Cubic parabola in railway applications

Cubic parabola is used in transition curves of the railway. The cubic parabola function is $y = kx^3$ (1)

The "main" elements in railway transition curve are: The radius of curvature at the end of transition, the length L of the curve, the length l of its projection on x axis and the coefficient k. Two of these elements must be given in order for the curve to be defined, the other two can be calculated as described in this article.

The 'secondary elements" of the curve are: the coordinates Rx and Ry of the center of curvature at the end of the transition, the shift f between the curve R and the x axis, the angle τ of the tangent to the curve at the end of the transition and and Y_l, the y coordinate of the curve at length l. When the "main" elements are known, the "secondary" elements can easily be calculated. (Fig. 1).

The radius of curvature r at a point of the curve y=f(x) is given by:

$$r = \frac{\sqrt{\left(1 + f'(x)^2\right)}}{f''(x)}$$

For the cubic parabola therefore: $$r = \frac{\sqrt{\left(1 + 9k^2 x^4\right)^3}}{6kx}. \qquad (2)$$

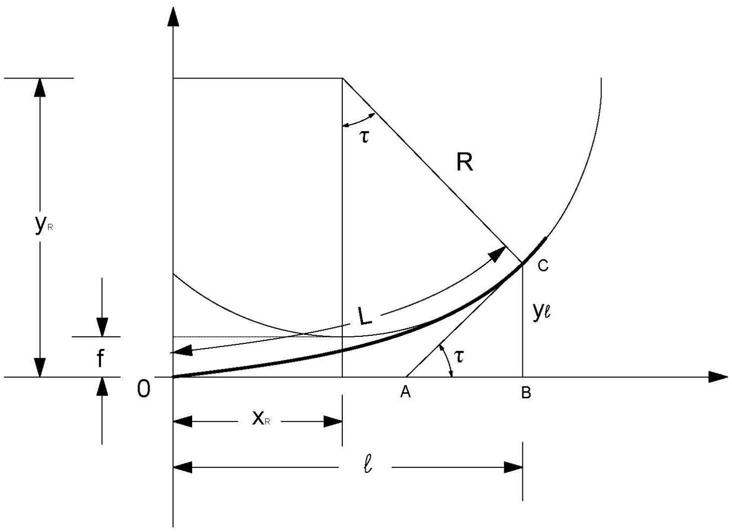

Fig. 1. Elements of transition curve

The centre of curvature in a point (x,y) of a curve y=f(x) has the following coordinates:

$$x_c = x - f'(x) \frac{1 + f'(x)^2}{f''(x)} \qquad y_c = y + \frac{1 + f'(x)^2}{f'(x)}$$

For the cubic parabola these coordinates are:

$$x_c = \frac{x\left(1 - 9k^2 x^4\right)}{2} \qquad y_c = \frac{15k^2 x^4 + 1}{6kx} \qquad (3)$$

The length s of a curve y=f(x) from 0 to x is:

$s=\int_0^x \sqrt{1+f'(x)^2}\,dx$ and for the cubic parabola: $s=\int_0^x \sqrt{1+9\mathrm{k}^2 x^4}\,dx$ (4)

The tangent to the parabola at a point x (which is perpendicular to the corresponding radius r) forms an angle φ with x-axis whose tangent is equal to the derivative at point x: $tan\ \varphi=(kx^3)'$ ή: $tan\ \varphi=3\mathrm{kx}^2$ (5).

Equalities (2), (3), (4),(5) are especially useful at the end of the transition curve. At this point, the radius of curvature is R, the x coordinate is l, the projection of the curve on x-axis and the total length of the transition curve is L. For this particular point the following equations apply:

$$R=\frac{\sqrt{(1+9\mathrm{k}^2 l^4)^3}}{6\mathrm{kl}}$$ (6),

$$x_R=\frac{l(1-9\mathrm{k}^2 l^4)}{2}$$ $$y_R=\frac{15\mathrm{k}^2 l^4+1}{6\mathrm{kl}}$$ (7)

$$L=\int_0^l \sqrt{1+9\mathrm{k}^2 x^4}\,dx$$ (8)

The angle of the tangent at this point can be defined by : $tan\ \tau=3\mathrm{k}\,l^2$ (9).
An interesting property of the cubic parabola can be proved as follows:
$tan\ \tau=3\mathrm{k}\,l^2=\frac{BC}{AB}=\frac{kl^3}{AB}$, so: $AB=\frac{l}{3}$. From (9) it follows $k=\frac{\varepsilon\varphi\tau}{3\mathrm{l}^2}$.

The relations (1), (6), (7), (8), (9) are use to calculate the elements of the transition curve. Usually R and l are given.

