Linear Dynamical Systems – Part 1

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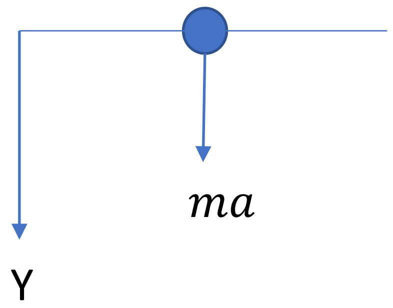
Outline

- Recap from the last dynamics lecture
- Dynamical Systems
- Converting higher order dynamical Systems in first order
- Linear dynamical systems
- Discrete time systems
- Finding the discrete time system for CT systems.

Some Definitions…

- **Kinematics**: The motion of rigid bodies (displacement, velocity etc.) regardless of the actual forces acting on the system such as friction E.g.: a trajectory of a particle in a frictionless plane.
- **Dynamics**: The motion of rigid bodies (displacement, velocity etc.) without referring to the actual forces such as friction. E.g.: how would the trajectory change if we apply friction?
- **Control**: Manipulating the actuators of a physical system to result in the desired behavior by accounting for the dynamics. E.g: changing the angle of the plane, so the particle would stay in place.
- **Degrees of Freedom:** The number of independent parameters we can use to define the configuration (State) of a system. E.g.: The particle has 2 DoF (X,Y) with respect to the plane.

Simulating a Particle



Initial Conditions:
$v(0) = 0, y(0) = 0, a(0)$

- First consider the change of velocity over an infinitesimal time period dt

$$v(t + dt) - v(t) = dt * a$$

$$\frac{v(t + dt) - v(t)}{dt} = a$$

- Taking the limit on both sides:

$$\lim_{dt \to 0} \left\{ \frac{v(t + dt) - v(t)}{dt} \right\} = a$$

The definition of the time derivative.

$$\dot{v} = a$$

- Do the same for the position.

$$\dot{y} = v$$

Simulating a Particle

Initial Conditions:

$$v(0) = 0,$$
$$y(0) = 0,$$
$$a(0) = g$$

- Consider a particle in space, assuming the only force working on it is gravity, we can write the following equations for the downward motion.

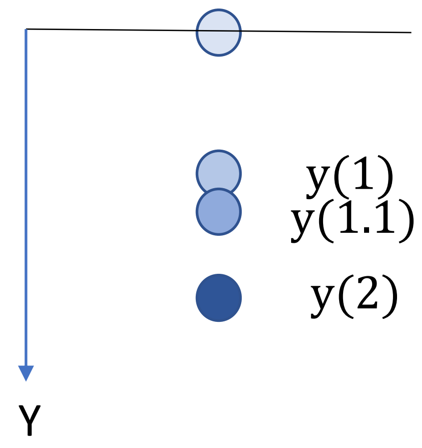
$$\dot{v} = g$$
$$\dot{y} = v$$

- But these equations only provide us how the system would change over an infinitesimal time.

- To see how these small steps accumulate over time, we have to integrate the system over a desired period.

- As the initial conditions change, the system behavior would also change.

- For that, we use Numerical integration.



Simulating a Particle

New system configuration are recursively passed inside.

```
def f(y, t):
    """this is the rhs of the ODE to integrate, i.e. dy/dt=f(y,t)"""
    y_, v_ = y

    fdot = [v_, 9.82]
    return fdot

y0 = [0, 0]             # initial value
a = 0              # integration limits for t
b = 2

t = N.arange(a, b, 0.01)  # values of t for
                          # which we require
                          # the solution y(t)
f_val = odeint(f, y0, t)  # actual computation of y(t)
```

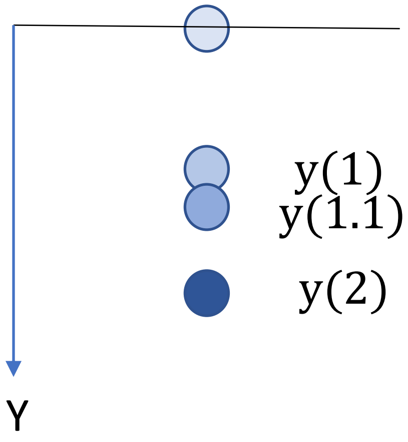
Code your equations here by reading from the array.

