Causal Set Theory and Fluctuating Cosmological Constant*

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One of the most influential motivations to formulate discrete general relativity is to recover the continuum information from the discrete spacetime. In the mid-seventies, Stephen Hawking and David Malament developed more rigorous metric recovery theorems under the realm of general relativity. Using these ideas, numerous discrete approaches are introduced to quantize gravity – causal set program is one of those. This project report aims at the phenomenology of causal set theory in the dark energy sector. A variety of observations indicate that the universe is accelerating, and dark energy is thought to be the simplest candidate of cosmological constant (Λ), responsible for the positive acceleration, but the problems of fine-tuning and coincidence are associated with it. To solve these problems, a fluctuating and time-dependent cosmological constant of the right order of magnitude was predicted by R. D. Sorkin (1991) using the ideas from causal set theory and a more detailed phenomenological model was simulated numerically by M. Ahmed (2004) for the spatially flat universe. This study is the continuation of the model for a closed universe, and its sensitivity is analyzed under different orders of the radii of curvature.

I. WHY DO WE NEED "QUANTUM GRAVITY"?

A. Introduction

In the Proceedings of the Royal Society of London, in 1929, one of the giants of the twentieth-century scientific community, Paul Dirac narrated [1] at a moment: "The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known." However, the advancements in technology and innovative mathematical theories to study nature at its fundamentals made this statement quite vague, in essence, with the passage of time. People also stated the same when the discovery of the Higgs boson completed the mysterious puzzle of the standard model (SM) and made heuristic additions to our understanding of the universe, but then cosmology played its role to change the game and showed that the successful SM explains only around four percent of the universe while the nature and dynamics of the rest of the twenty-six percent dark matter and the seventy percent dark energy cannot be explained under its realm as shown in figure 1. These and some other compelling arguments again convinced the scientists to think of a more fundamental formalism to describe the underlying phenomena of nature, that is, a *theory of quantum gravity* – the merger of classical general relativity and quantum mechanics.

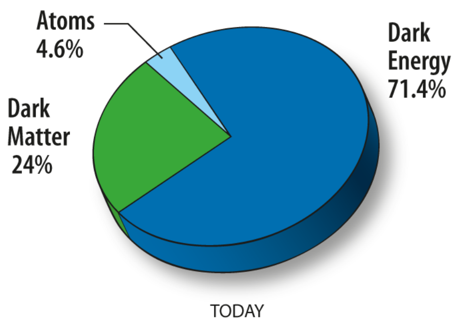

FIG. 1: Pie Chart of the content of the Universe (*Credits:* NASA/WMAP Science Team)

B. The Basic Idea

Since the formulation of Einstein's general relativity and quantum mechanics, both theories were (and are) successfully tested millions of times, which depicts how much important these theories are to modern day science. In the formalism and ideas, both are far apart. General relativity is classical in nature, while quantum mechanics upholds the idea of quantization and interference at the subatomic scale. Quantum (field) theory successfully accommodates all the interactions of nature in its framework but gravity. This framework is called the Standard Model of Particle Physics. However, general relativity is the only available tool to study all the manifestations of the gravitational field. Despite all the efforts, we are unable to wed general relativity to quantum theory. Anyhow some questions instantly pop out regarding their merger:

(i) Why do we want to unify these two theories?

(ii) Is not it possible that gravity is the classical theory and has no quantum roots?

(iii) Is it really challenging to quantize gravity?

Skip the first two questions for a moment and come to the very last question. The answer is no. It is not difficult to quantize gravity. We know how to canonically quantize fields. Canonical quantization, introduced by Dirac [2], is one of the most direct and simplest ways to quantize classical theories.

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People have already done it for non-gravitational fields such as electromagnetic, weak, and strong nuclear. It is exquisite and simple machinery.

(i) Write down the classical Lagrangian density in terms of fields.

(ii) Calculate momentum and Hamiltonian densities as fields.

(iii) Treat these field densities as operators.

(iv) Impose commutation relations to make them quantum mechanical.

(v) Expand the fields in terms of ladder operators.

And that's it. The classical field has been quantized successfully.

The same trick can also be applied to the gravitational field. So, then? Would not this be the desired theory of quantum gravity? If it is straightforward, why physicists are struggling? To answer this, let us visit the first two questions in the next section.

C. Motivations for Quantum Gravity

Before proceeding, it would be essential to note that certain 'uniqueness theorems' say whatever theory of quantum gravity we develop must approximate to Einstein's GR (or Einstein-Cartan theory) in some limit. It is important to note that Einstein-Cartan theory is the natural generalization of GR, written in the language of tensors and spinors to accommodate fermions and quarks in gravity.

General relativity is a remarkably successful theory in its realm. Still, there are some rudimentary observational pieces of evidence that point out discrepancies in theoretical models and suggest a more fundamental viewpoint about gravity. Some of the primary motivations for quantum gravity are:

- **Unification:** Physicists are reductionists. The history of science shows that reductionism has been proven very successful. Standard Model is one of its prime examples where all the non-gravitational interactions can only be studied by only one gauge group, $SU(3) \times SU(2) \times U(1)$. In the last century, we have seen the unification of electromagnetic and weak nuclear force (electroweak). Why would there be a reason gravity cannot be unified to others? It should be.
  Interestingly, there are also attempts to construct semi-classical frameworks in which all other forces are quantum while gravity remains essentially classical. However, all of these efforts have failed miserably. It indicates that classical and quantum concepts are likely incompatible, as inferred in the early days of quantum mechanics.

- **Universe:** All the attempts of gravity quantization face one severe problem – these are not UV complete. They break at high-energy scales. From the evolution history of universe, it is known that the present classical infrastructure of gravity tells us nothing about the initial conditions near the big bang or what happens at the sub–Planckian scale and final stages of black hole evolution. Let us discuss one example from cosmology and one from black holes in some detail.

  – **Cosmology:** There are cosmological solutions, by agreeing to the observations, which successfully predict an accelerated expansion of the universe. From this result, it can be extrapolated that this expansion was started, at some zero cosmic time, from a highly dense point (or singularity) with no spatial dimensions. It is where classical GR fails to yield any feasible result(s). This inability to extract any information from this singularity and the subsequent Planck era signifies the need for the theory of quantized gravity.

  – **Black Holes:** Annoyingly mysterious black holes also qualify to be a testable region for quantum gravity. In the early seventies, Bekenstein interpreted the entropy of a black hole in terms of the area of event horizon [3] and Hawking conjectured that black holes emit thermal radiations having a black body spectrum [4]. It brings about an identicalness in the thermal behavior of black holes and the conventional laws of thermodynamics. Classical GR fails to explain it well because entropy is described under the notion of discrete states of a quantum system in traditional thermodynamics. It makes black hole entropy phenomenologically important in search of quantized gravity.

- **Time:** It is one of the most radical issues in the pursuit of QG. Quantum and generally covariant theories (as GR) have drastically different concepts of time. They are incompatible. In quantum theory, time is available as an external parameter. It is not described by an operator. It is kinematical. It is absolute and universal. Even in relativistic quantum field theory, external Minkowski spacetime plays the role of absolute time. While in general relativity, time is a dynamical object. It is relative and non-absolute. It is not available as an external parameter. It needs to be constructed *naturally* by the theory itself. It is quite clear that the definition of time must need to be modified to develop a fully-fledged theory of quantum gravity.

- **Divergence:** Gravity is non-renormalizable. It breaks at high energy scales and yields infinities. It is still an open issue how to cure these divergences. There should be no divergences for any quantum version of gravity as there are none in quantum field theory.

D. Cut-Off Scales of Quantum Gravity

Length, mass, and time are fundamental, and the most basic physical quantities, and the speed of light (c), gravitational constant (G), and quantum of action ($\hbar$) are the fundamental constants of nature. Can we form these quantities from the constants of general relativity, c and G, or the constants of quantum theory, c and $\hbar$? No! There is no possible combination of these two sets of universal constants that can produce any of the fundamental quantities. But can we derive these quantities using all of these constants in any combination? Yes. We can. In honor of Max Planck, these units are called Planck units. (For the sake of completeness - Planck units are defined exclusively in terms of four universal constants. Fourth one is the Boltzmann constant, k_{B}. Using the four constants, one can also define Planck units of temperature, density, the fine structure constant of gravity and charge.) These units are:

$$\text{Planck length} = l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33}\,\mathrm{cm}, \tag{1}$$

$$\text{Planck time} = t_{\mathrm{P}} = \frac{l_{\mathrm{P}}}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44}\,\mathrm{s}, \tag{2}$$

$$\text{Planck mass} = m_{\mathrm{P}} = \frac{\hbar}{l_{\mathrm{P}} c} = \sqrt{\frac{\hbar c}{G}} \approx 2.18 \times 10^{-5}\,\mathrm{g}. \tag{3}$$

One may be bewildered by noticing that Planck mass is horrendously a larger quantity than Planck length and Planck time. But it is worth noting that m_{P} must be concentrated in a linear dimension l_{P}; otherwise, we will not be able to observe quantum effects of gravity. At this scale, classical properties become an emergent phenomenon of quantum theory. These scales provide the cut-offs to observe quantum gravity. We can easily realize that these scales are nearly impossible to detect and observe with the present technology. However, these scales do not provide stringent limits on gravity. These are just the most natural possible cut-offs. The unified theory of forces may also contain other parameters; for instance, in string theory, fundamental length scale is the string length, l_{S}, rather than Planck length, l_{P}. Nevertheless, it is an educated guess to set l_{S} equals to l_{P}.

E. Approaches Quantum Gravity

Since 1930s, there is no shortage of proposed theories for quantization of gravitational fields. No idea is complete; no theory is entirely consistent with quantum fields. No theory provides a way to put it to experimental tests; every approach suffers from conceptual problems. Every theory meets observational discrepancies. Issues are there, but even today, the charm of quantum gravity is not faded away. Nothing is more fascinating than the problems of gravity. It is challenging to review all the approaches of quantum gravity in just one section; however, we will briefly discuss its main research directions.

1. Three-and-Half Directions

Research in quantum gravity can be mapped onto three main and some *neglected* lines:

- covariant,
- canonical,
- sum-over-histories,
- others.

This list does not qualify to be a precise way to describe research directions since these names are somewhat misleading and sometimes are used interchangeably. Still, they possess specific scientific rationality, developmental logic, and methodological unity. Often, one approach becomes a hot topic, and others get neglected, and this is repeatedly happening over the years. However, all of these approaches share some common features, and many things cannot fit into any of these directions.

- **Covariant Approach:** Started in the early thirties by Rosenfeld, Fierz, and Pauli, it is one of the most dominant research directions in the pursuit of quantum gravity. It is an attempt to construct a quantum field theory of gravity. In covariant approaches, fluctuations of the metric element are studied over a four-dimensional Lorentzian flat spacetime or some other appropriate background metric. Rules and equations were established in the sixties by Feynman, Faddeev, and others, while the approach was proved to be non-renormalizable by 't Hooft and others in the seventies. The search for renormalizability gave birth to string theory in the late eighties, whose sun is not set to date.

- **Canonical Approach:** Initiated by Dirac and pioneered by Wheeler, DeWitt, and Ashtekar, it is the Hamiltonian (or canonical) formulation of general relativity. In this approach, phase space variables are promoted to quantum operators on usual Hilbert space, and these operators correspond to the full or some functions of the metric. The canonical approach started gaining attention when the framework of loop quantum gravity (LQG) was established. LQG is also one of the most significant competitors of string theory.

