Linear Dynamical Systems - 2

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* Examples and codes have been adopted from the book “Data Driven Science & Engineering - Machine Learning, Dynamical Systems, and Control” by Steve Brunton and Nathan Kutz.

Outline

- Feedback Control Systems
- Control Laws
- Linearization of Nonlinear Dynamical Systems
- Stability of Linear Dynamical Systems
- Inverted Pendulum Example
- Optimal control and Linear Quadratic Regulator

Why Feedback Control?

- To **stabilize** an unstable system.
- To account for **external disturbances**.
- To account for system inherent disturbances such as
  - Unmodeled dynamics,
  - Measurement errors,
  - Model uncertainties.
- To inhibit some **desired behavior** with dynamical systems.
  i.e.: trajectory tracking
- By calculating “some gain” parameters (linear control law) we can perform feedback control.

Control Laws

- A control law is a set of parameters that is used to regulate the control input for a system.

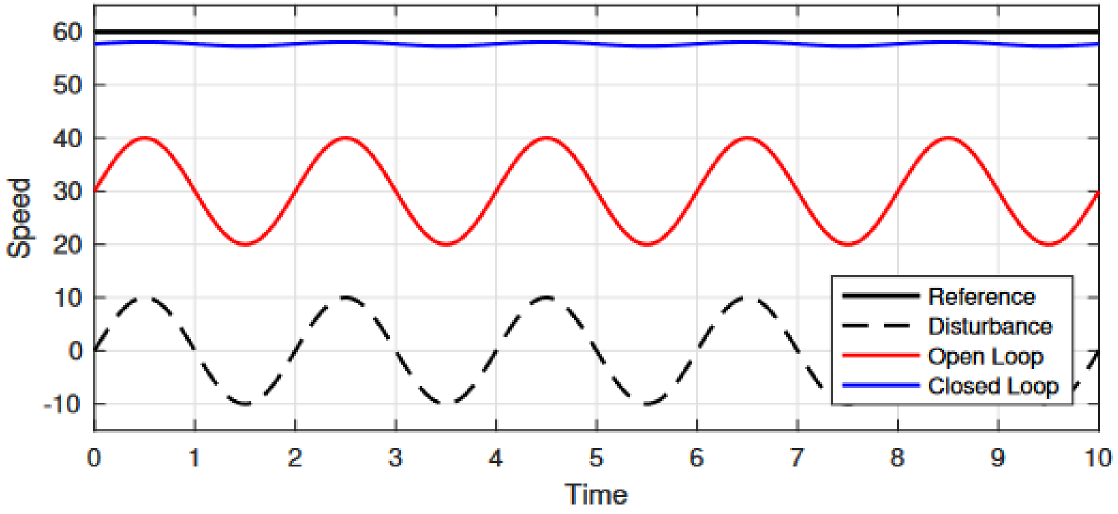
- Consider the control law $u = K(r - x)$ for a linear dynamical system. Here, r is the desired state of the system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}(\mathbf{r} - \mathbf{x})$$
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{r} - \mathbf{B}\mathbf{K}\mathbf{x}$$
$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{r}$$

- Here, A, B are inherent to the system. Thus, by choosing K appropriately, we can bring the system to stabilization, aka to r.

Cruise Control Example

Cruise control with closed (feedback) and open loop control

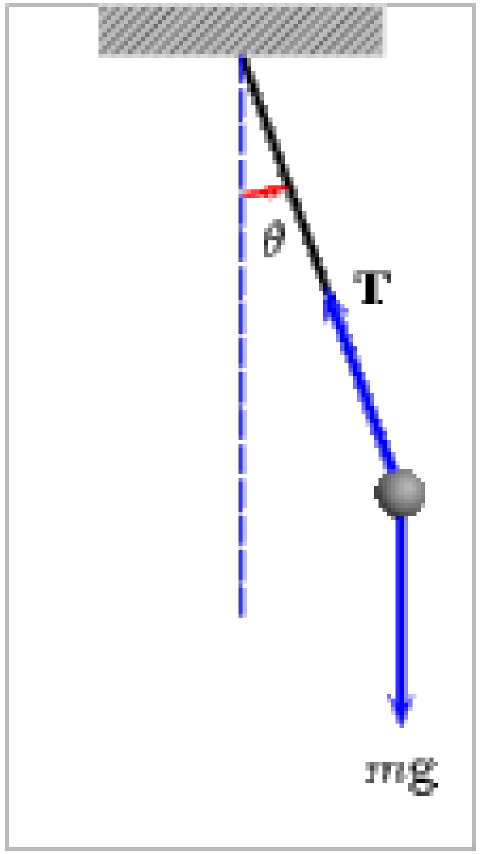


In the open loop control, we set the accelerator to a fixed value. Closed loop control is a result of P control.

$$u = k(r - x),\ k = 50, r = 60$$

What are Fixed Points?