The first derivative of (2)as a function of x is: $r'=\frac{(45\mathrm{k}^2 x^4-1)\sqrt{1+9\mathrm{k}^2 x^4}}{6\mathrm{kx}^2}$

When, therefore, $45\mathrm{k}^2 x^2=1$ or $\mathrm{k}=\frac{1}{\sqrt{(45)}\,x^2}$ or $\mathrm{x}=\sqrt[4]{\frac{1}{45}k^2}$, radius r is at an absolute minimum. Substituting these values in (2) one gets: $R_{min}=\frac{6\sqrt{6}}{10}x$.
At the end of transition curve, x equals l, so the radius R cannot be less than: $\frac{6\sqrt{6}}{10}l$, while in combination with (5), for the maximum angle of the tangent at the end of the transition one gets: $max\ tan\ \tau=3\sqrt{\frac{1}{45}}$ or τ≈24,09°.
Beyond this limit, cubic parabola is of no use for railway transition curve.

Other useful equations for the cubic parabola elements are:

$R_x=l-R\eta\mu\tau,$ $f=Y_l+R\sigma\upsilon\nu\tau-R,$ $\mathrm{k}=\frac{1}{6\mathrm{Rl}\,\sigma\upsilon\nu^3\tau}$

Calculation of k when l and R are known

In this case relation (6) as a function of k is used: $\dfrac{\sqrt{\left(1+9k^2 l^4\right)^3}}{6kl}-R=0$. The derivative of this function for k is: $\dfrac{\left(18k^2 l^4-1\right)\sqrt{1+9k^2 l^4}}{6k^2 l}$ (10) which has a minimum when $18k^2 l^4=1$ in other words when: $k=\dfrac{1}{\sqrt{18}\,l^2}$. Since the value of (10) for k=0 is +∞, if the value of (10) for $k=\dfrac{1}{\sqrt{18}\,l^2}$ (11) (which is the minimum) is:

>0: there is no solution for k (non compatible combination of R-l)
=0: there is only one solution for k (as above)
<0: there are two solutions for k, one less than the value in (11) and one more than it.

But, since k must be less than $\dfrac{1}{\sqrt{45}\,l^2}$ (in order for the cubic parabola to be usable as transition curve) all solutions for k that fall between $\dfrac{1}{\sqrt{45}\,l^2}$ and $\dfrac{1}{\sqrt{18}\,l^2}$ are not valid.

Calculation of l when k and R are known

Equation (2)as a function of x is used: $\dfrac{\sqrt{\left(1+9k^2 x^4\right)^3}}{6kx}-R=0$. The first derivative is: $\dfrac{\left(45k^2 x^4-1\right)\sqrt{1+9k^2 x^4}}{6kx^2}$ (12) which has a minimum when $x=\sqrt[4]{\dfrac{1}{45k^2}}$

Since the value of (12) for x=0 is $+\infty$, if the value of (12) for $x=\sqrt[4]{\dfrac{1}{45k^2}}$ (13) is:

>0: there is no solution for x (non compatible combination of R-k)
=0: there is one solution (given by (13))
<0: there are two solutions for x, one smaller and one greater than in (13)

Calculation of L when k and l are known

The length L of cubic parabola is given by: $L=\int_0^l \sqrt{1+9k^2x^4}\,dx$. Since there is no analytic formula for this integral, it can be calculated by the power series:

$$\sqrt{1+z}=1+\frac{1}{2}z-\frac{1.1}{2.4}z^2+\frac{1.1.3}{2.4.6}z^3-\frac{1.1.3.5}{2.4.6.8}z^4+\frac{1.1.3.5.7}{2.4.6.8.10}z^5-\frac{1.1.3.5.7.9}{2.4.6.8.10.12}z^6+\ldots..$$

This relation is valid for $z<1$. In our case z is equal to $9k^2x^4$ which always is less than 1, since its maximum value αφού η μέγιστη τιμή του είναι

$$9*\frac{1}{45}=\frac{1}{5}$$

Integrating the second part of this equation between 0 and 1 one gets:

$$\int_0^l \sqrt{1+9k^2x^4}=l\left(1+\frac{t}{2.5}-\frac{t^2}{2.4.9}+\frac{3t^3}{2.4.6.13}-\frac{3.5t^4}{2.4.6.8.17}+\frac{3.5.7t^5}{2.4.6.8.10.21}-\frac{3.5.7.9t^6}{2.4.6.8.10.12.25}+\ldots.\right)$$

where $t=9k^2l^4$. It can be observed that after the 2nd term of the series, every new term is derived from the previous one by multiplying it with

$$\frac{-(1+2v)}{(4+2v)}\frac{(5+4v)}{(9+4v)}t$$ (ν=0,1,2.., ν=0 for the 3rd term).

Since t has a maximum value of 1/5 (max(45* k^2*l^4) = 1 ⇒ max(9* k^2*l^4) = 1/5), for this value of t the power series converges to 1,01947959990638 (no mathematical proof could be derived), the maximum value of L with relation to l is είναι max L= l* 1.01947959990637971401 with an accuracy of 20 decimal digits.