Dynamic Optimization of an asset Portfolio with Quantum-Inspired Tensor Networks TN establishing the best trading trajectory over a period of time

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Abstract

In this paper, we shall investigate how an investment portfolio of assets optimizes dynamically establishing the optimal trading trajectory using quantum-inspired Tensor Networks TN. We will apply different formulation on our system using quantum computing.

1. Introduction

In 1952, Harry Markowitz introduced Modern Portfolio theory in an essay which predicts that an investor wants a return rate with maximum profit. Those portfolios are known as efficient portfolios if these portfolios meet the criteria of gaining higher expected return rate taking on more risk. So, brokers are confronted with a trade-off between expected and risk.

Portfolio optimization is helpful to choose a best portfolio specifying all possible assets distributions. There are some objective functions which optimizes using different techniques of portfolio optimization. This objective function tends to minimize financial risk and maximize expected returns. It becomes more problematic when a portfolio dynamically optimizes over a consecutive pattern of number of days. A good portfolio carries different aspects like transactions' costs and marked impact of transactions, that is why dynamic asset portfolio optimization gets complicated.

Our main target is focused around determining the optimal trading trajectory over a specific period of time in this dynamic portfolio optimization. An investor tends to take optimal decisions to maximize the return.

Analyzing above problem, it has been observed that solution of complex financial problem is quite helpful by quantum-inspired computing methodology. There are different methods which are developed and implemented to optimize portfolios but in this research we see the implementation of quantum-inspired algorithm for dynamic portfolio optimization.

2. Objective

It has been observed that other quantum inspired algorithms like D-Wave Hybrid quantum annealing, VQE constrained, Variational Quantum Eigensolver (VQE) on Quantum processor and a quantum-inspired Tensor Network (TN) for optimization algorithm. We will keep our focus on dynamically optimization of assets portfolio using quantum-inspired Tensor Network (TN) which is well known for its results' accuracy and short computing time. Quantum-inspired TN are efficiently for largest systems where calculations are made up to 1272 fully connected qubits for responsive purposes.

3. Optimal Dynamic Portfolio

A. Generalities

Modern portfolio theory explains that how weights are assigned to number of assets portfolios over a periodic time in dynamic versions. The idea is to minimize inclusive return at the end of period.

Consider N-dimension vector of weight $\bar{\mu}_t$ for N assets where its component $\bar{\mu}_n, t$ in weight of assets n at time t = $t_i,\ t_i + 1, \ldots\ldots, t_f$, where t_f and t_i are taken as final and initial times of trading times with number of trading steps $N_t = \ t_f - t_i + 1$.

Assets' forecast returns at any time t is defined by ω_t and assets' covariance is defined byΣ_t, where Σ_t is N × N matrix and ω_t is a N – length vector.

The overall return for a giving trading trajectory which is vectors set $\{ \bar{\mu}_{t_i}, \ldots\ldots\ldots, \bar{\mu}_{t_f} \}$, is given by

$$\text{Return} \equiv \sum_{t=\ t_i}^{t_f} \omega_t^T \bar{\mu}_t, \qquad (1$$

An asset portfolio carries a risk which is defined by

$$\text{Risk} \equiv \frac{1}{2} \sum_{t=\ t_i}^{t_f} \bar{\mu}_t{}^T \Sigma_t \bar{\mu}_t \qquad (2$$

Where $\bar{\mu}_t{}^T \Sigma_t \bar{\mu}_t$ is variance of portfolio return at time t.

More specifically eq. (2) explains the risk in dynamic setting making the optimization in a better way with prefactor $\frac{1}{2}$ a convention. We are focusing to determine the trajectory maximizing the return rate for a fixed risk in modern portfolio theory.

Total investment at any given time, i-e

$$\sum_{n=1}^{N} \bar{\mu}_{n,t} = K \ \ \forall \text{t}, \qquad (3$$

Where normalized weight $\mu_{n,t}$

$$\mu_{n,t} = \frac{\bar{\omega}_{n,t}}{K}. \qquad (4$$

Hamiltonian of the system can be written as

$$\text{H} = \sum_{t=t_i}^{t_f} -w_t^T u_t + \frac{\gamma}{2}\mu_t^T \Sigma_t \mu_t + \rho(\omega_{\mu_t}^T - 1)^2 \quad (5$$

Where above function has analogy to quantum mechanics, ρ is lagrange multiplier imposing constraints on eq. (3) and the zeal of investor to protest risky trajectories is repeated by γ as 'risk aversion'.

