Volume of the Closed Universe

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1 Spacetime Volume as a Causal Set Observable

In the context of causal set theory, spacetime volume plays a fundamental role in determining the number of elements within a causal region and, consequently, in the dynamical behavior of the cosmological constant. Since the effective cosmological constant in such models is expected to fluctuate inversely with spacetime volume, it becomes essential to obtain an explicit analytic expression for the invariant four–volume of a curved universe.

In this section, we derive the Lorentz–invariant four–volume of a closed Friedmann–Lemaître–Robertson–Walker (FLRW) universe, expressed in conformal coordinates. The calculation proceeds by integrating the determinant of the FLRW metric over the past light cone and systematically expanding the resulting expression in powers of curvature radius. This yields a hierarchy of curvature–dependent correction terms that generalize the flat–space causal volume to the case of a closed universe.

The resulting series expansion not only serves as a precise continuum analogue for discrete spacetime volumes in causal set theory but also provides the mathematical groundwork for connecting curvature effects to fluctuations in the cosmological constant. This formulation will be particularly useful in later sections where the causal set–based fluctuating–Λ framework is applied to cosmological models with compact spatial topology.

2 Volume Enclosed by the Past Light Cone

The homogeneous and isotropic FRLW metric of a closed universe in conformal coordinates is given by,

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-\left(\dfrac{r}{R_{\text{curv}}}\right)^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]. \tag{1}$$

For simplicity, we are taking $R = R_{\text{curv}}$. Since the metric is diagonal, so its determinant will simply be,,

$$g = a^8(\eta) \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{1-\left(\dfrac{r}{R}\right)^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix},$$

$$\sqrt{-g} = a^4(\eta)\frac{r^2\sin^2\theta}{\sqrt{1-\left(\dfrac{r}{R}\right)^2}}.$$

The general expression of Lorentz invariant four–volume, for both diagonal and non–diagonal metric, is

$$\mathcal{V} = \iiiint \sqrt{-g}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \tag{2}$$

So, for a closed universe, it becomes

$$\begin{aligned} \mathcal{V} &= \iiiint a^4(\eta)\frac{r^2\sin^2\theta}{\sqrt{1-\left(\dfrac{r}{R}\right)^2}}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \\ &= 4\pi\iint a^4(\eta)\frac{r^2\sin^2\theta}{\sqrt{1-\left(\dfrac{r}{R}\right)^2}}\,\mathrm{d}\eta\,\mathrm{d}r\,, \\ \mathcal{V} &= 4\pi\int_0^\eta a^4(\eta)\int_0^r \frac{r^2\sin^2\theta}{\sqrt{1-\left(\dfrac{r}{R}\right)^2}}\,\mathrm{d}r\,\mathrm{d}\eta\,, \end{aligned}$$

where θ runs from 0 to π and ϕ from 0 to 2π. The solution to above integral is

$$\mathcal{V} = 4\pi\int_0^\eta a^4(\eta)\frac{1}{2}\left[R^3\arcsin\frac{r}{R} - R^2 r\sqrt{1-\left(\frac{r}{R}\right)^2}\right]\mathrm{d}\eta\,. \tag{3}$$

Since in the Minkowski spacetime, apparently in an inertial frame of reference, along the path of a light ray, we have

$$\mathrm{d}s^2 = 0; \qquad \mathrm{d}\theta^2 = 0, \quad \mathrm{d}\phi^2 = 0. \tag{4}$$

Eq. (1) becomes now

$$
\begin{aligned}
0 &= a^2(\eta)^2 \left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-\left(\frac{r}{R}\right)^2} \right], \\
\mathrm{d}\eta^2 &= \frac{\mathrm{d}r^2}{1-\left(\frac{r}{R}\right)^2}, \\
\int_{\eta'}^{\eta} \mathrm{d}\eta &= \int_0^r \frac{\mathrm{d}r^2}{\sqrt{1-\left(\frac{r}{R}\right)^2}}, \\
\eta - \eta' &= R \arcsin \frac{r}{R}, \\
r &= R \sin \frac{\eta - \eta'}{R}.
\end{aligned}
$$

Now, substitute the above value of r in Eq. (3) and simply which will yield

$$
\mathcal{V}(\eta) = 2\pi \int_0^{\eta} a^4(\eta') \left[R^2(\eta - \eta') - \frac{R^3}{2} \sin \frac{2(\eta - \eta')}{R} \right] \mathrm{d}\eta' \,. \tag{5}
$$

Taylor expansion of sine gives:

$$
\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!} + \frac{x^{21}}{21!} - \frac{x^{23}}{23!} + \frac{x^{25}}{25!} - \frac{x^{27}}{27!} + \frac{x^{29}}{29!} - \cdots . \tag{6}
$$

Eq. (5) becomes now

$$
\begin{aligned}
\mathcal{V}(\eta) =& 2\pi \int_0^{\eta} a^4(\eta') \left[R^2(\eta - \eta') - \frac{R^3}{2} \left(\frac{2(\eta - \eta')}{R} - \frac{2^3(\eta - \eta')^3}{3!\, R^3} + \frac{2^5(\eta - \eta')^5}{5!\, R^5} - \frac{2^7(\eta - \eta')^7}{7!\, R^7} \right.\right. \\
&+ \frac{2^9(\eta - \eta')^9}{9!\, R^9} - \frac{2^{11}(\eta - \eta')^{11}}{11!\, R^{11}} + \frac{2^{13}(\eta - \eta')^{13}}{13!\, R^{13}} - \frac{2^{15}(\eta - \eta')^{15}}{15!\, R^{15}} + \frac{2^{17}(\eta - \eta')^{17}}{17!\, R^{17}} \\
&- \frac{2^{19}(\eta - \eta')^{19}}{19!\, R^{19}} + \frac{2^{21}(\eta - \eta')^{21}}{21!\, R^{21}} - \frac{2^{23}(\eta - \eta')^{23}}{23!\, R^{23}} + \frac{2^{25}(\eta - \eta')^{25}}{25!\, R^{25}} - \frac{2^{27}(\eta - \eta')^{27}}{27!\, R^{27}} \\
&\left.\left. + \frac{2^{29}(\eta - \eta')^{29}}{29!\, R^{29}} \right) \right] \mathrm{d}\eta' \,,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\mathcal{V}(\eta) =& 2\pi \left(-\frac{R^3}{2} \right) \int_0^{\eta} a^4(\eta') \left[-\frac{2^3(\eta - \eta')^3}{3!\, R^3} + \frac{2^5(\eta - \eta')^5}{5!\, R^5} - \frac{2^7(\eta - \eta')^7}{7!\, R^7} \right. \\
&+ \frac{2^9(\eta - \eta')^9}{9!\, R^9} - \frac{2^{11}(\eta - \eta')^{11}}{11!\, R^{11}} + \frac{2^{13}(\eta - \eta')^{13}}{13!\, R^{13}} - \frac{2^{15}(\eta - \eta')^{15}}{15!\, R^{15}} + \frac{2^{17}(\eta - \eta')^{17}}{17!\, R^{17}} \\
&- \frac{2^{19}(\eta - \eta')^{19}}{19!\, R^{19}} + \frac{2^{21}(\eta - \eta')^{21}}{21!\, R^{21}} - \frac{2^{23}(\eta - \eta')^{23}}{23!\, R^{23}} + \frac{2^{25}(\eta - \eta')^{25}}{25!\, R^{25}} - \frac{2^{27}(\eta - \eta')^{27}}{27!\, R^{27}} \\
&\left. + \frac{2^{29}(\eta - \eta')^{29}}{29!\, R^{29}} \right] \mathrm{d}\eta' \,.
\end{aligned} \tag{8}
$$

We can write it as:

$$\mathcal{V}(\eta) = V_{\rm F1} + V_{\rm F2} + V_{\rm F3} + V_{\rm F4} + V_{\rm F5} + V_{\rm F6} + V_{\rm F7} + V_{\rm F8} + V_{\rm F9} + V_{\rm F10} + V_{\rm F11} + V_{\rm F12} + V_{\rm F13} + V_{\rm F14}. \quad (9)$$

From Eq. (9), $V_{\rm F1}$ can be calculated as:

$$\begin{aligned} V_{\rm F1} &= 2\pi \int_0^\eta a^4(\eta')\frac{2}{3}(\eta-\eta')^3\,{\rm d}\eta'\,, \\ V_{\rm F1} &= \frac{4\pi}{3}\int_0^\eta a^4(\eta')(\eta-\eta')^3\,{\rm d}\eta'\,. \end{aligned} \quad (10)$$

The change in this integral can be evaluated as:

$$\begin{aligned} \frac{\Delta V_{\rm F1}}{\Delta\eta} &= \frac{3(4\pi)}{3}\int_0^\eta a^4(\eta')(\eta-\eta')^2\,{\rm d}\eta'\,, \\ \frac{\Delta V_{\rm F1}}{\Delta\eta} &= 4\pi\left[\eta^2\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 2\eta\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\right], \\ \Delta V_{\rm F1} &= 4\pi\left(\eta^2 I_{\rm 2F1} - 2\eta I_{\rm 1F1} + I_{\rm 0F1}\right)\Delta\eta, \end{aligned} \quad (11)$$

where

$$\begin{aligned} I_{\rm 2F1} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 2F1} = a^4(\eta)\Delta\eta \\ I_{\rm 1F1} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 1F1} = a^4(\eta)\eta\Delta\eta \\ I_{\rm 0F1} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 0F1} = a^4(\eta)\eta^2\Delta\eta \end{aligned}$$

