Computer Vision

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Textbook

Multiple View Geometry in Computer Vision, Hartley, R., and Zisserman

Richard Szeliski, **Computer Vision: Algorithms and Applications**, 2nd edition, 2022

Reference books

Readings for these lecture notes:

- Hartley, R., and Zisserman, A. **Multiple View Geometry in Computer Vision**, Cambridge University Press, 2004, Chapters 1-3.
- Forsyth, D., and Ponce, J. **Computer Vision: A Modern Approach**, Prentice-Hall, 2003, Chapter 2.
- **Linear Algebra and its application**
by David C Lay

These notes contain material c Hartley and Zisserman (2004), Forsyth and Ponce (2003), an Linear Algebra and its application by David C Lay

References

These notes are based

- Dr. Matthew N. Dailey's course: AT70.20: Machine Vision for Robotics and HCI

- Dr. Sohaib Ahmad Khan CS436 / CS5310 Computer Vision Fundamentals at LUMS

2D projective geometry

A model for the projective plane

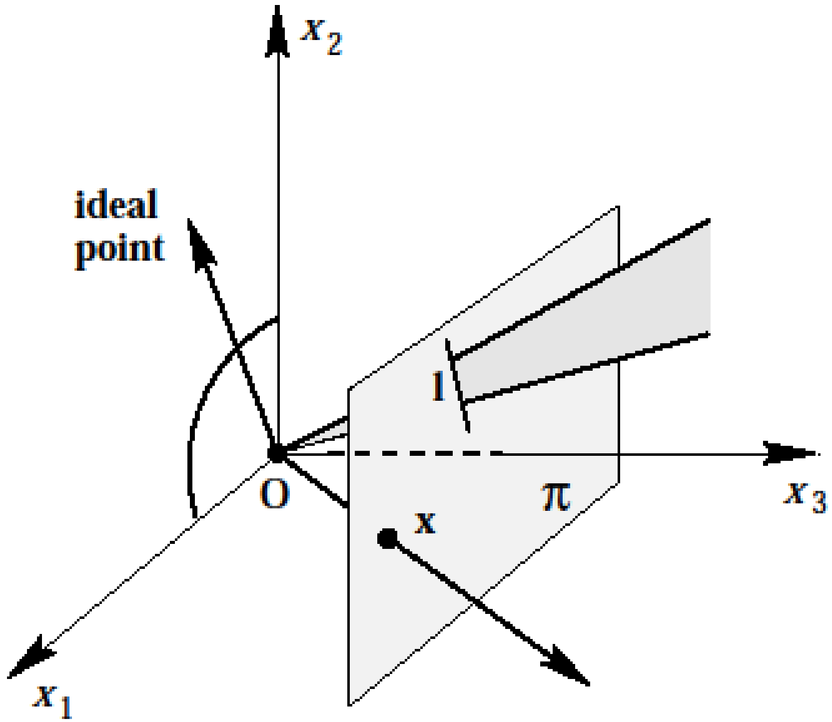


Fig 2.1 **A model of the projective plane**. **Points** and **lines** of $\mathbb{P}^2$ are represented by **rays** and **planes**, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the x_1 x_2 -plane represent **ideal points**, and the x_1 x_2 **-plane** represents $\vec{l}_\infty$ or $\mathbf{l}_\infty$.

2D projective geometry
A model for the projective plane

As illustrated in Fig 2.1 **the rays** representing **ideal points** and **the plane** representing $\vec{\mathrm{l}}_\infty$ or $\mathbf{l}_\infty$ are parallel to the **plane plane x_3 = 1**

Review $\mathbb{P}^2$

- **A point** is represented as a **homogeneous 3 vector** $(x_1, x_2, x_3)^T$ where $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^T$ gives the **corresponding point** in the **plane x_3 = 1**

- **A line** in the **plane x_3 = 1** is represented by a homogeneous vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ where **ax + by + c = 0**

- The vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ **can be interpreted as the normal to a plane in $\mathbb{R}^3$** ax_1 + bx_2 + cx_3 = 0

- The **intersection** of the **plane** ax_1 + bx_2 + cx_3 = 0 with the **plane** x_3 = 1 is the **line** ax + by + c = 0

Review $\mathbb{P}^2$

- If $\boldsymbol{x}^T \boldsymbol{l} = \mathbf{0}$ implies point $\boldsymbol{x}^T = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)^T$ lies on the line $\boldsymbol{l} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^T$

- Since $\boldsymbol{l}$ is a **line** but it is also **interpreted** as a **normal vector** to the **plane** that forms that **line**.

