Project 1

Malin Eriksen
(Dated: September 20, 2022)

https://github.com/malineri/fys3150

In this project we will look at the topic of the one-dimentional Poisson equation, which can be written as

$$-\frac{d^2u}{dx^2} = f(x) \tag{1}$$

Where $f(x)$ our source term is set to be

$$f(x) = 100e^{-10x}. \tag{2}$$

We will let x have a range $x \in [0, 1]$, and we use the boundry conditions $u(0) = 0$, and $u(1) = 0$.

PROBLEM 1

We will check analytically that the exact solution to equation (1) is given by equation (3) by deriving equation (3) two times, and compare it to our function $f(x)$.

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \tag{3}$$

We start by inserting this into equation (1).

$$\begin{aligned}
-\frac{d^2}{dx^2}(1 - (1 - e^{-10})x - e^{-10x}) &= f(x) \\
-\frac{d^2}{dx^2}\left(1 - (1 - e^{-10})x - e^{-10x}\right) &= 100e^{-10x} \\
-\frac{d}{dx}\left(10e^{-10x} + e^{-10} - 1\right) &= 100e^{-10x} \\
100e^{-10x} &= 100e^{-10x}
\end{aligned}$$

And we find that equation (3) is the exact solution to equation (1).

PROBLEM 2

Use a program language to find the corresponding Poisson values to the x values from 0 to 1. In this table we only present 6 steps, but in the corresponding graph we have used 101 steps.

TABLE I. Poisson: x values with corresponding u(x) values.

x	u(x)
0.00	0.00
0.20	0.66
0.40	0.58
0.60	0.40
0.80	0.20
1.00	$5.89 \cdot 10^{-18}$

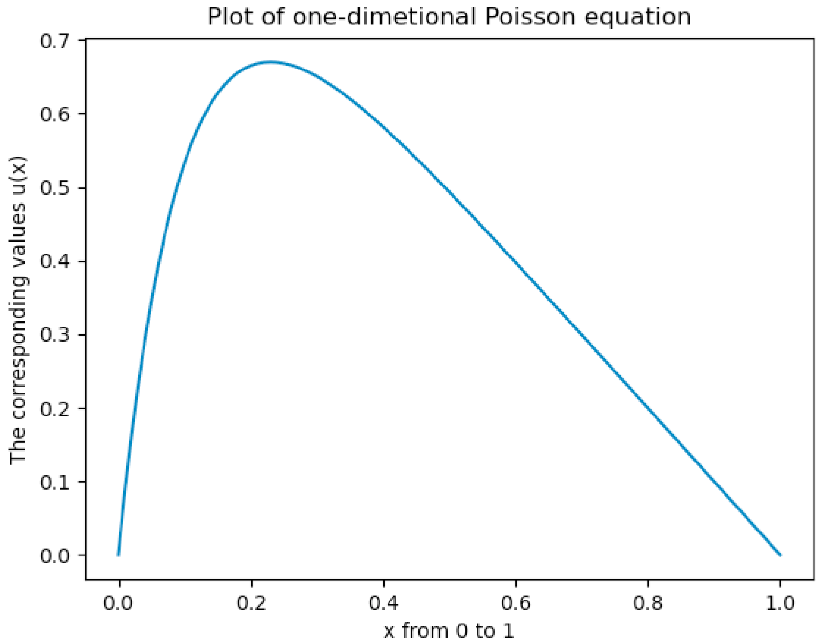

FIG. 1. Plot of the one-dimentional Poisson equation with what we found to be the solution to our function $f(x)$, $u(x)$ plotted against the values of $x \in [0, 1]$.