From Eq. (9), $V_{\rm F2}$ can be calculated as:

$$\begin{aligned} V_{\rm F2} &= -2\pi\int_0^\eta a^4(\eta')\frac{2}{15\,R^2}(\eta-\eta')^5\,{\rm d}\eta'\,, \\ V_{\rm F2} &= -\frac{4\pi}{15\,R^2}\int_0^\eta a^4(\eta')(\eta-\eta')^5\,{\rm d}\eta'\,. \end{aligned} \quad (12)$$

The change in this integral can be evaluated as:

$$\begin{aligned} \frac{\Delta V_{\rm F2}}{\Delta\eta} =& -\frac{5(4\pi)}{15\,R^2}\int_0^\eta a^4(\eta')(\eta-\eta')^4\,{\rm d}\eta'\,, \\ \frac{\Delta V_{\rm F2}}{\Delta\eta} =& -\frac{4\pi}{3\,R^2}\left[\eta^4\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 4\eta^3\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 6\eta^2\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\right. \\ & \left. -4\eta\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta'\right], \\ \Delta V_{\rm F2} =& -\frac{4\pi}{3\,R^2}\left(\eta^4 I_{\rm 2F2} - 4\eta^3 I_{\rm 1F2} + 6\eta^2 I_{\rm 0F2}\right. \\ & \left. -4\eta I_{\rm 4F2} + I_{\rm 5F2}\right)\Delta\eta, \end{aligned} \quad (13)$$

where

$$I_{2F2} = \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{2F2} = a^4(\eta)\Delta\eta$$
$$I_{1F2} = \int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{1F2} = a^4(\eta)\eta\Delta\eta$$
$$I_{0F2} = \int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{0F2} = a^4(\eta)\eta^2\Delta\eta$$
$$I_{4F2} = \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{4F2} = a^4(\eta)\eta^3\Delta\eta$$
$$I_{5F2} = \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{5F2} = a^4(\eta)\eta^4\Delta\eta$$

From Eq. (9), V_{F3} can be calculated as:

$$V_{F3} = \frac{4(2\pi)}{315\ R^4}\int_0^\eta a^4(\eta')(\eta-\eta')^7\,\mathrm{d}\eta'\,. \tag{14}$$

The change in this integral can be evaluated as:

$$\begin{aligned}
\frac{\Delta V_{F3}}{\Delta\eta} &= \frac{7(8\pi)}{315\ R^4}\int_0^\eta a^4(\eta')(\eta-\eta')^6\,\mathrm{d}\eta'\,,\\
\frac{\Delta V_{F3}}{\Delta\eta} &= \frac{8\pi}{45\ R^4}\left[\eta^6\int_0^\eta a^4(\eta')\,\mathrm{d}\eta' - 6\eta^5\int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' + 15\eta^4\int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta'\right.\\
&\quad \left. -20\eta^3\int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' + 15\eta^2\int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' - 6\eta\int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' + \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta'\right],\\
\Delta V_{F3} &= \frac{8\pi}{45\ R^4}\left(\eta^6 I_{2F3} - 6\eta^5 I_{1F3} + 15\eta^4 I_{0F3}\right.\\
&\quad \left. -20\eta^3 I_{4F3} + 15\eta^2 I_{5F3} - 6\eta I_{6F3} + I_{7F3}\right),
\end{aligned} \tag{15}$$

where

$$I_{2F3} = \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{2F3} = a^4(\eta)\Delta\eta$$
$$I_{1F3} = \int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{1F3} = a^4(\eta)\eta\Delta\eta$$
$$I_{0F3} = \int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{0F3} = a^4(\eta)\eta^2\Delta\eta$$
$$I_{4F3} = \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{4F3} = a^4(\eta)\eta^3\Delta\eta$$
$$I_{5F3} = \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{5F3} = a^4(\eta)\eta^4\Delta\eta$$
$$I_{6F3} = \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{6F3} = a^4(\eta)\eta^5\Delta\eta$$
$$I_{7F3} = \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{7F3} = a^4(\eta)\eta^6\Delta\eta$$

From Eq. (9), $V_{\rm F4}$ can be calculated as:

$$V_{\rm F4} = -\frac{2(2\pi)}{2835\ R^6}\int_0^\eta a^4(\eta')(\eta-\eta')^9\,{\rm d}\eta'\,. \tag{16}$$

The change in this integral can be evaluated as:

$$\begin{aligned}
\frac{\Delta V_{\rm F4}}{\Delta\eta} =& -\frac{9(4\pi)}{2835\ R^6}\int_0^\eta a^4(\eta')(\eta-\eta')^8\,{\rm d}\eta'\,,\\
\frac{\Delta V_{\rm F4}}{\Delta\eta} =& -\frac{4\pi}{315\ R^6}\Big[\eta^8\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 8\eta^7\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 28\eta^6\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\\
&- 56\eta^5\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 70\eta^4\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 56\eta^3\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta'\\
&+28\eta^2\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 8\eta\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta'\Big]\,,\\
\Delta V_{\rm F4} =& -\frac{4\pi}{315\ R^6}\big(\eta^8 I_{2F4} - 8\eta^7 I_{1F4} + 28\eta^6 I_{0F4}\\
&- 56\eta^5 I_{4F4} + 70\eta^4 I_{5F4} - 56\eta^3 I_{6F4}\\
&+28\eta^2 I_{7F4} - 8\eta I_{8F4} + I_{9F4}\big)\,\Delta\eta,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
I_{\rm 2F4} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 2F4} = a^4(\eta)\Delta\eta\\
I_{\rm 1F4} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 1F4} = a^4(\eta)\eta\Delta\eta\\
I_{\rm 0F4} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 0F4} = a^4(\eta)\eta^2\Delta\eta\\
\\
I_{\rm 4F4} &= \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 4F4} = a^4(\eta)\eta^3\Delta\eta\\
I_{\rm 5F4} &= \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 5F4} = a^4(\eta)\eta^4\Delta\eta\\
I_{\rm 6F4} &= \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 6F4} = a^4(\eta)\eta^5\Delta\eta\\
I_{\rm 7F4} &= \int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 7F4} = a^4(\eta)\eta^6\Delta\eta\\
I_{\rm 8F4} &= \int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 8F4} = a^4(\eta)\eta^7\Delta\eta\\
I_{\rm 9F4} &= \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 9F4} = a^4(\eta)\eta^8\Delta\eta
\end{aligned}$$

From Eq. (9), $V_{\rm F5}$ can be calculated as:

$$V_{\rm F5} = \frac{4(2\pi)}{155\,925\ R^8}\int_0^\eta a^4(\eta')(\eta-\eta')^{11}\,{\rm d}\eta'\,. \tag{18}$$

The change in this integral can be evaluated as:

$$\begin{aligned}
\frac{\Delta V_{\rm F5}}{\Delta\eta} &= \frac{11(4\pi)}{155\,925\ R^8}\int_0^\eta a^4(\eta')(\eta-\eta')^{10}\,{\rm d}\eta'\,,\\
\frac{\Delta V_{\rm F5}}{\Delta\eta} &= \frac{8\pi}{14\,175\ R^8}\left[\eta^{10}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 10\eta^9\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 45\eta^8\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\right.\\
&\quad - 120\eta^7\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 210\eta^6\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 252\eta^5\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta'\\
&\quad + 210\eta^4\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 120\eta^3\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 45\eta^2\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta'\\
&\quad \left.-10\eta\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta'\right],\\
\Delta V_{\rm F5} &= \frac{8\pi}{14\,175\ R^8}\left(\eta^{10}I_{2F5} - 10\eta^9 I_{1F5} + 45\eta^8 I_{0F5}\right.\\
&\quad - 120\eta^7 I_{4F5} + 210\eta^6 I_{5F5} - 252\eta^5 I_{6F5}\\
&\quad + 210\eta^4 I_{7F5} - 120\eta^3 I_{8F5} + 45\eta^2 I_{9F5}\\
&\quad \left.-10\eta I_{10F5} + I_{11F5}\right)\Delta\eta, \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
I_{\rm 2F5} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 2F5} &= a^4(\eta)\Delta\eta\\
I_{\rm 1F5} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 1F5} &= a^4(\eta)\eta\Delta\eta\\
I_{\rm 0F5} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 0F5} &= a^4(\eta)\eta^2\Delta\eta\\
I_{\rm 4F5} &= \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 4F5} &= a^4(\eta)\eta^3\Delta\eta\\
I_{\rm 5F5} &= \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 5F5} &= a^4(\eta)\eta^4\Delta\eta\\
I_{\rm 6F5} &= \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 6F5} &= a^4(\eta)\eta^5\Delta\eta\\
I_{\rm 7F5} &= \int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 7F5} &= a^4(\eta)\eta^6\Delta\eta\\
I_{\rm 8F5} &= \int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 8F5} &= a^4(\eta)\eta^7\Delta\eta\\
I_{\rm 9F5} &= \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 9F5} &= a^4(\eta)\eta^8\Delta\eta\\
I_{\rm 10F5} &= \int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 10F5} &= a^4(\eta)\eta^9\Delta\eta\\
I_{\rm 11F5} &= \int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 11F5} &= a^4(\eta)\eta^{10}\Delta\eta
\end{aligned}$$

