Partially completed answer for part A.

1. **Question:** Derive the kinematic equations for a differential drive robot model. Let x, y, α denote the X, Y Cartesian coordinate of the robot's center and the heading angle as measured from $X+$.

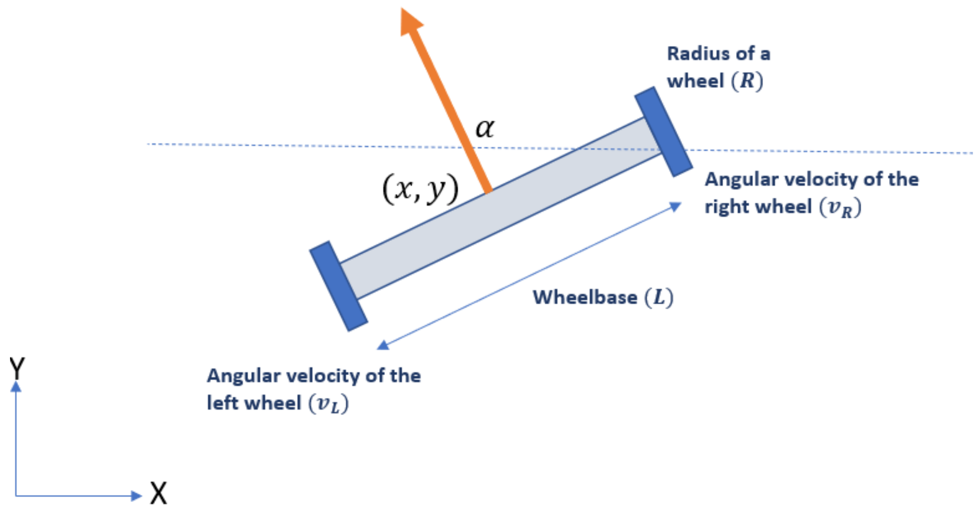

Figure 1: Differential drive robot model.

$$\dot{x} = \frac{R}{2}(v_L + v_R)cos(\alpha) \tag{1}$$

$$\dot{y} = \frac{R}{2}(v_L + v_R)sin(\alpha) \tag{2}$$

$$\dot{\alpha} = \frac{R}{L}(v_L - v_R) \tag{3}$$

The equation

$$v = r\omega \tag{4}$$

from circular motion plays a key role in this derivation. Here, v is the instantaneous linear (tangential) velocity of a particle moving on a circle at an angular velocity ω. The radius of the circle is denoted by r.

2. **Partial Answer:** Let's consider the wheels' angular velocities v_L, v_R are such that $v_L > v_R$. (The same procedure can be used for $v_L < v_R$ and $v_L = v_R$ cases too.)

   Therefore, the robot is **rotating and moving** clockwise. By looking at the equations, we can identify that the first two equations define the *translational motion* of the robot over the 2-D Cartesian plane and the third defines the *rotation motion* around its own axis (*yaw*). So, let's try to derive them separately.

   (a) **For the translational motion**

   For simplicity, we use α as an *acute* angle (See Fig. 3). By considering two instantaneous snapshots of the robot, we can identify that the robot is moving on a curvature. The center of this curvature is called the *Instantaneous Center of Curvature* or *ICC*.

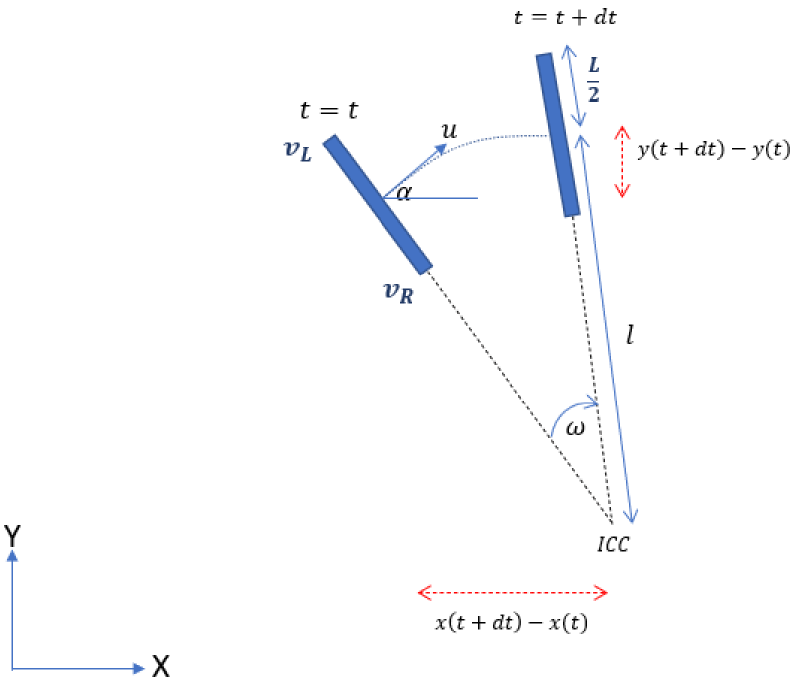

Figure 2: Two snapshots of the robot when $v_L > v_R$. The robot is moving clockwise on a curvature. The pose of the robot at the beginning ($t = t$) and after a dt time ($t = t + dt$) is shown.

   Consider the angular velocity of this motion is ω and the curvature's radius is l as shown above.

   i. Our goal is to find the instantaneous linear velocity (or tangential velocity) u, of the robot which can then be used to calculate the new positions of the robot $x(t + dt)$ and $y(t + dt)$.

   ii. By considering the rotation of the wheels, we can apply (4) to the instantaneous tangential velocities of the wheels. Thus, we get $v_L R$ and $v_R R$ for left and right wheels, respectively. In other words, this gives us the tangential velocity of two points at left and right wheels.

   iii. Now apply (4) to the circular motion of the robot's center around ICC using (4).

$$v_L R = \left(l + \frac{L}{2}\right)\omega \tag{5}$$

$$v_R R = \left(l - \frac{L}{2}\right)\omega \tag{6}$$

iv. Use (5) and (6) to solve for ω and l.

v. Now apply (4) to the robot's circular motion around ICC with l and ω to find u. It should look like

$$u = \frac{\left(v_R + v_L\right)R}{2}. \tag{7}$$

vi. Use $X-$ and $Y-$ directional components of u to find the new coordinates of the robot's centroid as

$$x(t + dt) = x(t) + u\cos(\alpha)dt, \tag{8}$$

$$y(t + dt) = y(t) + u\sin(\alpha)dt. \tag{9}$$

vii. Find $\left(x(t + dt) - x(t)\right)/dt$ and, take the limit $dt \to 0$ to find the rate of change of x. This will get rid of dt and yield the differential equation for $\dot{x}$. Ditto to $\dot{y}$. See the particle falling example on the slides for a reference.

(b) **For rotational motion**

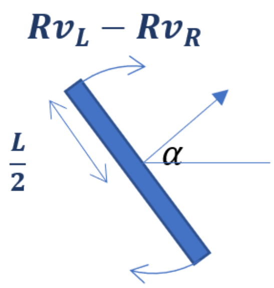

Figure 3: The rotational motion of the robot around its own axis (yaw). This is caused by the difference of the wheels' angular velocities.

i. Consider the rotation of the robot around its own axis caused by the difference of the wheels' angular velocities.

ii. Apply (4) to this motion and find the angular velocity of the rotation. This is the rate α is changing, aka $\dot{\alpha}$.