Friedman Equations from The Einstein Field Equations

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1 FRLW Metric

Working in units where $c = 1$, the homogeneous and isotropic FRLW metric in conformal coordinates is given by,

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \frac{a^2(t)\,\mathrm{d}r^2}{1-kr^2} - a^2(t)r^2\,\mathrm{d}\theta^2 - a^2(t)r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{1}$$

Or, it can also be written in the standard metric tensor form as:

$$g_{\mu\nu} = \begin{pmatrix} a^2(t) & 0 & 0 & 0 \\ 0 & -\dfrac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix}. \tag{2}$$

Evaluating the Einstein field equation for this metric gives the Friedman equations. This begins by calculating the nonzero Christoffel symbols:

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{3}$$

For $\rho = \sigma = 0$ and $\mu = \nu = 1$,

$$\begin{aligned} \Gamma^{0}{}_{11} &= \frac{1}{2}g^{00}(\partial_1 g_{10} + \partial_1 g_{10} - \partial_0 g_{11}), \\ &= -\frac{1}{2}g^{00}\partial_0 g_{11}, \\ &= -\frac{1}{2}\left[\frac{\partial_t a^2}{1-kr^2}\right], \end{aligned}$$

$$\boxed{\Gamma^{0}{}_{11} = \frac{a\dot{a}}{1-kr^2}.} \tag{4}$$

For $\rho = \sigma = 0$ and $\mu = \nu = 2$,

$$\begin{aligned}
{\Gamma^0}_{22} &= \frac{1}{2} g^{00} (\partial_2 g_{20} + \partial_2 g_{20} - \partial_0 g_{22}), \\
&= -\frac{1}{2} g^{00} \partial_0 g_{22}, \\
&= -\frac{1}{2} \left[\partial_t a^2 r^2 \right], \\
&\boxed{{\Gamma^0}_{22} = a\dot{a} r^2.}
\end{aligned} \tag{5}$$

For $\rho = \sigma = 0$ and $\mu = \nu = 3$,

$$\begin{aligned}
{\Gamma^0}_{33} &= \frac{1}{2} g^{00} (\partial_3 g_{30} + \partial_3 g_{30} - \partial_0 g_{33}), \\
&= -\frac{1}{2} g^{00} \partial_0 g_{33}, \\
&= -\frac{1}{2} \left[\partial_t a^2 r^2 \sin^2 \theta \right], \\
&\boxed{{\Gamma^0}_{33} = a\dot{a} r^2 \sin^2 \theta.}
\end{aligned} \tag{6}$$

For $\rho = \sigma = 1$ and $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1$,

$$\begin{aligned}
{\Gamma^1}_{01} = {\Gamma^1}_{10} &= \frac{1}{2} g^{11} (\partial_1 g_{01} + \partial_0 g_{11} - \partial_1 g_{10}), \\
&= \frac{1}{2} g^{11} \partial_0 g_{11}, \\
&= \frac{1}{2} \left(-\frac{1 - kr^2}{a^2} \right) \partial_t \frac{a^2}{1 - kr^2}, \\
&\boxed{{\Gamma^1}_{01} = {\Gamma^1}_{10} = \frac{\dot{a}}{a}.}
\end{aligned} \tag{7}$$

For $\rho = \sigma = 2$ and $\mu = 2, \nu = 0$ or $\mu = 0, \nu = 2$,

$$\begin{aligned}
{\Gamma^2}_{02} = {\Gamma^2}_{20} &= \frac{1}{2} g^{22} (\partial_2 g_{02} + \partial_0 g_{22} - \partial_2 g_{20}), \\
&= \frac{1}{2} g^{22} \partial_0 g_{22}, \\
&= \frac{1}{2} \left(-\frac{1}{a^2 r^2} \right) \partial_t (-a^2 r^2), \\
&\boxed{{\Gamma^2}_{02} = {\Gamma^2}_{20} = \frac{\dot{a}}{a}.}
\end{aligned} \tag{8}$$

For $\rho = \sigma = 3$ and $\mu = 3, \nu = 0$ or $\mu = 0, \nu = 3$,

$$\begin{aligned}
\Gamma^3{}_{03} = \Gamma^3{}_{30} &= \frac{1}{2} g^{33} (\partial_3 g_{03} + \partial_0 g_{33} - \partial_3 g_{30}), \\
&= \frac{1}{2} g^{33} \partial_0 g_{33}, \\
&= \frac{1}{2} \left(-\frac{1}{a^2 r^2 \sin^2 \theta} \right) \partial_t (-a^2 r^2 \sin^2 \theta), \\
\Aboxed{\Gamma^3{}_{03} = \Gamma^3{}_{30} &= \frac{\dot{a}}{a}.}
\end{aligned} \tag{9}$$

For $\rho = \sigma = 1$ and $\mu = \nu = 1$,

$$\begin{aligned}
\Gamma^1{}_{11} &= \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}), \\
&= \frac{1}{2} g^{11} \partial_1 g_{11}, \\
&= \frac{1}{2} \left(-\frac{1 - kr^2}{a^2} \right) \partial_r \left(-\frac{a^2}{1 - kr^2} \right), \\
\Aboxed{\Gamma^1{}_{11} &= \frac{kr}{1 - kr^2}.}
\end{aligned} \tag{10}$$

