Spacetime Diagrams and Einstein's Theory

For

Dummies

Instructor's Solution Manual

Dr. Muhammad Ali Yousuf
Assistant Program Manager
CTY Summer Programs
And
Physics and Mathematics Instructor
CTY Online Programs
Johns Hopkins Center for Talented Youth
mali@jhu.edu

Dr. Igor Woiciechowski
Associate Professor of Mathematics
Alderson-Broaddus University
woiciechowskiia@ab.edu

Version 3.0, 11/2/2016

JOHNS HOPKINS
CENTER for TALENTED YOUTH

Contact Dr. M. Ali Yousuf at mali@jhu.edu for electronic copies of this document

Contents

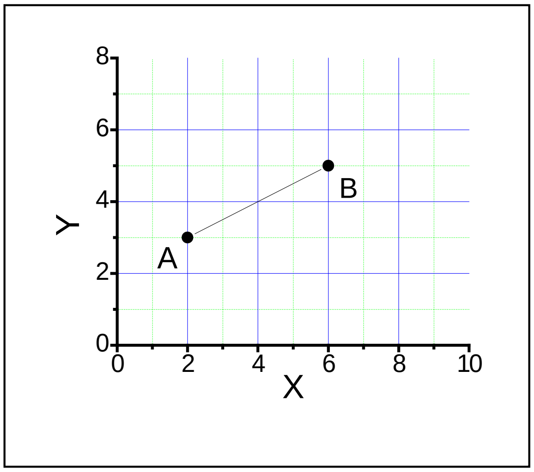
Lesson 1 – Instructor's Solution

Lesson 1 Topic: Description of Motion, Galilean Relativity Principle

Objective: To learn the Galilean Relativity Principle and the notion of invariances

Work:

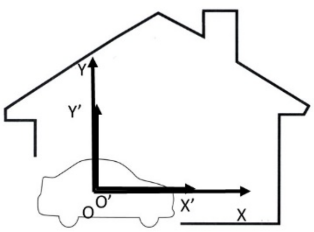
1. Write the coordinates of the points A and B. Then, using any method, calculate the distance between the points A and B in the coordinate system XOY below. The scale is given in meters.

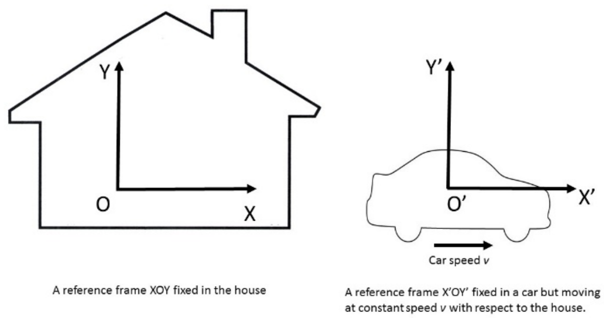


Answer: A(2, 2.5), B(6, 5.5), |AB| = $\sqrt{(6-2)^2 + (5.5-2.5)^2} = 5$

2. As shown below, a car is first at rest with respect to a house and their coordinates systems are identical. If the car now starts moving with constant speed (hence is an inertial frame) with respect to the house, which quantities must be the same in the two inertial frames of reference (house and car)? Which of the quantities may not be the same?

   - Speed of an object
   - Electric charge of an electron
   - Kinetic energy of a particle
   - Time interval between two events
   - Order of the elements in the Periodic Table



