L I P 6 C N R S

# Automatic Theorem Proving

*Tableau Method*

**Resolution in Propositional Logic**

**Unification**

**Resolution in First Order Logic**

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# Automatic Theorem Proving

*Tableau Method*

**Resolution in Propositional Logic**

*Unification*

*Resolution in First Order Logic*

ACASA – Faculté des sciences de Sorbonne Université

# Resolution Rule in Propositional Logic

- Been given two <u>clauses</u> $C_1$ and $C_2$
- Been given an atomic proposition such that $A \in C_1$ and $\neg A \in C_2$
- The resolution of $C_1$ and $C_2$ by A and $\neg A$ is:
  $C = res(C_1, C_2; A, \neg A)$
  $= [C_1 - \{A\}] \vee [C_2 - \{\neg A\}]$

*Example*: if $C_1 = \neg man \vee mortal$,
$C_2 = \neg socrate \vee man$, $C_3 =$ socrate and
$C_4 = \neg mortal$

$C = res(C_3, C_2; socrate, \neg socrate) = man$

$C' = res(C_1, C_4 ; mortal, \neg mortal) = \neg man$

# Resolution (*example*)

$C_1 = \neg man \vee mortal = man \supset mortal$

$C_2 = \neg socrate \vee man = socrate \supset man$

$C_3 = socrate$, $C_4 = \neg mortal$,

$C = res(C_3, C_2; socrate, \neg socrate) = man$

$C' = res(C_1, C_4 ; mortal, \neg mortal) = \neg man$

- The resolution is more general than the *Modus Ponens* ($A, A \supset B / B$) and the *Modus Tolens* ($\neg B, A \supset B / \neg A$)

# Generality of the Resolution

- To prove $S \models C$, it is sufficient to prove that $S \cup \{\neg C\}$ is contradictory, i.e. that $S \cup \{\neg C\} \models \square$

$\square$ denotes the empty clause, i.e. the false

- **Theorem**: $S \models C$ if and only if it is possible to derive the empty clause by iterative application of the resolution rule on $S \cup \{\neg C\}$

# Example

- Consider S = {$C_1$, $C_2$, $C_3$}

$C_1$ = ¬man ∨ mortal = man ⊃ mortal

$C_2$= ¬socrate ∨ man = socrate ⊃ man

$C_3$= socrate, $C_4$= ¬mortal,

- To prove S ⊨ mortel, it is sufficient to derive the empty clause, i.e. □, from S ∪{$C_4$}

$C_5$: ¬man  res($C_1$, $C_4$; mortal, ¬mortal)

$C_6$: ¬socrate  res($C_2$, $C_5$; man, ¬man)

$C_5$: □  res($C_3$, $C_6$; socrate, ¬socrate)

QED

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*Resolution in First Order Logic*

# Substitution

*Definition*:

- Being $\mathcal{V}$ the set of variables.
- Being $\mathcal{F}$ the set of functions.
- Being $\mathcal{T}$ the set of terms built on $\mathcal{V}$ and $\mathcal{F}$.

A *substitution* $\sigma$ is an application from $\mathcal{V}$ to the set of terms $\mathcal{T}$ which is the identity *almost everywhere*

A *substitution* is characterized by a finite set of pairs "$x_i$ / $t_i$", $x_i$ being a variable and $t_i$ a term.

It is denoted $\sigma = \{v_1/t_1, \ldots, v_n/t_n\}$

*Example*: $\sigma = \{x/3, y/(u - 7)\}$

# Term instance

*Definition*:

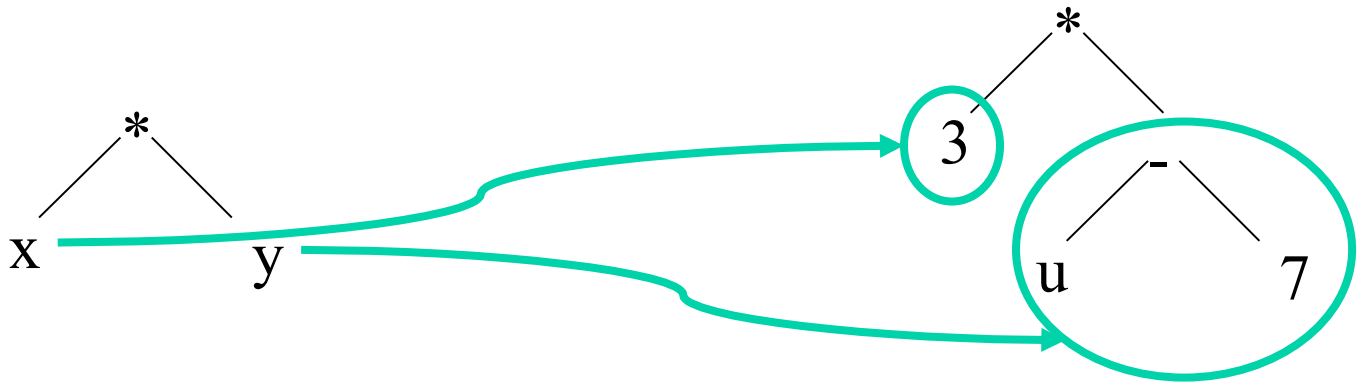
- Being E a term of $\mathcal{T}$ (set of terms built on $\mathcal{V}$ and $\mathcal{F}$)
- Being a substitution $\sigma$ of variables $\mathcal{V}$ by terms of $\mathcal{T}$
  $\sigma = \{v_1/t_1, \ldots, v_n/t_n\}$