From Eq. (9), $V_{\rm F6}$ can be calculated as:

$$
\begin{aligned}
V_{\rm F6} &= 2\pi \left(-\frac{R^3}{2}\right) \int_0^\eta a^4(\eta') \frac{2^{13}(\eta-\eta')^{13}}{13!\ R^{13}}\, {\rm d}\eta' \,, \\
V_{\rm F6} &= -\frac{8\pi}{6\,081\,075\ R^{10}} \int_0^\eta a^4(\eta')(\eta-\eta')^{13}\, {\rm d}\eta' \,.
\end{aligned} \tag{20}
$$

The change in this integral can be evaluated as:

$$
\begin{aligned}
\frac{\Delta V_{\rm F6}}{\Delta\eta} =& -\frac{13(8\pi)}{6\,081\,075\ R^{10}} \int_0^\eta a^4(\eta')(\eta-\eta')^{12}\, {\rm d}\eta' \,, \\
\frac{\Delta V_{\rm F6}}{\Delta\eta} =& -\frac{32\pi}{2\,027\,025\ R^{10}} \left[\eta^{12}\int_0^\eta a^4(\eta')\, {\rm d}\eta' - 12\eta^{11}\int_0^\eta a^4(\eta')\eta'\, {\rm d}\eta' + 66\eta^{10}\int_0^\eta a^4(\eta')\eta'^2\, {\rm d}\eta' \right. \\
& - 220\eta^9 \int_0^\eta a^4(\eta')\eta'^3\, {\rm d}\eta' + 495\eta^8 \int_0^\eta a^4(\eta')\eta'^4\, {\rm d}\eta' - 792\eta^7 \int_0^\eta a^4(\eta')\eta'^5\, {\rm d}\eta' \\
& + 924\eta^6 \int_0^\eta a^4(\eta')\eta'^6\, {\rm d}\eta' - 792\eta^5 \int_0^\eta a^4(\eta')\eta'^7\, {\rm d}\eta' + 495\eta^4 \int_0^\eta a^4(\eta')\eta'^8\, {\rm d}\eta' \\
& - 220\eta^3 \int_0^\eta a^4(\eta')\eta'^9\, {\rm d}\eta' + 66\eta^2 \int_0^\eta a^4(\eta')\eta'^{10}\, {\rm d}\eta' - 12\eta \int_0^\eta a^4(\eta')\eta'^{11}\, {\rm d}\eta' \\
& \left. + \int_0^\eta a^4(\eta')\eta'^{12}\, {\rm d}\eta' \right] , \\
\Delta V_{\rm F6} =& -\frac{32\pi}{2\,027\,025\ R^{10}} \left(\eta^{12} I_{2F6} - 12\eta^{11} I_{1F6} + 66\eta^{10} I_{0F6} \right. \\
& - 220\eta^9 I_{4F6} + 495\eta^8 I_{5F6} - 792\eta^7 I_{6F6} \\
& + 924\eta^6 I_{7F6} - 792\eta^5 I_{8F6} + 495\eta^4 I_{9F6} \\
& - 220\eta^3 I_{10F6} + 66\eta^2 I_{11F6} - 12\eta I_{12F6} \\
& \left. + I_{13F6}\right) \Delta\eta,
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
I_{\text{2F6}} &= \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{2F6}} = a^4(\eta)\Delta\eta \\
I_{\text{1F6}} &= \int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{1F6}} = a^4(\eta)\eta\Delta\eta \\
I_{\text{0F6}} &= \int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{0F6}} = a^4(\eta)\eta^2\Delta\eta \\
I_{\text{4F6}} &= \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{4F6}} = a^4(\eta)\eta^3\Delta\eta \\
I_{\text{5F6}} &= \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{5F6}} = a^4(\eta)\eta^4\Delta\eta \\
I_{\text{6F6}} &= \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{6F6}} = a^4(\eta)\eta^5\Delta\eta \\
I_{\text{7F6}} &= \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{7F6}} = a^4(\eta)\eta^6\Delta\eta \\
I_{\text{8F6}} &= \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{8F6}} = a^4(\eta)\eta^7\Delta\eta \\
I_{\text{9F6}} &= \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{9F6}} = a^4(\eta)\eta^8\Delta\eta \\
I_{\text{10F6}} &= \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{10F6}} = a^4(\eta)\eta^9\Delta\eta \\
I_{\text{11F6}} &= \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{11F6}} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{\text{12F6}} &= \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{12F6}} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{\text{13F6}} &= \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{13F6}} = a^4(\eta)\eta^{12}\Delta\eta
\end{aligned}
$$

From Eq. (9), V_{F7} can be calculated as:

$$
\begin{aligned}
V_{\text{F7}} &= 2\pi\left(\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{15}(\eta-\eta')^{15}}{15!\;R^{15}}\,\mathrm{d}\eta'\,, \\
V_{\text{F7}} &= \frac{16\pi}{638\,512\,875\;R^{12}}\int_0^\eta a^4(\eta')(\eta-\eta')^{15}\,\mathrm{d}\eta'\,. \qquad (22)
\end{aligned}
$$

The change in this integral can be evaluated as:

$$\frac{\Delta V_{\rm F7}}{\Delta\eta} = \frac{15(16\pi)}{638\,512\,875\ R^{12}} \int_0^\eta a^4(\eta')(\eta-\eta')^{14}\,{\rm d}\eta'\,,$$

$$\begin{aligned}\frac{\Delta V_{\rm F7}}{\Delta\eta} = \frac{16\pi}{42\,567\,525\ R^{12}} \Bigg[&\eta^{14}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 14\eta^{13}\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 91\eta^{12}\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \\ &- 364\eta^{11}\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 1001\eta^{10}\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 2002\eta^9\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \\ &+ 3003\eta^8\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 3432\eta^7\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 3003\eta^6\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \\ &- 2002\eta^5\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + 1001\eta^4\int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' - 364\eta^3\int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta' \\ &+91\eta^2\int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' - 14\eta\int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta'\Bigg]\,,\end{aligned}$$

$$\begin{aligned}\Delta V_{\rm F7} = \frac{16\pi}{42\,567\,525\ R^{12}} \big(&\eta^{14}I_{2F7} - 14\eta^{13}I_{1F7} + 91\eta^{12}I_{0F7} \\ &- 364\eta^{11}I_{4F7} + 1001\eta^{10}I_{5F7} - 2002\eta^9 I_{6F7} \\ &+ 3003\eta^8 I_{7F7} - 3432\eta^7 I_{8F7} + 3003\eta^6 I_{9F7} \\ &- 2002\eta^5 I_{10F7} + 1001\eta^4 I_{11F7} - 364\eta^3 I_{12F7} \\ &+91\eta^2 I_{13F7} - 14\eta I_{14F7} + I_{15F7}\big)\,\Delta\eta,\end{aligned} \tag{23}$$

where

$$I_{\rm 2F7} = \int_0^\eta a^4(\eta')\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 2F7} = a^4(\eta)\Delta\eta$$

$$I_{\rm 1F7} = \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 1F7} = a^4(\eta)\eta\Delta\eta$$

$$I_{\rm 0F7} = \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 0F7} = a^4(\eta)\eta^2\Delta\eta$$

$$I_{\rm 4F7} = \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 4F7} = a^4(\eta)\eta^3\Delta\eta$$

$$I_{\rm 5F7} = \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 5F7} = a^4(\eta)\eta^4\Delta\eta$$

$$I_{\rm 6F7} = \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 6F7} = a^4(\eta)\eta^5\Delta\eta$$

$$I_{\rm 7F7} = \int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 7F7} = a^4(\eta)\eta^6\Delta\eta$$

$$I_{\rm 8F7} = \int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 8F7} = a^4(\eta)\eta^7\Delta\eta$$

$$I_{\rm 9F7} = \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 9F7} = a^4(\eta)\eta^8\Delta\eta$$

$$
\begin{aligned}
I_{10\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{10\text{F}7} = a^4(\eta)\eta^9\Delta\eta \\
I_{11\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{11\text{F}7} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{12\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{12\text{F}7} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{13\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{13\text{F}7} = a^4(\eta)\eta^{12}\Delta\eta \\
I_{14\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{14\text{F}7} = a^4(\eta)\eta^{13}\Delta\eta \\
I_{15\text{F}7} &= \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{15\text{F}7} = a^4(\eta)\eta^{14}\Delta\eta
\end{aligned}
$$

From Eq. (9), $V_{\text{F}8}$ can be calculated as:

$$
\begin{aligned}
V_{\text{F}8} &= 2\pi\left(-\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{17}(\eta-\eta')^{17}}{17!\;R^{17}}\,\mathrm{d}\eta'\,, \\
V_{\text{F}8} &= -\frac{4\pi}{10\,854\,718\,875\;R^{14}}\int_0^\eta a^4(\eta')(\eta-\eta')^{17}\,\mathrm{d}\eta'\,.
\end{aligned} \tag{24}
$$