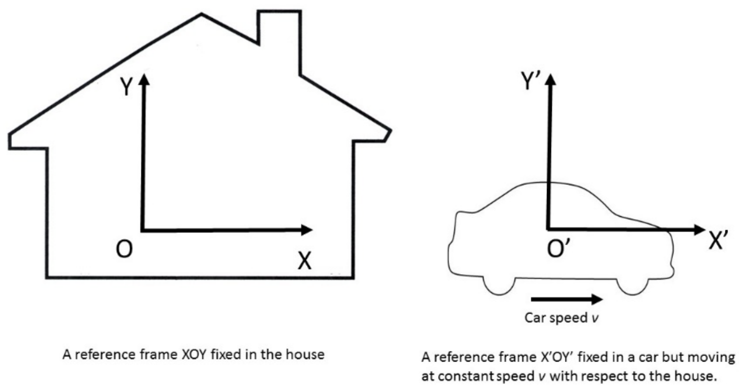


Initial Configuration – Car and house frames at the same point

After a few seconds, car has moved forward

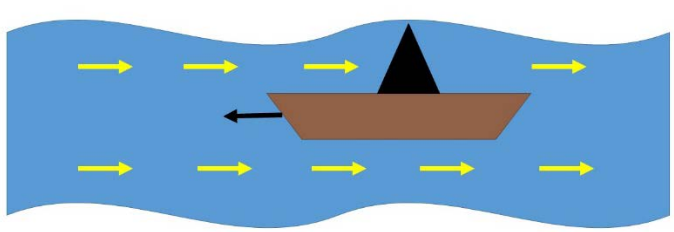
Answer: Electric charge of an electron, Order of the elements in The Periodic Table do not depend on a frame of reference. All other quantities are relative, and depend on a frame of reference.

3. To describe motion of an object we need a clock along with an appropriate frame of reference. Let us assume that the frame of reference X'O'Y' starts moving with constant speed v relatively to the frame of reference XOY along the X axis. Write equations connecting the coordinates of the point B in the moving system and in the system at rest. Time is absolute and running identically in the both frames of reference.



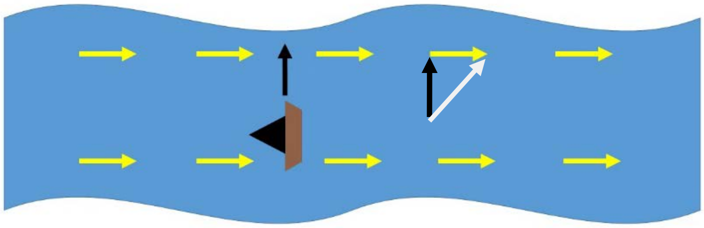
Answer: $x = x' + vt;\ y = y'$

4. A boat travelling upstream at 5 km/h relative to the shore. If there is a current of 7 km/h which direction the boat is moving with respect to water? What is the boat's speed relative to the water?



Answer: Using the law of velocity addition, -5 km/h + 7 km/h = 2 km/h, the velocity of the boat, moving to the right.

5. A sailboat is on a heading of due West at 5 m/s while crossing the Gulf Stream current, which is moving 4 m/s due North. What is the sailboat actual speed and heading?



Answer: You can use the Pythagorean Theorem for the velocity triangle. The speed relative to the ground is $\sqrt{5^2 + 4^2} = 6.4$ m/s. The direction is $\tan^{-1}\frac{4}{5} = 38.7°$ North West.

6. A jet plane travelling horizontally at 1200 km/h relative to the ground fires a rocket forwards at 1100 km/h relative to itself. What is the speed of the rocket relative to the ground?

Answer: Using The Law of Velocity Addition the speed of the rocket relative to the ground is 1200 km/h + 1100 km/h = 2300 km/h

Lesson 2 – Instructor's Solution

Lesson 2 Topic: Introduction to Spacetime Diagrams and the Light Cone

Objective:
To learn spacetime diagrams and the light cone.
Work:

Convert time or distance to meters

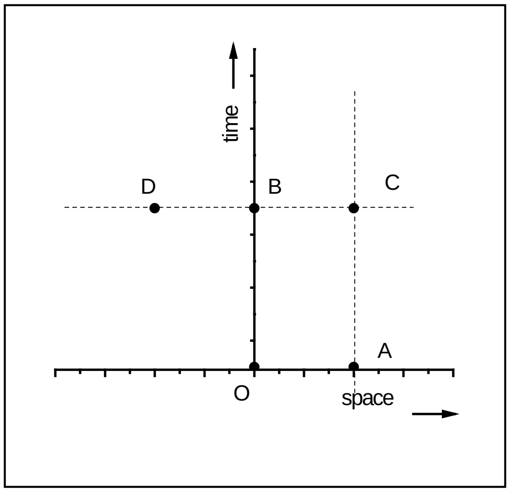
5 nanoseconds = 5 ns · 10^{-9} s/ns · 3 · 10^{8} m/s = 1.5 m