- **Sum-Over-Histories Approach:** Feynman pioneered path integral or sum-over-histories formulation, and it proved itself as one of the most successful formalisms of quantum theory. In quantum world, there are infinite ways (paths) to go from some initial point to final point, and the probabilities are given by adding contributions from all the possible trajectories (or histories). Similarly, in gravity, this sum is taken over all the possible geometries of spacetime. Some discrete causal approaches to quantum gravity are based on different versions of SOH, for instance, spinfoam formalism or causal set approach.

- **Other Approaches:** These include effective field theories (such as noncommutative geometry), Penrose's twistor theory, Regge calculus, causal dynamical triangulation, causal sets, and many more. Some of the alternate approaches have suffered from the serious and fundamental crisis. Some of these are too mathematical to describe physical reality. Some of these ideas also provide an astonishing insight into the phenomenology of quantum gravity and bridge the gap between theoretical and observational discrepancies.

It is also important to note that only string theory claims to be the unified version of all fundamental forces or, loosely speaking, the theory of everything, while all other approaches provide different ways to merge quantum theory with general relativity. None of these approaches have been delved into fully-fledged quantum versions of gravity. All ideas are still in development. Some of the proposed versions of gravitational theories have an established kinematical side, but they lack a dynamical framework. We have briefly reviewed the idea of quantum gravity, its need, present problems, and some of the possible directions. In the upcoming sections, we will discuss the problem of cosmological constant via causal set approach, an idea initiated by Rafael Sorkin in the early eighties.

II. PROBLEM OF DARK ENERGY

Story of the dark energy dates back to 1917 when Einstein included the cosmological constant (represented as Λ in this work) in his field equations to balance the gravitational effects of matter. Λ does this by generating repulsive gravity but in turn leads to an unstable static universe [5]. Einstein was of the view that without the cosmological constant, his theory of general relativity would not lead to an isotropic and homogeneous universe. In 1922, however, Friedmann derived a metric which allowed for an isotropic and homogeneous universe that did evolve in time and had three possible topological structures (open, flat and closed) [6]. Also in 1929, Edwin Hubble discovered that the universe is expanding [7], so Einstein abandoned this constant in 1931 and regarded it as "the greatest blunder" of his life [8]. Λ has refused to die, nonetheless, and has been resurfacing in cosmology ever since [9].

Very little is settled about the nature of the dark matter and the dark energy. Λ is a very natural and simplest explanation of the nature of the dark energy [10], but it is definitely not the only one. The details of this claim are unavoidably technical and we cannot elaborate them in any meaningful way without perusing the mathematical jargon of cosmology. Despite all the uncertainty about the nature of the dark energy, its existence is now beyond debate. A direct affirmation of its existence came in 1998 when two different teams of researchers were observing distant type–Ia supernova in a hope to observe some kind of cosmic deceleration but shockingly they found *cosmic acceleration* instead, that is, the universe is speeding up in its expansion rate [11–13]. This acceleration definitely requires an energy component which has negative pressure and hence confirms the existence of something other than matter. This component with negative pressure, permeates all of the space and interacts, most probably, only gravitationally with the known matter.

It should be noted that there are also pre–supernovae observation arguments such as the *cosmic age problem* [14] which does require a non–zero cosmological constant. It states that without the cosmological constant, the age of the universe is nearly 9 billion years but the astronomers of that time were already aware of cosmic objects which are older than this predicted age [14]. Other serious arguments about the existence of dark energy stem from the structure formation scenario in the ΛCDM model [15]. The analysis of angular power spectrum of spatial distribution of galaxies, and of the nearly uniform but fluctuating temperature of cosmic microwave background radiations (CMBR) and above mentioned supernovae observations indicate [16–18]:

1. the energy density of the universe is very close to the critical density (hence it is either flat or very nearly flat),

2. the matter amounts to about 30% of the total energy density,

3. the dark energy constitutes about 70% of the total energy density.

The constraints coming from all these data are summarized in figure 2.

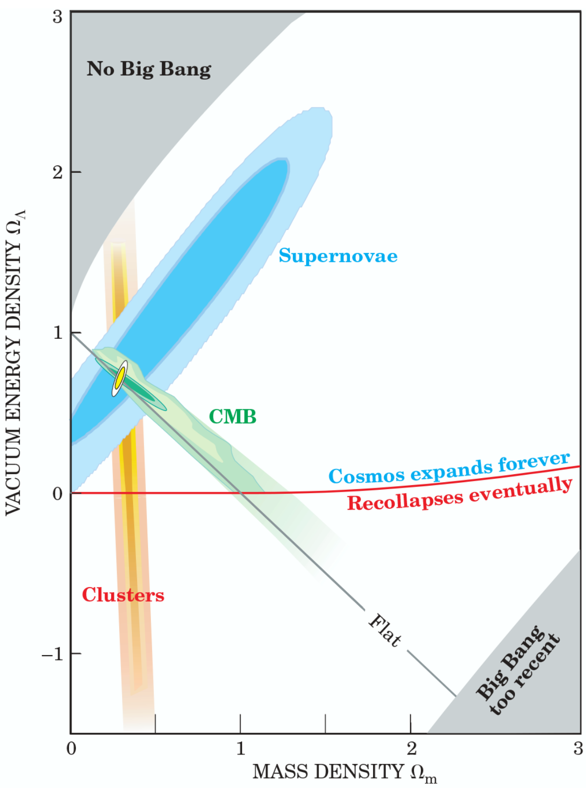


FIG. 2: Constraint on the density of Λ (Credits: S. Perlmutter, 2003)

A. The Cosmological Constant

In Einstein's theory of general relativity, the geometry of spacetime and the energy density of the universe are intertwined with one another. The energy density can be linked to two sources, the most obvious matter and radiation and the less obvious vacuum. Any change in the energy contents of the universe would lead to a change in the geometric properties of spacetime, which is thus very responsive and sensitive to the vacuum energy density or the cosmological constant. It makes vacuum a physical quantity and according to the quantum field theory, physical quantities usually have a tendency to fluctuate. In the Standard Model of particle physics, vacuum is the lowest energy state having the least but not necessarily zero energy density. Vacuum energy or cosmological constant is simply the energy present in the vacuum. A negative cosmological constant will slow down the expansion rate of the universe while a positive Λ will allow the universe to expand at an ever increasing rate. Despite of the effectiveness of the cosmological constant to model the dark energy, there are two serious problems associated with it, the naturalness problem and the fine–tuning puzzle.

The naturalness problem arises because of the imbalance between the densities of the dark energy and the visible baryonic matter. Figure 3 shows that at very early times, Ω_Λ was close to zero and the universe was dominated by the radiation but it diluted quickly as it scales as a^{-4}, so the matter–radiation crossover happened at a redshift of around 1100. This started the matter–dominated era of the universe which lasted for very long time since matter dilutes as a^{-3}. But at later times, the ratio of dark energy to matter became close to unity and it drove the universe into the matter–lambda transition era. Observations suggested that the density of the dark energy is still changing but very slowly and the universe is continually expanding. So, the question arises here that why the ratio of dark energy to matter is so close to unity? Are we privileged enough to witness an era of matter–lambda transition? This is what we call the *coincidence puzzle* [19]. In the upcoming sections, we will discuss a mechanism which drives the density of the dark energy in such a way so it becomes almost equal to the matter density today.

The underlying discrepancy in the predicted value of the cosmological constant from particle physics and the observed value from the cosmological observations of CMBR, supernovae and structure formation is the root cause of the naturalness problem. At Planck scale, the density of the vacuum fluctuations is nearly $10^{93}\,\mathrm{g\,cm^{-3}}$ and the observed value of the vacuum energy density (the critical density of the universe) is $10^{-29}\,\mathrm{g\,cm^{-3}}$. It can be seen that the observed energy density is smaller than its predicted value by nearly 120 orders of magnitude. This is by far regarded as the "worst prediction in the history of physics" [20]. This discrepancy in the magnitude is solely based on the cut–off that we are applying to our theory. At Planck scale, it is around 122 orders of magnitude but if we consider the available energies at Large Hadron Collider, this discrepancy will only be up to fifty orders of magnitude [21].

On the basis of the observations, the fine–tuning problem is classified into two types: the *old* and the *new* [19]. Old

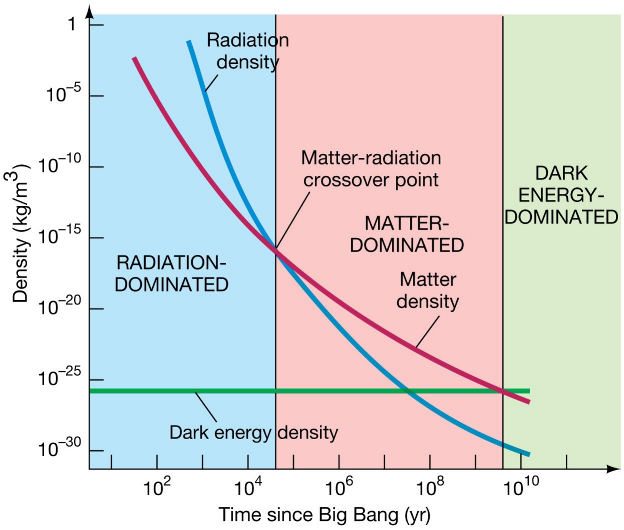


FIG. 3: Evolution of the contents of the universe with the scale factor (Credits: 2011 Pearson Education. Inc.)

cosmological constant puzzle argued that Λ should be zero for the static universe, that is, all of the vacuum fluctuations must be canceled out by some mechanism. But the recent advancements in the theory has changed this old puzzle into the new one, that is, Λ is not effectively zero but very small. There still has to be some mechanism to cancel out all the vacuum fluctuations but must leave a residual cosmological constant, which is needed for the accelerating universe [19, 22]. In the upcoming sections, we will see shortly that causal set theory also proposes a residual cosmological constant but a fluctuating constant rather than a 'true positive constant'.

To be conclusive, we can safely say that the problem of the cosmological constant (or the dark energy) directly affects the fate of the universe. It makes it one of the extremely important problems of the theory of quantum gravity which needs to be answered [23] and a complete theory of quantum gravity will not only be able to enhance our understanding about how things behave on the very fundamental level but also lead us to comprehend the nature of the very early universe.

III. INTRODUCTION TO CAUSAL SET THEORY

Causal set (causet) program was initiated in early eighties by Rafael Sorkin as a first and direct acknowledgement to the metric recovery theorems proposed by Hawking [24] and Malament [25] in the mid seventies. The idea of causet theory solely relies on two aesthetic concepts of physics: discreteness and causality. Informally, discreteness of spacetime means that space (or the three–dimensional line elements) is composed of a finite numbers of points where each point is assumed to be the smallest fundamental spatial point in nature and time is framed as the shortest possible time interval. Causality is the exquisite essence of the theory in such a way so as any event of nature can influence any other event if it is to the future of the previous since spacelike events remain uninfluential to each other – such events are also thought to be the part of different universes but it is quite early to say anything until the causet dynamics develop completely. This is the direct consequence of cause–and–effect relationship and important feature of causet theory since it prohibits time travel and does not allow closed timelike curves.

A. On the Axioms of Causal Set Theory

In causet theory, spacetime points are replaced by the fundamental events or spacetime atoms (also called 'satoms'). It is assumed that these satoms neither have inner structure nor any intrinsic identity. On their own, such satoms do not convey any sort of physical information. It is the causal relation among satoms, imposed by a binary relation $\prec$ which puts information in this discrete structure. Foundations of causet theory rely on a set of axioms that corresponds to two different, but roughly equivalent, formulations [26]:

(i) partial order formulation,

(ii) irreflexive (or strict partial order) formulation.