For $\rho = \sigma = 1$ and $\mu = \nu = 2$,

$$\begin{aligned}
\Gamma^1{}_{22} &= \frac{1}{2} g^{11} (\partial_2 g_{21} + \partial_2 g_{21} - \partial_1 g_{22}), \\
&= -\frac{1}{2} g^{11} \partial_1 g_{22}, \\
&= \frac{1}{2} \left(-\frac{1 - kr^2}{a^2} \right) \partial_r \left(-a^2 r^2 \right), \\
\Aboxed{\Gamma^1{}_{22} &= -r(1 - kr^2).}
\end{aligned} \tag{11}$$

For $\rho = \sigma = 1$ and $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^1{}_{33} &= \frac{1}{2} g^{11} (\partial_3 g_{31} + \partial_3 g_{21} - \partial_1 g_{33}), \\
&= -\frac{1}{2} g^{11} \partial_1 g_{33}, \\
&= \frac{1}{2} \left(-\frac{1 - kr^2}{a^2} \right) \partial_r \left(-a^2 r^2 \sin^2 \theta \right), \\
\Aboxed{\Gamma^1{}_{33} &= -r(1 - kr^2) \sin^2 \theta.}
\end{aligned} \tag{12}$$

For $\rho = \sigma = 2$ and $\mu = \nu = 3$,

$$\begin{aligned}
{\Gamma^2}_{33} &= \frac{1}{2} g^{22} (\partial_3 g_{32} + \partial_3 g_{32} - \partial_2 g_{33}), \\
&= -\frac{1}{2} g^{22} \partial_2 g_{33}, \\
&= -\frac{1}{2} \left(-\frac{1}{a^2 r^2} \right) \partial_\theta \left(-a^2 r^2 \sin^2 \theta \right), \\
&\boxed{{\Gamma^2}_{33} = -\sin\theta \cos\theta.}
\end{aligned} \tag{13}$$

For $\rho = \sigma = 2$ and $\mu = 1, \nu = 2$ or $\mu = 2, \nu = 1$,

$$\begin{aligned}
{\Gamma^2}_{12} = {\Gamma^2}_{21} &= \frac{1}{2} g^{22} (\partial_2 g_{12} + \partial_1 g_{22} - \partial_2 g_{21}), \\
&= \frac{1}{2} g^{22} \partial_1 g_{22}, \\
&= \frac{1}{2} \left(-\frac{1}{a^2 r^2} \right) \partial_r \left(-a^2 r^2 \right), \\
&\boxed{{\Gamma^2}_{12} = {\Gamma^2}_{21} = \frac{1}{r}.}
\end{aligned} \tag{14}$$

For $\rho = \sigma = 3$ and $\mu = 1, \nu = 3$ or $\mu = 3, \nu = 1$,

$$\begin{aligned}
{\Gamma^3}_{13} = {\Gamma^3}_{31} &= \frac{1}{2} g^{33} (\partial_3 g_{13} + \partial_1 g_{33} - \partial_3 g_{31}), \\
&= \frac{1}{2} g^{33} \partial_1 g_{33}, \\
&= \frac{1}{2} \left(-\frac{1}{a^2 r^2 \sin^2\theta} \right) \partial_r \left(-a^2 r^2 \sin^2 \theta \right), \\
&\boxed{{\Gamma^3}_{13} = {\Gamma^3}_{31} = \frac{1}{r}.}
\end{aligned} \tag{15}$$

For $\rho = \sigma = 3$ and $\mu = 2, \nu = 3$ or $\mu = 3, \nu = 2$,

$$\begin{aligned}
{\Gamma^3}_{23} = {\Gamma^3}_{32} &= \frac{1}{2} g^{33} (\partial_3 g_{32} + \partial_2 g_{33} - \partial_3 g_{32}), \\
&= \frac{1}{2} g^{33} \partial_2 g_{33}, \\
&= \frac{1}{2} \left(-\frac{1}{a^2 r^2 \sin^2\theta} \right) \partial_\theta \left(-a^2 r^2 \sin^2 \theta \right), \\
&\boxed{{\Gamma^3}_{23} = {\Gamma^3}_{32} = \cot\theta.}
\end{aligned} \tag{16}$$

Ricci tensor can be defined as:

$$R_{\mu\nu} = \partial_\rho {\Gamma^\rho}_{\mu\nu} - \partial_\nu {\Gamma^\rho}_{\mu\rho} + {\Gamma^\sigma}_{\mu\nu} {\Gamma^\rho}_{\sigma\rho} - {\Gamma^\sigma}_{\mu\rho} {\Gamma^\rho}_{\sigma\nu}. \tag{17}$$

For $\mu = \nu = 0$, Eq (17) becomes

$$R_{00} = \partial_\rho \Gamma^\rho{}_{00} - \partial_0 \Gamma^\rho{}_{0\rho} + \Gamma^\sigma{}_{00}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{0\rho}\Gamma^\rho{}_{\sigma 0}. \tag{18}$$

Now, we will be expand the sum over ρ for $0, 1, 2, 3$. (Each line of every equation will correspond to each value of ρ.)