The change in this integral can be evaluated as:

$$
\begin{aligned}
\frac{\Delta V_{\text{F}8}}{\Delta\eta} =& -\frac{17(4\pi)}{10\,854\,718\,875\;R^{14}}\int_0^\eta a^4(\eta')(\eta-\eta')^{16}\,\mathrm{d}\eta'\,, \\
\frac{\Delta V_{\text{F}8}}{\Delta\eta} =& -\frac{4\pi}{638\,512\,875\;R^{14}}\left[\eta^{16}\int_0^\eta a^4(\eta')\,\mathrm{d}\eta' - 16\eta^{15}\int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' + 120\eta^{14}\int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta'\right. \\
&- 560\eta^{13}\int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' + 1820\eta^{12}\int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' - 4368\eta^{11}\int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \\
&+ 8008\eta^{10}\int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' - 11\,440\eta^9\int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' + 12\,870\eta^8\int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' \\
&- 11\,440\eta^7\int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' + 8008\eta^6\int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' - 4368\eta^5\int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' \\
&+ 1820\eta^4\int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' - 560\eta^3\int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' + 120\eta^2\int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' \\
&\left.-16\eta\int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' + \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta'\right], \\
\Delta V_{\text{F}8} =& -\frac{4\pi}{638\,512\,875\;R^{14}}\left(\eta^{16}I_{2F8} - 16\eta^{15}I_{1F8} + 120\eta^{14}I_{0F8}\right. \\
&- 560\eta^{13}I_{4F8} + 1820\eta^{12}I_{5F8} - 4368\eta^{11}I_{6F8} \\
&+ 8008\eta^{10}I_{7F8} - 11\,440\eta^9 I_{8F8} + 12\,870\eta^8 I_{9F8} \\
&- 11\,440\eta^7 I_{10F8} + 8008\eta^6 I_{11F8} - 4368\eta^5 I_{12F8} \\
&+ 1820\eta^4 I_{13F8} - 560\eta^3 I_{14F8} + 120\eta^2 I_{15F8} \\
&\left.-16\eta I_{16F8} + I_{17F8}\right)\Delta\eta,
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
I_{\rm 2F8} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 2F8} = a^4(\eta)\Delta\eta \\
I_{\rm 1F8} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 1F8} = a^4(\eta)\eta\Delta\eta \\
I_{\rm 0F8} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 0F8} = a^4(\eta)\eta^2\Delta\eta \\
I_{\rm 4F8} &= \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 4F8} = a^4(\eta)\eta^3\Delta\eta \\
I_{\rm 5F8} &= \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 5F8} = a^4(\eta)\eta^4\Delta\eta \\
I_{\rm 6F8} &= \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 6F8} = a^4(\eta)\eta^5\Delta\eta \\
I_{\rm 7F8} &= \int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 7F8} = a^4(\eta)\eta^6\Delta\eta \\
I_{\rm 8F8} &= \int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 8F8} = a^4(\eta)\eta^7\Delta\eta \\
I_{\rm 9F8} &= \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 9F8} = a^4(\eta)\eta^8\Delta\eta \\
I_{\rm 10F8} &= \int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 10F8} = a^4(\eta)\eta^9\Delta\eta \\
I_{\rm 11F8} &= \int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 11F8} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{\rm 12F8} &= \int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 12F8} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{\rm 13F8} &= \int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 13F8} = a^4(\eta)\eta^{12}\Delta\eta \\
I_{\rm 14F8} &= \int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 14F8} = a^4(\eta)\eta^{13}\Delta\eta \\
I_{\rm 15F8} &= \int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 15F8} = a^4(\eta)\eta^{14}\Delta\eta \\
I_{\rm 16F8} &= \int_0^\eta a^4(\eta')\eta'^{15}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 16F8} = a^4(\eta)\eta^{15}\Delta\eta \\
I_{\rm 17F8} &= \int_0^\eta a^4(\eta')\eta'^{16}\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 17F8} = a^4(\eta)\eta^{16}\Delta\eta
\end{aligned}
$$

From Eq. (9), $V_{\rm F9}$ can be calculated as:

$$
\begin{aligned}
V_{\rm F9} &= 2\pi\left(\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{19}(\eta-\eta')^{19}}{19!\;R^{19}}\,{\rm d}\eta'\,, \\
V_{\rm F9} &= \frac{8\pi}{1\,856\,156\,927\,625\;R^{16}}\int_0^\eta a^4(\eta')(\eta-\eta')^{19}\,{\rm d}\eta'\,. \qquad (26)
\end{aligned}
$$

The change in this integral can be evaluated as:

$$
\begin{aligned}
\frac{\Delta V_{\text{F9}}}{\Delta\eta} &= \frac{19(8\pi)}{1\,856\,156\,927\,625\ R^{16}} \int_0^\eta a^4(\eta')(\eta-\eta')^{18}\,\mathrm{d}\eta'\,,\\
\frac{\Delta V_{\text{F9}}}{\Delta\eta} &= \frac{8\pi}{97\,692\,469\,875\ R^{16}} \left[\eta^{18}\int_0^\eta a^4(\eta')\,\mathrm{d}\eta' - 18\eta^{17}\int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' + 153\eta^{16}\int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta'\right.\\
&\quad - 816\eta^{15}\int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' + 3060\eta^{14}\int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' - 8568\eta^{13}\int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta'\\
&\quad + 18\,564\eta^{12}\int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' - 31\,824\eta^{11}\int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' + 43\,758\eta^{10}\int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta'\\
&\quad - 48\,620\eta^{9}\int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' + 43\,758\eta^{8}\int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' - 31\,824\eta^{7}\int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta'\\
&\quad + 18\,564\eta^{6}\int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' - 8568\eta^{5}\int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' + 3060\eta^{4}\int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta'\\
&\quad - 816\eta^{3}\int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' + 153\eta^{2}\int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' - 18\eta\int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta'\\
&\quad \left. + \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta'\right],\\
\Delta V_{\text{F9}} &= \frac{8\pi}{97\,692\,469\,875\ R^{16}} \left(\eta^{18}I_{2F9} - 18\eta^{17}I_{1F9} + 153\eta^{16}I_{0F9}\right.\\
&\quad - 816\eta^{15}I_{4F9} + 3060\eta^{14}I_{5F9} - 8568\eta^{13}I_{6F9}\\
&\quad + 18\,564\eta^{12}I_{7F9} - 31\,824\eta^{11}I_{8F9} + 43\,758\eta^{10}I_{9F9}\\
&\quad - 48\,620\eta^{9}I_{10F9} + 43\,758\eta^{8}I_{11F9} - 31\,824\eta^{7}I_{12F9}\\
&\quad + 18\,564\eta^{6}I_{13F9} - 8568\eta^{5}I_{14F9} + 3060\eta^{4}I_{15F9}\\
&\quad - 816\eta^{3}I_{16F9} + 153\eta^{2}I_{17F9} - 18\eta I_{18F9}\\
&\quad \left. + I_{19F9}\right)\Delta\eta, \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{\text{2F9}} &= \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{2F9}} &= a^4(\eta)\Delta\eta\\
I_{\text{1F9}} &= \int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{1F9}} &= a^4(\eta)\eta\Delta\eta\\
I_{\text{0F9}} &= \int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{0F9}} &= a^4(\eta)\eta^2\Delta\eta\\
I_{\text{4F9}} &= \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{4F9}} &= a^4(\eta)\eta^3\Delta\eta\\
I_{\text{5F9}} &= \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{5F9}} &= a^4(\eta)\eta^4\Delta\eta\\
I_{\text{6F9}} &= \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\text{6F9}} &= a^4(\eta)\eta^5\Delta\eta
\end{aligned}
$$

$$
\begin{aligned}
I_{\text{7F9}} &= \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{7F9}} = a^4(\eta)\eta^6\Delta\eta \\
I_{\text{8F9}} &= \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{8F9}} = a^4(\eta)\eta^7\Delta\eta \\
I_{\text{9F9}} &= \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{9F9}} = a^4(\eta)\eta^8\Delta\eta \\
I_{\text{10F9}} &= \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{10F9}} = a^4(\eta)\eta^9\Delta\eta \\
I_{\text{11F9}} &= \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{11F9}} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{\text{12F9}} &= \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{12F9}} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{\text{13F9}} &= \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{13F9}} = a^4(\eta)\eta^{12}\Delta\eta \\
I_{\text{14F9}} &= \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{14F9}} = a^4(\eta)\eta^{13}\Delta\eta \\
I_{\text{15F9}} &= \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{15F9}} = a^4(\eta)\eta^{14}\Delta\eta \\
I_{\text{16F9}} &= \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{16F9}} = a^4(\eta)\eta^{15}\Delta\eta \\
I_{\text{17F9}} &= \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{17F9}} = a^4(\eta)\eta^{16}\Delta\eta \\
I_{\text{18F9}} &= \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{18F9}} = a^4(\eta)\eta^{17}\Delta\eta \\
I_{\text{19F9}} &= \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{\text{19F9}} = a^4(\eta)\eta^{18}\Delta\eta
\end{aligned}
$$

From Eq. (9), V_{F10} can be calculated as:

$$
\begin{aligned}
V_{\text{F10}} &= 2\pi\left(-\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{21}(\eta-\eta')^{21}}{21!\;R^{21}}\,\mathrm{d}\eta'\,, \\
V_{\text{F10}} &= -\frac{8\pi}{194\,896\,477\,400\,625\;R^{18}}\int_0^\eta a^4(\eta')(\eta-\eta')^{20}\,\mathrm{d}\eta'\,.
\end{aligned}
\tag{28}
$$