Constraint on eq. (3) is made more compact by introducing N-diemenional vector μ, with $\mu_n = 1\ \forall_n$.

Eq. (5) can also be written as

$$\text{H} = \sum_{t=t_i}^{t_f} \hbar_t \quad (6$$

Where

$$\hbar_t = -\omega_t^T u_t + \frac{\gamma}{2}\mu_t^T \Sigma_t \mu_t + \rho(\omega_{\mu_t}^T - 1)^2 \quad (7$$

Thus, H is diagonal in time.

Above problem can be solved analytically. Note, If we take $\mu_{n,t} = 0.1$ means we are investing 10% of our total amount K in asset time, so

$$\omega_{n,t} \leq K'/JK$$

Sharp ratio which defines the quality of the portfolio is given as

$$\text{Sharp} \equiv \frac{\sum_t^{t_f} \omega_t^T \mu_t}{\sqrt{\sum_{t=t_i}^{t_f} \mu_t^T \Sigma_t \mu_t}} \quad (8$$

Which emphasize the amount of return per unit risk of trading trajectory in eq. (8) which imposes removing all possible costs. Large return for risk is measured by large sharp and vice versa and negative ratio means less profits.

4. Continuous formulations for the Portfolio Optimization

In our work we are facing to determine the trading trajectory by dynamically optimization an asset portfolio.

4.1 Continuous asset allocation

Problem becomes easy and solvable when transaction costs involving in portfolios are zero. To understand any system we should know the Hamiltonian and Lagrangian of the system, we can solve our problem of optimization of portfolio by solving Hamiltonian and Lagrangian of the system.

So, partially differentiating eq. (6) with respect to μ_t, so eq. (6) becomes

$$\frac{\partial H}{\partial \mu_t^T} = \frac{\partial h_t}{\partial \mu_t^T} = 0 \ \forall t \quad (9$$

By using

$$\omega_t^T \mu_t = \frac{(\omega_t^T \mu_t + \mu_t^T \omega_t)}{2}, \text{ we get}$$

$$-\frac{1}{2}\omega_t - \rho\omega + \left(\frac{\gamma}{2}\Sigma_t + \rho\omega \, . \, \omega^T\right) . \, \mu_t = 0 \quad (10$$

Hence by solving eq. (10) we get optimal solution which is given as

$$\mu_t = \left(\frac{\gamma}{2}\Sigma_t + \rho\omega \, . \, \omega^T\right)^{-1} \left(\frac{1}{2}\omega_t + \rho\omega\right) \quad (11$$

and dynamic optimal portfolio of a trajectory of time is just concatenation of optimal portfolio at each time t.

Consider eq(6) again and partially differentiating w.r.t ρ

$$\frac{\partial H}{\partial \rho} = \frac{\partial h_t}{\partial \rho} = 0 \ \forall t$$

(13
We get at the end

$$\omega^T \mu_t = 1 \quad (12$$

By using value of μ_t from eq. (12) in eq. (14), we get

$$\omega^T \left(\frac{\gamma}{2}\Sigma_t + \rho\omega \, . \, \omega^T\right)^{-1} \left(\frac{1}{2}\omega_t + \rho\omega\right) = 1 \quad (13$$

Hence, we can see degree of accuracy by knowing ρ.

It is important to note that when transactions costs get involved into the problem while dynamically optimizing the portfolios over trading trajectory over a period of time.

So that we make set of partial differentian equations (PDEs), we notice that increment in time steps is $\Delta t = 1$ in discrete formulation which implies that continuous time limit can be taken as $\Delta t \to 0$, Hence

$$\sum_{t=t_i}^{t_f} \Delta t \to \int_{t_i}^{t_f} dt \, , \ \frac{\Delta \mu_t}{\Delta t} \to \omega \quad (14$$

Integrating Langrange in terms of μ the time derivative from initial time to final time, μ in terms of time derivative and vector μ is asset allocation, which is known as vector field. the cost function is described by equation below,

$$S = \int_{t_i}^{t_f} L(\mu, \dot{\mu}, t)dt \quad (15$$

Eq. (16) S is referred as action and $L(\mu, \dot{\mu}, t)$ as Langrangian making it analogy with physics. Langrangian $L(\mu, \dot{\mu}, t)$ is defined by taking continous time limit of cost function H from eq. (5).

Thus our problem optimal trading trajectory becomes conventional functional minimizing problem.