Being $E\sigma$ the expression built by replacing each occurrence of free variables $v_i$ of E by $t_i$.

$E\sigma$ is called an *instance* of E

# Term instance (following)

*Example*: if $\sigma = \{x/3, y/(u - 7)\}$ et $E = (x*y)$
then $E\sigma = (3 * (u - 7))$ is an instance of E



# Substitution composition

- Being $\theta=\{t_1/x_1,\ldots, t_n/x_n\}$ and $\lambda=\{u_1/y_1,\ldots, u_m/y_m\}$ two substitutions.

- The *composition* of $\theta$ and $\lambda$ is the substitution $\lambda\circ\theta$ which is obtained from the set $\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\}$ by deleting all the $t_j\lambda/x_j$ such that $t_j\lambda=x_j$ and all the $u_i/y_i$ such that $y_i\in\{x_1, \ldots, x_n\}$

# Substitution composition

- The *composition* of two substitution $\theta$ and $\lambda$ is the substitution $\lambda\circ\theta$ which is obtained from $\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\}$ by deleting all the $t_j\lambda/x_j$ such that $t_j\lambda=x_j$ and all the $u_i/y_i$ such that $y_i\in\{x_1,\ldots, x_n\}$

*Example*: $\theta=\{t_1/x_1, t_2/x_n\}=\{f(y)/x, z/y\}$ et
$\lambda=\{u_1/y_1, u_2/y_2, u_3/y_3\} = \{a/x, b/y, y/z\}$

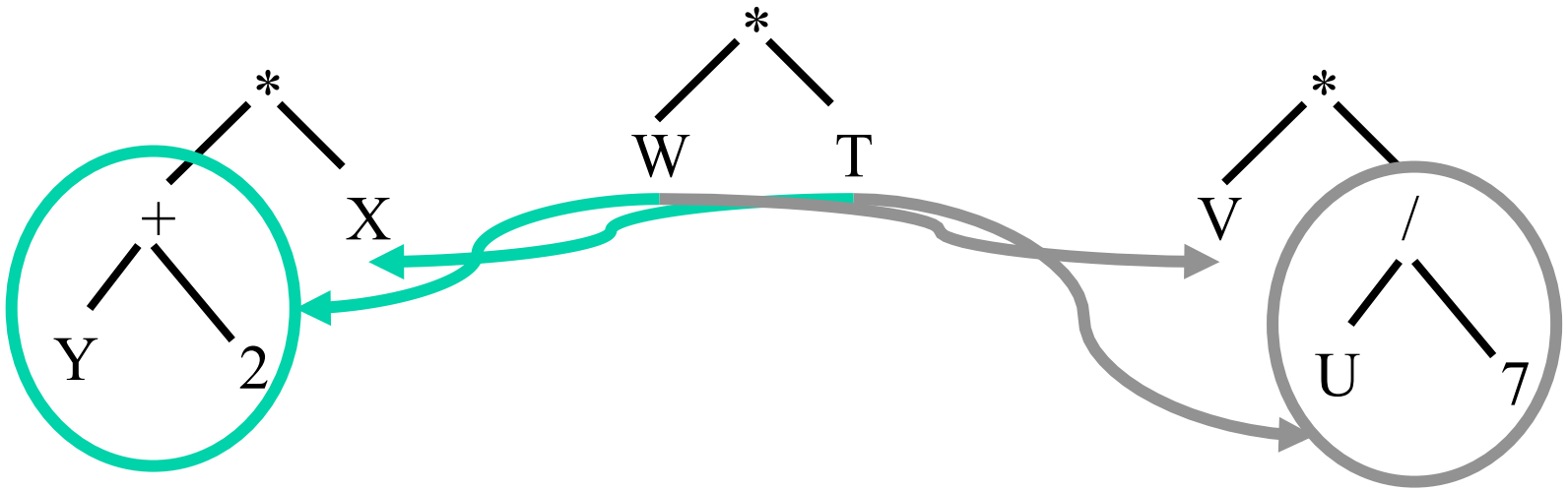
$\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\} =$
$\{f(b)/x, \underline{y/y}, \underline{a/x}, \underline{b/y}, y/z\} = \{f(b)/x, y/z\}$