In partial order formulation, "a causal set is defined to be a locally finite, partially ordered set". Rigorously, it means that a causal set is defined as an ordered pair $(C, \prec)$ where C is the causal set consist of satoms with induced causal relations. More precisely, the axioms of partial order formulation can be stated as:

(i) $\prec$ imposes a partial order on causet C where some of its elements (not necessarily all) are related to each other via this binary relation of precedence. Such a poset satisfies the following conditions:

- $(C, \prec)$ is reflexive i.e., $\forall\, x \in C,\ x \prec x$,
- $(C, \prec)$ is transitive i.e., $\forall\, x, y, z \in C$, if $x \prec y$ and $y \prec z$, then $x \prec z$.,
- $(C, \prec)$ obeys antisymmetricity i.e., $\forall\, x, y \in C$ if $x \prec y$ and $y \prec x$, then $x = y$,

(ii) $(C, \prec)$ is locally finite i.e., $\forall\ x, z \in C$

$$\mathrm{card}\{y \in C \mid x \prec y \prec z\} < \infty, \tag{4}$$

(iii) causet C is countable.

Reflexivity is an 'axiom of convenience' which only shows how an element is related to itself. Transitivity and antisymmetricity imply the exclusion of closed causal curves in causal sets. It directly comes from the unitary direction of causal relation that an effect cannot precede its cause. Transitivity of partial order implies the Sorkin's perception of 'order' in causal sets. Finiteness and countability are implicit conditions to make causal set discrete or to make each neighborhood of a satom finite. It is from where the notion of 'number' comes in causal sets.

In irreflexive formulation, reflexivity of satoms in a causet is just replaced by irreflexivity i.e., $\forall\, x \in C,\ x \nprec x$. Here, transitivity with irreflexivity imply *acyclicity* which rules out the possibility of closed timelike curves i.e., the effect of a spacetime event contribute neither directly nor indirectly to its very own cause.

It is important to see that both formulations eliminate the likelihood of closed timelike curves (or informally, do not allow time machines). Without imposing the acyclicity, order among satoms won't make any sense. In such a case, satoms, under an ordered relation, would be indistinguishable and it might be possible that a bunch of satom represents just a single element (or satom). Informally, it can be argued that in such a situation, a number of 'effects' would be because of a single 'cause' or a whole lot of 'causes' would lead to a single 'effect'.

B. Representation of Causal Sets

Causal sets can be represented in multitude of ways but commonly directed acyclic graphs (DAG) or acyclic digraphs are used. Since we are adhering to the partial order approach, we will particularly use a special kind of DAG, the Hasse diagram. Hasse representation is a pictorial way to draw partially ordered sets. Formally, these are the representations of (self–contained) *classical universes* or (extended) *histories* which are modeled via causal sets. In Hasse diagrams, satoms are represented by dots or nodes and the causal relations among these satoms are drawn by lines or vertices. Since, the growth process of causal sets is thought to be positively sequential, analogous to Minkowski spacetime diagrams, so the arrows are not drawn on the lines connecting satoms.

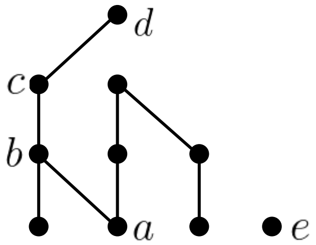
Figure 4a shows the Hasse diagram of a simplest two–element causal set $C = \{x, y\}$ under a binary relation of $x \prec y$ where cause (x) precedes effect (y). Such representations are quite analogous to spacetime diagrams of special relativity. Conventionally, in causet theory, time flows in the upward or forward direction, that is, representation for the elimination of the closed causal curves, so no two or more causets can have same Hasse diagram. Figure 4a represents that x is the *parent* element while y is the *child* or x is to the past of y or x is an ancestor of y so on and so forth, since there are number of genealogical ways to comprehend such relations.

Figure 4b shows one of the possible representations of a ten–element causet. Here, d is the only maximal element with a past subset of a, b and c. Element e is not causally linked to any other element and is spacelike to other three minimal elements including a. This representation only shows links which directly means that relations by transitivity are implicit and these are not shown by Hasse diagrams.

C. From Discrete Spacetime to Continuum Approximation

One of the most influential motivations in the pursuit of discrete theory of spacetime is to recover the continuum infromation from the discrete spacetime. This impelling hypothesis is known as the causal metric hypothesis (CMH) which states that "the properties of the physical universe are manifestations

(a) Two–element causet



(b) A possible representation of ten–element causet

FIG. 4: Hasse representation of different causets

of causal structure" [26]. A more rigorous and important connection, in the context of causal theories of spacetime, was developed in the mid seventies by Hawking [24] and Malament [25] in which they related the conformal structure of continuous manifold and causal anatomy of discrete spacetime under the realm of general relativity.

In [24], Hawking conjectured that topological structure of the manifold can be determined from its conformal structure and in turn, Malament [25] showed that such a topological structure can be ascertained from the causal structure of spacetime. On the basis of these recovery theorems, Malament argued that if two distinguishable spacetime manifolds (by 'spacetime manifold', it is meant to have connected and four–dimensional smooth manifolds with a smooth Lorentzian metric). $\mathcal{M}$ and $\mathcal{M}'$ have the same topological and causal structure, then their metrical structure $g_{\alpha\beta}$ and $\widetilde{g_{\alpha\beta}}$ would also be conformally equivalent such that

$$\widetilde{g_{\alpha\beta}} = \Gamma^2 g_{\alpha\beta}, \tag{5}$$

where Γ is the positively smooth conformal factor. So the causal structure can itself specify the metric element of spacetime manifold since the same causal structure is possessed by all of the conformally equivalent spacetime manifolds. In a more abstract way, it can be said that if the causal structure remains invariant under a mapping between spacetime manifolds then such a map will also leave the continuous geometrical structure intact but again only up to a conformal (or missing natural scale) factor.

Since the causal structure of satoms alone encodes all of the information for the approximation of metric element of continuum spacetime excluding a conformal factor which can be introduced directly by computing the volume element [27]. In Sorkin's version of discrete causal theory, condition of finiteness implies that this intrinsic spacetime volume can be directly linked with the number of spacetime elements such that,

$$\mathcal{V} = l_{\mathrm{P}}^4 \mathcal{N}. \tag{6}$$

For simplicity, it is assumed that unit volume corresponds to unit satom with no inner structure where l_{P}^4 is thought to be some sort of fundamental (Planck) length (in natural units, which we use most often, Planck length is taken to be unity.). This is aesthetically so simple that all macroscopic properties of continuum, such as length, volume, area, angle etc., simply reduce to the process of counting. One simply has to count, count and find the possible causal relations among discrete elements. All of this can simply be summed up into Sorkin's famous slogan:

$$\text{Geometry} = \text{Order} + \text{Number}. \tag{7}$$

D. Classical Sequential Growth

Classical Sequential Growth (CSG) is one of the widely accepted models for the growth of causet dynamics, however, there is no present consensus that how this model would behave when a fully–fledged quantum causet dynamics (QCD) will be developed. CSG is a stochastic process which assigns probabilities to each new transition. figure 5 shows the sequential growth or the CSG tree of a three–element causet. CSG depicts a birth process of satoms in which succession is preserved via the causal relation, so we choose any 'natural labeling' to preserve the growth process. Let us label these different colored satoms as *red* ($\mathsf{r} = 0$), *blue* ($\mathsf{b} = 1$), and green ($\mathsf{g} = 2$). Since the birth of these satoms behaves like a time–maker, so any different labelling would not alter the underlying physics. Let the first element ($\mathsf{r} = 0$) comes out of nothing with a unit probability. The element ($\mathsf{b} = 1$) born right after either via transition p and creates a chain or via q by creating an antichain. As we add another element, the number of transitions also increases such that there are three possibilities to form a three–element causet from a chain of two–element and four possible outcomes from a two–element antichain. Transitivity implies that if $0 \prec 1$ and $1 \prec 2$ then 0 will also precede 2 but this thing is pretty implicit in Hasse representation.

Technically, we can say that if we have two causets $N \in C_n$ and $M \in C_m$ where n and m are the cardinalities of the respective causets such that $n < m$. It is one of the possible outcomes that satom M may come out of satom N, irrespective that both belongs to the different causal sets, by the inclusion of some suitable number of satoms with causal relation among them. Such a partially order set is termed as *poscau*, a neology for "poset of causal sets". From the holy principle of general covariance, the probability to reach the terminal satom M will be the same along each path.

Growth of causets via CSG can be visualized in a more comprehensible way in terms of poscau $\mathcal{P}$. Figure 6 shows a partial execution of a poscau which is rendered for all causets of less than five elements. Since causets are represented by Hasse diagram, so a poscau is a "Hasse diagram of Hasse diagrams" in which each causet is understood to be a single 'element' of the poscau. As the CSG model represents a positively sequential growth, so any natural labeling would mean to initiate from an empty causet, with unity transition probability q_0, to a terminal five–element causet. Each causet, $\mathcal{C} \in \mathcal{P}$, shows a partial universe, so the poscau reaching the same causet via different paths would approach the identical transition probability, that is, each path will correspond to the same partial universe.

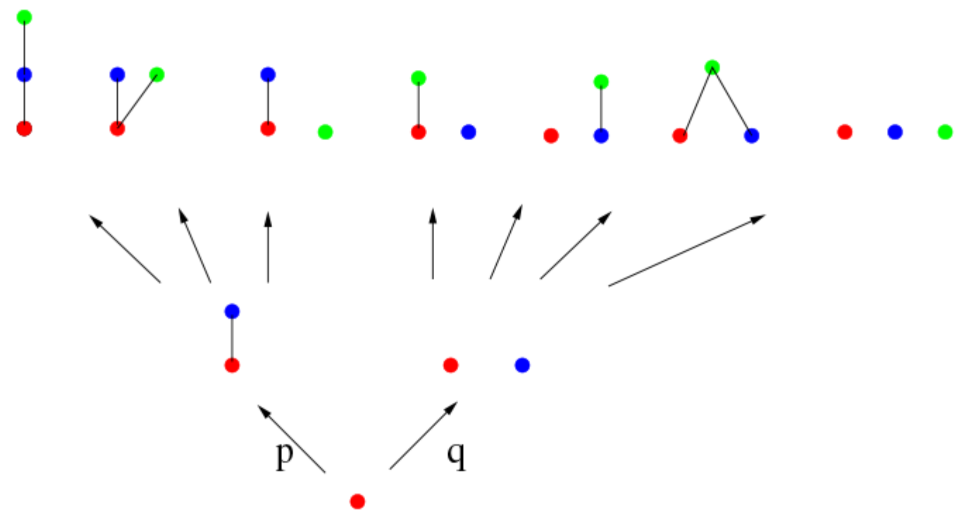


FIG. 5: Growth of three–element causet via CSG (Credits: Rideout et al. (2000))