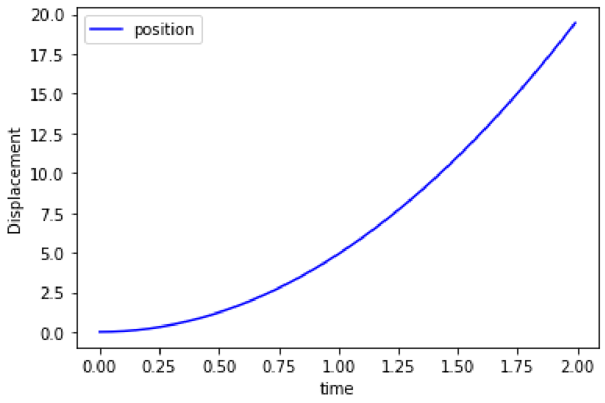
Stack all the LHS values in an array.

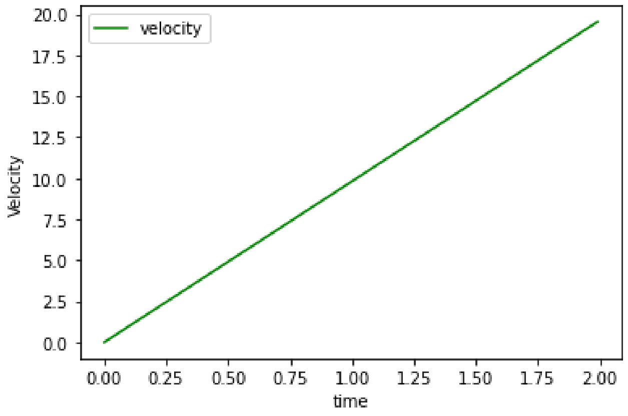
Pass the initial conditions, the time period and the integration function to the solver

Initial Conditions:

$v(0) = 0,$
$y(0) = 0,$
$a(0) = g$







Do the calculations by hand and verify if they are correct for a few timesteps.

Simulating a particle on a 2D Plane

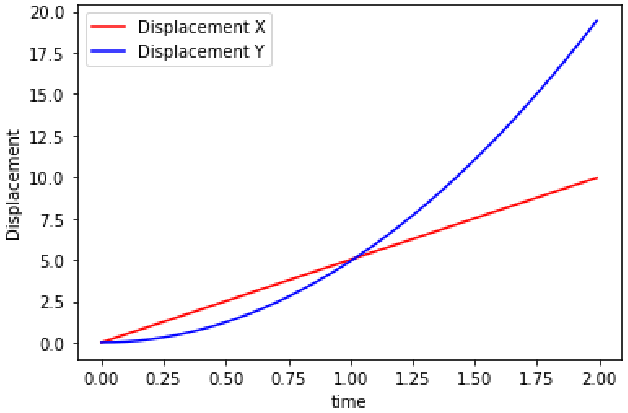
- Differential equations for each dimension:

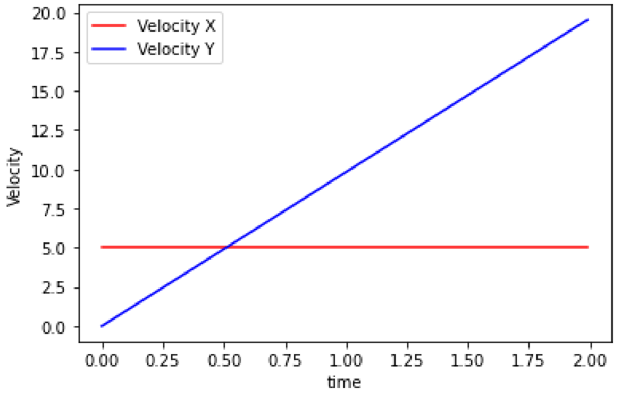
$$\dot{v_x} = 0$$

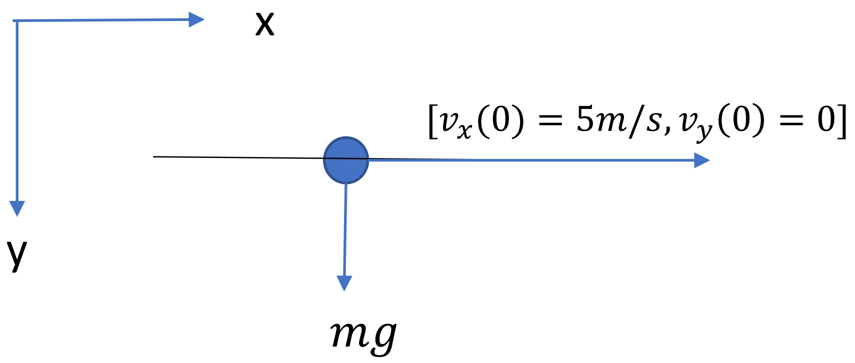
$$\dot{v_y} = g$$

$$\dot{d_x} = v_x$$

$$\dot{d_y} = v_y$$







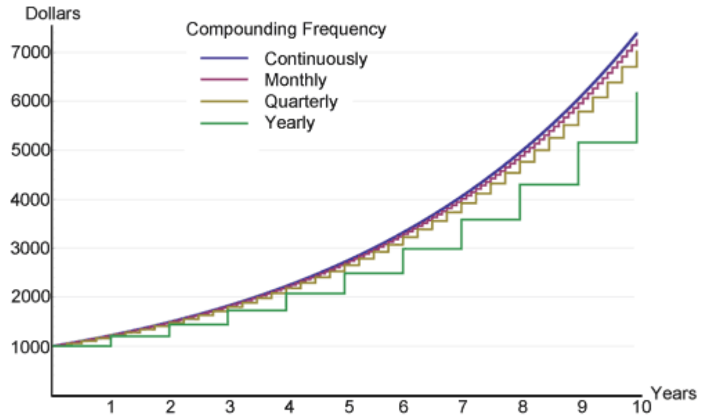
$$\textit{Initial conditions: } v(0) = [v_x(0), v_y(0)],$$
$$\mathrm{d}(0) = [0,0],$$
$$a(0) = [0, g]$$

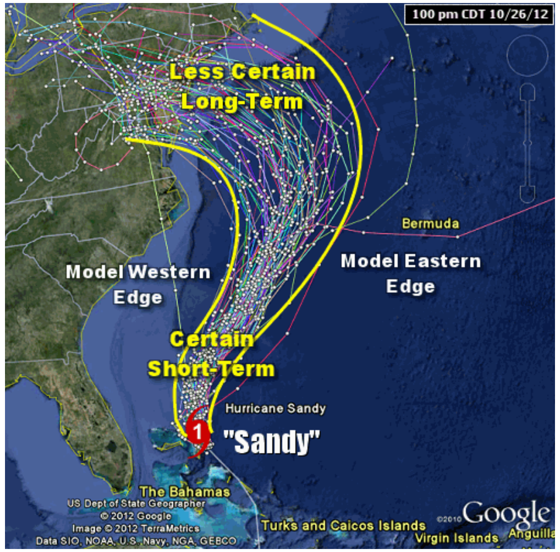
What are Dynamical Systems?

- A function a that defines *some system's* evolution over time.

$$\dot{x} = f(x(t))$$

- These systems could be
  - bank balance,
  - population of some species (population dynamics),
  - trajectory of a hurricane,
  - motion of a robot.
- In other words, dynamical systems *model* the system behavior.
- In practice, we find many discrete and continuous time dynamical systems.
- We can use difference and differential equations to model the dynamical systems.