- A fixed point is any point where our dynamical system is at an equilibrium (or stabilized).
- Formally, $\bar{x}$ is a fixed point if $\dot{x} = f(\bar{x}) = 0$.
- i.e.: A pendulum has two fixed points, $\theta = 0$, $\theta = \Pi$.
- In trajectory tracking, a fixed point can be any point on the trajectory, that we would like the robot to converge.



Linearization of Nonlinear Dynamics

- Most dynamical systems in practice are nonlinear.
  i.e.: Car, Alphabot, Population dynamic models etc.

- However, linear dynamics have been well studied and established. But nonlinear dynamics are not.

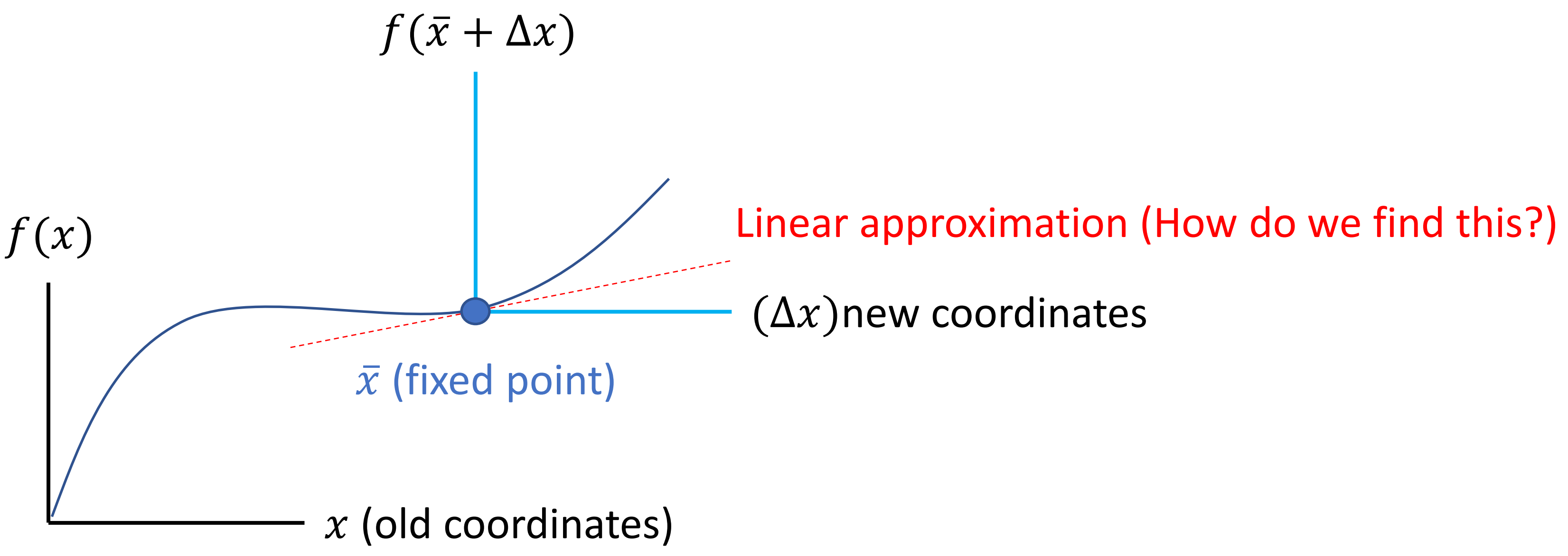
- To analyze nonlinear dynamical systems, we linearize them around their fixed (aka equilibrium/desired) points.

- We assume these linear approximations are valid for a finite neighborhood of points around the fixed points.

- If we look carefully, *most nonlinear dynamical systems are linear for a short duration*.

Linearization

- The key idea of linearization is to **find a linear approximation** of the nonlinear system that is valid for a **close neighborhood around a fixed point**.
- This also defines a new coordinate system whose origin is a **fixed point**. Any measurement on the new coordinate system denotes the deviation of the system from the fixed point.



Linearization of Nonlinear Dynamics

- Consider some nonlinear system of equations in the form $\dot{x} = f(x, u)$.
  i.e.: $\dot{x}_1 = f_1(x, u), \dot{x}_2 = f_2(x, u)$, … etc.

- Identify the fixed point $(\bar{x}, \bar{u})$. Here $\bar{u}$ is the control input at the fixed point $\bar{x}$.