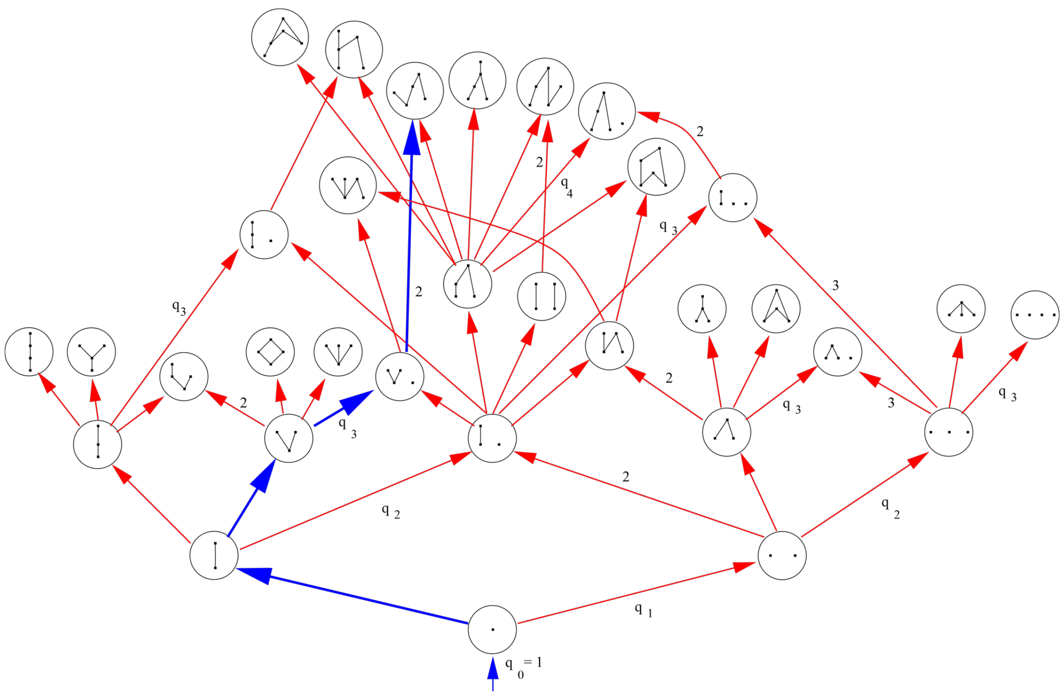


FIG. 6: A partial execution of poscau for all causets of less than five satoms (Credits: David P. Rideout [28])

The main theme of classical sequential growth model for the causet dynamics is to assign probabilities to each transition, but to get any kind of physical information from this growth, following set of certain conditions must have been obeyed [28].

Condition of Internal Temporality According to this condition, all newly born elements must be spacelike or future–like to the pre–existing elements. This condition is intrinsically obvious from the sequential growth of causal elements since the appearance of past-like events to already present elements would be contrary to the intrinsic ordering.

Condition of General Covariance This condition implies that the ordering of birth of satoms is important while labeling itself does not. Labeling does not carry any physical information. This condition also implies the net probability to be the same, that is, the product of all probabilities to reach causet C_{n+1} via different paths originating from a parent causet C_n would be the same in a given poscau.

Condition of Bell Causality It is a 'strict' extension to the condition of internal temporality that an element in a causet C would exclusively be influenced only by those elements which lie to its past while rest will remain uninfluential to the given causet.

Markov Chain Rule Rideout–Sorkin model of causet growth ensures this dynamics to be a Markov process. According to this process, Markov sum rule states that the sum of all transition probabilities from a given parent causet to all of its possible 'distinguishable' children must be unity.

IV. FLUCTUATING LAMBDA IN CAUSAL SET THEORY

A. Integrants

Two out of the four tenets of the fluctuating lambda [29] has already been discussed in the previous chapter. First argument comes directly from the discreteness of spacetime, that is, a finite spacetime volume will accommodate finite number of satoms while the second desideratum is the relationship of spacetime volume and number of satoms. This relation is assumed to be linear at the Planck level, that is, $\mathcal{V} = l_{\mathrm{P}}^4\mathcal{N}$. This linearity between $\mathcal{V}$ and $\mathcal{N}$ is the simplest and most natural but not the only one. On large scales, there might be other relations as well with quite different behavior. Both of these requirements are the generic and genuine features of any discrete causal theory and are not special to the causal set program.

Third input to the fluctuating lambda is the conjugacy of spacetime volume and cosmological constant. At the fundamental level, the spacetime structure can be written via sum–over–causet approach which can be well approximated to gravitational path integral at the macroscopic scale under the sum over only those classes of four–geometries which obey the constraint of fixed spacetime volume. Such a modification of general relativity is regarded as the "unimodular gravity". In such a theory, Λ and $\mathcal{V}$ always come as the product, $\Lambda\mathcal{V}$, in the gravitational action:

$$-\Lambda \int \sqrt{-g}\,\mathrm{d}^4x = -\Lambda\mathcal{V}. \tag{8}$$

This conjugacy is quite like the conjugacy of energy and time in the non–relativistic quantum mechanics, such that, in natural units,

$$\Delta\Lambda\,\Delta\mathcal{V} \sim 1. \tag{9}$$

The fourth integrant of fluctuating lambda, that is Lorentz invariance, is quite peculiar and most important to the causal set approach. Let's visit this integral integrant of local Lorentz invariance in detail in the upcoming section.

B. Lorentz Invariance and "Sprinkled" Causal Sets

Causal sets are proposed to be the discrete analogue of classical spacetime but still there is no known dynamical mechanism to generate actually "manifold–like" directed sets in general. The intricacies in embedding causal sets to a manifold can be overcome by getting over this problem the other way round. Causal set theory proposes the idea of "sprinkling" to begin with. Assume a $(3+1)$–dimensional Minkowski spacetime and sprinkle points on some part of spacetime region randomly. These randomly sprinkled points can be finite as well as infinite but for brevity, we assume to sprinkle satoms in proportion to the volume of the region. Causal relation will then be induced on these satoms via causal order. To maintain Lorentz invariance, this sprinkling of points must be done randomly using a Poisson process, where the probability of finding k satoms in region $\mathcal{V}$ is:

$$P(k) = \frac{(\rho\mathcal{V})^k e^{-\rho\mathcal{V}}}{k!}. \tag{10}$$

Here, ρ is the density on the Planckian scale and $\rho\mathcal{V}$ gives the number of average elements sprinkled in the spacetime volume. Although sprinkling does not carry any physical information in itself, yet it helps to assign causal sets, a continuum approximation, that is, a faithfully causet, C, can be originated from a Lorentzian spacetime manifold, $(\mathcal{M}, g)$.

Poisson sprinkling successfully overrules the violation of Lorentz invariance in discrete causal theories since no preferred frame is picked. Assume that we sprinkle points randomly onto a spacetime manifold, $\mathcal{M}$, and apply a Lorentz boost to it in a particular direction. Since, the points were sprinkled randomly with unit density, so they do not pick a preferred frame. After applying Lorentz transformation, old manifold, $\mathcal{M}$, will be transformed to a new manifold, $\mathcal{M}'$, with randomly sprinkled points of unit probability. It helps to preserve local behavior of the manifold, that is, what happens around an individual satom. Graphically, this can be illustrated via the following diagram, taken from [30].

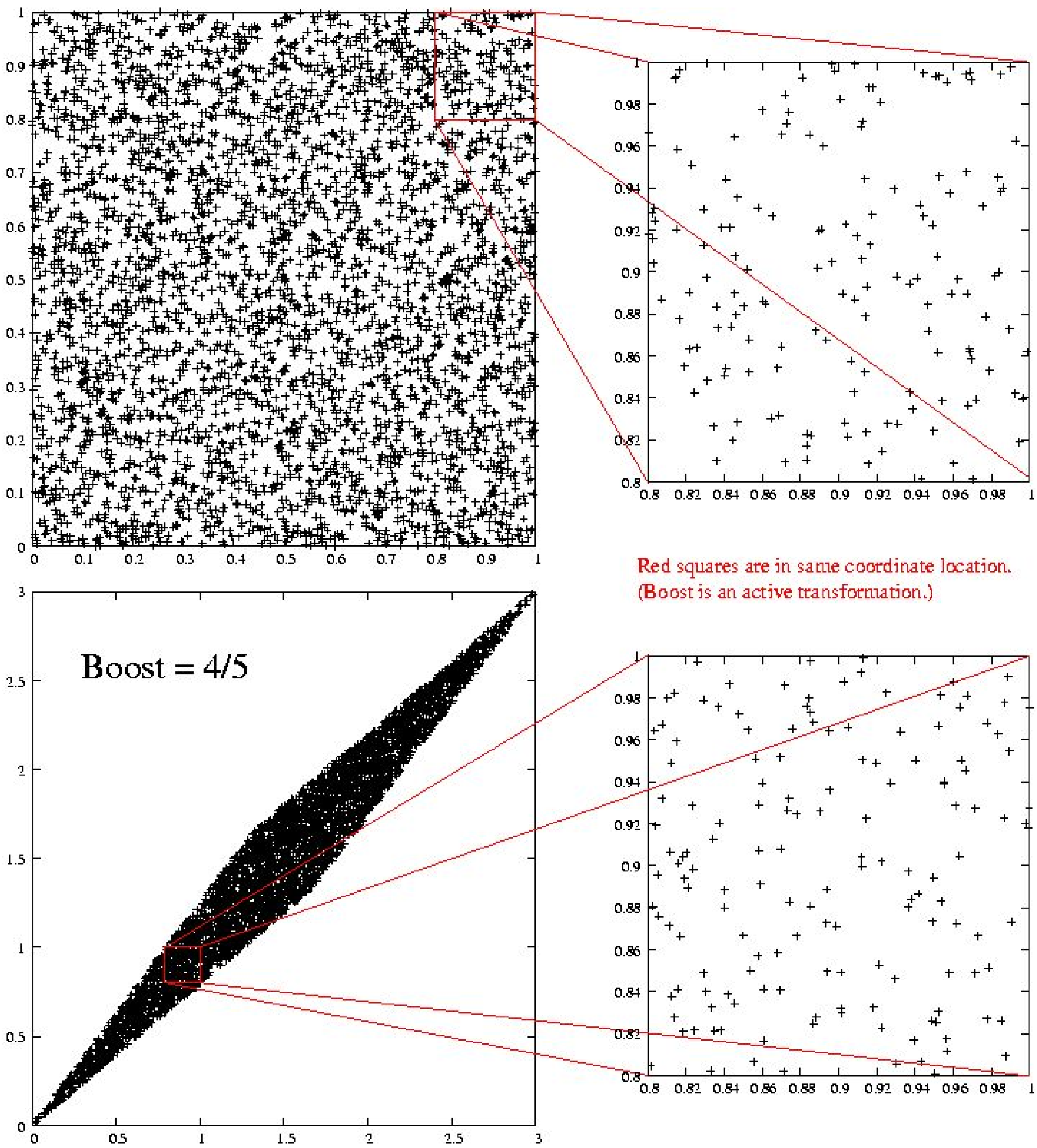


FIG. 7: Graphical Visualization of Poisson sprinkling to preserve Lorentz invariance in causal set theory

In figure 7, a region of (1 + 1)–dimensional Minkowski spacetime, as seen by an observer with velocity $v = 0$, is endowed with 4096 sprinkled spacetime atoms (or points). To the upper right corner, a blown up part of the original region is shown. The lower left image shows the original region, as observed by an observer moving with velocity $v = 0.8$, so the region will be contracted in the direction of the boost and the lower right image shows the same part of the region (in the observer $v = 0$ coordinates). Blown up regions in both cases show that the locations of all points are not exactly the same, but still both shows the random Poisson distributions of the same density.

This depiction of Lorentz invariance in causet theory constrains that the linear relationship of satoms ($\mathcal{N}$) and spacetime volume ($\mathcal{V}$) is not exact, but subject to some kind of Poisson fluctuations.