The change in this integral can be evaluated as:

$$\frac{\Delta V_{\rm F10}}{\Delta\eta} = -\frac{21(8\pi)}{194\,896\,477\,400\,625\ R^{18}}\int_0^\eta a^4(\eta')(\eta-\eta')^{20}\,{\rm d}\eta'\,,$$

$$\begin{aligned}\frac{\Delta V_{\rm F10}}{\Delta\eta} = &-\frac{8\pi}{9\,280\,784\,638\,125\ R^{18}}\left[\eta^{20}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 20\eta^{19}\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 190\eta^{18}\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\right.\\
&- 1140\eta^{17}\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 4845\eta^{16}\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 15\,504\eta^{15}\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta'\\
&+ 38\,760\eta^{14}\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 77\,520\eta^{13}\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 125\,970\eta^{12}\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta'\\
&- 167\,960\eta^{11}\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + 184\,756\eta^{10}\int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' - 167\,960\eta^9\int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta'\\
&+ 125\,970\eta^8\int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' - 77\,250\eta^7\int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' + 38\,760\eta^6\int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta'\\
&- 15\,504\eta^5\int_0^\eta a^4(\eta')\eta'^{15}\,{\rm d}\eta' + 4845\eta^4\int_0^\eta a^4(\eta')\eta'^{16}\,{\rm d}\eta' - 1140\eta^3\int_0^\eta a^4(\eta')\eta'^{17}\,{\rm d}\eta'\\
&\left.+190\eta^2\int_0^\eta a^4(\eta')\eta'^{18}\,{\rm d}\eta' - 20\eta\int_0^\eta a^4(\eta')\eta'^{19}\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^{20}\,{\rm d}\eta'\right],\end{aligned}$$

$$\begin{aligned}\Delta V_{\rm F10} = &-\frac{8\pi}{9\,280\,784\,638\,125\ R^{18}}\left(\eta^{20}I_{2F10} - 20\eta^{19}I_{1F10} + 190\eta^{18}I_{0F10}\right.\\
&- 1140\eta^{17}I_{4F10} + 4845\eta^{16}I_{5F10} - 15\,504\eta^{15}I_{6F10}\\
&+ 38\,760\eta^{14}I_{7F10} - 77\,520\eta^{13}I_{8F10} + 125\,970\eta^{12}I_{9F10}\\
&- 167\,960\eta^{11}I_{10F10} + 184\,756\eta^{10}I_{11F10} - 167\,960\eta^9 I_{12F10}\\
&+ 125\,970\eta^8 I_{13F10} - 77\,250\eta^7 I_{14F10} + 38\,760\eta^6 I_{15F10}\\
&- 15\,504\eta^5 I_{16F10} + 4845\eta^4 I_{17F10} - 1140\eta^3 I_{18F10}\\
&\left.+190\eta^2 I_{19F10} - 20\eta I_{20F10} + I_{21F10}\right)\Delta\eta,\end{aligned} \tag{29}$$

where

$$I_{\rm 2F10} = \int_0^\eta a^4(\eta')\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 2F10} = a^4(\eta)\Delta\eta$$

$$I_{\rm 1F10} = \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 1F10} = a^4(\eta)\eta\Delta\eta$$

$$I_{\rm 0F10} = \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 0F10} = a^4(\eta)\eta^2\Delta\eta$$

$$I_{\rm 4F10} = \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 4F10} = a^4(\eta)\eta^3\Delta\eta$$

$$I_{\rm 5F10} = \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 5F10} = a^4(\eta)\eta^4\Delta\eta$$

$$I_{\rm 6F10} = \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 6F10} = a^4(\eta)\eta^5\Delta\eta$$

$$
\begin{aligned}
I_{7\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{7\text{F}10} = a^4(\eta)\eta^6\Delta\eta \\
I_{8\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{8\text{F}10} = a^4(\eta)\eta^7\Delta\eta \\
I_{9\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{9\text{F}10} = a^4(\eta)\eta^8\Delta\eta \\
I_{10\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{10\text{F}10} = a^4(\eta)\eta^9\Delta\eta \\
I_{11\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{11\text{F}10} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{12\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{12\text{F}10} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{13\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{13\text{F}10} = a^4(\eta)\eta^{12}\Delta\eta \\
I_{14\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{14\text{F}10} = a^4(\eta)\eta^{13}\Delta\eta \\
I_{15\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{15\text{F}10} = a^4(\eta)\eta^{14}\Delta\eta \\
I_{16\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{16\text{F}10} = a^4(\eta)\eta^{15}\Delta\eta \\
I_{17\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{17\text{F}10} = a^4(\eta)\eta^{16}\Delta\eta \\
I_{18\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{18\text{F}10} = a^4(\eta)\eta^{17}\Delta\eta \\
I_{19\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{19\text{F}10} = a^4(\eta)\eta^{18}\Delta\eta \\
I_{20\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{20\text{F}10} = a^4(\eta)\eta^{19}\Delta\eta \\
I_{21\text{F}10} &= \int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{21\text{F}10} = a^4(\eta)\eta^{20}\Delta\eta
\end{aligned}
$$

From Eq. (9), $V_{\text{F}11}$ can be calculated as:

$$
\begin{aligned}
V_{\text{F}11} &= 2\pi\left(\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{23}(\eta-\eta')^{23}}{23!\;R^{23}}\,\mathrm{d}\eta'\,, \\
V_{\text{F}11} &= \frac{16\pi}{49\,308\,808\,782\,358\,125\;R^{20}}\int_0^\eta a^4(\eta')(\eta-\eta')^{23}\,\mathrm{d}\eta'\,. \qquad (30)
\end{aligned}
$$

The change in this integral can be evaluated as:

$$\frac{\Delta V_{\rm F11}}{\Delta\eta} = \frac{23(16\pi)}{49\,308\,808\,782\,358\,125\ R^{20}} \int_0^\eta a^4(\eta')(\eta-\eta')^{22}\,{\rm d}\eta'\,,$$

$$\begin{aligned}
\frac{\Delta V_{\rm F11}}{\Delta\eta} = \frac{16\pi}{2\,143\,861\,251\,406\,875\ R^{20}} \Bigg[& \eta^{22}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 22\eta^{21}\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 231\eta^{20}\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \\
& - 1540\eta^{19}\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 7315\eta^{18}\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 26\,334\eta^{17}\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \\
& + 74\,613\eta^{16}\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 170\,544\eta^{15}\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 319\,770\eta^{14}\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \\
& - 497\,420\eta^{13}\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + 646\,646\eta^{12}\int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' - 705\,432\eta^{11}\int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta' \\
& + 646\,646\eta^{10}\int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' - 497\,420\eta^{9}\int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' + 319\,770\eta^{8}\int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta' \\
& - 170\,544\eta^{7}\int_0^\eta a^4(\eta')\eta'^{15}\,{\rm d}\eta' + 74\,613\eta^{6}\int_0^\eta a^4(\eta')\eta'^{16}\,{\rm d}\eta' - 26\,334\eta^{5}\int_0^\eta a^4(\eta')\eta'^{17}\,{\rm d}\eta' \\
& + 7315\eta^{4}\int_0^\eta a^4(\eta')\eta'^{18}\,{\rm d}\eta' - 1540\eta^{3}\int_0^\eta a^4(\eta')\eta'^{19}\,{\rm d}\eta' + 231\eta^{2}\int_0^\eta a^4(\eta')\eta'^{20}\,{\rm d}\eta' \\
& - 22\eta\int_0^\eta a^4(\eta')\eta'^{21}\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^{22}\,{\rm d}\eta' \Bigg]\,,
\end{aligned}$$

$$\begin{aligned}
\Delta V_{\rm F11} = \frac{16\pi}{2\,143\,861\,251\,406\,875\ R^{20}} \big(& \eta^{22} I_{2F11} - 22\eta^{21} I_{1F11} + 231\eta^{20} I_{0F11} \\
& - 1540\eta^{19} I_{4F11} + 7315\eta^{18} I_{5F11} - 26\,334\eta^{17} I_{6F11} \\
& + 74\,613\eta^{16} I_{7F11} - 170\,544\eta^{15} I_{8F11} + 319\,770\eta^{14} I_{9F11} \\
& - 497\,420\eta^{13} I_{10F11} + 646\,646\eta^{12} I_{11F11} - 705\,432\eta^{11} I_{12F11} \\
& + 646\,646\eta^{10} I_{13F11} - 497\,420\eta^{9} I_{14F11} + 319\,770\eta^{8} I_{15F11} \\
& - 170\,544\eta^{7} I_{16F11} + 74\,613\eta^{6} I_{17F11} - 26\,334\eta^{5} I_{18F11} \\
& + 7315\eta^{4} I_{19F11} - 1540\eta^{3} I_{20F11} + 231\eta^{2} I_{21F11} \\
& - 22\eta I_{22F11} + I_{23F11} \big)\,\Delta\eta, \qquad (31)
\end{aligned}$$

where

$$\begin{aligned}
I_{\rm 2F11} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 2F11} = a^4(\eta)\Delta\eta \\
I_{\rm 1F11} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 1F11} = a^4(\eta)\eta\Delta\eta \\
I_{\rm 0F11} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 0F11} = a^4(\eta)\eta^2\Delta\eta \\
I_{\rm 4F11} &= \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 4F11} = a^4(\eta)\eta^3\Delta\eta \\
I_{\rm 5F11} &= \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' && \Rightarrow && \Delta I_{\rm 5F11} = a^4(\eta)\eta^4\Delta\eta
\end{aligned}$$