- Find the deviation from the fixed point.

$$\Delta x = x(t) - \bar{x}$$
$$\Delta u = u(t) - \bar{u}$$

- Now write the Taylor expansion for the system around the fixed point.

$$\mathrm{f}(\bar{\mathrm{x}} + \Delta\mathrm{x}, \bar{\mathrm{u}} + \Delta\mathrm{u}) = \mathrm{f}(\bar{\mathrm{x}}, \bar{\mathrm{u}}) + \underbrace{\left.\frac{\mathrm{df}}{\mathrm{dx}}\right|_{(\bar{\mathrm{x}}, \bar{\mathrm{u}})}}_{\mathbf{A}} \cdot\Delta\mathrm{x} + \underbrace{\left.\frac{\mathrm{df}}{\mathrm{du}}\right|_{(\bar{\mathrm{x}}, \bar{\mathrm{u}})}}_{\mathbf{B}} \cdot\Delta\mathrm{u} + \cdots .$$

- For small deviations from the fixed points, the higher order terms become negligible.

Gist

- We can obtain the linear dynamical system matrices from a nonlinear differential equations system as below.

- Evaluate the two matrices at the fixed point $(\bar{x}, \bar{u})$ to get the *numerical* linearized matrices at any point.

$$A = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \dots \\ \vdots & \vdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \Bigg|_{\bar{x}} \qquad B = \frac{df}{du} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots \\ \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_2} & \dots \\ \vdots & \vdots & \frac{\partial f_n}{\partial u_l} \end{bmatrix} \Bigg|_{\bar{u}}$$

- In trajectory following, the fixed point could be any *“close enough”* point in the given **trajectory**.

Linearization - Pendulum Example

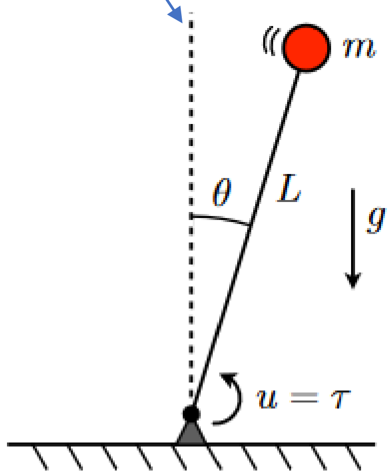
- Let's consider an inverted pendulum attached to a light rod, angling slightly away from a **<u>fixed point</u>**.

- Write the equations of motion for this system.

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) + u.$$

- But this doesn't fit our formula. Let's write them as first order differential equations. (Recall the example from last lecture).

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \Longrightarrow \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin(x_1) + u \end{bmatrix}$$



- Call these functions f_1, f_2.

- Now do the partial differentiation to get the Jacobian matrices.

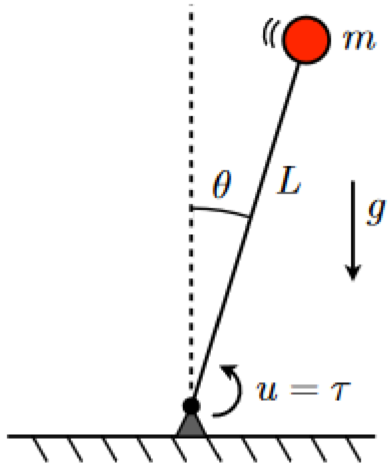
Linearization - Pendulum Example

- Let’s do the partial differentiation according to our formula.

$$\frac{\mathbf{df}}{\mathbf{dx}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x_1) & 0 \end{bmatrix}, \qquad \frac{\mathbf{df}}{\mathbf{du}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Now evaluate this system at the **fixed points**. Since we have two fixed points, this will yield two A matrices.

$$A_D = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \Big| (\bar{x} = [\Pi, 0]) \qquad A_U = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \Big| (\bar{x} = [0,0])$$