V. FLUCTUATING LAMBDA TERM

The route to fluctuating lambda has been paved as the four inputs, mentioned in the section (IV A) indicate. First input argues that the continuum spacetime is an approximation to a more fundamental discrete structure in which events are related to each other via an imposed causal ordering while the second argument defines a relationship between the elements of discrete spacetime and four–volume of the continuum. This relation simply tells the number of satoms of which the spacetime region is composed. For the sake of simplicity, this relation is assumed to be linear, that is, number of satoms ($\mathcal{N}$) is the direct reflection of spacetime volume ($\mathcal{V}$).

The third input, which comes from unimodular gravity, shows that its gravitational path integral makes lambda and spacetime volume conjugate variable and hence introduces an uncertainty relation between them, such that,

$$\Delta\Lambda \sim \frac{1}{\Delta\mathcal{V}}. \tag{11}$$

Since, causet theory links the spacetime volume with the number of satoms, so if $\mathcal{N}$ is fixed, then $\mathcal{V}$ can only be fixed up to "some" level and will subject to Poisson fluctuations of the order of $\pm\sqrt{\mathcal{V}}$, that is,

$$\Delta\Lambda \sim \pm\frac{1}{\sqrt{\mathcal{V}}}. \tag{12}$$

It is suggested that these fluctuations are around a mean value of zero, though the exact reasons to this are still to be comprehended. Fluctuations in $\mathcal{V}$, correspondingly, will produce fluctuations in the value of lambda,

$$\Delta\Lambda \sim \frac{1}{0 \pm \sqrt{\mathcal{V}}}. \tag{13}$$

And by using the first two inputs to the argument of fluctuating lambda, Eq. (13) can be translated to,

$$\Delta\Lambda \sim \frac{1}{0 \pm \sqrt{\mathcal{N}}} \sim \pm\frac{1}{\sqrt{\mathcal{N}}}. \tag{14}$$

Eq. (12) readily answers the question that why the value of lambda is very close to zero but not exactly zero? That is, as the volume of spacetime gets larger and larger, fluctuations in lambda becomes smaller and smaller but these fluctuations will be around zero mean value, $\langle\Lambda\rangle = 0$. Since, the volume of spacetime roughly equals to the fourth power of Hubble radius, so the density of lambda becomes

$$\Lambda = \rho_\Lambda \sim H^2, \tag{15}$$

$$\rho_\Lambda \sim \rho_\mathrm{c}, \tag{16}$$

that is, the density of cosmological term (or dark energy) is equal to the square of Hubble's constant or of the order of critical density. This prediction is well consistent with the observational data and depicts that density of lambda is in the right order of magnitude. Also, we expect lambda to be of the order of ρ_c, not only today but at all times too, which makes lambda, the *everpresent lambda* [31].

A. Cosmology with Fluctuating Lambda

In section (V), we studied the idea of fluctuating cosmological term, proposed by the causal set theory, where Λ fluctuates around a mean value (assumed to be zero) and the magnitude of these fluctuations can be found from the conjugacy of lambda and four–volume. Now the problem arises that how to devise a process to incorporate this idea into the numerically implementable cosmological solutions of Einstein field equations, which are,

$$\mathbf{G} + \Lambda\mathbf{g} = \mathbf{T}. \tag{17}$$

The divergence of Eq. (17) yields Λ a constant function, since Einstein tensor is divergenceless and $\mathbf{T}$ also disappears as matter is, separately, conserved. For a spatially homogeneous and isotropic universe, these equations reduce to the following ordinary differential equations:

$$H^2 = \frac{\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{18}$$

and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \Lambda = -p, \tag{19}$$

where the *dot* shows that these equations are written in comoving coordinates, H is the Hubble constant, ρ is the density of all energy components, excluding lambda, with pressure p and curvature constant k. Eq. (18) and (19) are referred as HCEq and AccEq respectively. While doing cosmology with non–fluctuating lambda, AccEq adds no information to the HCEq except when acceleration becomes zero but for the case of fluctuating Λ, both equations independently or their any possible linear combination can be used although there are some issues of stability of these equations which imposes a question that which equation should be favored over the other.

In the present work, we will go with the HCEq since it is first order in time and numerically easy to solve. But one of the serious problems associated with HCEq is that it becomes invalidated when its right hand side becomes negative. Although it is not an issue in standard cosmology where lambda is a constant parameter and retains right hand side positive always. But in causal set cosmology, due to random fluctuations, at times, it becomes negative enough to make the total energy density negative and thus invalidates the equation.

B. Model with Fluctuating Lambda

To model the fluctuating lambda, two things are required to do viable cosmology, one is the dynamical equation and the other is the measure of the four–volume. Out of the possible thicket of equations, we preferred HCEq over the other choices because of its simplicity to model. Since, the Eq. (14) shows that fluctuations in lambda decrease as the number of satoms or the spacetime volume gets larger and larger. It makes lambda a function of satom but in causet theory, according to CSG, satoms play the role of time parameter. So to obtain a dynamical equation, we simply put lambda as a function of time or satoms, by hand, such that HCEq becomes:

$$H^2 = \frac{\rho}{3} + \frac{\Lambda(t)}{3} - \frac{k}{a^2}. \tag{20}$$

The other important parameter, to govern the magnitude of these fluctuations, is the measure of the volume. Volume of the universe can be measured by computing the volume of past lightcone, at two different epochs as shown in figure 8.

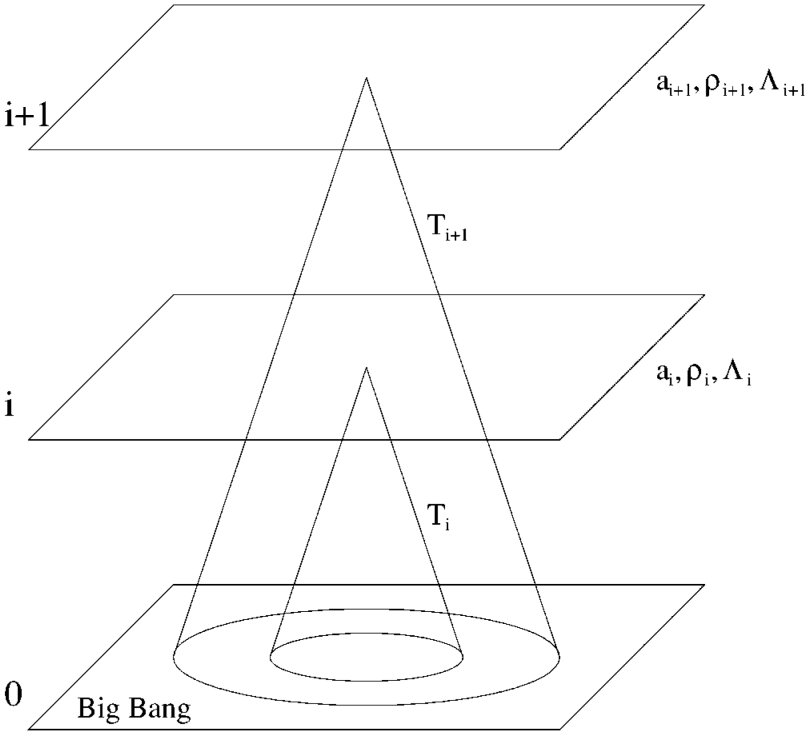


FIG. 8: Schematic representation of past lightcone at two different epoch [31]

In the gravitational action of unimodular gravity, from the conjugacy of lambda and spacetime volume, Λ can be interpreted as,

$$\Lambda = \frac{\mathcal{S}}{\mathcal{V}}, \tag{21}$$

that is, lambda as the action per unit four–volume of spacetime. But in causal set theory, $\mathcal{V}$ is just the number of satoms, so the Eq. (21) becomes:

$$\Lambda = \frac{\mathcal{S}}{\mathcal{N}}, \tag{22}$$

or simply the action per satom. Since, in CSG, the growth of satoms is interpreted as a time parameter, so at each epoch when a satom born, it contributes 'a unit' to the action of spacetime. At any point, on the hypersurface of homogeneity, all contributions from the satoms to the past of that point can be summed up to get the total contribution to the action. But a problem arises here is that as the number of satoms increases, contribution to action becomes huge and in turn $\mathcal{S}/\mathcal{N}$ or Λ remains small or quite near to zero. It requires a cancellation mechanism and uniformly random contribution of Λ. It can be done by generating a string of Gaussian number, ξ, with mean 0 and standard deviation 1. When it is multiplied by $\sqrt{\delta\mathcal{N}_n}$, we get a number with mean 0 and standard deviation $\sqrt{\delta\mathcal{N}_n}$. Now, we can calculate the change for the new Λ using the formula of action. New $\rho_{\Lambda,n+1}$ is the new action action divided by the new number of elements which is equal to the old action plus the change in the action,

$$\Lambda_{n+1} = \frac{\mathcal{S}_{n+1}}{\mathcal{N}_{n+1}} = \frac{\mathcal{S} + \alpha\xi_{n+1}\sqrt{\delta\mathcal{N}_n}}{\mathcal{N}_n + \delta\mathcal{N}_n}, \tag{23}$$

where α is the free parameter of the theory which governs the amplitude of these fluctuations. The change in action is calculated by central limit theorem which says that for a large enough number of elements (as 10^{240} in our case), their average contribution will be the same as the contribution by a Gaussian distribution. And in turn, it yields the present lambda, which is of the right order of magnitude.

The prediction of fluctuation lambda is the fundamental cornerstone in the causal set phenomenology it was was first successfully proposed by Ahmed et al. in [31]. In the next section, we will review the results of this "everpresent lambda" for a flat universe.

VI. FLUCTUATING LAMBDA IN FLAT UNIVERSE

The metric element for flat universe in conformal coordinates can be written as,

$$\mathrm{d}s^2 = a(\eta)^2 \left(-\,\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{24}$$

and HCEq becomes

$$\mathrm{d}a' = a^2\sqrt{\frac{\rho_{\text{matter}} + \rho_{\text{radiation}} + \Lambda}{3}}, \tag{25}$$

$$\Delta a = a^2\sqrt{\frac{\rho_{\text{matter}} + \rho_{\text{radiation}} + \Lambda}{3}}\Delta\eta, \tag{26}$$

where $\mathrm{d}a'$ denotes the differentiation with respect to η in conformal coordinates. Spacetime volume of such a flat universe in conformal coordinates can be computed via past lightcone as

$$\mathcal{V}(\eta) = \frac{4\pi}{3}\int_0^{\eta} \mathrm{d}\eta'\, a(\eta')^4(\eta-\eta')^3. \tag{27}$$

For convenience, it is preferred to find the change in volume, $\Delta\mathcal{V}$, corresponding to change in conformal time, $\Delta\eta$, and add it to the previous volume rather than to re–evaluate the integral at each step. Such a change in volume is given as

$$\Delta\mathcal{V} = 4\pi\left(\eta^2 I_2 - 2\eta I_1 + I_0\right)\Delta\eta, \tag{28}$$

with,

$$I_2 = \int_0^{\eta} \mathrm{d}\eta'\, a(\eta')^4 \quad \Longrightarrow \quad \Delta I_2 = \Delta\eta a(\eta)^4, \tag{29}$$

$$I_1 = \int_0^{\eta} \mathrm{d}\eta'\, a(\eta')^4\eta' \quad \Longrightarrow \quad \Delta I_1 = \Delta\eta a(\eta)^4\eta, \tag{30}$$

$$I_0 = \int_0^{\eta} \mathrm{d}\eta'\, a(\eta')^4\eta'^2 \quad \Longrightarrow \quad \Delta I_0 = \Delta\eta a(\eta)^4\eta^2. \tag{31}$$

Using these equations, the contribution of satoms to lambda can be evaluated numerically by using Eq. (23).