Linear Equations

consider system of linear equations

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

can be written in matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear Dynamical Systems

- Linear dynamical system is some system whose evolution can be modeled using a set of **linear equations**.
- Although, most of the real-world systems are non-linear, we can find linear approximations to them that are accurate to a finite time duration.
- We can represent such linear dynamical systems in the following form.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

State space form for a **continuous time invariant linear** dynamical system

$\mathbf{A}$	system matrix	$\mathbf{x}$	state vector
$\mathbf{B}$	input matrix	$\mathbf{u}$	input vector
$\mathbf{C}$	output matrix	$\mathbf{y}$	output vector
$\mathbf{D}$	feedthrough matrix		

State Space Form

"A **state-space representation** is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations or difference equations."

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"The internal state variables are the smallest possible subset of system variables that can represent the entire state of the system at any given time."

Why do we need two equations?

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"A **state-space representation** is a mathematical model of a physical system as a set of input, output and state variables related by **first-order** differential equations or difference equations."

"The internal **state variables** are the smallest possible subset of system variables that can represent the entire state of the system at any given time."

Great for simulations and representing MIMO systems.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
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- Why do we need two equations?
  - Not all the system states are observable thus, we need a mapping between the observable states to the system states.
  - We don't need to observe all the system states in the design.

State Space Form (Example)

Write the following dynamical system in the state space form.

$$\dddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u$$

(1) This is a 3rd order differential equation.
First convert it to the first order form.

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$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \end{aligned}$$

(2) These are our system states. Write them in terms of x in the original equation.

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$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \dddot{x} = -5\ddot{x} - 3\dot{x} - 2x + u \end{aligned}$$

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$$\dddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u$$

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$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \ddot{x}$$

(2) These are our system states. Write them in terms of x in the original equation.

(3) By variable substituting,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \dddot{x} = -5\ddot{x} - 3\dot{x} - 2x + u \longrightarrow \dot{x}_3 = -5x_3 - 3x_2 - 2x_1 + u$$

State Space Form (Example)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix}$$

- These are our new system equations in the first order form.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \text{-}5x_3 - 3x_2 - 2x_1 + u$$

- Let's convert them to the $\dot{x} = Ax + Bu$ format by finding A and B matrices. The LHS becomes,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

- The dimensions of A matrix become (3x3). Since we have a single input u, dimensions of B matrix become (3x1).

State Space Form (Example)

- Therefore, our system in the state space form becomes,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Examples

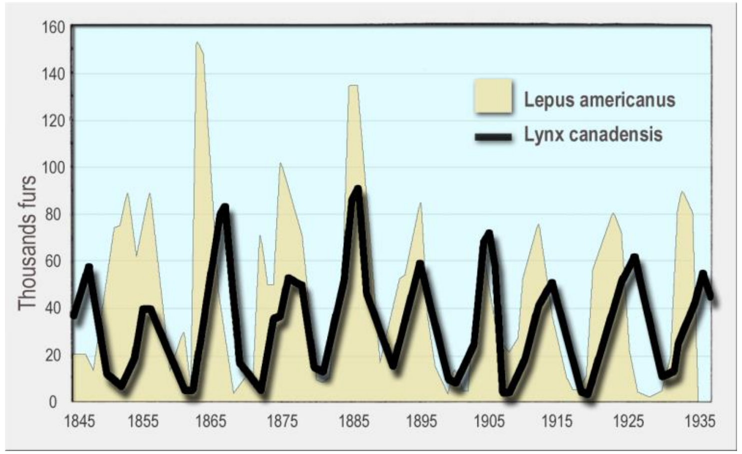
Lotka-Voltera population dynamics

x: Population of preys
y: Population of predators

$$\frac{dx}{dt} = \alpha x - \beta x y,$$
$$\frac{dy}{dt} = \delta x y - \gamma y,$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f\big(t, x(t), y(t)\big) = \begin{bmatrix} \alpha & -\beta x y \\ \delta x y & -\gamma y \end{bmatrix}$$

This is not a linear dynamical system.