$$\begin{aligned}
R_{00} = &+ \partial_0 \Gamma^0{}_{00} - \partial_0 \Gamma^0{}_{00} + \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} \\
&+ \partial_1 \Gamma^1{}_{00} - \partial_0 \Gamma^1{}_{01} + \Gamma^\sigma{}_{00}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{01}\Gamma^1{}_{\sigma 0} \\
&+ \partial_2 \Gamma^2{}_{00} - \partial_0 \Gamma^2{}_{02} + \Gamma^\sigma{}_{00}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{02}\Gamma^2{}_{\sigma 0} \\
&+ \partial_3 \Gamma^3{}_{00} - \partial_0 \Gamma^3{}_{03} + \Gamma^\sigma{}_{00}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{03}\Gamma^3{}_{\sigma 0}, \\
R_{00} = &+ \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} \\
&- \partial_0 \Gamma^1{}_{01} + \Gamma^\sigma{}_{00}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{01}\Gamma^1{}_{\sigma 0} \\
&- \partial_0 \Gamma^2{}_{02} + \Gamma^\sigma{}_{00}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{02}\Gamma^2{}_{\sigma 0} \\
&- \partial_0 \Gamma^3{}_{03} + \Gamma^\sigma{}_{00}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{03}\Gamma^3{}_{\sigma 0}, \\
R_{00} = &- \partial_0 \Gamma^1{}_{01} - \partial_0 \Gamma^2{}_{02} - \partial_0 \Gamma^3{}_{03} \\
&+ \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} \\
&+ \Gamma^\sigma{}_{00}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{01}\Gamma^1{}_{\sigma 0} \\
&+ \Gamma^\sigma{}_{00}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{02}\Gamma^2{}_{\sigma 0} \\
&+ \Gamma^\sigma{}_{00}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{03}\Gamma^3{}_{\sigma 0}.
\end{aligned} \tag{19}$$

Now, we will be expand the sum over σ for $0, 1, 2, 3$, but first rewrite the equation. (Each line of every equation will correspond to each value of σ.)

$$\begin{aligned}
R_{00} = &- \partial_0 \Gamma^1{}_{01} - \partial_0 \Gamma^2{}_{02} - \partial_0 \Gamma^3{}_{03} \\
&+ \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{00}\Gamma^0{}_{\sigma 0} + \Gamma^\sigma{}_{00}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{01}\Gamma^1{}_{\sigma 0} + \Gamma^\sigma{}_{00}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{02}\Gamma^2{}_{\sigma 0} + \Gamma^\sigma{}_{00}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{03}\Gamma^3{}_{\sigma 0}, \\
R_{00} = &- \partial_0 \Gamma^1{}_{01} - \partial_0 \Gamma^2{}_{02} - \partial_0 \Gamma^3{}_{03} \\
&+ \Gamma^0{}_{00}\Gamma^0{}_{00} - \Gamma^0{}_{00}\Gamma^0{}_{00} + \Gamma^0{}_{00}\Gamma^1{}_{01} - \Gamma^0{}_{01}\Gamma^1{}_{00} + \Gamma^0{}_{00}\Gamma^2{}_{02} - \Gamma^0{}_{02}\Gamma^2{}_{00} + \Gamma^0{}_{00}\Gamma^3{}_{03} - \Gamma^0{}_{03}\Gamma^3{}_{00} \\
&+ \Gamma^1{}_{00}\Gamma^0{}_{10} - \Gamma^1{}_{00}\Gamma^0{}_{10} + \Gamma^1{}_{00}\Gamma^1{}_{11} - \Gamma^1{}_{01}\Gamma^1{}_{10} + \Gamma^1{}_{00}\Gamma^2{}_{12} - \Gamma^1{}_{02}\Gamma^2{}_{10} + \Gamma^1{}_{00}\Gamma^3{}_{13} - \Gamma^1{}_{03}\Gamma^3{}_{10} \\
&+ \Gamma^2{}_{00}\Gamma^0{}_{20} - \Gamma^2{}_{00}\Gamma^0{}_{20} + \Gamma^2{}_{00}\Gamma^1{}_{21} - \Gamma^2{}_{01}\Gamma^1{}_{20} + \Gamma^2{}_{00}\Gamma^2{}_{22} - \Gamma^2{}_{02}\Gamma^2{}_{20} + \Gamma^2{}_{00}\Gamma^3{}_{23} - \Gamma^2{}_{03}\Gamma^3{}_{20} \\
&+ \Gamma^3{}_{00}\Gamma^0{}_{30} - \Gamma^3{}_{00}\Gamma^0{}_{30} + \Gamma^3{}_{00}\Gamma^1{}_{31} - \Gamma^3{}_{01}\Gamma^1{}_{30} + \Gamma^3{}_{00}\Gamma^2{}_{32} - \Gamma^3{}_{02}\Gamma^2{}_{30} + \Gamma^3{}_{00}\Gamma^3{}_{33} - \Gamma^3{}_{03}\Gamma^3{}_{30}, \\
R_{00} = &- \partial_0 \Gamma^1{}_{01} - \partial_0 \Gamma^2{}_{02} - \partial_0 \Gamma^3{}_{03} \\
&+ 0 \\
&- \Gamma^1{}_{01}\Gamma^1{}_{10} \\
&- \Gamma^2{}_{02}\Gamma^2{}_{20} \\
&- \Gamma^3{}_{03}\Gamma^3{}_{30},
\end{aligned}$$

By inserting values, we get

$$
\begin{aligned}
R_{00} =& - \partial_t \left(\frac{\dot{a}}{a}\right) - \partial_t \left(\frac{\dot{a}}{a}\right) - \partial_t \left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}\right), \\
R_{00} =& - 3\partial_t \left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2, \\
R_{00} =& - 3\frac{\ddot{a}a - \dot{a}^2}{a^2} - 3\left(\frac{\dot{a}}{a}\right)^2, \\
R_{00} =& - 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2, \\
&\boxed{R_{00} = -3\frac{\ddot{a}}{a}.}
\end{aligned}
\tag{20}
$$