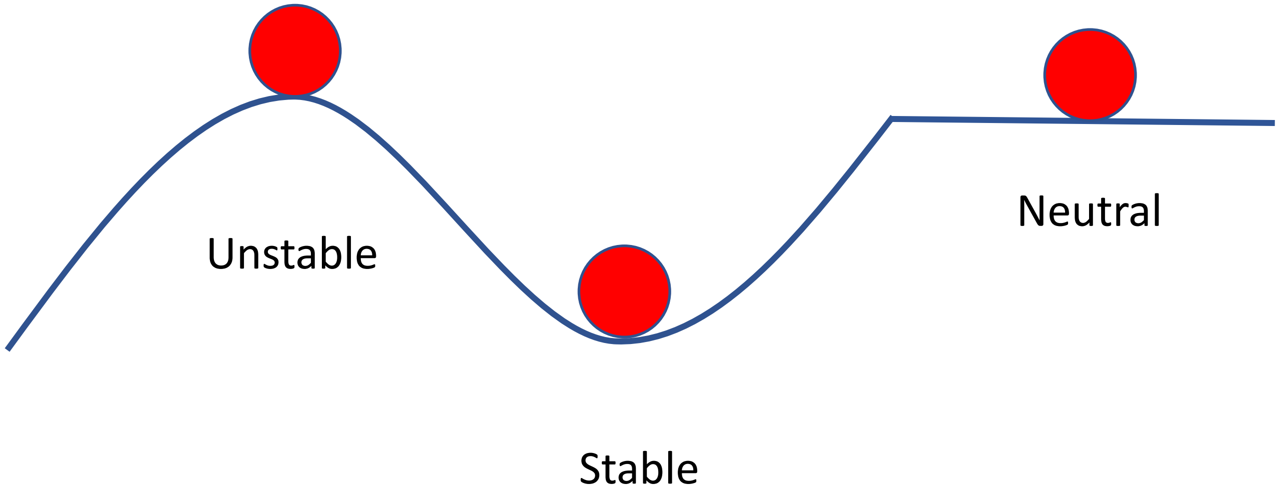


- Substituting g, L we will get the corresponding linear system near the fixed points.

- **Linearization is done!!!!**

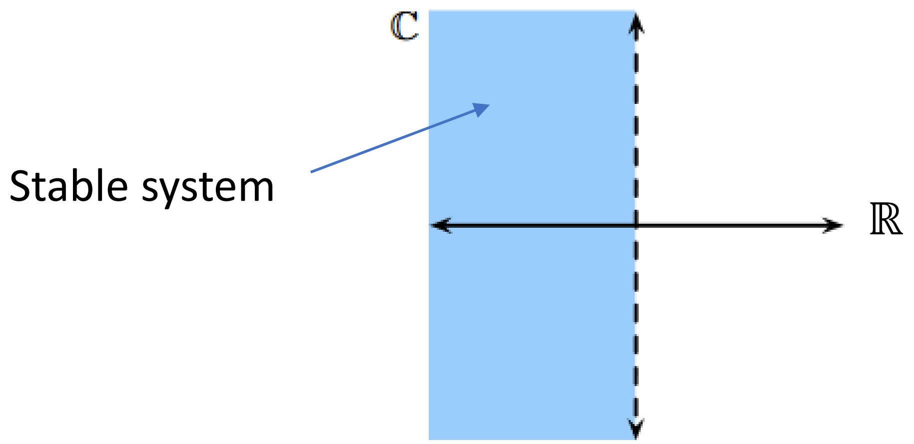
Stability of Linear Systems

- Consider a linear system $\dot{x} = Ax$ with drift ($A \neq 0$) near fixed points.
- Will the system return to its original state after a little perturbation?



Stability of Linear Systems

- We can analyze the A matrix to check if a system is stable at a fixed point.
- For any **continuous time** dynamical system,
  - If the all the **Eigenvalues** of A has negative real parts → system is stable at the fixed point.
  - If any of the **Eigenvalues** has positive real parts → unstable at the fixed point.



Stability - Pendulum Example

- Let's consider the linearized A matrices of the pendulum system.

- Add a small damping to the system to make sure it has some resistance.

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- The system is unstable near the upper fixed point.

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How Can We Stabilize It?

- Let's consider the control law $u = K(x)$ again.

- After applying this control, our system becomes,

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \underbrace{(\mathbf{A} - \mathbf{B}\mathbf{K})}_{\text{New } A \text{ Matrix}}\mathbf{x}.$$

- Choose $K = [4,4]$ for the system at the upper fixed point.

$$\dot{x} = [A - BK]x$$

$$\dot{x} = \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 4 & 4 \end{bmatrix}\right]x \qquad = \qquad \left[\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\right]x$$

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- New Eigenvalues for the matrix becomes $\lambda = -1, \lambda = -3$.
  This means we can stabilize this system using this control law at an unstable fixed point!!!.

How Can We Find K?

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Linear Quadratic Regulator FTW!

Linear Quadratic Regulator (LQR)

- An optimal control mechanism for **linear dynamical systems** by optimizing a quadratic **cost function**.

- The canonical form for the cost function takes form

$$C = (x - \bar{x})^T Q (x - \bar{x}) + (u - \bar{u})^T R (u - \bar{u})$$

  where, $Q \succcurlyeq 0$ and $R \succ 0$ are two weight matrices.

- The weights lets us quantitate the contributions of the state’s error and control amount.

- The objective is to find the control input that minimizes this cost and **drives the system to the stability** (or to the desired state).

- The output of the LQR is a gain matrix K that can be used to calculate the optimal control with the control law $u = K(\bar{x} - x)$.

Why Optimal Control?

- To stabilize a system with **minimal control effort**.
- Classical controllers such as PID are much harder to tune when the states are tightly coupled.
- Allows arbitrating between competing goals.
  i.e.: convergence time vs control effort.
- Allows finding the optimal gain matrix (control law) accounting for the system model by using **mathematical optimization**.
- Much more **robust** to disturbances.
- *Example: Navigating a drone to a certain location while minimizing the required thrust.*