3 ms = 3 ms ·10^{-3} s/ms · 3 · 10^{8} m/s = 9 · 10^{5} m

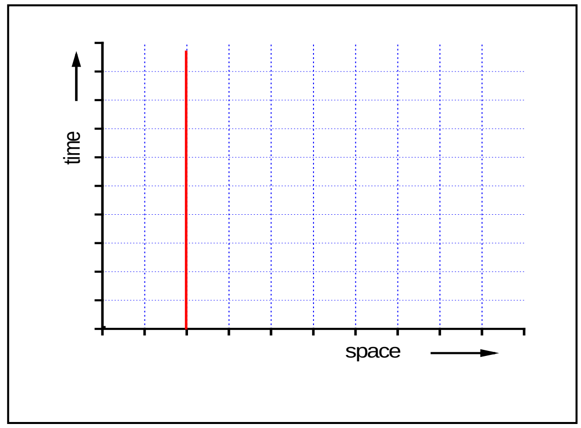
1 light-second = 1 light-second · 3 · 10^{8} m/s = 3 · 10^{8} m

1. In this diagram, which events occur at the same time? Which events occur at the same place?

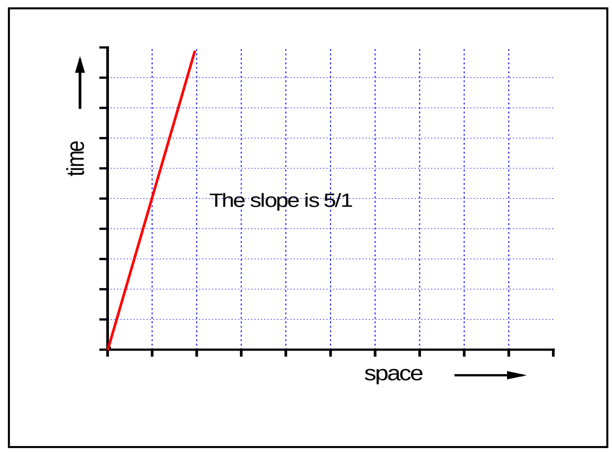
   Answer: Events D, B, and C occur at the same time. Events A and C occur at the same place.



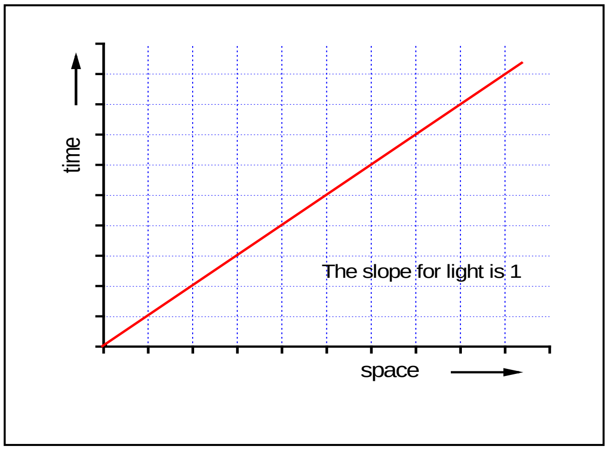
2. One division of the space axis corresponds to 1 meter. Construct a world line of the particle that is resting at 2 m from the reference event.



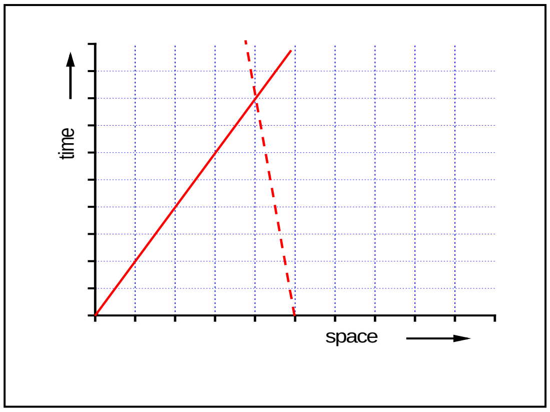
3. Time and distance are measured in meters construct the world line of the particle that is moving along the x-axis with the speed 0.2 m/m.