For $\mu = \nu = 1$, Eq (17) becomes

$$
R_{11} = \partial_\rho \Gamma^\rho{}_{11} - \partial_1 \Gamma^\rho{}_{1\rho} + \Gamma^\sigma{}_{11}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{1\rho}\Gamma^\rho{}_{\sigma 1}. \tag{21}
$$

Now, we will be expand the sum over ρ for $0, 1, 2, 3$. (Each line of every equation will correspond to each value of ρ.)

$$
\begin{aligned}
R_{11} =& + \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^0{}_{10} + \Gamma^\sigma{}_{11}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{10}\Gamma^0{}_{\sigma 1} \\
& + \partial_1 \Gamma^1{}_{11} - \partial_1 \Gamma^1{}_{11} + \Gamma^\sigma{}_{11}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{11}\Gamma^1{}_{\sigma 1} \\
& + \partial_2 \Gamma^2{}_{11} - \partial_1 \Gamma^2{}_{12} + \Gamma^\sigma{}_{11}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{12}\Gamma^2{}_{\sigma 1} \\
& + \partial_3 \Gamma^3{}_{11} - \partial_1 \Gamma^3{}_{13} + \Gamma^\sigma{}_{11}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{13}\Gamma^3{}_{\sigma 1}, \\
R_{11} =& + \partial_0 \Gamma^0{}_{11} - \Gamma^\sigma{}_{10}\Gamma^0{}_{\sigma 1} \\
& + 0 \\
& - \partial_1 \Gamma^2{}_{12} + \Gamma^\sigma{}_{11}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{12}\Gamma^2{}_{\sigma 1} \\
& - \partial_1 \Gamma^3{}_{13} + \Gamma^\sigma{}_{11}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{13}\Gamma^3{}_{\sigma 1}.
\end{aligned}
\tag{22}
$$

Now, we will be expand the sum over σ for $0, 1, 2, 3$, but first rewrite the equation. (Each line of every equation will correspond to each value of σ.)

$$\begin{aligned}
R_{11} = &+ \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^2{}_{12} - \partial_1 \Gamma^3{}_{13} \\
&- \Gamma^\sigma{}_{10}\Gamma^0{}_{\sigma 1} + \Gamma^\sigma{}_{11}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{12}\Gamma^2{}_{\sigma 1} + \Gamma^\sigma{}_{11}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{13}\Gamma^3{}_{\sigma 1}, \\
R_{11} = &+ \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^2{}_{12} - \partial_1 \Gamma^3{}_{13} \\
&- \Gamma^0{}_{10}\Gamma^0{}_{01} + \Gamma^0{}_{11}\Gamma^2{}_{02} - \Gamma^0{}_{12}\Gamma^2{}_{01} + \Gamma^0{}_{11}\Gamma^3{}_{03} - \Gamma^0{}_{13}\Gamma^3{}_{01} \\
&- \Gamma^1{}_{10}\Gamma^0{}_{11} + \Gamma^1{}_{11}\Gamma^2{}_{12} - \Gamma^1{}_{12}\Gamma^2{}_{11} + \Gamma^1{}_{11}\Gamma^3{}_{13} - \Gamma^1{}_{13}\Gamma^3{}_{11} \\
&- \Gamma^2{}_{10}\Gamma^0{}_{21} + \Gamma^2{}_{11}\Gamma^2{}_{22} - \Gamma^2{}_{12}\Gamma^2{}_{21} + \Gamma^2{}_{11}\Gamma^3{}_{23} - \Gamma^2{}_{13}\Gamma^3{}_{21} \\
&- \Gamma^3{}_{10}\Gamma^0{}_{31} + \Gamma^3{}_{11}\Gamma^2{}_{32} - \Gamma^3{}_{12}\Gamma^2{}_{31} + \Gamma^3{}_{11}\Gamma^3{}_{33} - \Gamma^3{}_{13}\Gamma^3{}_{31}, \\
R_{11} = &+ \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^2{}_{12} - \partial_1 \Gamma^3{}_{13} \\
&+ \Gamma^0{}_{11}\Gamma^2{}_{02} + \Gamma^0{}_{11}\Gamma^3{}_{03} \\
&- \Gamma^1{}_{10}\Gamma^0{}_{11} + \Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^1{}_{11}\Gamma^3{}_{13} \\
&- \Gamma^2{}_{12}\Gamma^2{}_{21} \\
&- \Gamma^3{}_{13}\Gamma^3{}_{31}.
\end{aligned} \tag{23}$$

