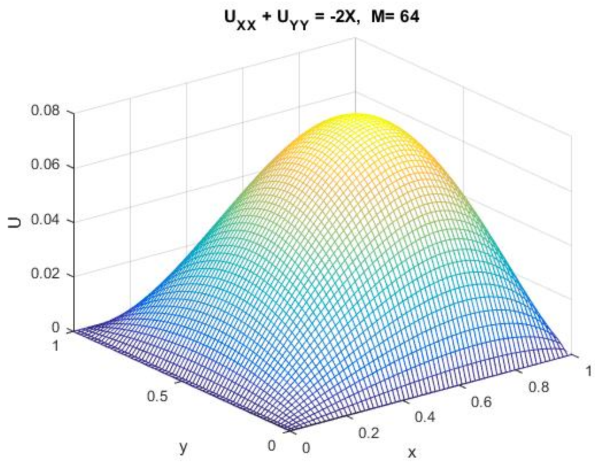


Figure 1: Poisson Equation with $N_x = N_y = 64$

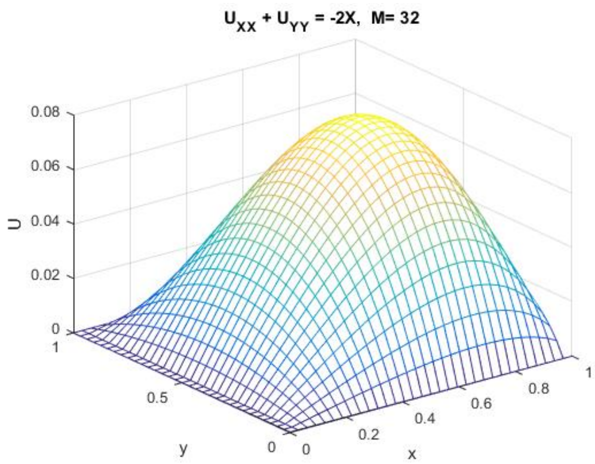


Figure 2: Poisson Equation with $N_x = N_y = 32$

The number of operations required to solve this problem using FFTs is $O(N_{n_y} logN)$ for both the $f_{i,j}$ transform and the $\hat{T}_{n,j}$ inversion, with a further $O(N_{n_y})$ operations for the tridiagonal solver.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \hdotsfor{5} \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix}$$

The have been two plots set up, with grid size 32 and 64. Then, in order to evaluate the accuracy an exact solution is approximated with a grid size 4096. Further for computation of the norm, the matrix dimension is scaled to similar sizes by means of interpolation. The norm of plot with grid size 64 is lower than that of grid size 32, hence in accordance with theory.

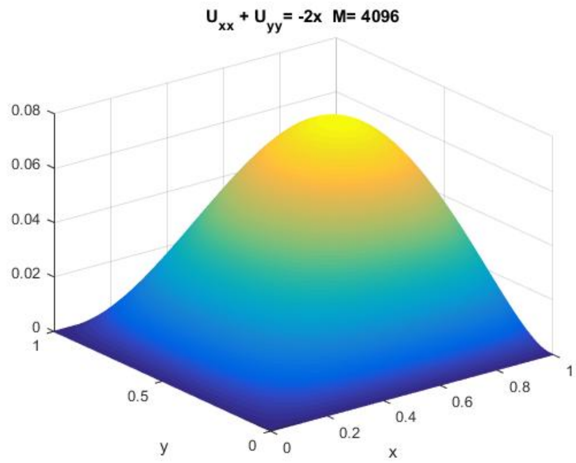


Figure 3: Approximated Exact Solution with $N_x = N_y = 4096$

Question 2

The second question of the assignment involves developing a function which implements the preconditioned gradient method and the preconditioned conjugate gradient method. The preconditioner should be of the form:

$$M_2 M_1 Z = r,$$

therefore a two-step process is implemented to calculate z . If M_2 is not set, identity matrix is considered; whereas for non-conditioned both M1 and M2 are not set , then they should be set to the identity matrix. Altogether, there are 8 cases; steepest descent method and the conjugate gradient method, using no preconditioning and preconditioning using Jacobi iteration, symmetric GaussSiedel and symmetric SOR.