4. Construct the world line of a light ray (a photon) emitted at the origin and propagating along the x-axis.



5. Make a sketch of two particles that starting from different places move toward each other with constant velocities (not necessary the same magnitudes), and meet at some point of space.



Lesson 3 – Instructor's Solution

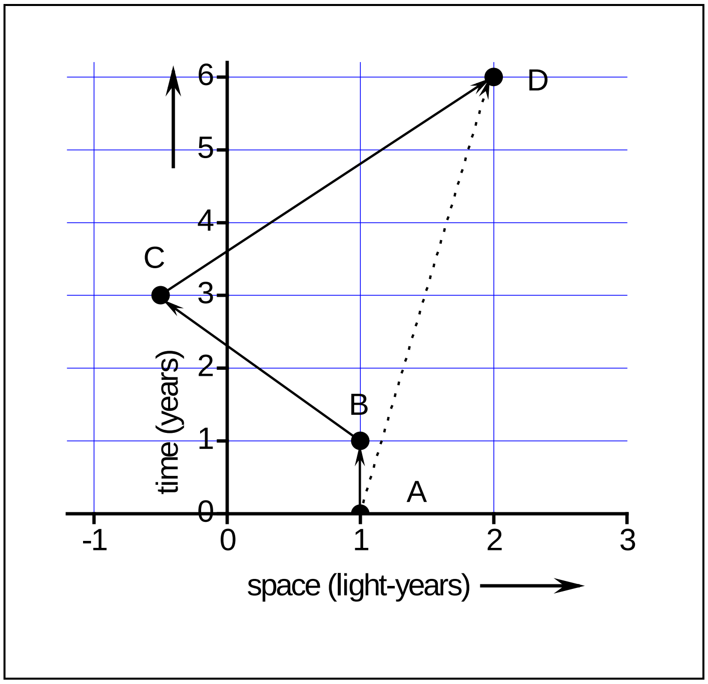
Lesson 3 Topic: Space and Spacetime Diagrams

Objective: To calculate and classify intervals on spacetime diagrams

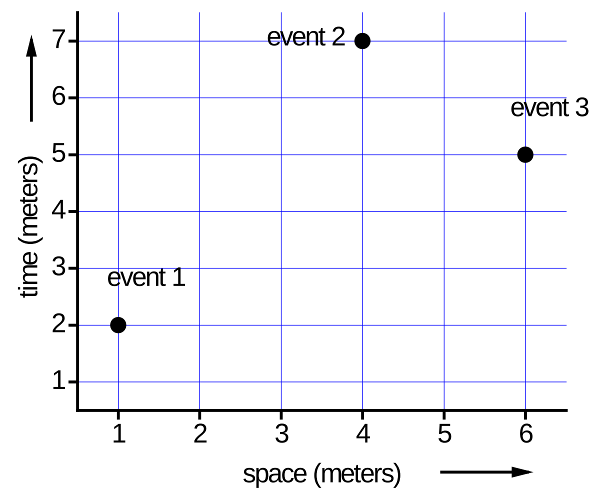
Work:

1. Interval in a spacetime map is defined as $(\text{interval})^2 = (\text{time separation})^2 - (\text{space separation})^2$. For any two events the interval is invariant for all inertial reference frames. If the space separation is zero what reference frame this interval corresponds to?
   Answer: The events occur at the same location. The object is at rest relatively to the frame of reference. Such a frame of reference is called the laboratory frame of reference.

2. In the spacetime diagram, time and distance are measured in years. Calculate the time increase on the traveler's clock while she travels from the point A to the point D through the points B and C. Calculate the wristwatch time if the traveler moves directly from the event A to event D. Compare the two times. What can you conclude?
   Answer: $t_{ABCD} = t_{AB} + t_{BC} + t_{CD} = 1 + \sqrt{2^2 - 1.5^2} + \sqrt{3^2 - 2.5^2} = 3.98$ years, $t_{AD} = \sqrt{6^2 - 1^1} = 5.92$ years. Wrist time along a straight line on a spacetime diagram is the longest among all possible paths between the two events.