By inserting values, we get

$$\begin{aligned}
R_{11} = &+ \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^2{}_{12} - \partial_1 \Gamma^3{}_{13} + \Gamma^0{}_{11}\Gamma^2{}_{02} + \Gamma^0{}_{11}\Gamma^3{}_{03} \\
&- \Gamma^1{}_{10}\Gamma^0{}_{11} + \Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^1{}_{11}\Gamma^3{}_{13} - \Gamma^2{}_{12}\Gamma^2{}_{21} - \Gamma^3{}_{13}\Gamma^3{}_{31}, \\
R_{11} = &+ \partial_t \left(\frac{a\dot{a}}{1-kr^2}\right) - \partial_r \left(\frac{1}{r}\right) - \partial_r \left(\frac{1}{r}\right) + \left(\frac{a\dot{a}}{1-kr^2}\right)\left(\frac{\dot{a}}{a}\right) + \left(\frac{a\dot{a}}{1-kr^2}\right)\left(\frac{\dot{a}}{a}\right) \\
&- \left(\frac{a\dot{a}}{1-kr^2}\right)\left(\frac{\dot{a}}{a}\right) + \left(\frac{kr}{1-kr^2}\right)\left(\frac{1}{r}\right) + \left(\frac{kr}{1-kr^2}\right)\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\left(\frac{1}{r}\right), \\
R_{11} = &+ \partial_t \left(\frac{a\dot{a}}{1-kr^2}\right) - 2\partial_r \left(\frac{1}{r}\right) + \frac{\dot{a}^2}{1-kr^2} + \frac{2k}{1-kr^2} - 2\left(\frac{1}{r}\right)^2, \\
R_{11} = &+ \frac{a\ddot{a}}{1-kr^2} + \frac{2\dot{a}^2}{1-kr^2} + \frac{2k}{1-kr^2}, \\
&\boxed{R_{11} = +\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}.}
\end{aligned} \tag{24}$$

For $\mu = \nu = 2$, Eq (17) becomes

$$R_{22} = \partial_\rho \Gamma^\rho{}_{22} - \partial_2 \Gamma^\rho{}_{2\rho} + \Gamma^\sigma{}_{22}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{2\rho}\Gamma^\rho{}_{\sigma 2}. \tag{25}$$

Now, we will be expand the sum over ρ for $0, 1, 2, 3$. (Each line of every equation will correspond to each value of ρ.)

$$\begin{aligned}
R_{22} = &+ \partial_0 \Gamma^0{}_{22} - \partial_2 \Gamma^0{}_{20} + \Gamma^\sigma{}_{22}\Gamma^0{}_{\sigma 0} - \Gamma^\sigma{}_{20}\Gamma^0{}_{\sigma 2} \\
&+ \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^1{}_{21} + \Gamma^\sigma{}_{22}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{21}\Gamma^1{}_{\sigma 2} \\
&+ \partial_2 \Gamma^2{}_{22} - \partial_2 \Gamma^2{}_{22} + \Gamma^\sigma{}_{22}\Gamma^2{}_{\sigma 2} - \Gamma^\sigma{}_{22}\Gamma^2{}_{\sigma 2} \\
&+ \partial_3 \Gamma^3{}_{22} - \partial_2 \Gamma^3{}_{23} + \Gamma^\sigma{}_{22}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{23}\Gamma^3{}_{\sigma 2}, \\
R_{22} = &+ \partial_0 \Gamma^0{}_{22} - \Gamma^\sigma{}_{20}\Gamma^0{}_{\sigma 2} \\
&+ \partial_1 \Gamma^1{}_{22} + \Gamma^\sigma{}_{22}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{21}\Gamma^1{}_{\sigma 2} \\
&+ 0 \\
&- \partial_2 \Gamma^3{}_{23} + \Gamma^\sigma{}_{22}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{23}\Gamma^3{}_{\sigma 2},
\end{aligned}$$

Now, we will be expand the sum over σ for $0, 1, 2, 3$, but first rewrite the equation. (Each line of every equation will correspond to each value of σ.)

$$\begin{aligned}
R_{22} = &+ \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^3{}_{23} \\
&- \Gamma^\sigma{}_{20}\Gamma^0{}_{\sigma 2} + \Gamma^\sigma{}_{22}\Gamma^1{}_{\sigma 1} - \Gamma^\sigma{}_{21}\Gamma^1{}_{\sigma 2} + \Gamma^\sigma{}_{22}\Gamma^3{}_{\sigma 3} - \Gamma^\sigma{}_{23}\Gamma^3{}_{\sigma 2}, \\
R_{22} = &+ \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^3{}_{23} \\
&- \Gamma^0{}_{20}\Gamma^0{}_{02} + \Gamma^0{}_{22}\Gamma^1{}_{01} - \Gamma^0{}_{21}\Gamma^1{}_{02} + \Gamma^0{}_{22}\Gamma^3{}_{03} - \Gamma^0{}_{23}\Gamma^3{}_{02} \\
&- \Gamma^1{}_{20}\Gamma^0{}_{12} + \Gamma^1{}_{22}\Gamma^1{}_{11} - \Gamma^1{}_{21}\Gamma^1{}_{12} + \Gamma^1{}_{22}\Gamma^3{}_{13} - \Gamma^1{}_{23}\Gamma^3{}_{12} \\
&- \Gamma^2{}_{20}\Gamma^0{}_{22} + \Gamma^2{}_{22}\Gamma^1{}_{21} - \Gamma^2{}_{21}\Gamma^1{}_{22} + \Gamma^2{}_{22}\Gamma^3{}_{23} - \Gamma^2{}_{23}\Gamma^3{}_{22} \\
&- \Gamma^3{}_{20}\Gamma^0{}_{32} + \Gamma^3{}_{22}\Gamma^1{}_{31} - \Gamma^3{}_{21}\Gamma^1{}_{32} + \Gamma^3{}_{22}\Gamma^3{}_{33} - \Gamma^3{}_{23}\Gamma^3{}_{32}, \\
R_{22} = &+ \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^3{}_{23} \\
&+ \Gamma^0{}_{22}\Gamma^1{}_{01} + \Gamma^0{}_{22}\Gamma^3{}_{03} \\
&+ \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^1{}_{22}\Gamma^3{}_{13} \\
&- \Gamma^2{}_{20}\Gamma^0{}_{22} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&- \Gamma^3{}_{23}\Gamma^3{}_{32}.
\end{aligned} \tag{26}$$