Figure 9 shows the evolution of energy densities of different contents of the universe for $\alpha = 0.01$. The thick line, with engrossed fluctuations, represents the absolute ρ_Λ. Early on, it scales as a^{-4}, that is, radiation and later on, it traces matter, as a^{-3}, which makes it always of the order of dominant energy component. Here, the negative values on the x–axis show the point of expansion as compared to the present size, for example, '-8' represents a cosmic scale when universe was nearly 0.1 billionth of its present size.

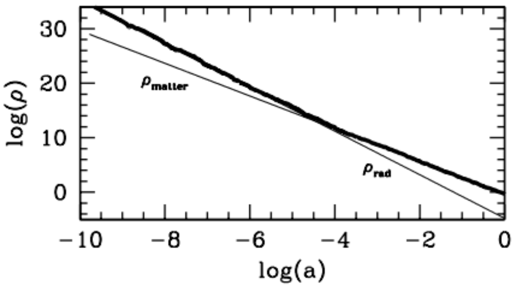


FIG. 9: Evolution of energy densities of different contents for $\alpha = 0.01$ (Credits: Ahmed et al. (2004))

Figure 10 shows another run for $\alpha = 0.02$ with the present scale factor of the order of 31. Here, the dashed wiggly line shows that absolute ρ_Λ tracks the absolute total energy density and the bottom figure shows a magnified part of radiation–matter crossover near the scale factor of nearly 28 on the logarithmic scale.

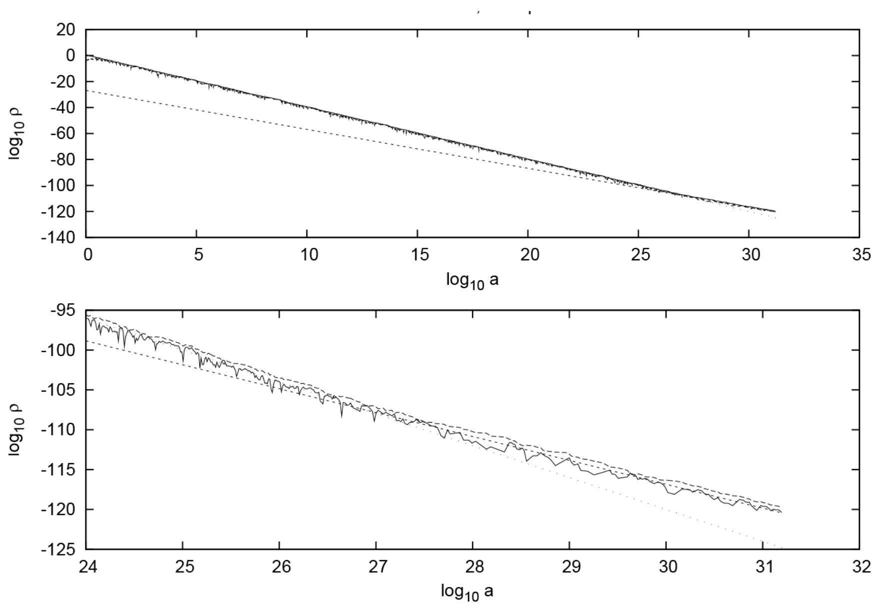


FIG. 10: Evolution of energy densities of different contents for $\alpha = 0.02$ (Credits: Maqbool Ahmed (2006))

Figure 11 represents, Ω_Λ, the ratio of absolute energy density of cosmological constant to the total energy density of the contents of the universe as a function of scale factor. It shows that Λ fluctuates around a mean value of zero with an amplitude of nearly 0.5, that is, Λ accounts for the fifty percent of the present energy density and even it changes very little, about 0.4, at a redshift of one ($a = 0.5$). This shows that density of Λ is consistent with the observational data. Here, the present scale factor is normalized to unity.

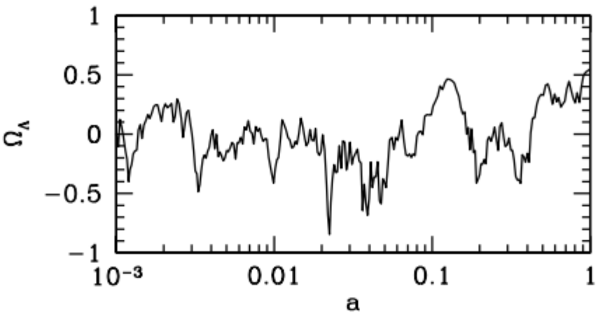


FIG. 11: The ratio of $|\rho_\Lambda|$ to the $\rho_{\rm total}$ as a function of scale factor, a, for $\alpha = 0.02$ (Credits: Ahmed et al. (2004))

It is obvious from the above results that universe cannot have such an excess energy at a redshift of 1×10^9 ($a \sim 1 \times 10^{-9}$), otherwise the predictions of BBN will be on stake since during BBN lighter nuclei, such as helium, formed in the universe and with this huge amount of energy, lighter elements would have been decayed instantaneously and universe could not reach to its present size. Also, to be consistent with the observational constraints, ρ_Λ cannot exactly scale the matter density today. But the proposed model is consistent with the BBN predictions, since the figure 11 shows that lambda fluctuates uniformly around zero and for half of the time, it remains negative and thus reduces the total energy density of the universe.

Histograms of final values of Ω_Λ show the probability of the particular event. Two such realizations have been shown in figures 12 and 13 which depict that final values of Ω_Λ are nicely distributed around the mean value of zero. From the theory, lambda should spend nearly equal time with the positive and negative values but in both realizations, it can be seen that Λ is slightly outweighed towards positive values which shows that a universe with positive lambda is more probable to reach the present size rather than negative ones. However, the model is stochastic in nature, so it might be possible that some other realization would produce quite different results but on the average, it favors more positive values of lambda than the negative, particularly for smaller α.

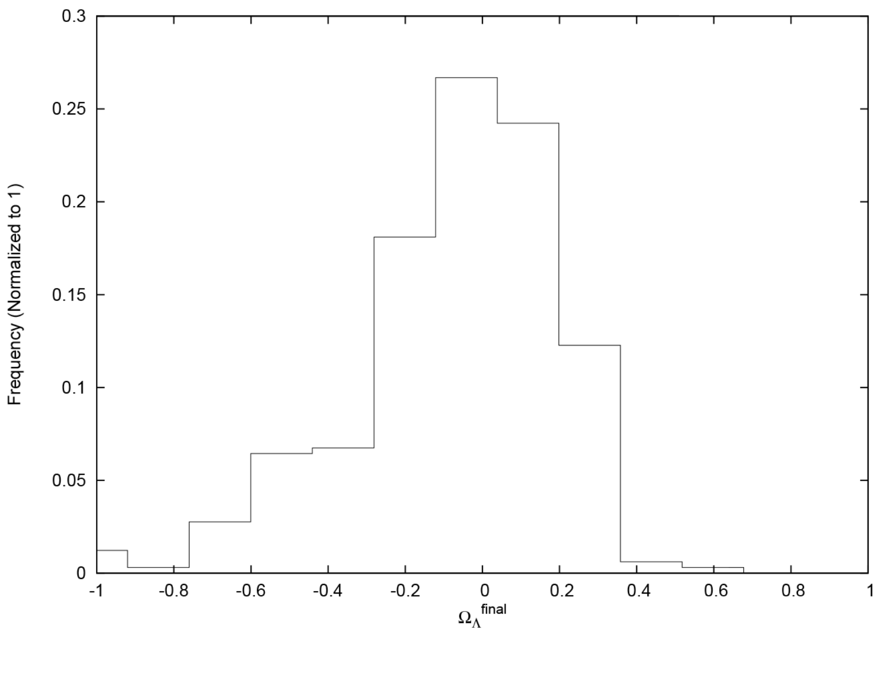


FIG. 12: Histogram of final values of Ω_Λ, for $\alpha = 0.01$, that reaches the present day universe (Credits: Ahmed et al. (2004))

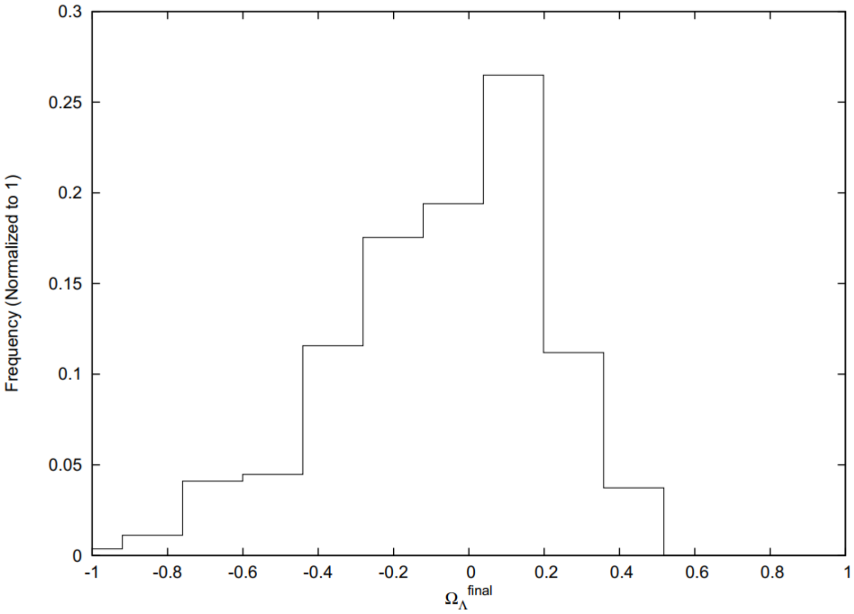


FIG. 13: Histogram of final values of Ω_Λ, for $\alpha = 0.02$, that reaches the present day universe (Credits: Ahmed et al. (2004))

For flat universe [29, 31], free parameter α lies in the range of 0.01 to 0.02. For $\alpha = 0.02$, only one out of seven runs hit the present universe while this ratio increases more for 0.01. But if we keep α too low, then it will be very improbable that Λ would be large enough to respect the observational constraints on its density. And if we keep on increasing α, then it is very unlikely for the simulation to reach the size of present universe since a larger α will in turn make ρ_{total} negative and thus invalidate the HCEq, that is, simulation will stop every time when $\rho_{\text{total}} < 0$. That is why, it is very uncertain for α to be equal to its natural value of unity since no run in that case would end up with positive total energy density.

VII. FLUCTUATING LAMBDA IN CLOSED UNIVERSE

A. Simulations

The homogeneous and isotropic FRLW metric of a closed universe ($k = +1$) in conformal coordinates is given by,

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-\left(\frac{r}{R_{\text{curv}}}\right)^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right], \tag{32}$$

where $a(\eta)$ is the conformal scale factor and η is the conformal time. R_{curv} is the radius of curvature where for spherical (closed) universe, it simply corresponds to the radius of the sphere. For simplicity, from now on, we will take $R_{\text{curv}} = R$. In natural units, HCEq for matter, radiation, lambda and curvature can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2R^2}, \tag{33}$$

where $\rho = \rho_M + \rho_R + \Lambda$. For spatially closed universe, Eq. (34) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{1}{a^2R^2}, \tag{34}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = a\sqrt{\frac{\rho}{3} - \frac{1}{a^2R^2}}, \tag{35}$$

or simply the change in scale factor, in conformal coordinates, would then be

$$\Delta a = a^2\sqrt{\frac{\rho}{3} - \frac{1}{a^2R^2}}\Delta\eta. \tag{36}$$

In our simulations, we fixed the step size to 0.01, for each next iteration. Although $\frac{\Delta a}{a}$ of 0.01 is very large step size to study the behavior of fluctuating lambda at the fundamental scale. It should be around 0.001. But due to some hardware issues, we are restricted to keep it 0.01.