3. Events 1, 2 and 3 all have laboratory locations $y = z = 0$. Their x and t measurements are shown on the laboratory spacetime map.



a) Classify the interval between the events 1 and 2 as timelike, spacelike, or lightlike.
Answer: It is a timelike interval because the slope of the line event 1 – event 2 is greater than 45°.

b) Classify the interval between the events 1 and 3
Answer: It is a spacelike interval because the slope of the line event 1 – event 3 is less than 45°.

c) Classify the interval between the events 2 and 3
Answer: It is a lightlike interval because the slope of the line event 2 – event 3 is 45°.

d) Is it possible that one of the events caused the other event?
Answer: Event 1 can cause event 2, event 3 (light emission) can cause event 2 when light reaches this point.

e) What is the proper time between two events?

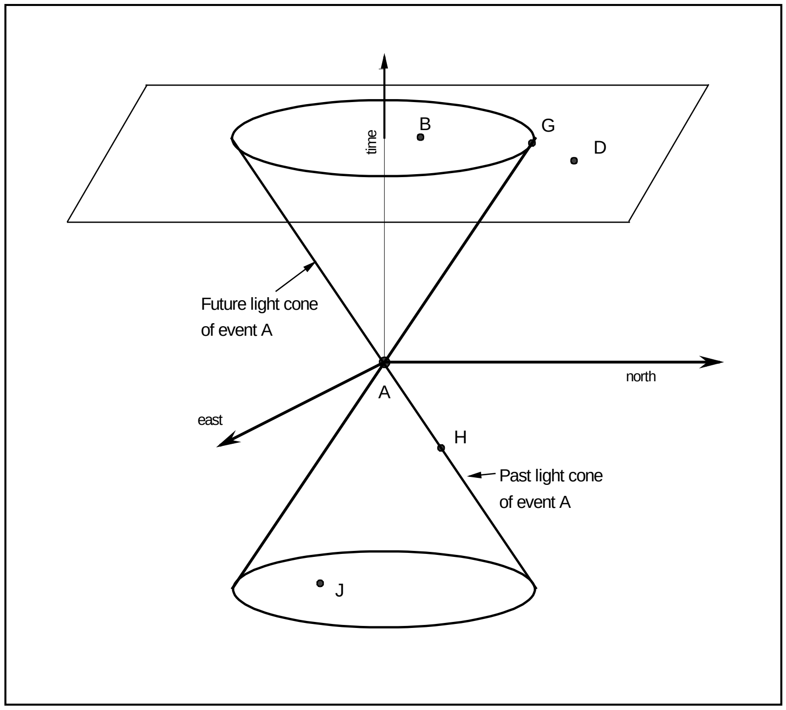
Answer: $\tau_{12} = \sqrt{(7-2)^2 - (4-1)^2} = \sqrt{25-9} = 4$ meters

$\tau_{23} = 0$ meters

f) For the timelike pair of the events, find the speed and direction of a rocket frame with respect to which the two events occurred at the same place (optional)

Answer: The rocket is moving to the right with speed $v = \frac{3}{5} = 0.6\mathrm{c}$.

4. Three dimensional spacetime map showing eastward, northward, and time locations of events occurring on a flat plane in space is shown on the picture. The light cone shows the partition in spacetime.



a) Can a material particle emitted at A affect what is going to happen at B?

   Answer: Yes, the interval AB is a timelike interval.

b) Can a light ray emitted at A affect what is going to happen at G?

   Answer: Yes, the interval AG is a lightlike interval.

c) Can no effect whatever produced at A affect what happens at D?

   Answer: No, the interval AD is a spacelike interval. The signal should move faster than speed of light.

d) Can a material particle emitted at J affect what is happening at A?

   Answer: Yes, the interval JA is a timelike interval.

e) Can a light ray emitted at H affect what is happening at A?

   Answer: Yes, the interval HA is a lightlike interval.