By inserting values, we get

$$\begin{aligned}
R_{22} = &+ \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^3{}_{23} \\
&+ \Gamma^0{}_{22}\Gamma^1{}_{01} + \Gamma^0{}_{22}\Gamma^3{}_{03} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^1{}_{22}\Gamma^3{}_{13} - \Gamma^2{}_{20}\Gamma^0{}_{22} - \Gamma^2{}_{21}\Gamma^1{}_{22} - \Gamma^3{}_{23}\Gamma^3{}_{32}, \\
R_{22} = &+ \partial_t(a\dot{a}r^2) + \partial_r[-r(1-kr^2)] - \partial_\theta(\cot\theta) \\
&+ (a\dot{a}r^2)\frac{\dot{a}}{a} + (a\dot{a}r^2)\frac{\dot{a}}{a} - r(1-kr^2)\frac{kr}{1-kr^2} - r(1-kr^2)\frac{1}{r} - \frac{\dot{a}}{a}(a\dot{a}r^2) + r(1-kr^2)\frac{1}{r} - \cot^2\theta, \\
R_{22} = &+ \partial_t(a\dot{a}r^2) + \partial_r[-r(1-kr^2)] - \partial_\theta(\cot\theta) + \dot{a}^2r^2 - kr^2 - \cot^2\theta, \\
R_{22} = &+ r^2(a\ddot{a} + \dot{a}^2) + 3kr^2 - 1 + \csc^2\theta + \dot{a}^2r^2 - kr^2 - \cot^2\theta,
\end{aligned}$$

$$\boxed{R_{22} = +(a\ddot{a} + 2\dot{a}^2 + 2k)r^2.} \tag{27}$$

For $\mu = \nu = 3$, Eq (17) becomes

$$R_{33} = \partial_\rho \Gamma^\rho{}_{33} - \partial_3 \Gamma^\rho{}_{3\rho} + \Gamma^\sigma{}_{33}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{3\rho}\Gamma^\rho{}_{\sigma 3}. \tag{28}$$

Now, we will be expand the sum over ρ for $0, 1, 2, 3$. (Each line of every equation will correspond to each value of ρ.)

$$\begin{aligned}
R_{33} = &+ \partial_0 {\Gamma^0}_{33} - \partial_3 {\Gamma^0}_{30} + {\Gamma^\sigma}_{33}{\Gamma^0}_{\sigma 0} - {\Gamma^\sigma}_{30}{\Gamma^0}_{\sigma 3} \\
&+ \partial_1 {\Gamma^1}_{33} - \partial_3 {\Gamma^1}_{31} + {\Gamma^\sigma}_{33}{\Gamma^1}_{\sigma 1} - {\Gamma^\sigma}_{31}{\Gamma^1}_{\sigma 3} \\
&+ \partial_2 {\Gamma^2}_{33} - \partial_3 {\Gamma^2}_{32} + {\Gamma^\sigma}_{33}{\Gamma^2}_{\sigma 2} - {\Gamma^\sigma}_{32}{\Gamma^2}_{\sigma 3} \\
&+ \partial_3 {\Gamma^3}_{33} - \partial_3 {\Gamma^3}_{33} + {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3} - {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3}, \\
R_{33} = &+ \partial_0 {\Gamma^0}_{33} - {\Gamma^\sigma}_{30}{\Gamma^0}_{\sigma 3} \\
&+ \partial_1 {\Gamma^1}_{33} + {\Gamma^\sigma}_{33}{\Gamma^1}_{\sigma 1} - {\Gamma^\sigma}_{31}{\Gamma^1}_{\sigma 3} \\
&+ \partial_2 {\Gamma^2}_{33} + {\Gamma^\sigma}_{33}{\Gamma^2}_{\sigma 2} - {\Gamma^\sigma}_{32}{\Gamma^2}_{\sigma 3} \\
&+ {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3} - {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3}.
\end{aligned}$$

Now, we will be expand the sum over σ for $0, 1, 2, 3$, but first rewrite the equation. (Each line of every equation will correspond to each value of σ.)

