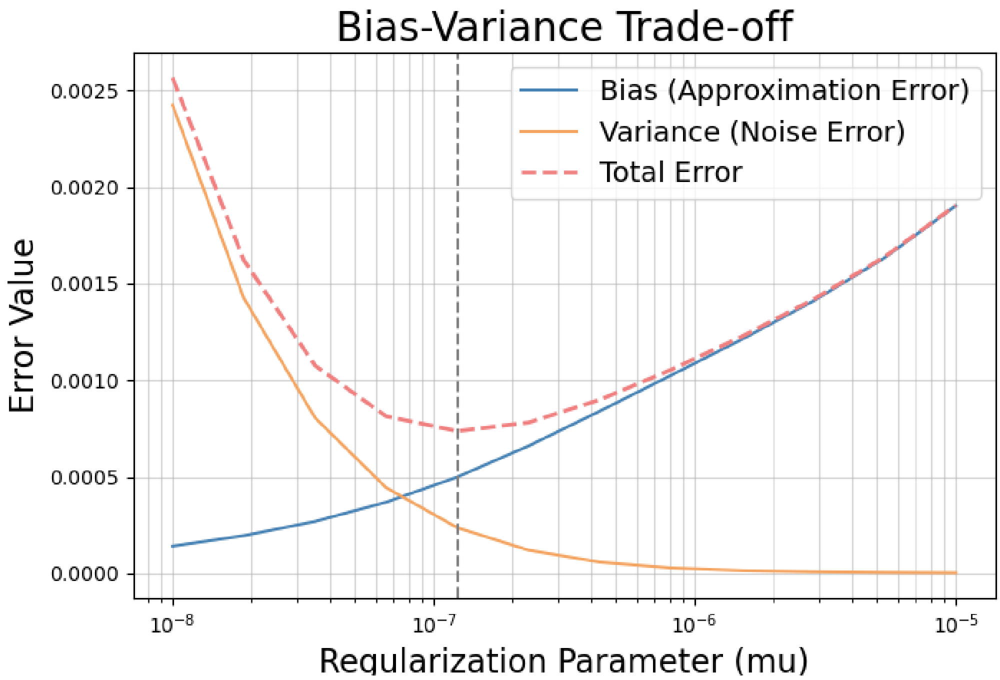


Figure 6: The plot of bias, variance and total error, zoomed-in near the error optimum. (H1 penalty, SNR=30, out-of-focus blur with R=5). The minimum of total error is removed slightly to the right of the intersection point, due to the faster decay of the variance component.