**$\theta\circ\lambda = \{f(b)/x, y/z\}$**

# Pattern matching

## Is a term an instance of another?

**Pattern matching**: the term $t_1$ match with the term $t_2$ if and only if there exists a substitution $\sigma$ such that: $\mathbf{t_1\sigma = t_2}$




# Pattern matching algorithm

filtering(t1, t2)

**If** t1 = t2 **then** σ := {} **return;**

**If** t1 = () or t2 = () **then** σ := fail; **return;**

**If** var(t1) **then** σ := {{t1 / t2}}; **return;**

**If** atom(t1) or atom(t2) **then** σ := fail; **return;**

$\sigma_1$ := filtering( head (t1) , head(t2));

**If** $\sigma_1$ = fail **then** σ := fail ; **return;**

**else** $\sigma_2$ := filtering( substitute($\sigma_1$ , tail(t1)) , tail(t2));

**If** $\sigma_2$ = fail **then** σ := fail; **return;**

**else** σ := composition ($\sigma_1$ , $\sigma_2$ )

# Representing a term with a list

- **S-expression**:
  - An *S-expression* is either an *atom (character string)*
  - Or a *sequence* (possibly empty) of S-expression
- *Examples*:
  - []
  - ‘a’, ‘Salut’, ‘Insoluble’, ‘Rien!’, 421
  - [‘a’, [‘Salut’, ‘à’, ‘toi’], 421, [‘Rien’, [‘vraiment rien!’]]]
- **Term**:
  - **Definition**: set of functions $\mathcal{F}$, set of variables $\mathcal{V}$
  - **Representation with a S-expression**: (3 * (?x + (?y - ?x)))
    [‘*’, 3, [‘+’, ‘?x’, [‘-’, ‘?y’, ‘?x’]]]

```
def filtrage_(t1, t2, v1):
    if t1 == t2:
      return { }
   if var(t1):
      if t1 in v1:
         sigma = {}
         sigma[t1] = t2
      else: sigma = 'echec'
      return sigma
   if ( t1 == [] or t2 == []):
      return 'echec'
   if ( type(t1) <> type([]) or  type(t2) <> type([])):
      return 'echec'
   sigma1 = filtrage_(t1[0], t2[0], v1)
   if (sigma1 <> 'echec' and len(t1) > 1 and  len(t2) > 1 ):
      sigma2 = filtrage_(appliquer_substitution(t1[1:], sigma1), t2[1:], v1)
      return composer_substitution(sigma2, sigma1)
   elif t1[1:] <> t2[1:]:
      return 'echec'
   else: return sigma1
```

)

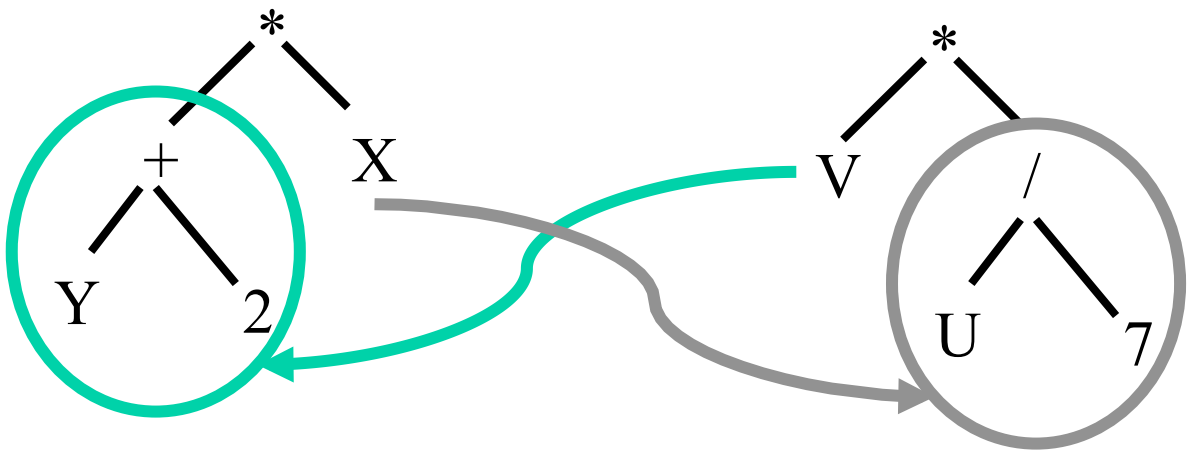
2, vars1)