$$\begin{aligned}
R_{33} = &+ \partial_0 {\Gamma^0}_{33} + \partial_1 {\Gamma^1}_{33} + \partial_2 {\Gamma^2}_{33} \\
&- {\Gamma^\sigma}_{30}{\Gamma^0}_{\sigma 3} + {\Gamma^\sigma}_{33}{\Gamma^1}_{\sigma 1} - {\Gamma^\sigma}_{31}{\Gamma^1}_{\sigma 3} + {\Gamma^\sigma}_{33}{\Gamma^2}_{\sigma 2} - {\Gamma^\sigma}_{32}{\Gamma^2}_{\sigma 3} + {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3} - {\Gamma^\sigma}_{33}{\Gamma^3}_{\sigma 3}, \\
R_{33} = &+ \partial_0 {\Gamma^0}_{33} + \partial_1 {\Gamma^1}_{33} + \partial_2 {\Gamma^2}_{33} \\
&- {\Gamma^0}_{30}{\Gamma^0}_{03} + {\Gamma^0}_{33}{\Gamma^1}_{01} - {\Gamma^0}_{31}{\Gamma^1}_{03} + {\Gamma^0}_{33}{\Gamma^2}_{02} - {\Gamma^0}_{32}{\Gamma^2}_{03} + {\Gamma^0}_{33}{\Gamma^3}_{03} - {\Gamma^0}_{33}{\Gamma^3}_{03} \\
&- {\Gamma^1}_{30}{\Gamma^0}_{13} + {\Gamma^1}_{33}{\Gamma^1}_{11} - {\Gamma^1}_{31}{\Gamma^1}_{13} + {\Gamma^1}_{33}{\Gamma^2}_{12} - {\Gamma^1}_{32}{\Gamma^2}_{13} + {\Gamma^1}_{33}{\Gamma^3}_{13} - {\Gamma^1}_{33}{\Gamma^3}_{13} \\
&- {\Gamma^2}_{30}{\Gamma^0}_{23} + {\Gamma^2}_{33}{\Gamma^1}_{21} - {\Gamma^2}_{31}{\Gamma^1}_{23} + {\Gamma^2}_{33}{\Gamma^2}_{22} - {\Gamma^2}_{32}{\Gamma^2}_{23} + {\Gamma^2}_{33}{\Gamma^3}_{23} - {\Gamma^2}_{33}{\Gamma^3}_{23} \\
&- {\Gamma^3}_{30}{\Gamma^0}_{33} + {\Gamma^3}_{33}{\Gamma^1}_{31} - {\Gamma^3}_{31}{\Gamma^1}_{33} + {\Gamma^3}_{33}{\Gamma^2}_{32} - {\Gamma^3}_{32}{\Gamma^2}_{33} + {\Gamma^3}_{33}{\Gamma^3}_{33} - {\Gamma^3}_{33}{\Gamma^3}_{33}, \\
R_{33} = &+ \partial_0 {\Gamma^0}_{33} + \partial_1 {\Gamma^1}_{33} + \partial_2 {\Gamma^2}_{33} \\
&+ {\Gamma^0}_{33}{\Gamma^1}_{01} + {\Gamma^0}_{33}{\Gamma^2}_{02} \\
&+ {\Gamma^1}_{33}{\Gamma^1}_{11} + {\Gamma^1}_{33}{\Gamma^2}_{12} \\
&+ 0 \\
&- {\Gamma^3}_{30}{\Gamma^0}_{33} - {\Gamma^3}_{31}{\Gamma^1}_{33} - {\Gamma^3}_{32}{\Gamma^2}_{33},
\end{aligned}$$

By inserting values, we get

$$
\begin{aligned}
R_{33} =& + \partial_0 \Gamma^0{}_{33} + \partial_1 \Gamma^1{}_{33} + \partial_2 \Gamma^2{}_{33} + \Gamma^0{}_{33}\Gamma^1{}_{01} + \Gamma^0{}_{33}\Gamma^2{}_{02} \\
& + \Gamma^1{}_{33}\Gamma^1{}_{11} + \Gamma^1{}_{33}\Gamma^2{}_{12} \\
& - \Gamma^3{}_{30}\Gamma^0{}_{33} - \Gamma^3{}_{31}\Gamma^1{}_{33} - \Gamma^3{}_{32}\Gamma^2{}_{33}, \\
R_{33} =& + \partial_t(a\dot{a}r^2 \sin^2\theta) + \partial_r[-r(1-kr^2)\sin^2\theta] - \partial_\theta(\sin\theta\cos\theta) + (a\dot{a}r^2\sin^2\theta)\frac{\dot{a}}{a} + (a\dot{a}r^2\sin^2\theta)\frac{\dot{a}}{a} \\
& - r(1-kr^2)\sin^2\theta\frac{kr}{1-kr^2} + [-r(1-kr^2)\sin^2\theta]\frac{1}{r} \\
& - \frac{\dot{a}}{a}(a\dot{a}r^2\sin^2\theta) + \frac{1}{r}[-r(1-kr^2)\sin^2\theta] + \cot\theta\sin\theta\cos\theta, \\
R_{33} =& + r^2\sin^2\theta\partial_t(a\dot{a}) - \sin^2\theta[\partial_r(r) - \partial_r(kr^3)] - \partial_\theta(\sin\theta\cos\theta) + \dot{a}^2r^2\sin^2\theta - kr^2\sin^2\theta + \cos^2\theta, \\
R_{33} =& + r^2\sin^2\theta(a\ddot{a} + \dot{a}^2) - \sin^2\theta(1-3kr^2) + \sin^2\theta - \cos^2\theta + \dot{a}^2r^2\sin^2\theta - kr^2\sin^2\theta + \cos^2\theta, \\
R_{33} =& + r^2\sin^2\theta(a\ddot{a} + \dot{a}^2) + 2kr^2\sin^2\theta + \dot{a}^2r^2\sin^2\theta,
\end{aligned}
$$

$$
\boxed{R_{33} = +(a\ddot{a} + 2\dot{a}^2 + 2k)r^2\sin^2\theta.} \tag{29}
$$

Finally, we can write Ricci tensor in matrix form as

$$
R_{\mu\nu} = \begin{pmatrix}
-3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\
0 & \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} & 0 & 0 \\
0 & 0 & (a\ddot{a} + 2\dot{a}^2 + 2k)r^2 & 0 \\
0 & 0 & 0 & (a\ddot{a} + 2\dot{a}^2 + 2k)r^2\sin^2\theta
\end{pmatrix}. \tag{30}
$$