Numbers of snowshoe hare (yellow, background) and Canada lynx (black line, foreground) furs sold to the Hudson's Bay Company.

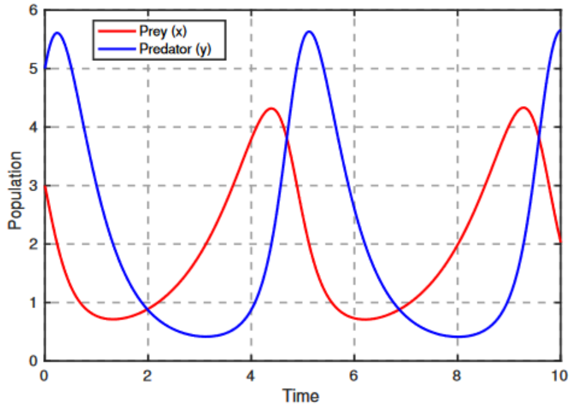


Figure 1: Population of preys (red) and the predators (blue) against time.

Examples

Omni directional robot assuming velocity inputs.

$$\left.\begin{aligned}\dot{x} &= v_x\\ \dot{y} &= v_y\\ \dot{\theta} &= v_\theta\end{aligned}\right\}\ \text{Differentially flat dynamics}$$

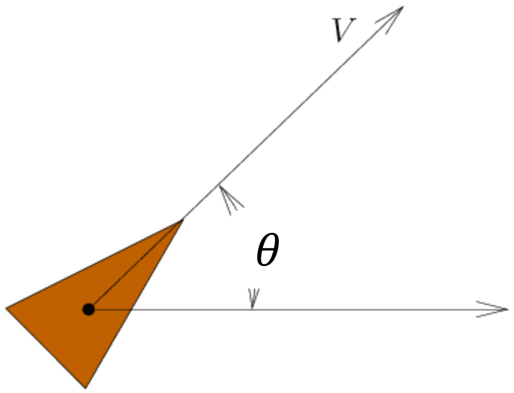
This is a linear dynamical system with $A = 0$.

Remark: This is a very high-level abstraction of the physical system. Not a good description if we want to analyze the system's stability.

Examples

Unicycle Model

Assume we can control the linear (v) and angular velocity (ω) of the system independently.



$$\left.\begin{aligned} \dot{x} &= v cos\theta \\ \dot{y} &= v sin\theta \\ \dot{\theta} &= \omega \end{aligned}\right\}$$ Differential driven dynamics

This is a non-linear dynamical system $A = 0, B = \begin{bmatrix} cos\theta & 0 \\ sin\theta & 0 \\ 0 & 1 \end{bmatrix} = B(x_t)$.

For controlling, we need to linearize non-linear dynamical systems around a **fixed point**.

Discrete Time Dynamical Systems

- Not all the dynamical systems evolve in time in continuous manner. i.e.: Bank interest.
- Computational cost for integration increases as the sampling time becomes finer.

$$\dot{x} = f(x(t))$$

$$x_{k+1} = F(x_k)$$

$$x_{k+1} = x_k + \int_{k(dt)}^{(k+1)dt} f\big(x(\tau)\big)d\tau$$

This is called the flow map. Any continuous time dynamical system can be written as a discrete time system using flow map.

Continuous Time → Discrete Time

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- Use inverse Laplace transform.

$$x[(k+1)T] = Gx[kT] + Hu[kT]$$

$$y[kT] = Cx[kT] + Du[kT]$$

Where, $G = e^{AT} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$ and, $H = \int_0^T e^{A\lambda} d\lambda\, B$

Here T is the sampling period, I is the identity matrix, λ is an integrating parameter. $\mathcal{L}^{-1}$ is the inverse Laplace transformation.