PROBLEM 3

In this problem we would like to discretize the Poisson equation. We start this by declaring the notation for this operation. Where we let $x \to x_i$, $u(x) \to u(x_i) \equiv u_i$, and in the end we use the term $u(x \pm h) \to u(x_i \pm h) \equiv u_{i \pm h}$. We will also use the step size h, where this is defined as the distance between two steps.

$$h = \frac{x_n - x_0}{n}. \tag{4}$$

Where n is the number of steps we do, and therefore x_n is our maximum value. We can also then use this to define each step.

$$x_i = x_0 + ih, \qquad (i = 0, 1, ..., n). \tag{5}$$

Now we will look at the Taylor approximation to find our second derivative, we use the Taylor approximation with the cases $u(x + h)$ and $u(x - h)$.

$$u(x + h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O(h^4)$$
$$u(x - h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O(h^4)$$

Now we add the two terms $u(x + h) + u(x - h)$, so that we are left with one expression, that we rearrange, so that we get,

$$u(x + h) + u(x - h) = 2u(x) + u''(x)h^2 + O(h^4)$$
$$u''(x) = \frac{u(x + h) + u(x - h) + 2u(x)}{h^2} + O(h^2)$$

Now we discretize this expression using the notation we presented earlier. We know h is one step, so $\pm h$ can be represented as ± 1 step. Then we find that the second derivative to be,

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2). \tag{6}$$

Which for the poisson equation then will be,

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) = f_i. \tag{7}$$

We can do an approximation, leaving out the rest of the Taylor terms left in the $O(h^2)$ term, and change the notation from u to v, and multiply by h, so that we get a clean expression for the one-dimentional poisson equation.

$$-v_{i+1} + 2v_i - v_{i-1} = h^2 f_i. \tag{8}$$

We can insert our function $f(x)$ to get the special occation we are looking at,

$$-v_{i+1} + 2v_i - v_{i-1} = h^2 \cdot 100e^{-10x_i}. \tag{9}$$

PROBLEM 4

Now we can use the discretized formula (8) we ended up with in problem 3 to create a matrix $\mathbf{A}\vec{v} = \vec{g}$. We use our 5 points that we used for the table in problem 2, letting i go from 0 to 5, x_0 to x_5. We already know the endpoints, $u(0) = 0$ and $u(1) = 0$, so our remaining unknown steps is the four middle points. We write these lines down and rearrange them slightly so that we get the equations,

$$\begin{aligned}
(i = 1) &: 2v_1 - v_2 = h^2 f_1 + v_0 \\
(i = 2) &: -v_1 + 2v_2 - v_3 = h^2 f_2 \\
(i = 3) &: -v_2 + 2v_3 - v_4 = h^2 f_3 \\
(i = 4) &: -v_3 + 2v_4 = h^2 f_4 v_5
\end{aligned}$$

Now if we let v be a vector containing the values $\vec{v} = (v_1, v_2, v_3, v_4)$. We can multiply this with the numbers we have in the equations in a matrix equation. We rearrange the right hand side of the equation as well by adding the various f values to a new vector we call g, $\vec{g} = (h^2 f_1 + v_0, h^2 f_2, h^2 f_3, h^2 f_4 v_5) = (g_1, g_2, g_3, g_4)$. Now we rearrange it all onto the form $\mathbf{A}\vec{v} = \vec{g}$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix}$$

PROBLEM 6

For problem 5 we try to implement this matrix in a c++ code, the matrices.cpp. This program is still under work, and I didnt manage to make it run. I have the trouble of "Armadillo not found" which I tried to use in this program. I just tried with adding one element to the matrix, and I would work furter on this code to make the program solving the equation. In python however I do find the vector v, but I do get an error message when trying to solve the number of FLOPs that has been used. This program is named matrices.py.

PROBLEM 7

We use the python program written above to determine the values of v(x), we have written these results out with their corresponding x value, and added them to table two. I tried making a program from the formulas I get when I do backward and forward substitution on the matrix A, but I have a failure in my $\tilde{b}$ and $\tilde{g}$ vectors that leaves me with a zero vector. Therefore I just made the plot with the 6 points I did in the simplified program, which we can see in figure 2. The figure in itself looks like it is an inverse of the figure we found in problem 2, which means that when finding v, we might have done something wrong. If we study the two tables we might also see that for x incresing u(x) in problem 2 seems to decrease, while for v(x) it is increasing. Which doesnt completely makes sence.