# Unification

**Unification**: the terms $t_1$ and $t_2$ unify if and only if there exists a substitution $\sigma$ such that:

$t_1\sigma = t_2\sigma$

**Pattern matching ⇒ Unification**




# Most General "Unifier"

**Definition**: a substitution σ is an *unifier* of the set of terms $\{t_1, t_2, \ldots t_n\}$ if and only if

$t_1 \sigma = t_2 \sigma = \ldots = t_n \sigma$

**Definition**: a unifier σ of a set of terms $\{t_1, t_2, \ldots t_n\}$ is *the most general unifier* (*mgu*) if and only if for all unifier θ of this set, there exists λ such that $\theta = \sigma \circ \lambda$

# Most General "Unifier"

*Example*: S = {P(g(z), y), P(x, f(u))}

**S is unifiable** because the substitution
θ = {x/g(a), y/f(b), z/a, u/b} is a unifier S

**θ is not the most general unifier** because there exists a unifier ζ = {x/g(z), y/f(b), u/b} such that θ = ζ ° {z/a}

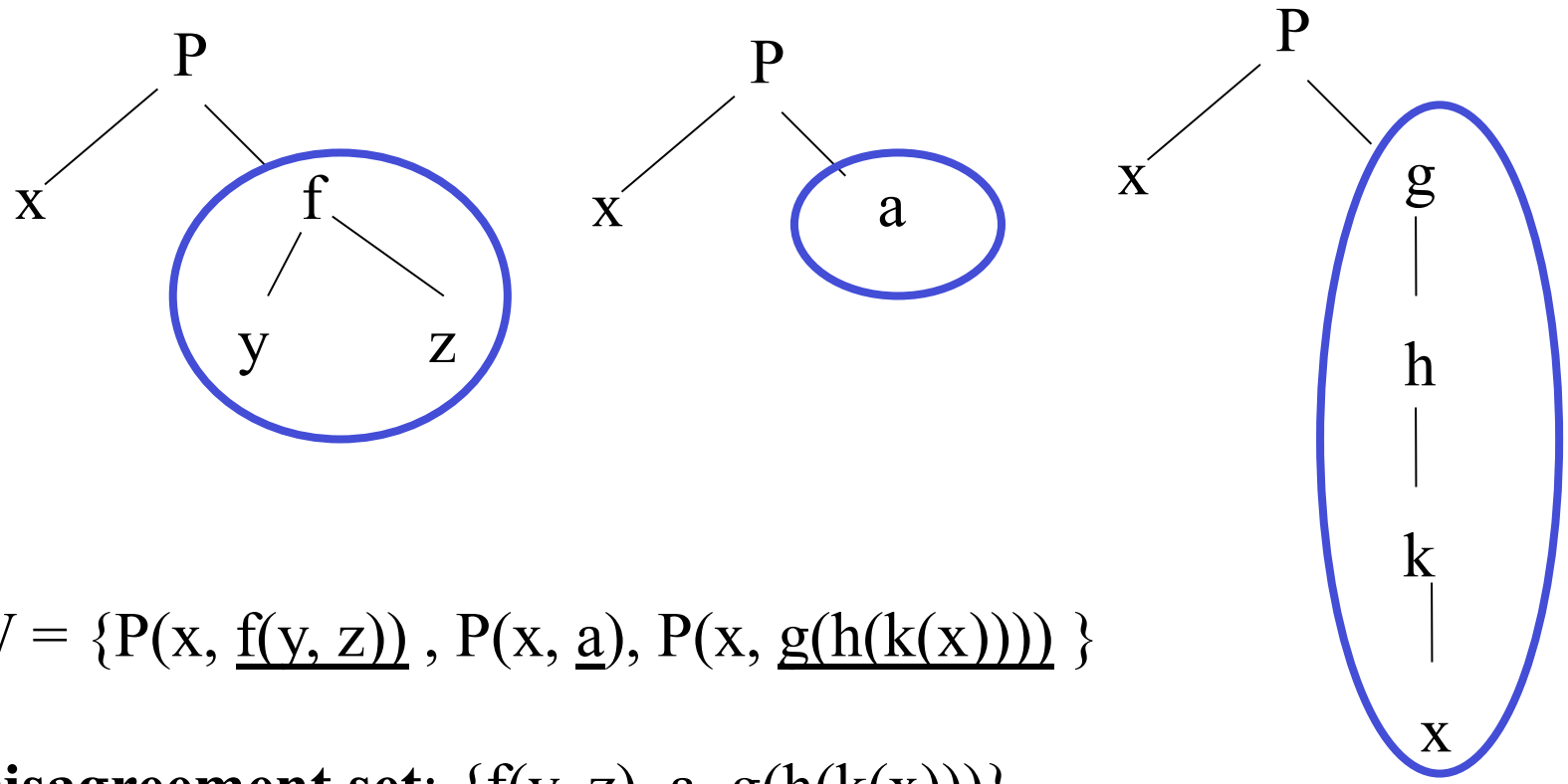
**The unifier σ = {x/g(z), y/f(u)} is the *most general unifier*** of this set

# Unification Algorithm (*preliminaries*)

**Definition**: *a disagreement set* of a set of terms W is obtained by detecting the first difference in non identical terms (depth first search).