In conformal coordinates, the volume of closed universe, using the observationally available past lightcone, becomes

$$\mathcal{V}(\eta) = 2\pi\int_0^\eta a^4(\eta')\left[R^2(\eta-\eta') - R^3\sin\frac{2(\eta-\eta')}{R}\right]\mathrm{d}\eta'. \tag{37}$$

For a given change in $\Delta\eta$, we can compute the change in spacetime volume at a given scale factor using

$$\Delta\mathcal{V} = 2\pi R^2 \left(I_2 - I_1 - I_0\right) \Delta\eta,$$

where the integrals and change in these integrals will be

$$I_2 = \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' \,, \tag{38}$$

$$I_1 = \int_0^\eta a^4(\eta') \cos\frac{2\eta}{R} \cos\frac{2\eta'}{R}\,\mathrm{d}\eta' \,, \tag{39}$$

$$I_0 = \int_0^\eta a^4(\eta') \sin\frac{2\eta}{R} \sin\frac{2\eta'}{R}\,\mathrm{d}\eta' \,, \tag{40}$$

and

$$\Delta I_2 = a^4(\eta)\Delta\eta, \tag{41}$$

$$\Delta I_1 = a^4(\eta)\cos^2\frac{2\eta}{R}\Delta\eta, \tag{42}$$

$$\Delta I_0 = a^4(\eta)\sin^2\frac{2\eta}{R}\Delta\eta. \tag{43}$$

This technique will help to get rid of computing the complete volume (or re–evaluate the integral) at each step such that new volume can simply be computed by adding $\Delta\mathcal{V}$ to the previous volume. From the relation relation of linearity between number of satoms ans spacetime volume, we have

$$\delta\mathcal{N}_n = \mathcal{N}_{n+1} - \mathcal{N}_n,$$
$$\delta\mathcal{N}_n = \frac{\mathcal{V}_{n+1} - \mathcal{V}_n}{l_\mathrm{P}^4}.$$

Now by using the above equations of scale factor and spacetime volume, density of lambda at each step can simply be found out with the aid of Eq. (23).

VIII. RESULTS AND CONCLUSION

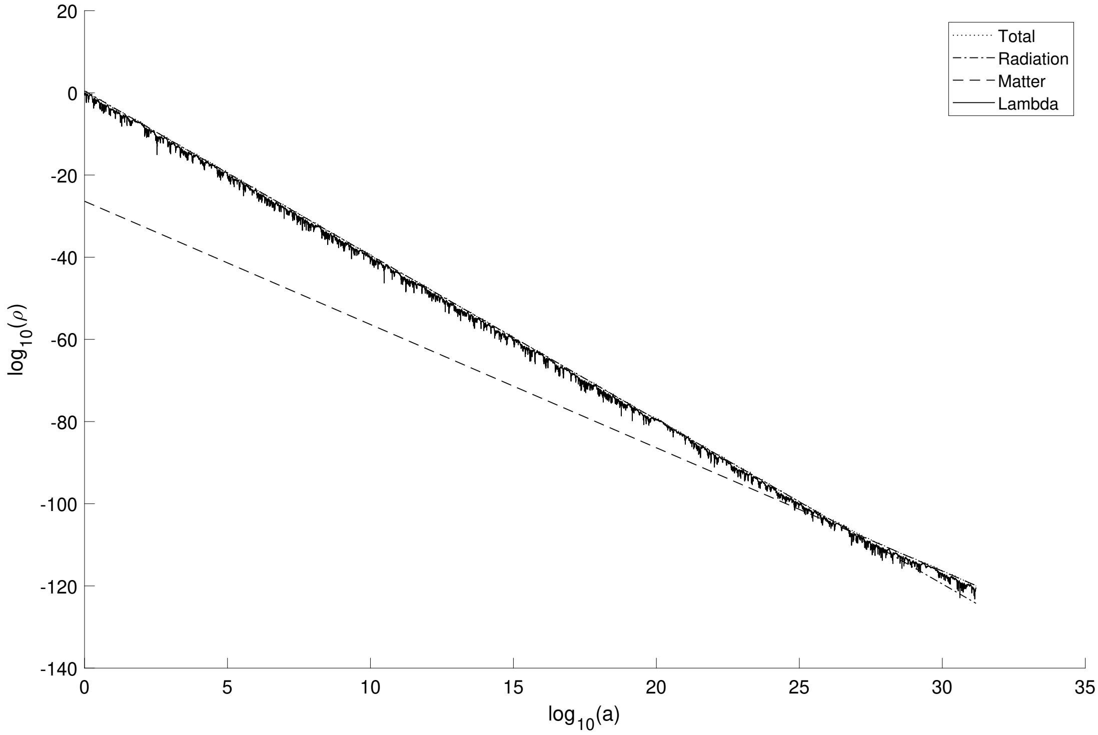
Fine–tuning puzzle seems to be solved as the model successfully removes the discrepancy of about 120 orders of magnitude in the closed universe, for α from 0.01 to 0.02. It is quite close to unity and it is just a “mild” fine–tuning.

Figure (14) shows the tracking behavior of the total energy density from the big bang to the present universe. Near the radiation–matter crossover, it nicely transits from the radiation domination (a^{-4} scaling) to matter dominion (a^{-3} scaling). Also, the effective energy density of lambda tracks the total energy density and switches its scale from a^{-4} to a^{-3} as well.

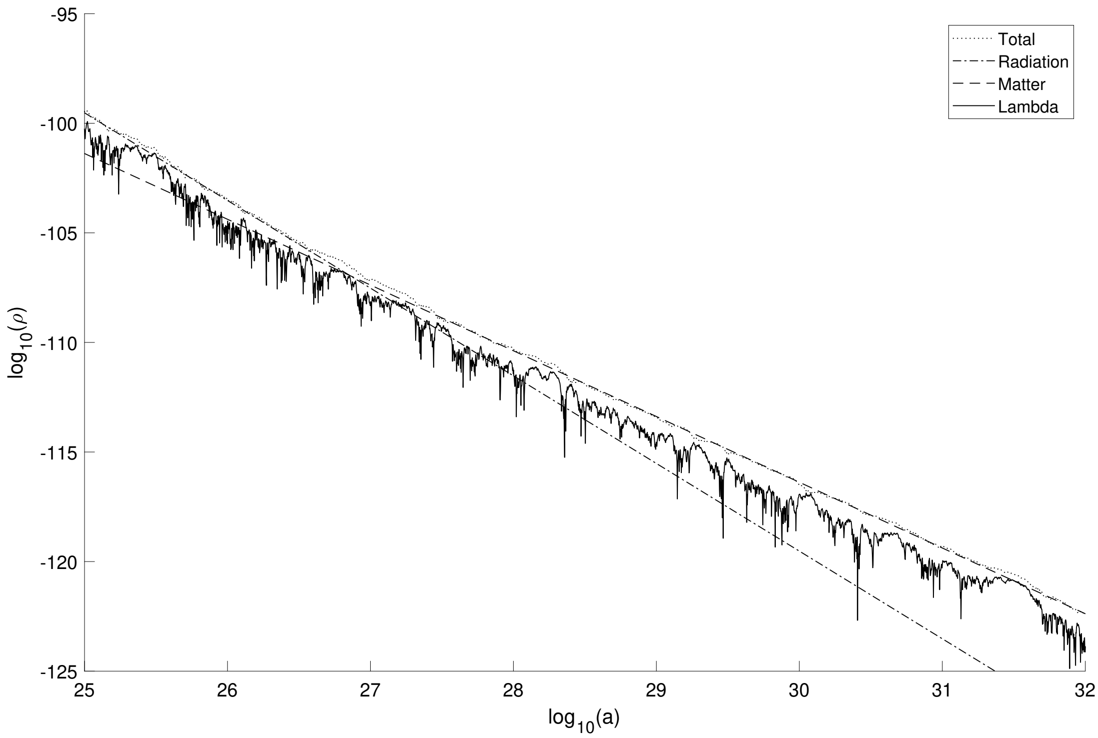
Figure (15) shows that the simulated age is little more than the classical age (with constant Λ). For strong curvature, it is less than the classical age and it could have ruled out the model, but evidently, this is unfavored by the observations. If the curvature becomes very large which indicates, the model might have the potential to solve the horizon puzzle.

Figure (16) shows that naturalness problem has been answered satisfactorily since the amplitude of fluctuations in Ω_Λ remains of the order of ambient density at all past epochs.

Figure (17) depicts for strong curvature and before the radiation-matter crossover, $\Omega_\Lambda^\mathrm{final}$ rarely end on the negative side, and positive values completely outweigh the negative ones thus drives the universe with more positive energy but soon after that, distribution of $\Omega_\Lambda^\mathrm{final}$ indicates that a universe universe with positive lambda is more probable to reach the present size. The energy density of the universe is very close to the critical density (either flat or very nearly flat).