TABLE II. x values to corresponding v values, using the matrix equation $\mathbf{A}\vec{v} = \vec{g}$.

x	v(x)
0.00	0.00
0.20	0.68
0.40	0.82
0.60	0.89
0.80	0.94
1.00	0.0

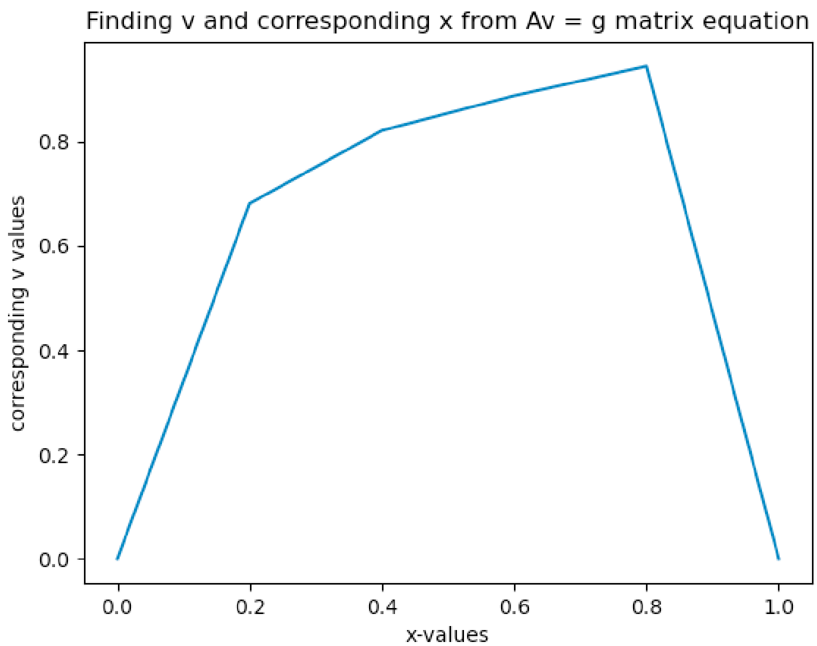

FIG. 2. Plot of the one-dimentional Poisson equation where we use the matrix equation $\mathbf{A}\vec{v} = \vec{g}$ to find v.

PROBLEM 8

In this problem I added my u values and v values and found the absolute error x and the relative error, w, from the equations,

$$x = |u_i - v_i|$$
$$w = \left|\frac{u_i - v_i}{u_i}\right|$$

And made a program error.cpp, where I find the values of u, v, x and w, which we can see in table three. These values if I had time I would be taking the logarithm of and made plots of, but I didn't have enough time to complete either this or the last two tasks of this problem set. I am aware that this problem set has loads of holes, and if it weren't for the fact that I have to deliver now I would definitely finished this in a better way. Hopefully there is enough within this problem set to let me pass. I haven't had much days completing this task and haven't had the time to get help from group lessons due to time, and therefore loads of my programs have bits from googling issues, which if I had time I should have written down the links from. Hopefully this will not give me any issues!

TABLE III. u-values, v-values, the relative error w and the absolute value x

u	v	w	x
0.0000000000e+00	0.0000000000e+00	nan	0.0000000000e+00
6.6000000000e-01	6.8000000000e-01	3.0303030303e-02	2.0000000000e-02
5.8000000000e-01	8.2000000000e-01	4.1379310345e-01	2.4000000000e-01
4.0000000000e-01	8.9000000000e-01	1.2250000000e+00	4.9000000000e-01
2.0000000000e-01	9.4000000000e-01	3.7000000000e+00	7.4000000000e-01
5.8900000000e-18	0.0000000000e+00	1.0000000000e+00	5.8900000000e-18