Lesson 4 – Instructor's Solution

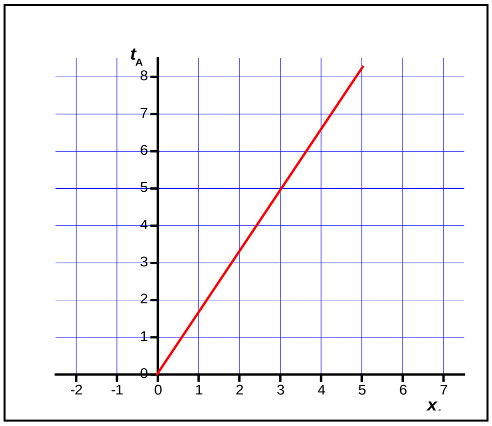
Lesson 4 Topic: Space and Spacetime Diagrams

Objective:

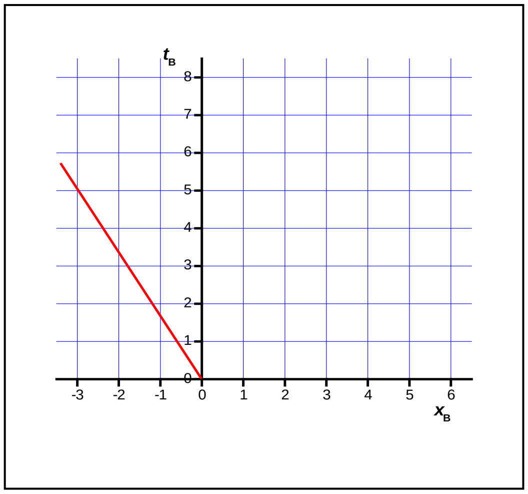
To construct spacetime diagrams for moving reference frames. To understand that simultaneity depends on a frame of reference. To understand time dilation and length contruction.

Work:

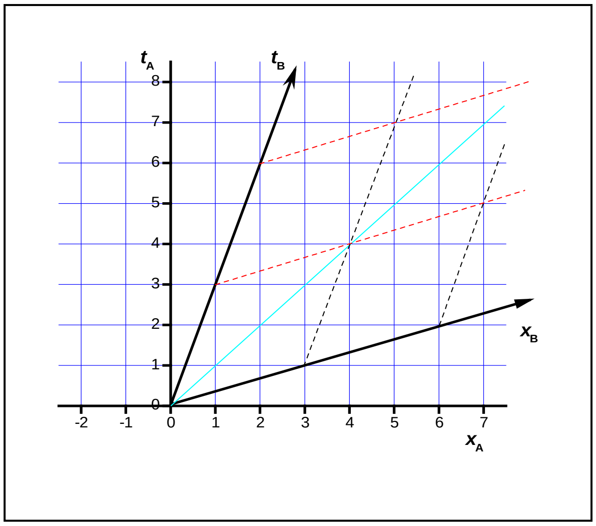
1. Alice is standing on the Earth. She is an observer. Bob is passing by Alice in a spaceship that has speed v = 0.6c. At time 0 meters (both time and distance are measured in meters) Alice and Bob are at the origin of the spacetime map, construct the world line of the Bob's spaceship. At what time is he is 5 m away from Alice?
   Answer: About 8.3 meters.



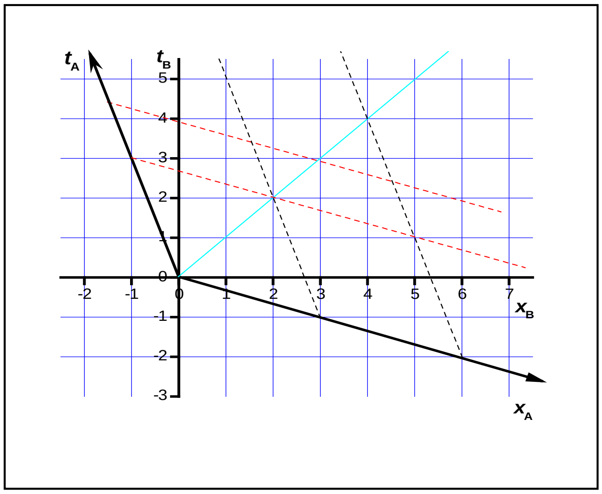
2. Bob's spaceship passing by Alice, who is on Earth, with speed v = 0.6c. He is now an observer. Construct the world line of Alice in the spacetime map of Bob.