We have to find time path of μ which minimize the S known as action. We take first derivative to minimize any function, taking derivative of eq. (16) with respect to μ and making it equal to zero, such as

$$\frac{\delta S}{\delta \mu} = 0 \quad (16$$

Euler-Langrange of our system can be written as

$$\frac{\partial \mathrm{L}}{\partial \mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mu}} = 0 \quad (17$$

In our case we want to find asset allocations $\mu_n(t)$ over the period of time for the portfolio, we write Langrange resulting set of coupled PDEs for each asset allocation in limit of t = t_i.

This is how we can solve our problem of dynamic portfolio optimization utilizing various algorithms for system of PDEs.

In our case, Langrangian of the system can be written as

$$\mathrm{L}(\mu, \dot{\mu}, t) = -\omega^T \mu + \frac{\gamma}{2}\mu^T \Sigma \mu + \lambda \dot{\mu}^2 + \rho(\omega^T \mu - 1)^2 \quad (18$$

with μ, Σ, $\dot{\mu}$ and ω being time-independent.

5. Discrete formulations for the Portfolio Optimization

We will use discrete variables in our portfolio optimization problem because investments trade in large amount so which seems to be relevant to big industry brokers. We will see how asset of portfolios will be allocated with initial t to final t.

5.1 Discrete asset allocation

It is in common practice that trade assets are funded in large scale which is discrete packages in industry, so rebalancing is maintained at discrete steps time t.
In this kind of problem binary encoding is chosen as each variable $\mu_{n,t}$ in terms of N_q bits $x_{n,t,q}$.
Considering binary encoding

$$\mu_{n,t} = \frac{1}{k} \sum_{q=0}^{N_{q-1}} 2^q x_{n,t,q} \tag{19}$$

Where $x_{n,t,q}$ = 0.1.

Thus, Ground state of classical Hamiltonian over the minimum variables $\{x_{n,t,q}\}$ is equivalent to finding the optimal weights so

$$\mathrm{H} = x^T Q_x \tag{20}$$

with x ∈ $\{0,1\}^N$.

We quantize eq. (10) to solve above problem on a quantum computer using operator $\{\hat{x}_{n,,t,q}\}$ with eigenstates | 0 > and | 1 >.

5.2 Discrete problem complexity

Ground state of an Ising spin glass can be found by applying transformation $x_i = \frac{(1+S_i)}{2}$ on eq. (9), which is given by

$$\mathrm{H} = \sum_{i,j=0}^{n_{tot-1}} J_{i,j=0} S_i S_j \tag{21}$$

Where $S_i = \pm 1$ are taken as spin variables, where coupling is J_{ij} which can be derived from eq. (9).

Note, Ising spin glasses are known asNP-Hard then optimal trading trajectory becomes concatenation of portfolios at each step time t by solving N_t independent optimization problems.

6. Tensor Networks' technique to Optimize Portfolio

We have discussed the formation of Hamiltonian of the system and finding of ground states. In TNs system, complex quantum states are discussed which are based on their local entanglement structure.

Wave function of system can be expressed in terms of its coefficient O (2^n) on computational basis. So these coefficients are taken as a tensor with n indices, where possible values for each index are two (say, 1 or 2). It is under consideration of making this huge set of tensors into less complicated form by interconnecting tensor with fewer coefficients. We assume that rand of interconnecting indices is upper-bound which is known as ‘bond dimension’ represented by D.

Likewise, interconnecting indices are also known as ‘bond indices’ and provide layout of many-body in quantum state. Any D > 1 gives an entangled quantum state. It is a well-known established fact that TNs are taken as a tool to solve optimizing problems.

We can map Hamiltonian eigen value problems to mapping optimization problems and then we can use tensor techniques and algorithms to solve optimization of portfolios at hand.

In our case, the strategy of TN is implemented over Matrix product states (MPS).

7. Results

We observe that methodology of quantum inspired Tensor Networks TN tends to address problem’s global minimum more authentically than D-Wave Hybrid in few systems. In case datasets of different sizes, for instance, there is larger sharp ratio or large profits for our TN algorithm.

Moreover, improvement in the algorithm is under process for XXL dataset by introducing various hyper parameters and also we can improve its accuracy and reduce at least $\approx 10\ \times$ the run time of Tensor Networks algorithm.

Is is concluded that largest systems can efficiently be handled with TN algorithm wher1272 full-connected qubits are connected for demonstrative purposes.

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