OR

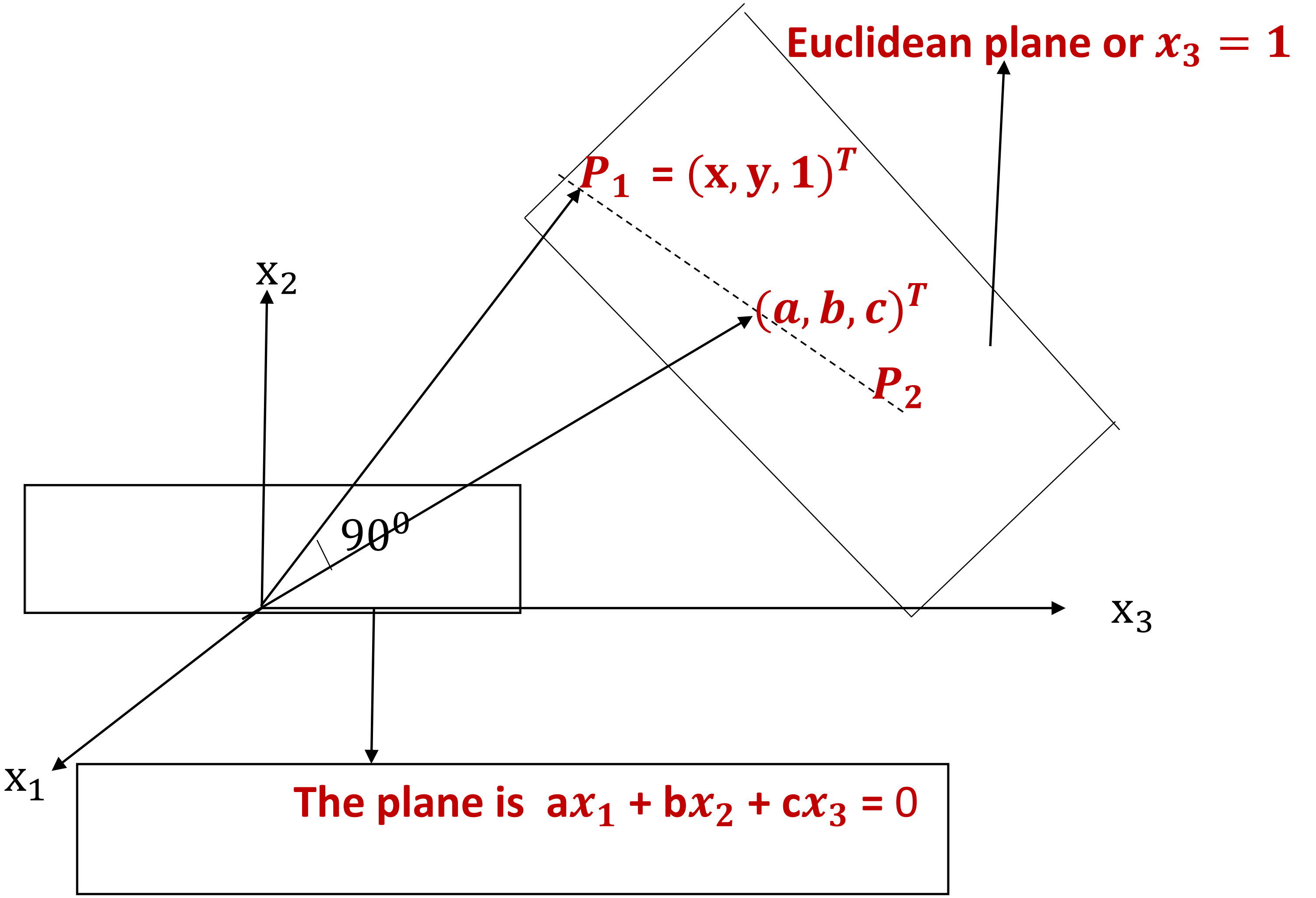
- **Points** on **a line** must be **orthogonal** to the **vector** that is **orthogonal** to the **plane** containing **that line**.

The vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ representing a **line** in the **Euclidean plane (i.e., $\boldsymbol{x_3 = 1}$),** when **interpreted as a vector** in $\mathbb{R}^{\mathbf{3}}$ is **orthogonal** to the $\mathbb{R}^{\mathbf{3}}$ plane representing the line in $\mathbb{P}^{\mathbf{2}}$.

The **line** representation $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ in $\mathbb{R}^{\mathbf{3}}$ is a **vector orthogonal** to the **plane** formed by the **points** on line $\boldsymbol{l}$ and the **origin.**

Proof:

Lines in $\mathbb{P}^2$ is described by $(\mathbf{a}, \mathbf{b}, \mathbf{c})^\mathbf{T}$. The vector, $(\mathbf{a}, \mathbf{b}, \mathbf{c})^\mathbf{T}$ in $\mathbb{R}^3$ is interpreted as being a **normal vector** to some plane in $\mathbb{R}^3$ through the **origin**.


Euclidean plane or $x_3 = 1$
$P_1 = (x, y, 1)^T$
x_2
$(a, b, c)^T$
P_2
90^0
x_3
x_1
The plane is $ax_1 + bx_2 + cx_3 = 0$

- The vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is **orthogonal** to the **plane** $\mathbf{ax_1 + bx_2 + cx_3 = 0}$. Because $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is describing a **normal vector** of the **plane** $\mathbf{ax_1 + bx_2 + cx_3 = 0}$ to the **origin**.

- It means $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ describes **some plane**. We would like to establish $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is also a **line** describing the **points**.

- Suppose we take some arbitrary point $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$**.** If the point $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$ lies on $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ then their **dot product** must be **zero** i.e., $\mathbf{(x, y, 1)}\ (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}} \mathbf{= 0}$

- If we **think geometrically** in $\mathbb{R}^3$ we must ask: *what is the **relationship between these vectors*** $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$ ***and*** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ in three-dimensional space?.

- Let us consider the vector $(\mathbf{x}, \mathbf{y}, \mathbf{1})^{\mathbf{T}}$**.** If this vector is **orthogonal** to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$**it implies that:**

  $(\boldsymbol{x}, \boldsymbol{y}, \mathbf{1})^{\boldsymbol{T}}.\ (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}} = 0,$

  which leads to the equation:

  a$\boldsymbol{x}$ + b$\boldsymbol{y} + \mathbf{1}.\boldsymbol{c} = \mathbf{0}$.

- This is precisely the equation of a **plane through the origin** in $\mathbb{R}^3$:

  $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$.

- It also means the point $(x, y, 1)^T$ must lies on the **plane** $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$.

- We already know $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ is **orthogonal to the plane** $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 = \mathbf{0}$.

- Since $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ represents **the normal vector**, it defines **a plane** that passes **through the origin**.

- **Note:** If a plane can be described by its **normal vector**, then the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ uniquely defines a **plane passing through the origin**, with $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ being **orthogonal to every point on that plane**.

- If we take any point $(x_1, x_2, x_3)^T$ on the **plane** **ax_1 + bx_2 + cx_3 = 0** and we **normalize** it then what we will get?

- We get a point that is **orthogonal** to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. It means the **point** satisfy the equation. That means the **point** is on the **line** in the **Euclidean plane**.

- Take any **two points** on the **line** described by $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. These two points (i.e., **P_1** and **P_2**) that lie on the **line** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ **described by the plane** x_3=1.

- If a **point** lies on the **line** $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ then it must lie on the **plane through the origin** that is described by the $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$. $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ is **orthogonal** to **every point** on the **plane**.