$$\omega = 2 - \frac{2 * \pi}{n_x}$$

The Steepest Descent Method is a first-order iterative optimization algorithm that involves find a local minimum of a function using gradient descent, one takes steps proportional to the negative of the gradient of the function at the current point. Only 1 matrix-vector multiplication at each step takes place such that:

$$r^{(k)} = b - Tx^{(k)} = b - T(x^{(k-1)} + \alpha_{k-1} r^{k+1}) = r^{(k-1)} - \alpha_{k-1} T r^{(k-1)} \tag{8}$$

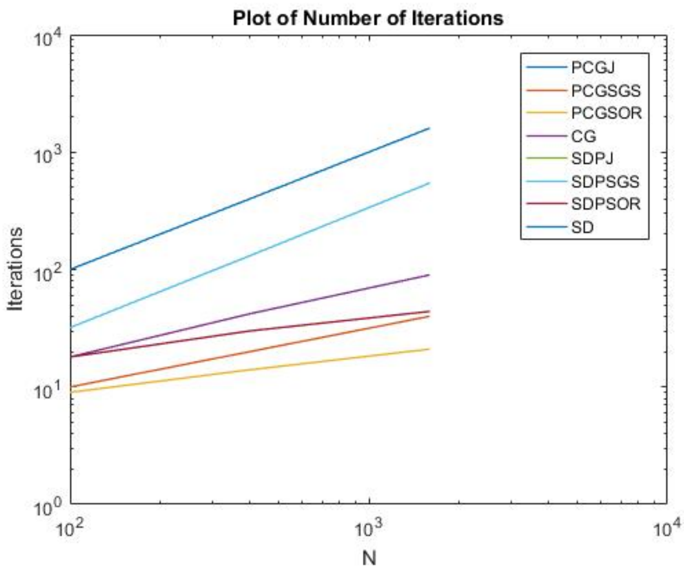
The problem with the method of steepest descent or gradient is that the intuitive direction for performing a line search is not necessarily the best direction. Therefore in Conjugate Gradient Method we choose the conjugate vectors p_k carefully, to obtain a good approximation to the solution x^*.

$$p^{(k)} = r^{(k-1)} + \beta_{(k-1)} p^{(k-1)}$$

where

$$\beta_{(k-1)} = \frac{(p^{(k-1)})^T T r^{(k-1)}}{(p^{(k-1)})^T T p^{(k-1)}}$$

In most cases, preconditioning is necessary to ensure fast convergence of the conjugate gradient method. The preconditioned conjugate gradient method takes the following form:



The improvement is typically linear and its speed is determined by the condition number $\kappa(A)$ of the system matrix A: the larger $\kappa(A)$ is, the slower the improvement.If $\kappa(A)$ is large, preconditioning is used to replace the original system $\mathbf{Ax} - \mathbf{b} = 0$ with $\mathbf{M}^{-1}(\mathbf{Ax} - \mathbf{b}) = 0$ $\mathbf{M}^{-1}(\mathbf{Ax} - \mathbf{b}) = 0$ such that $\kappa(\mathbf{M}^{-1}\mathbf{A})$ is smaller than $\kappa(\mathbf{A})$.

The results confirm Steepest descent is relatively slow close to the minimum: technically, its asymptotic rate of convergence is inferior to many other methods.Steepest descent benefits from preconditioning, but still unsatisfactory. The speed of convergence of gradient descent depends on the ratio of the maximum to minimum eigenvalues of T, while the speed of convergence of conjugate gradients has a more complex dependence on the eigenvalues.

Theoretically, the conjugate gradient method produces the exact solution after a finite number of iterations, which is not larger than the size of the matrix. However, the conjugate gradient method is unstable with respect to even small perturbations, e.g., most directions are not in practice conjugate, and the exact solution is never obtained. It provides monotonically improving approximations to the exact solution, which may reach the required tolerance after a relatively small number of iterations.

But, after reviewing the results it is clear preconditioning is necessary to ensure fast convergence of the conjugate gradient method; in fact preconditioned conjugate gradient method - SOR is the fast method to obtain results.