Ricci scalar can be defined as:

$$
\begin{aligned}
R &= R_{\mu\nu}g^{\mu\nu}, \qquad (31) \\
&= R_{00}g^{00} + R_{11}g^{11} + R_{22}g^{22} + R_{33}g^{33}, \\
&= -3\frac{\ddot{a}}{a} + \left(\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}\right)\left(\frac{1-kr^2}{-a^2}\right) \\
&\quad - (a\ddot{a} + 2\dot{a}^2 + 2k)r^2\left(\frac{1}{-a^2r^2}\right) + (a\ddot{a} + 2\dot{a}^2 + 2k)r^2\sin^2\theta\left(\frac{1}{-a^2r^2\sin^2\theta}\right), \\
&= -3\frac{\ddot{a}}{a} - 3\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{a^2},
\end{aligned}
$$

$$
\boxed{R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).} \tag{32}
$$

In order to model the large scale behaviour of the universe, so that Einstein equations are satisfied, we model the energy and matter in the universe by a perfect fluid. A perfect fluid has no heat conduction or viscosity and is characterized by its mass density ρ and pressure p. The particles in the fluid are galaxy clusters. In comoving coordinates, $\dot{t} = 1$ and $\dot{x}_i = 0$, the fourvelocity $u^\mu = (1,0,0,0)$, the stress-energy tensor becomes:

$$T_{\mu\nu} = (\rho + p)\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} - p g_{\mu\nu}, \tag{33}$$

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}. \tag{34}$$

Further

$$T_{00} = \rho, \tag{35}$$

$$\begin{aligned} T_{11} &= (\rho + p)\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} - p g_{11}, \\ &= -p\left(-\frac{a^2}{1-kr^2}\right), \end{aligned}$$

$$T_{11} = \frac{pa^2}{1-kr^2}, \tag{36}$$

and its trace is

$$T = \rho - 3p. \tag{37}$$

The 00-component of the Einstein tensor is:

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}R g_{00}, \\ &= -3\frac{\ddot{a}}{a} - \frac{1}{2}(-6)\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \\ &= -3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \end{aligned} \tag{38}$$

$$\boxed{G_{00} = 3\frac{\dot{a}^2 + k}{a^2}.} \tag{39}$$

Now the 00-component of the Einstein field equation gives:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{40}$$

$$\begin{aligned} G_{00} &= 8\pi G T_{00} + \Lambda g_{00}, \\ 3\frac{\dot{a}^2 + k}{a^2} &= 8\pi G\rho + \Lambda, \\ \frac{\dot{a}^2 + k}{a^2} &= \frac{8\pi G\rho + \Lambda}{3}. \end{aligned} \tag{41}$$

Or by inserting c, it becomes

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3} \tag{42}$$

And the 11-component of the trace inverted Einstein field equation gives:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{43}$$
$$R_{11} - \frac{1}{2}Rg_{11} = 8\pi G T_{11} + \Lambda g_{11},$$

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} - \frac{1}{2}\left[-6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\right]\frac{a^2}{1-kr^2} = 8\pi G p \frac{a^2}{1-kr^2} + \Lambda\frac{a^2}{1-kr^2},$$
$$(a\ddot{a} + 2\dot{a}^2 + 2k) + 3a^2\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G p a^2 + \Lambda a^2,$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{44}$$

Or by inserting c, it becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{45}$$

We supplement these Friedman equations by the equation of conservation of energy using the stress-energy-momentum tensor:

$$T^{\mu\nu}{}_{;\mu} = 0. \tag{46}$$

Considering the conservation of energy equation using the energy-momentum tensor, we get

$$T^{\mu 0}{}_{;\mu} = 0,$$
$$\partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\mu\nu} T^\nu{}_0 - \Gamma^\nu{}_{\mu 0} T^\mu{}_\nu = 0. \tag{47}$$

As the energy-momentum tensor is diagonal, so it reduces to

$$\begin{aligned}
0 = &+ \partial_0 T^0{}_0 + \partial_1 T^1{}_0 + \partial_2 T^2{}_0 + \partial_3 T^3{}_0 \\
&+ \Gamma^0{}_{00} T^0{}_0 + \Gamma^1{}_{10} T^0{}_0 + \Gamma^2{}_{20} T^0{}_0 + \Gamma^3{}_{30} T^0{}_0 \\
&- \Gamma^0{}_{00} T^0{}_0 - \Gamma^1{}_{10} T^1{}_1 - \Gamma^2{}_{20} T^2{}_2 - \Gamma^3{}_{30} T^3{}_3, \\
0 = &+ \partial_t \rho + 0 + 0 + 0 \\
&+ 0 + \frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}\rho \\
&- 0 - \frac{\dot{a}}{a}p - \frac{\dot{a}}{a}p - \frac{\dot{a}}{a}p,
\end{aligned}$$
$$\boxed{\frac{\partial\rho}{\partial t} = -3\frac{\dot{a}}{a}(\rho + p).} \tag{48}$$

To proceed further, we have to choose an equation of state, a relationship between ρ and p. The perfect fluid relevant to cosmology obeys the simple equation of state

$$p = \omega\rho, \tag{49}$$

where ω is a constant independent of time. In view of this result, we re-write the energy conservation equation as

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + \omega\rho),$$
$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + \omega),$$

which on integration yields

$$\rho \propto a^{-3(1+\omega)}. \tag{50}$$