- Conversely **every point** on the **plane ($ax_1 + bx_2 + cx_3 = 0$)**. is describing a **homogeneous representation** of a point that is on the **line in the Euclidean plane (plane x_3=1)**.

If [x y 1] $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ = 0 is true then it tells us two things

- **[x y 1]** lies on the line $[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]^{\mathbf{T}}$ in the Euclidean plane.

- The vector that represents this point in $\mathbb{R}^3$ is necessarily orthogonal the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\boldsymbol{T}}$ in $\mathbb{R}^{\mathbf{3}}$.

- Therefore, it is lying on the **plane through the origin** and through the **line in the Euclidean plane.** These are some interesting facts about lines and points in $\mathbb{p}^{\mathbf{2}}$

- If the equation

$$[x\ y\ 1]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

holds true, it reveals two key insights:

1. The point **[x y 1]** lies on the **line** defined by the vector $[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]^{\mathbf{T}}$ in the **Euclidean plane.**

2. The vector representing this point in $\mathbb{R}^3$ is orthogonal to the vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})^T$ in $\mathbb{R}^3$.

This implies that **the point** lies on the **plane through the origin** in $\mathbb{R}^3$, which contains the **entire line** in the Euclidean plane.

Now we start seeing these relationship:

- Vector describing **points**
- Vectors describing **lines**
- Vectors describing **planes**

They are closely related to each other.

Conic section

What is conic section?

- In mathematics, a **conic section** (or simply **conic**) is the curve formed by the intersection of a **plane** with the **surface of a cone**. There are three fundamental types of conic sections: the **hyperbola**, the **parabola**, and the **ellipse**. The **circle** is a special case of the ellipse, and due to its unique properties and historical importance, it is sometimes considered a fourth distinct type of conic section.

OR

- A conic section is the intersection of **a plane** and **a cone**.

OR

- A **conic** is a two-dimensional shape formed by slicing a cone in a particular way. In Euclidean geometry, the resulting curves—known as **conic sections** that **include circles, parabolas, hyperbolas**, and **ellipses**.

Conic section

conic sections can be written in matrix form as:

$\vec{x}^T C \, \vec{x} = 0$

$$\text{Where } C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

We can represent any **conic section** by the **3×3 matrix**

- Matrix C is a **symmetric matrix**, meaning that it contains **nine elements** in total, but only **six** of them are unique **due to symmetry. As a result, the matrix has six degrees of freedom (DoF).**

- Since C is a **homogeneous matrix**, we could scale it by any factor, and we still satisfy the equation, i.e., $\mathbf{x}^T C \; \mathbf{x} = 0$. Intuitively, it means we **lose one dof**.

Conic section

- **For example**, if we fix the scale of the conic say, by normalizing the matrix C such that all its elements are divided by a **nonzero scalar f**, then the conic effectively has **five degrees of freedom instead of six**.

- Understanding and managing **degrees of freedom (DoF)** is central to computer vision, as it governs the number of **independent parameters** we must estimate for various objects of interest.

Conic section

- It means, if we want to **estimate the parameters** of a **conic**, then we have to **constraint five parameters**.

- We have **five parameters** based linear object, and we like to **estimate the parameters** of this object.

Conic section

- For a conic, we **need six equations** to fully constraint the **five dof of this conic**.

- If we have **three points** on a parabola, then that's all we need to define it. In the case of a circle, three points fully define it. We need 3 points to fit an ellipse and so on.

- All conic sections can be defined by finding **three points** on that **particular conic**. We need **five equations** to define a conic section up to **scale uniquely**.

Inhomogeneous representation of conic section:

Any conic section in **inhomogeneous coordinates** can be represented as

$ax^2 + bxy + cy^2 + dx + ey + f = 0$------------(1) (any quadratic equation in xy-plane)

Trick: Homogenizing, we write the point

$(x_1, x_2, x_3)^T$ as $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^T$.