*Example*: W = {P(x, f(y, z)) , P(x, a), P(x, g(h(k(x)))) }

# Disagreement set: *example*




W = {P(x, f(y, z)) , P(x, a), P(x, g(h(k(x)))) }

**Disagreement set**: {f(y, z), a, g(h(k(x)))}

# Unification algorithm for a set of terms W

*Step 1*: $k := 0$, $W_k := W$, $\sigma_k := \varepsilon$

*Step 2*: if $W_k$ is a singleton then stop; $\sigma_k$ is the MGU (*most general unifier)* of W.
else, find the disagreement set $D_k$ of $W_k$

*Step 3*: if there exists a variable $v_k$ and a term $t_k$ belonging to $D_k$ such that $v_k$ does not appear in $t_k$, then go to step 4.
else, stop; W is not unifiable

*Step 4*: $\sigma_{k+1} := \sigma_k \{v_k / t_k\}$ ; $W_{k+1} := W_k \{v_k / t_k\}$ *[note that* $W_{k+1} = W\sigma_{k+1}$ *]*

*Step 5*: $k := k+1$; go to step 2

# Unification algorithm: *example*

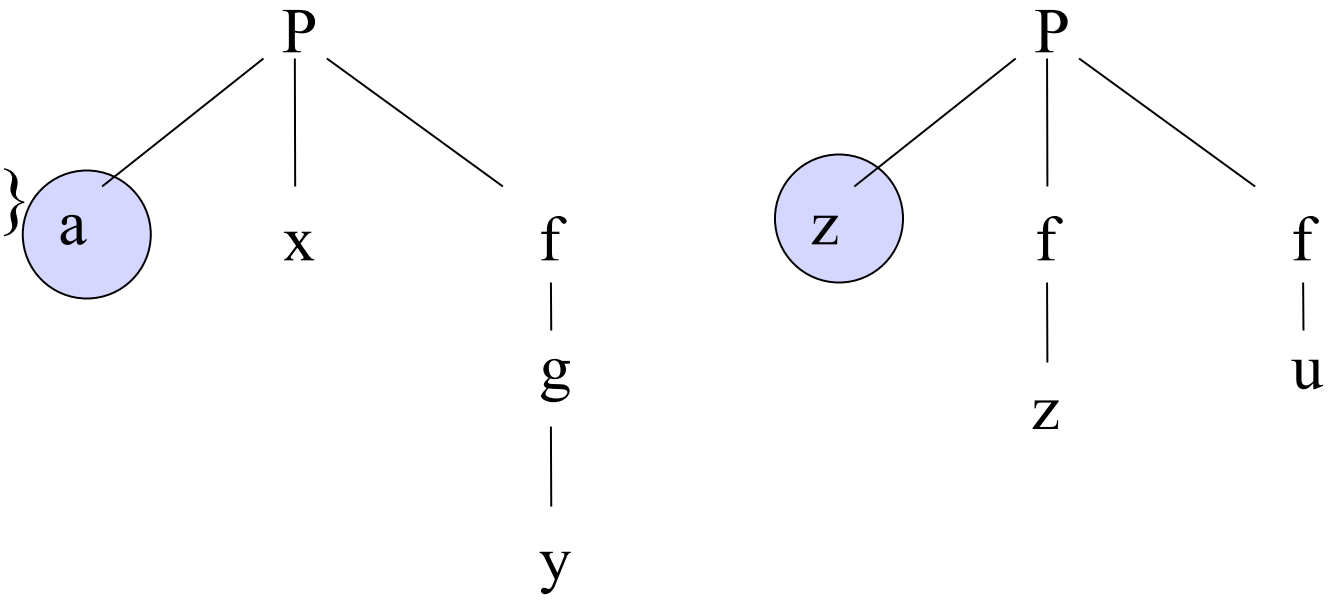
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