$$
\begin{aligned}
I_{6\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{6\text{F}11} = a^4(\eta)\eta^5\Delta\eta \\
I_{7\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{7\text{F}11} = a^4(\eta)\eta^6\Delta\eta \\
I_{8\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{8\text{F}11} = a^4(\eta)\eta^7\Delta\eta \\
I_{9\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{9\text{F}11} = a^4(\eta)\eta^8\Delta\eta \\
I_{10\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{10\text{F}11} = a^4(\eta)\eta^9\Delta\eta \\
I_{11\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{11\text{F}11} = a^4(\eta)\eta^{10}\Delta\eta \\
I_{12\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{12\text{F}11} = a^4(\eta)\eta^{11}\Delta\eta \\
I_{13\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{13\text{F}11} = a^4(\eta)\eta^{12}\Delta\eta \\
I_{14\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{14\text{F}11} = a^4(\eta)\eta^{13}\Delta\eta \\
I_{15\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{15\text{F}11} = a^4(\eta)\eta^{14}\Delta\eta \\
I_{16\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{16\text{F}11} = a^4(\eta)\eta^{15}\Delta\eta \\
I_{17\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{17\text{F}11} = a^4(\eta)\eta^{16}\Delta\eta \\
I_{18\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{18\text{F}11} = a^4(\eta)\eta^{17}\Delta\eta \\
I_{19\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{19\text{F}11} = a^4(\eta)\eta^{18}\Delta\eta \\
I_{20\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{20\text{F}11} = a^4(\eta)\eta^{19}\Delta\eta \\
I_{21\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{21\text{F}11} = a^4(\eta)\eta^{20}\Delta\eta \\
I_{22\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{21}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{22\text{F}11} = a^4(\eta)\eta^{21}\Delta\eta \\
I_{23\text{F}11} &= \int_0^\eta a^4(\eta')\eta'^{22}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{23\text{F}11} = a^4(\eta)\eta^{22}\Delta\eta
\end{aligned}
$$

From Eq. (9), $V_{\text{F}12}$ can be calculated as:

$$
\begin{aligned}
V_{\text{F}12} &= 2\pi\left(-\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{25}(\eta-\eta')^{25}}{25!\,R^{25}}\,\mathrm{d}\eta'\,, \\
V_{\text{F}12} &= -\frac{8\pi}{3\,698\,160\,658\,676\,859\,375\;R^{22}}\int_0^\eta a^4(\eta')(\eta-\eta')^{25}\,\mathrm{d}\eta'\,. \qquad (32)
\end{aligned}
$$

The change in this integral can be evaluated as:

$$
\begin{aligned}
\frac{\Delta V_{\rm F12}}{\Delta\eta} =& -\frac{25(8\pi)}{3\,698\,160\,658\,676\,859\,375\ R^{22}}\int_0^\eta a^4(\eta')(\eta-\eta')^{24}\,{\rm d}\eta'\,,\\
\frac{\Delta V_{\rm F12}}{\Delta\eta} =& -\frac{8\pi}{147\,926\,426\,347\,074\,375\ R^{22}}\left[\eta^{24}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 24\eta^{23}\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 276\eta^{22}\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta'\right.\\
&- 2024\eta^{21}\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 10\,626\eta^{20}\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 42\,504\eta^{19}\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta'\\
&+ 134\,596\eta^{18}\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 346\,104\eta^{17}\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 735\,471\eta^{16}\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta'\\
&- 1\,307\,504\eta^{15}\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + 1\,961\,256\eta^{14}\int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' - 2\,496\,144\eta^{13}\int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta'\\
&+ 2\,704\,156\eta^{12}\int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' - 2\,496\,144\eta^{11}\int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' + 1\,961\,256\eta^{10}\int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta'\\
&- 1\,307\,504\eta^{9}\int_0^\eta a^4(\eta')\eta'^{15}\,{\rm d}\eta' + 735\,471\eta^{8}\int_0^\eta a^4(\eta')\eta'^{16}\,{\rm d}\eta' - 346\,104\eta^{7}\int_0^\eta a^4(\eta')\eta'^{17}\,{\rm d}\eta'\\
&+ 134\,596\eta^{6}\int_0^\eta a^4(\eta')\eta'^{18}\,{\rm d}\eta' - 42\,504\eta^{5}\int_0^\eta a^4(\eta')\eta'^{19}\,{\rm d}\eta' + 10\,626\eta^{4}\int_0^\eta a^4(\eta')\eta'^{20}\,{\rm d}\eta'\\
&- 2024\eta^{3}\int_0^\eta a^4(\eta')\eta'^{21}\,{\rm d}\eta' + 276\eta^{2}\int_0^\eta a^4(\eta')\eta'^{22}\,{\rm d}\eta' - 24\eta\int_0^\eta a^4(\eta')\eta'^{23}\,{\rm d}\eta'\\
&\left.+\int_0^\eta a^4(\eta')\eta'^{24}\,{\rm d}\eta'\right],\\
\Delta V_{\rm F12} =& -\frac{8\pi}{147\,926\,426\,347\,074\,375\ R^{22}}\left(\eta^{24}I_{2F12} - 24\eta^{23}I_{1F12} + 276\eta^{22}I_{0F12}\right.\\
&- 2024\eta^{21}I_{4F12} + 10\,626\eta^{20}I_{5F12} - 42\,504\eta^{19}I_{6F12}\\
&+ 134\,596\eta^{18}I_{7F12} - 346\,104\eta^{17}I_{8F12} + 735\,471\eta^{16}I_{9F12}\\
&- 1\,307\,504\eta^{15}I_{10F12} + 1\,961\,256\eta^{14}I_{11F12} - 2\,496\,144\eta^{13}I_{12F12}\\
&+ 2\,704\,156\eta^{12}I_{13F12} - 2\,496\,144\eta^{11}I_{14F12} + 1\,961\,256\eta^{10}I_{15F12}\\
&- 1\,307\,504\eta^{9}I_{16F12} + 735\,471\eta^{8}I_{17F12} - 346\,104\eta^{7}I_{18F12}\\
&+ 134\,596\eta^{6}I_{19F12} - 42\,504\eta^{5}I_{20F12} + 10\,626\eta^{4}I_{21F12}\\
&- 2024\eta^{3}I_{22F12} + 276\eta^{2}I_{23F12} - 24\eta I_{24F12}\\
&\left.+I_{25F12}\right)\Delta\eta, \qquad (33)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{\rm 2F12} &= \int_0^\eta a^4(\eta')\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 2F12} &= a^4(\eta)\Delta\eta\\
I_{\rm 1F12} &= \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 1F12} &= a^4(\eta)\eta\Delta\eta\\
I_{\rm 0F12} &= \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \qquad &\Rightarrow \qquad \Delta I_{\rm 0F12} &= a^4(\eta)\eta^2\Delta\eta
\end{aligned}
$$

$$I_{4\text{F}12} = \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{4\text{F}12} = a^4(\eta)\eta^3\Delta\eta$$

$$I_{5\text{F}12} = \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{5\text{F}12} = a^4(\eta)\eta^4\Delta\eta$$

$$I_{6\text{F}12} = \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{6\text{F}12} = a^4(\eta)\eta^5\Delta\eta$$

$$I_{7\text{F}12} = \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{7\text{F}12} = a^4(\eta)\eta^6\Delta\eta$$

$$I_{8\text{F}12} = \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{8\text{F}12} = a^4(\eta)\eta^7\Delta\eta$$

$$I_{9\text{F}12} = \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{9\text{F}12} = a^4(\eta)\eta^8\Delta\eta$$

$$I_{10\text{F}12} = \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{10\text{F}12} = a^4(\eta)\eta^9\Delta\eta$$

$$I_{11\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{11\text{F}12} = a^4(\eta)\eta^{10}\Delta\eta$$

$$I_{12\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{12\text{F}12} = a^4(\eta)\eta^{11}\Delta\eta$$

$$I_{13\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{13\text{F}12} = a^4(\eta)\eta^{12}\Delta\eta$$

$$I_{14\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{14\text{F}12} = a^4(\eta)\eta^{13}\Delta\eta$$

$$I_{15\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{15\text{F}12} = a^4(\eta)\eta^{14}\Delta\eta$$

$$I_{16\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{16\text{F}12} = a^4(\eta)\eta^{15}\Delta\eta$$

$$I_{17\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{17\text{F}12} = a^4(\eta)\eta^{16}\Delta\eta$$

$$I_{18\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{18\text{F}12} = a^4(\eta)\eta^{17}\Delta\eta$$

$$I_{19\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{19\text{F}12} = a^4(\eta)\eta^{18}\Delta\eta$$

$$I_{20\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{20\text{F}12} = a^4(\eta)\eta^{19}\Delta\eta$$

$$I_{21\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{21\text{F}12} = a^4(\eta)\eta^{20}\Delta\eta$$

$$I_{22\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{21}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{22\text{F}12} = a^4(\eta)\eta^{21}\Delta\eta$$

$$I_{23\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{22}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{23\text{F}12} = a^4(\eta)\eta^{22}\Delta\eta$$

$$I_{24\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{23}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{24\text{F}12} = a^4(\eta)\eta^{23}\Delta\eta$$

$$I_{25\text{F}12} = \int_0^\eta a^4(\eta')\eta'^{24}\,\mathrm{d}\eta' \quad \Rightarrow \quad \Delta I_{25\text{F}12} = a^4(\eta)\eta^{24}\Delta\eta$$