We have a homogeneous point$(x_1, x_2, x_3)^T$. This point is mapped to the **plane x_3 = 1.**

Replace x by x_1/x_3 and y by x_2/x_3 in equation (1)

$a(x_1/x_3)^2 + b(x_1/x_3)(x_2/x_3) + c(x_2/x_3)^2 + d(x_1/x_3) + e(x_2/x_3) + f = 0$

$\Rightarrow$ **$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$**

This is the equation of conic section in **homogenous coordinates**.

Example 1

Let the equation of parabola is

$y = (x - 2)^2 + 1$

$x^2 - y - 4x + 5 = 0$ --(1)

We know equation of conic section in **homogeneous coordinates**

$\mathbf{ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0}$ ----------------(2)

Replace **x** by $\mathbf{x_1/x_3}$ and **y** by $\mathbf{x_2/x_3}$ in equation (1)

$(x_1/x_3)^2 - x_2/x_3 - 4(x_1/x_3) + 5 = 0$

$\Rightarrow x_1^2 - x_2x_3 - 4x_1x_3 + 5x_3^2 = 0$ ------------------------------------(3)

Comparing equations (2) and (3), we get

$\Rightarrow a = 1, b = 0, c = 0, d = -4, e = -1, f = 5$

Given

a = 1, b = 0, c = 0, d = -4, e = -1, f = 5

$\vec{x}^T C \vec{x} = 0$

where

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 0/2 & -4/2 \\ 0/2 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix}$$

$\vec{x}^T C \vec{x} = 0$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_{\mathbf{1\times3}} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix}_{\mathbf{3\times3}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathbf{3\times1}} = 0$$

$$\begin{bmatrix} x_1 - 2x_3 & -\frac{1}{2}x_3 & -2x_1 - \frac{1}{2}x_2 + 2x_3 \end{bmatrix}_{\mathbf{1\times 3}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathbf{3\times 1}} = 0$$

$$x_1^2 - 2x_1x_3 - \frac{1}{2}x_2x_3 - 2x_1\,x_3 - \frac{1}{2}x_2x_3 + 5x_3^2 = 0$$

$$x_1^2 - 4x_1x_3 - x_2x_3 + 5x_3^2 = 0.$$

Which is the same equation.

Note: We need 5 equations to define a conic section upto scale.

Show that the point $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ satisfy the $\vec{x}^T C \vec{x} = 0$

$\vec{x}^T C \vec{x} = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4/2 \\ 0 & 0 & -1/2 \\ -4/2 & -1/2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$-\frac{1}{2} + \frac{1}{2} = 0$$

$$0 = 0$$

The point, $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ satisfies the equation.

Note: Any point satisfies the equation $\vec{x}^T C \vec{x} = 0$ lies on the conic

Example. Write the equation of circle about the origin with radius r in conic matrix form and verify the point $\begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix}$ lies on the circle $x^2 + y^2 = r^2$. Let r = 5.

$x^2 + y^2 = r^2$ --(1)

Replace x by $\frac{x_1}{x_3}$ and y **by $\frac{x_2}{x_3}$ in (1), we get**

$\Rightarrow (\frac{x_1}{x_3})^2 + (\frac{x_2}{x_3})^2 = r^2$

$\Rightarrow x_1^2 + x_2^2 = r^2 x_3^2$

$\Rightarrow x_1^2 + x_2^2 - r^2 x_3^2 = 0$ ------------------------------------(2)

We know equation of conic section in homogeneous coordinates

$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$-------(3)

Comparing (2) and (3), we get

$\Rightarrow a = 1, b = 0, c = 1, d = 0, f = - r^2$

$\vec{x}^T C \vec{x} = 0$

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\Rightarrow [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Suppose $x^2 + y^2 - 5^2 = 0$

$$\Rightarrow [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow [x_1 \quad x_2 \quad -25x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1^2 + x_2^2 - 25x_3^2 = 0$

$$\vec{x}^T C\, \vec{x} = 0$$

$$\Rightarrow \begin{bmatrix} \frac{5\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix} = 0$$

$$\vec{x}^T C\, \vec{x} = \begin{bmatrix} \frac{5\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} & -25 \end{bmatrix} \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$= \frac{50}{4} + \frac{50}{4} - 25$$

$$= 0$$