$k = 0$

$W_k = W,$

$\sigma_k = \varepsilon$

$D_k = \{a, z\}$



# Unification algorithm: *example*

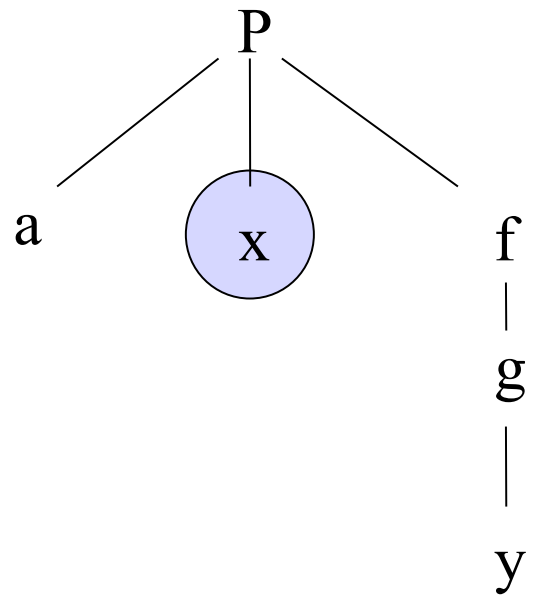
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

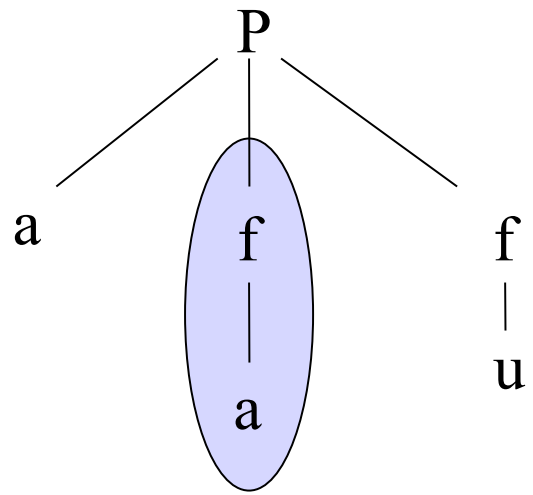
$k = 1$

$W_k = \{ P(a, x, f(g(y))), P(a, f(a), f(u))\}$

$\sigma_k = \{z / a\}$

$D_k = \{x, f(a)\}$




# Unification algorithm: *example*

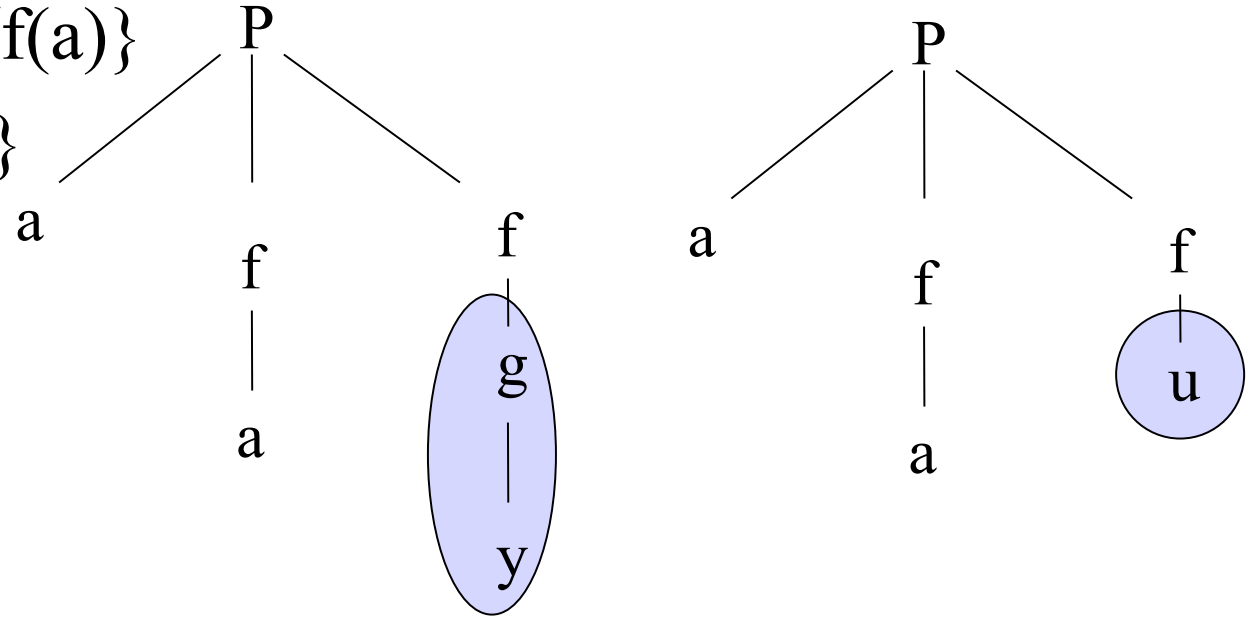
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