From Eq. (9), $V_{\rm F13}$ can be calculated as:

$$V_{\rm F13} = 2\pi \left(\frac{R^3}{2}\right) \int_0^\eta a^4(\eta') \frac{2^{27}(\eta-\eta')^{27}}{27!\ R^{27}}\, {\rm d}\eta' \,,$$
$$V_{\rm F13} = \frac{16\pi}{1\,298\,054\,391\,195\,577\,640\,625\ R^{24}} \int_0^\eta a^4(\eta')(\eta-\eta')^{27}\, {\rm d}\eta' \,. \tag{34}$$

The change in this integral can be evaluated as:

$$\begin{aligned}
\frac{\Delta V_{\rm F13}}{\Delta\eta} &= \frac{27(16\pi)}{1\,298\,054\,391\,195\,577\,640\,625\ R^{24}} \int_0^\eta a^4(\eta')(\eta-\eta')^{26}\, {\rm d}\eta' \,, \\
\frac{\Delta V_{\rm F13}}{\Delta\eta} &= \frac{16\pi}{48\,076\,088\,562\,799\,171\,875\ R^{24}} \Bigg[\eta^{26}\int_0^\eta a^4(\eta')\,{\rm d}\eta' - 26\eta^{25}\int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' + 325\eta^{24}\int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \\
&\quad - 2600\eta^{23}\int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' + 14\,950\eta^{22}\int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' - 65\,780\eta^{21}\int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \\
&\quad + 230\,230\eta^{20}\int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' - 657\,800\eta^{19}\int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' + 1\,562\,275\eta^{18}\int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \\
&\quad - 3\,124\,550\eta^{17}\int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' + 5\,311\,735\eta^{16}\int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' - 7\,726\,160\eta^{15}\int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta' \\
&\quad + 9\,657\,700\eta^{14}\int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' - 10\,400\,600\eta^{13}\int_0^\eta a^4(\eta')\eta'^{13}\,{\rm d}\eta' + 9\,657\,700\eta^{12}\int_0^\eta a^4(\eta')\eta'^{14}\,{\rm d}\eta' \\
&\quad - 7\,726\,160\eta^{11}\int_0^\eta a^4(\eta')\eta'^{15}\,{\rm d}\eta' + 5\,311\,735\eta^{10}\int_0^\eta a^4(\eta')\eta'^{16}\,{\rm d}\eta' - 3\,124\,550\eta^{9}\int_0^\eta a^4(\eta')\eta'^{17}\,{\rm d}\eta' \\
&\quad + 1\,562\,275\eta^{8}\int_0^\eta a^4(\eta')\eta'^{18}\,{\rm d}\eta' - 657\,800\eta^{7}\int_0^\eta a^4(\eta')\eta'^{19}\,{\rm d}\eta' + 230\,230\eta^{6}\int_0^\eta a^4(\eta')\eta'^{20}\,{\rm d}\eta' \\
&\quad - 65\,780\eta^{5}\int_0^\eta a^4(\eta')\eta'^{21}\,{\rm d}\eta' + 14\,950\eta^{4}\int_0^\eta a^4(\eta')\eta'^{22}\,{\rm d}\eta' - 2600\eta^{3}\int_0^\eta a^4(\eta')\eta'^{23}\,{\rm d}\eta' \\
&\quad + 325\eta^{2}\int_0^\eta a^4(\eta')\eta'^{24}\,{\rm d}\eta' - 26\eta\int_0^\eta a^4(\eta')\eta'^{25}\,{\rm d}\eta' + \int_0^\eta a^4(\eta')\eta'^{26}\,{\rm d}\eta'\Bigg] \,, \\
\frac{\Delta V_{\rm F13}}{\Delta\eta} &= \frac{16\pi}{48\,076\,088\,562\,799\,171\,875\ R^{24}} \big(\eta^{26} I_{2F13} - 26\eta^{25} I_{1F13} + 325\eta^{24} I_{0F13} \\
&\quad - 2600\eta^{23} I_{4F13} + 14\,950\eta^{22} I_{5F13} - 65\,780\eta^{21} I_{6F13} \\
&\quad + 230\,230\eta^{20} I_{7F13} - 657\,800\eta^{19} I_{8F13} + 1\,562\,275\eta^{18} I_{9F13} \\
&\quad - 3\,124\,550\eta^{17} I_{10F13} + 5\,311\,735\eta^{16} I_{11F13} - 7\,726\,160\eta^{15} I_{12F13} \\
&\quad + 9\,657\,700\eta^{14} I_{13F13} - 10\,400\,600\eta^{13} I_{14F13} + 9\,657\,700\eta^{12} I_{15F13} \\
&\quad - 7\,726\,160\eta^{11} I_{16F13} + 5\,311\,735\eta^{10} I_{17F13} - 3\,124\,550\eta^{9} I_{18F13} \\
&\quad + 1\,562\,275\eta^{8} I_{19F13} - 657\,800\eta^{7} I_{20F13} + 230\,230\eta^{6} I_{21F13} \\
&\quad - 65\,780\eta^{5} I_{22F13} + 14\,950\eta^{4} I_{23F13} - 2600\eta^{3} I_{24F13} \\
&\quad + 325\eta^{2} I_{25F13} - 26\eta I_{26F13} + I_{27F13}\big)\,\Delta\eta,
\end{aligned} \tag{35}$$

where

$$I_{2\text{F}13} = \int_0^\eta a^4(\eta')\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{2\text{F}13} = a^4(\eta)\Delta\eta$$

$$I_{1\text{F}13} = \int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{1\text{F}13} = a^4(\eta)\eta\Delta\eta$$

$$I_{0\text{F}13} = \int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{0\text{F}13} = a^4(\eta)\eta^2\Delta\eta$$

$$I_{4\text{F}13} = \int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{4\text{F}13} = a^4(\eta)\eta^3\Delta\eta$$

$$I_{5\text{F}13} = \int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{5\text{F}13} = a^4(\eta)\eta^4\Delta\eta$$

$$I_{6\text{F}13} = \int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{6\text{F}13} = a^4(\eta)\eta^5\Delta\eta$$

$$I_{7\text{F}13} = \int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{7\text{F}13} = a^4(\eta)\eta^6\Delta\eta$$

$$I_{8\text{F}13} = \int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{8\text{F}13} = a^4(\eta)\eta^7\Delta\eta$$

$$I_{9\text{F}13} = \int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{9\text{F}13} = a^4(\eta)\eta^8\Delta\eta$$

$$I_{10\text{F}13} = \int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{10\text{F}13} = a^4(\eta)\eta^9\Delta\eta$$

$$I_{11\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{11\text{F}13} = a^4(\eta)\eta^{10}\Delta\eta$$

$$I_{12\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{12\text{F}13} = a^4(\eta)\eta^{11}\Delta\eta$$

$$I_{13\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{13\text{F}13} = a^4(\eta)\eta^{12}\Delta\eta$$

$$I_{14\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{14\text{F}13} = a^4(\eta)\eta^{13}\Delta\eta$$

$$I_{15\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{15\text{F}13} = a^4(\eta)\eta^{14}\Delta\eta$$

$$I_{16\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{16\text{F}13} = a^4(\eta)\eta^{15}\Delta\eta$$

$$I_{17\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{17\text{F}13} = a^4(\eta)\eta^{16}\Delta\eta$$

$$I_{18\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{18\text{F}13} = a^4(\eta)\eta^{17}\Delta\eta$$

$$I_{19\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{19\text{F}13} = a^4(\eta)\eta^{18}\Delta\eta$$

$$I_{20\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{20\text{F}13} = a^4(\eta)\eta^{19}\Delta\eta$$

$$I_{21\text{F}13} = \int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{21\text{F}13} = a^4(\eta)\eta^{20}\Delta\eta$$

$$
\begin{aligned}
I_{22\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{21}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{22\text{F}13} = a^4(\eta)\eta^{21}\Delta\eta \\
I_{23\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{22}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{23\text{F}13} = a^4(\eta)\eta^{22}\Delta\eta \\
I_{24\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{23}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{24\text{F}13} = a^4(\eta)\eta^{23}\Delta\eta \\
I_{25\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{24}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{25\text{F}13} = a^4(\eta)\eta^{24}\Delta\eta \\
I_{26\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{25}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{26\text{F}13} = a^4(\eta)\eta^{25}\Delta\eta \\
I_{27\text{F}13} &= \int_0^\eta a^4(\eta')\eta'^{26}\,\mathrm{d}\eta' && \Rightarrow && \Delta I_{27\text{F}13} = a^4(\eta)\eta^{26}\Delta\eta
\end{aligned}
$$

From Eq. (9), $V_{\text{F}14}$ can be calculated as:

$$
\begin{aligned}
V_{\text{F}14} &= 2\pi\left(-\frac{R^3}{2}\right)\int_0^\eta a^4(\eta')\frac{2^{29}(\eta-\eta')^{29}}{29!\;R^{29}}\,\mathrm{d}\eta'\,, \\
V_{\text{F}14} &= -\frac{16\pi}{263\,505\,041\,412\,702\,261\,046\,875\;R^{26}}\int_0^\eta a^4(\eta')(\eta-\eta')^{29}\,\mathrm{d}\eta'\,. \qquad (36)
\end{aligned}
$$