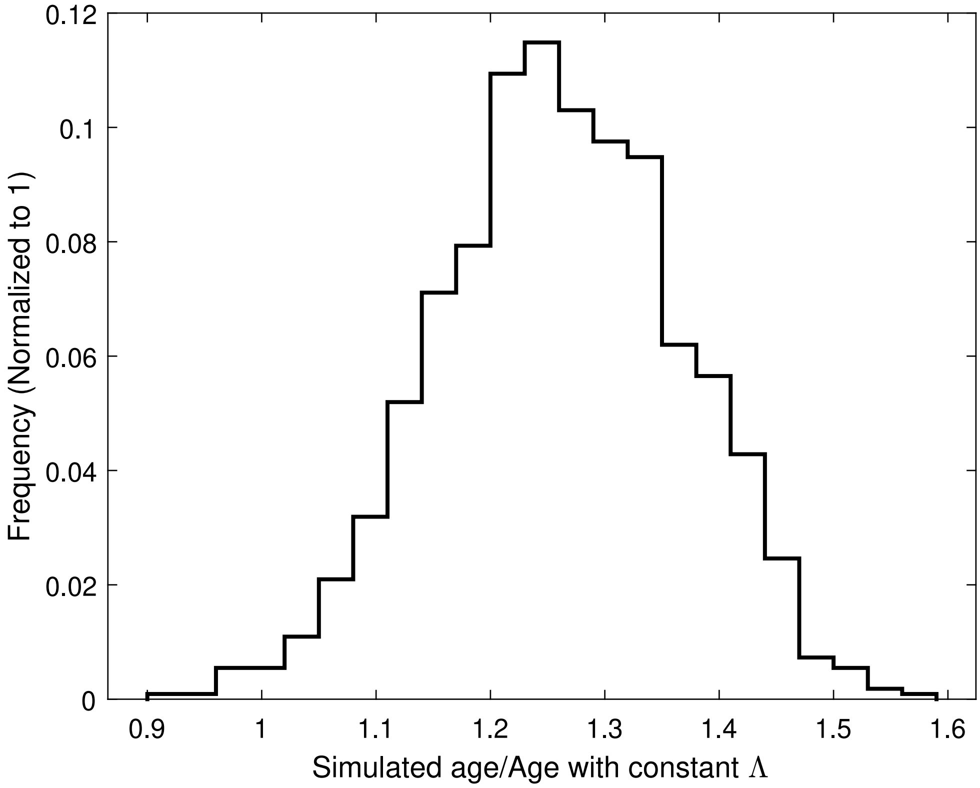


(a) From big bang to present

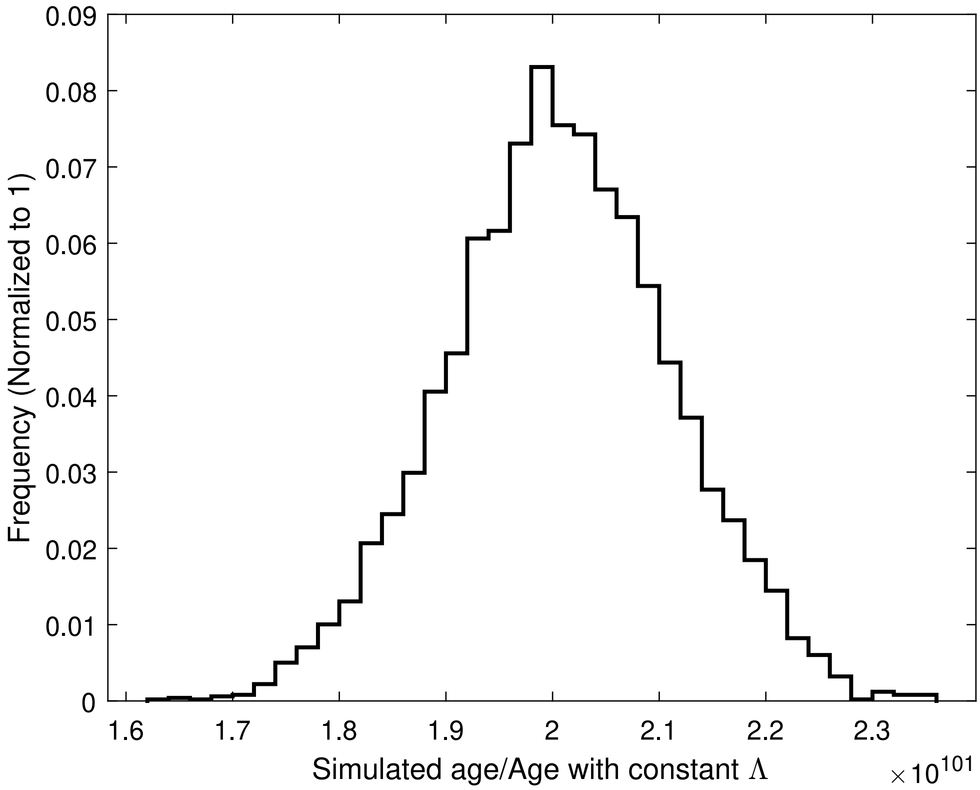


(b) Magnified view of radiation-matter transition

FIG. 14: Tracking behavior of Λ (a) $\alpha = 1/60,\ R = 10^{31}$ (b) $\alpha = 1/70,\ R = 10^{34}$

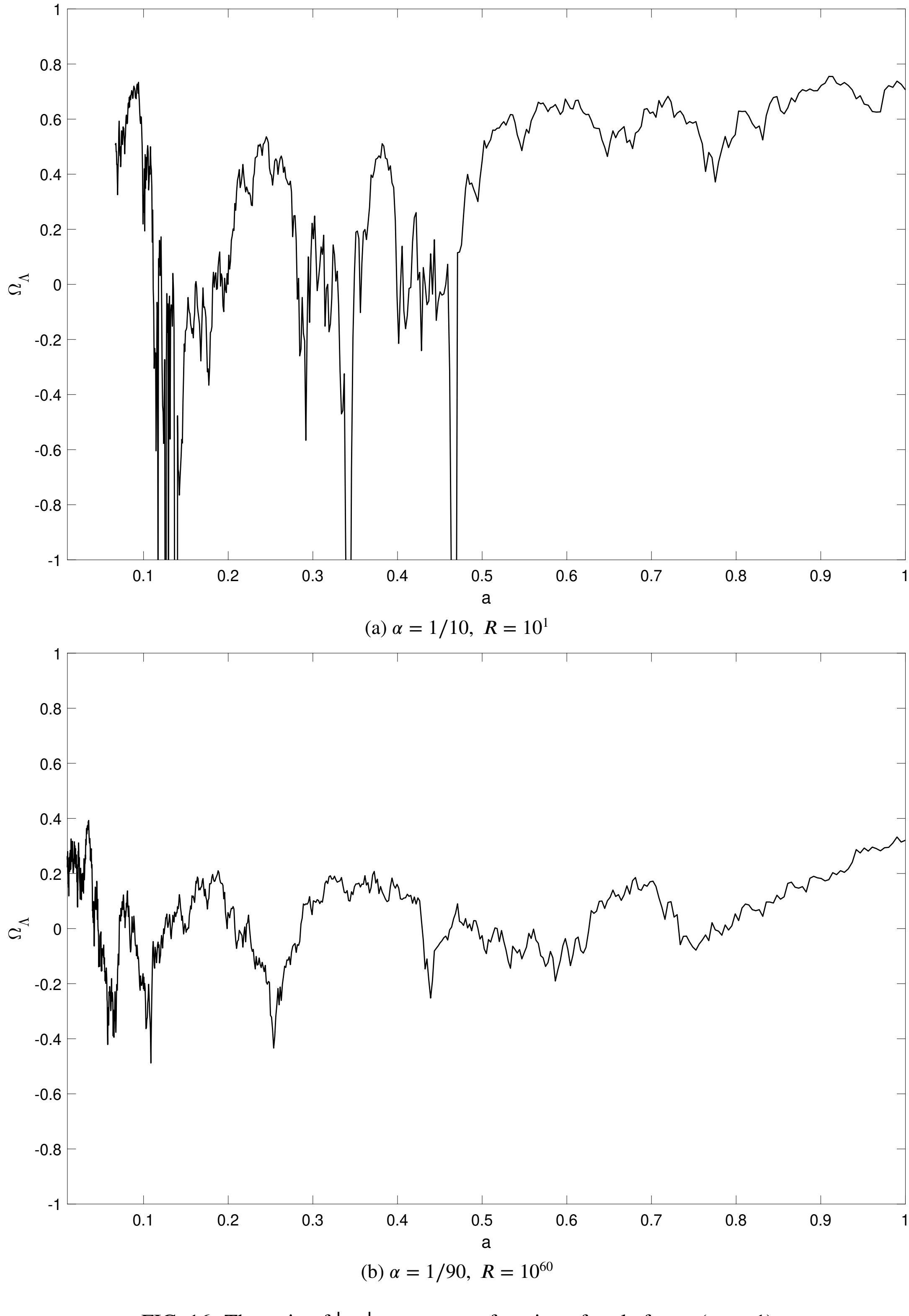


(a) For $\alpha = 1/40,\ R = 10^{31}$



(b) For $\alpha = 1/100,\ R = 10^{100}$

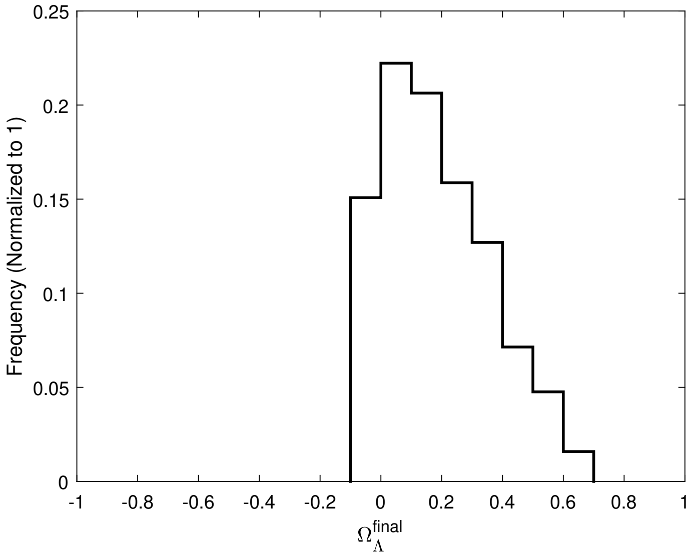
FIG. 15: A histogram of age of the quantum-to-classical universe



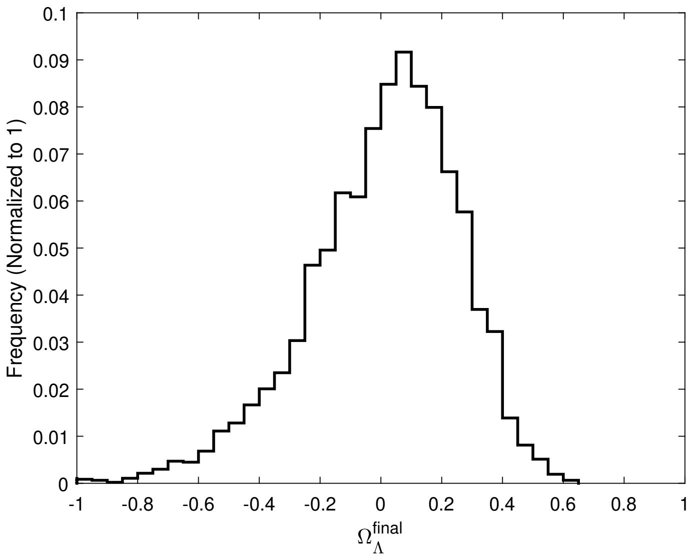
(a) $\alpha = 1/10,\ R = 10^1$

(b) $\alpha = 1/90,\ R = 10^{60}$

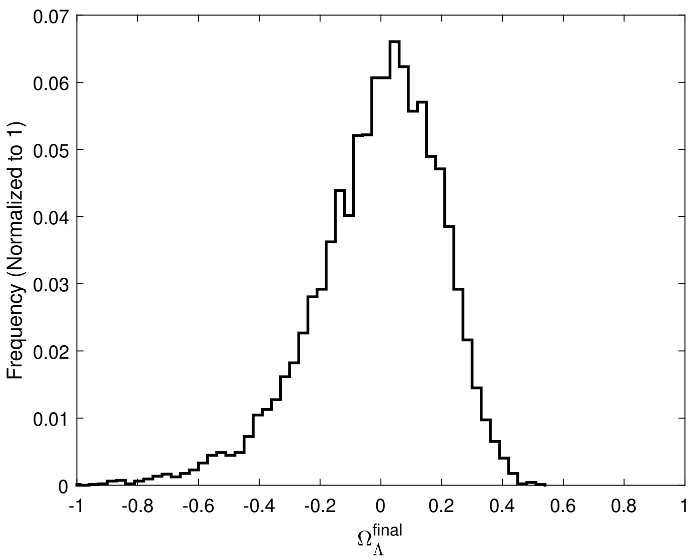
FIG. 16: The ratio of $|\rho_\Lambda|$ to ρ_{total} as a function of scale factor ($a_0 = 1$)



(a) For $\alpha = 1/30,\ R = 10^{15}$



(b) For $\alpha = 1/50,\ R = 10^{27}$



(c) For $\alpha = 1/80,\ R = 10^{40}$

FIG. 17: Histograms of Ω_Λ which reach present CMB temperature

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