$k = 2$

$W_k = \{ P(a, f(a), f(g(y))), P(a, f(a), f(u))\}$

$\sigma_k = \{z / a, x/f(a)\}$

$D_k = \{g(y), u \}$




# Unification algorithm: *example*

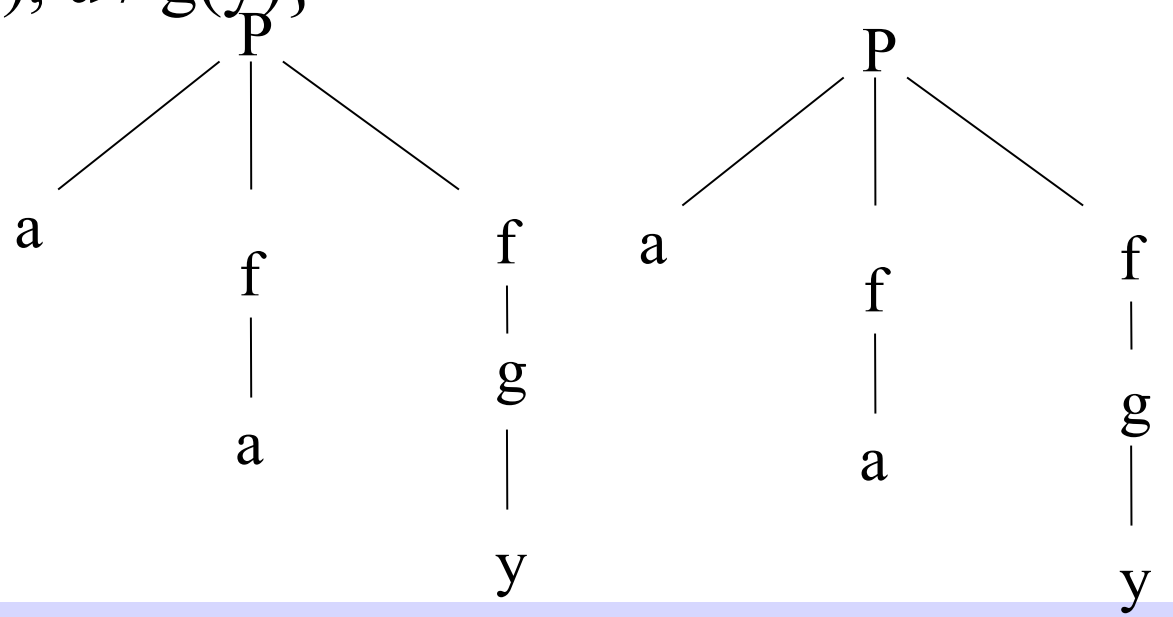
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

$k = 3$

$W_k = \{ P(a, f(a), f(g(y))) \}$

$\sigma_k = \{z / a, x/f(a), u / g(y)\}$

$D_k = \{\}$




# Unification Theorem

**Theorem**: if W is a non empty set of unifiable terms, then the algorithm always finishes at step 2 and the last substitution $\sigma_k$ is the most general unifier of W.

L I P 6 C N R S

# Automatic Theorem Proving

*Tableau Method*

*Resolution in Propositional Logic*

*Unification*

**Resolution in First Order Logic**

# General Resolution

**Definition**:

- Being given two clauses A and B of which variables are disjoint (if needed, rename variable),
- Being given a literal $L_1$ of A and a literal $L_2$ of B such that either $L_1$ and $\neg L_2$ or $\neg L_1$ and $L_2$ unify with the substitution $\sigma$.

**C is said to be the resolvant of A and B by $L_1$ and $L_2$**, which is noted:

$$\mathbf{C = Res(A, B; L_1, L_2) = (A\sigma - \{L_1\sigma\}) \cup (B\sigma - \{L_2\sigma\})}$$

*Examples*:

**Res**($\underline{R(x, y)}$ , $\underline{\neg R(u, v)}$ ; R(x, y), ¬R(u, v)) =∅

**Res**($\underline{P(x)}$ ∨ P(y), $\underline{\neg P(z)}$ ∨ ¬P(r); P(x), ¬P(z)) = P(y) ∨ ¬P(r)

**Res**($\underline{man(Socrates)}$, $\underline{\neg man(z)}$ ∨ mortal(z);
man(Socrates), ¬man(z) = mortal(Socrates)

# Resolution : example

$C_1$: vénéneux(X) ∨ ¬champignon(X) ∨ ¬non-comestible(X)

$C_2$: toxique(U) ∨ ¬vénéneux(U)