The change in this integral can be evaluated as:

$$
\begin{aligned}
\frac{\Delta V_{\text{F}14}}{\Delta\eta} &= -\frac{29(16\pi)}{263\,505\,041\,412\,702\,261\,046\,875\;R^{26}}\int_0^\eta a^4(\eta')(\eta-\eta')^{28}\,\mathrm{d}\eta'\,, \\
\frac{\Delta V_{\text{F}14}}{\Delta\eta} &= -\frac{16\pi}{9\,086\,380\,738\,369\,043\,484\,375\;R^{26}}\left[\eta^{28}\int_0^\eta a^4(\eta')\,\mathrm{d}\eta' - 28\eta^{27}\int_0^\eta a^4(\eta')\eta'\,\mathrm{d}\eta' + 378\eta^{26}\int_0^\eta a^4(\eta')\eta'^2\,\mathrm{d}\eta'\right. \\
&\quad - 3276\eta^{25}\int_0^\eta a^4(\eta')\eta'^3\,\mathrm{d}\eta' + 20\,475\eta^{24}\int_0^\eta a^4(\eta')\eta'^4\,\mathrm{d}\eta' - 98\,280\eta^{23}\int_0^\eta a^4(\eta')\eta'^5\,\mathrm{d}\eta' \\
&\quad + 376\,740\eta^{22}\int_0^\eta a^4(\eta')\eta'^6\,\mathrm{d}\eta' - 1\,184\,040\eta^{21}\int_0^\eta a^4(\eta')\eta'^7\,\mathrm{d}\eta' + 3\,108\,105\eta^{20}\int_0^\eta a^4(\eta')\eta'^8\,\mathrm{d}\eta' \\
&\quad - 6\,906\,900\eta^{19}\int_0^\eta a^4(\eta')\eta'^9\,\mathrm{d}\eta' + 13\,123\,110\eta^{18}\int_0^\eta a^4(\eta')\eta'^{10}\,\mathrm{d}\eta' - 21\,474\,180\eta^{17}\int_0^\eta a^4(\eta')\eta'^{11}\,\mathrm{d}\eta' \\
&\quad + 30\,421\,755\eta^{16}\int_0^\eta a^4(\eta')\eta'^{12}\,\mathrm{d}\eta' - 37\,442\,160\eta^{15}\int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' + 40\,116\,600\eta^{14}\int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' \\
&\quad - 37\,442\,160\eta^{13}\int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' + 30\,421\,755\eta^{12}\int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' - 21\,474\,180\eta^{11}\int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' \\
&\quad + 13\,123\,110\eta^{10}\int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' - 6\,906\,900\eta^9\int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' + 3\,108\,105\eta^8\int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' \\
&\quad - 1\,184\,040\eta^7\int_0^\eta a^4(\eta')\eta'^{21}\,\mathrm{d}\eta' + 376\,740\eta^6\int_0^\eta a^4(\eta')\eta'^{22}\,\mathrm{d}\eta' - 98\,280\eta^5\int_0^\eta a^4(\eta')\eta'^{23}\,\mathrm{d}\eta' \\
&\quad + 20\,475\eta^4\int_0^\eta a^4(\eta')\eta'^{24}\,\mathrm{d}\eta' - 3276\eta^3\int_0^\eta a^4(\eta')\eta'^{25}\,\mathrm{d}\eta' + 378\eta^2\int_0^\eta a^4(\eta')\eta'^{26}\,\mathrm{d}\eta' \\
&\quad \left. -28\eta\int_0^\eta a^4(\eta')\eta'^{27}\,\mathrm{d}\eta' + \int_0^\eta a^4(\eta')\eta'^{28}\,\mathrm{d}\eta'\right],
\end{aligned}
$$

$$\Delta V_{\rm F14} = -\frac{16\pi}{9\,086\,380\,738\,369\,043\,484\,375\ R^{26}}\left(\eta^{28}I_{2F14} - 28\eta^{27}I_{1F14} + 378\eta^{26}I_{0F14}\right.$$
$$- 3276\eta^{25}I_{4F14} + 20\,475\eta^{24}I_{5F14} - 98\,280\eta^{23}I_{6F14}$$
$$+ 376\,740\eta^{22}I_{7F14} - 1\,184\,040\eta^{21}I_{8F14} + 3\,108\,105\eta^{20}I_{9F14}$$
$$- 6\,906\,900\eta^{19}I_{10F14} + 13\,123\,110\eta^{18}I_{11F14} - 21\,474\,180\eta^{17}I_{12F14}$$
$$+ 30\,421\,755\eta^{16}I_{13F14} - 37\,442\,160\eta^{15}I_{14F14} + 40\,116\,600\eta^{14}I_{15F14}$$
$$- 37\,442\,160\eta^{13}I_{16F14} + 30\,421\,755\eta^{12}I_{17F14} - 21\,474\,180\eta^{11}I_{18F14}$$
$$+ 13\,123\,110\eta^{10}I_{19F14} - 6\,906\,900\eta^{9}I_{20F14} + 3\,108\,105\eta^{8}I_{21F14}$$
$$- 1\,184\,040\eta^{7}I_{22F14} + 376\,740\eta^{6}I_{23F14} - 98\,280\eta^{5}I_{24F14}$$
$$+ 20\,475\eta^{4}I_{25F14} - 3276\eta^{3}I_{26F14} + 378\eta^{2}I_{27F14}$$
$$\left.-28\eta I_{28F14} + I_{29F14}\right)\Delta\eta, \tag{37}$$

where

$$I_{\rm 2F14} = \int_0^\eta a^4(\eta')\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 2F14} = a^4(\eta)\Delta\eta$$
$$I_{\rm 1F14} = \int_0^\eta a^4(\eta')\eta'\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 1F14} = a^4(\eta)\eta\Delta\eta$$
$$I_{\rm 0F14} = \int_0^\eta a^4(\eta')\eta'^2\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 0F14} = a^4(\eta)\eta^2\Delta\eta$$
$$I_{\rm 4F14} = \int_0^\eta a^4(\eta')\eta'^3\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 4F14} = a^4(\eta)\eta^3\Delta\eta$$
$$I_{\rm 5F14} = \int_0^\eta a^4(\eta')\eta'^4\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 5F14} = a^4(\eta)\eta^4\Delta\eta$$
$$I_{\rm 6F14} = \int_0^\eta a^4(\eta')\eta'^5\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 6F14} = a^4(\eta)\eta^5\Delta\eta$$
$$I_{\rm 7F14} = \int_0^\eta a^4(\eta')\eta'^6\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 7F14} = a^4(\eta)\eta^6\Delta\eta$$
$$I_{\rm 8F14} = \int_0^\eta a^4(\eta')\eta'^7\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 8F14} = a^4(\eta)\eta^7\Delta\eta$$
$$I_{\rm 9F14} = \int_0^\eta a^4(\eta')\eta'^8\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 9F14} = a^4(\eta)\eta^8\Delta\eta$$
$$I_{\rm 10F14} = \int_0^\eta a^4(\eta')\eta'^9\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 10F14} = a^4(\eta)\eta^9\Delta\eta$$
$$I_{\rm 11F14} = \int_0^\eta a^4(\eta')\eta'^{10}\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 11F14} = a^4(\eta)\eta^{10}\Delta\eta$$
$$I_{\rm 12F14} = \int_0^\eta a^4(\eta')\eta'^{11}\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 12F14} = a^4(\eta)\eta^{11}\Delta\eta$$
$$I_{\rm 13F14} = \int_0^\eta a^4(\eta')\eta'^{12}\,{\rm d}\eta' \qquad \Rightarrow \qquad \Delta I_{\rm 13F14} = a^4(\eta)\eta^{12}\Delta\eta$$

$$I_{14\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{13}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{14\text{F}14} = a^4(\eta)\eta^{13}\Delta\eta$$

$$I_{15\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{14}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{15\text{F}14} = a^4(\eta)\eta^{14}\Delta\eta$$

$$I_{16\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{15}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{16\text{F}14} = a^4(\eta)\eta^{15}\Delta\eta$$

$$I_{17\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{16}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{17\text{F}14} = a^4(\eta)\eta^{16}\Delta\eta$$

$$I_{18\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{17}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{18\text{F}14} = a^4(\eta)\eta^{17}\Delta\eta$$

$$I_{19\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{18}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{19\text{F}14} = a^4(\eta)\eta^{18}\Delta\eta$$

$$I_{20\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{19}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{20\text{F}14} = a^4(\eta)\eta^{19}\Delta\eta$$

$$I_{21\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{20}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{21\text{F}14} = a^4(\eta)\eta^{20}\Delta\eta$$

$$I_{22\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{21}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{22\text{F}14} = a^4(\eta)\eta^{21}\Delta\eta$$

$$I_{23\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{22}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{23\text{F}14} = a^4(\eta)\eta^{22}\Delta\eta$$

$$I_{24\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{23}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{24\text{F}14} = a^4(\eta)\eta^{23}\Delta\eta$$

$$I_{25\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{24}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{25\text{F}14} = a^4(\eta)\eta^{24}\Delta\eta$$

$$I_{26\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{25}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{26\text{F}14} = a^4(\eta)\eta^{25}\Delta\eta$$

$$I_{27\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{26}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{27\text{F}14} = a^4(\eta)\eta^{26}\Delta\eta$$

$$I_{28\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{27}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{28\text{F}14} = a^4(\eta)\eta^{27}\Delta\eta$$

$$I_{29\text{F}14} = \int_0^\eta a^4(\eta')\eta'^{28}\,\mathrm{d}\eta' \qquad \Rightarrow \qquad \Delta I_{29\text{F}14} = a^4(\eta)\eta^{28}\Delta\eta$$