$C_3$: ¬toxique(a)

**Rés**($C_1$, $C_2$ ; vénéneux(X) , ¬vénéneux(U) ) =

**Rés**(**vénéneux(X)** ∨ ¬champignon(X) ∨ ¬non-comestible(X),
toxique(U) ∨ **¬vénéneux(U)**
; vénéneux(X) , ¬vénéneux(U) ) =

¬champignon(X) ∨ ¬non-comestible(X) ∨ toxique(X)

$C_1$ means **champignon(X) ∧ non-comestible(X) ⇒ vénéneux(X)**

$C_2$ means **vénéneux(U) ⇒ toxique(U)**

La resolvant of $C_1$ and $C_2$ means
**champignon(X) ∧ non-comestible(X) ⇒ toxique(X)**

# Resolution : example (following)

$C_1$: vénéneux(X) ∨ ¬champignon(X) ∨ ¬non-comestible(X)

$C_2$: toxique(U) ∨ ¬vénéneux(U)

$C_3$: ¬toxique(a)

**Res**($C_2$, $C_3$ ; toxique(U) , ¬toxique(a) ) =

**Res**( **toxique(U)** ∨ ¬vénéneux(U),

**¬toxique(a)** ,

; toxique(U) , ¬toxique(a) ) =

¬vénéneux(a),

$C_2$ means **vénéneux(U) ⇒ toxique(U)**, which is equivalent to **¬toxique(U) ⇒ ¬vénéneux(U)**,

$C_3$ means **¬toxique(a)** (in other words that a is not toxic)

The resolvant of $C_2$ and $C_3$ means **¬vénéneux(a)** in other words that a is not venenous.

# Resolution Theorem

**Resolution theorem**: being S a set of clauses,

*S is unsatisfiable if and only if there exists a R-refutation de S,*

in other words, if it is possible to derive the empty clause ( □ , i.e. false) by iteratively applying the resolution rule, which means that $S \vdash_{Res} \square$ .

# Automating the Proof

- Why?
  - Proof of programs
  - Artificial intelligence (*mathematical reasoning*)
  - Programming (*logic-based programming language*)
  - Knowledge representation (*artificial intelligence and data bases*)
- How?

# How to automatically prove a theorem?

- **Proof by refutation**:

  Being S a set of clauses and C a clause. To prove that $S \vdash C$ it is enough to prove that S and $\neg C$ are unsatisfiable, which is equivalent to refutate S and $\neg C$ i.e. to proof that $S, \neg C \vdash_{Res}$

- **Resolution**:

$$C_k = Res(C_i, C_j; L_{i1}, L_{j2}) = (C_i\sigma - \{L_{i1}\sigma\}) \cup (C_j\sigma - \{L_{j2}\sigma\})$$

- **Choices** $C_i$, $C_j$, $L_{i1}$, $L_{j2}$
  - Complexity at step k: $(|S| + k)^2 * |C_i| * |C_j|$

# Horn clause

Horn clause: *not more* than a positive literal

Positive Horn clause: *exactly* one positive literal

Also called "definite clause"

Negative Horn clause: *no* positive literal

Also called "Query"

*Notation*: « P :- $N_1$, $N_2$, …, $N_n$. » is a Horn clause.

« P » is the clause **head**

« $N_1$, $N_2$, …, $N_n$. » is the **body** or the **goal** part of the clause

# SLD-Resolution

**Definition**: (SLD: Selection rule, Linear resolution, Definite clauses)

**Selection**: choice of literals in the clauses

**Linear resolution**: one of the two clauses comes from the previous application of the resolution

**Definite clauses**: positive Horn clauses

# How to automatically prove a theorem?

- **Proof by refutation**:

  Being S a set of clauses and C a clause. To prove that $S \vdash C$ it is enough to prove that S and $\neg C$ are unsatisfiable, which is equivalent to refutate S and $\neg C$ i.e. to proof that $S, \neg C \vdash_{Res}$

- **Resolution**:

$$C_k = Res(C_i, C_j; L_{i1}, L_{j2}) = (C_i\sigma - \{L_{i1}\sigma\}) \cup (C_j\sigma - \{L_{j2}\sigma\})$$

- **Choices** $C_i$, $C_j$, $L_{i1}$, $L_{j2}$
  - **<u>Complexity at step k</u>**: $(|S| + k)^2 * |C_i| * |C_j|$

# Completeness of SLD resolution

SLD resolution is complete when the set of clauses S is composed of:

1. A negative Horn clause
2. A set of definite Horn clauses

*(which are necessarily satisfiable…)*

*Complexity at step k:* $|S|$ *instead* $(|S| + k)^2 * |C_i| * |C_j|$

# Completeness of the SLD refutation

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