3. The tilted line is the time axis of the Bob's frame in the Alice's frame of reference. In the system of units used the speed of light *c* = 1. What is the speed of Bob? The world line of a light flash is a bisector between the directions of time and space. Keeping this in mind, construct the space axis of the Bob's frame. What are the slopes of the Bob's axes in the Alice's frame? Make a sketch of lines of simultaneity and same position lines in the Bob's frame.
   Answer: The speed of Bob is 1/3*c*. The slope of the time axis is 3/1. The slope of the coordinate axis is 1/3. Dashed lines parallel to the t_B axis are the same position lines. The dashed lines parallel to x_B are the simultaneity lines.

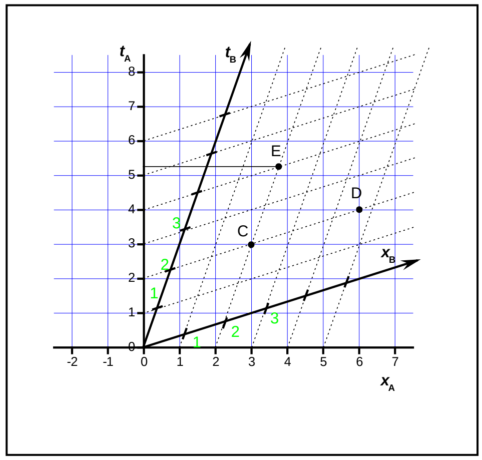


4. Construct the spacetime coordinate map of Alice in the Bob's frame of reference. You remember, Alice is standing on Earth. Bob is flying to the right on his spaceship. Make a sketch of simultaneity lines and same position lines.



5. On the spacetime map, at what instants do events C and D occur in the Alice's frame? In the Bob's frame? Distance and time are measured in meters. Are these events simultaneous?
   What is the interval between events C and E in the Bob's frame? In the Alice's frame? What clocks are running slow?

Take one unit of distance in the Bob's frame, for example, the distance between marks 1 and 2. What can you say about the one-unit distance in the Alice's frame?



Answer: In the Alice's frame of reference, event C occurs at 3 m, event D occurs at 4 m of time. In the Bob's frame, events C and D are simultaneous, occurring at 2 m of time. $\tau_{CE}(Bob) = \sqrt{(4-2)^2 - 0} = 2$ meters, $\tau_{CE}(Alice) = \sqrt{(5.3-3)^2 - (3.8-3)^2} = 2.2$ meters. Bob's clocks are running slow. One unit distance is shorter in the Alice's frame.

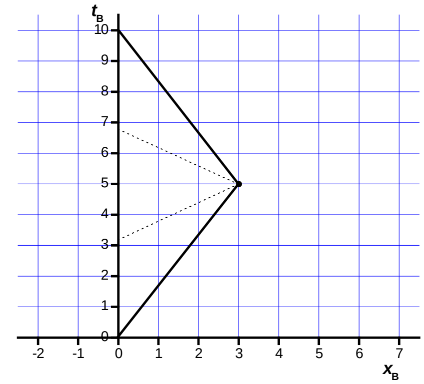
Lesson 5 – Instructor's Solution

Lesson 5 Topic: Resolving the Twin Paradox

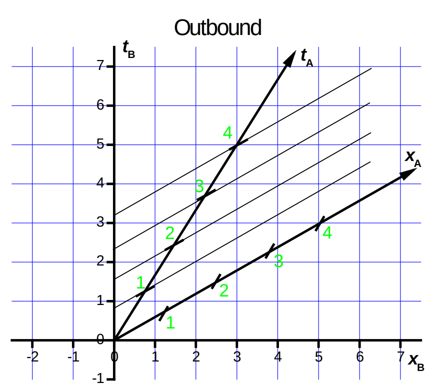
Objective:
To be able to resolve any "paradox" in the Special Theory of Relativity

Work:

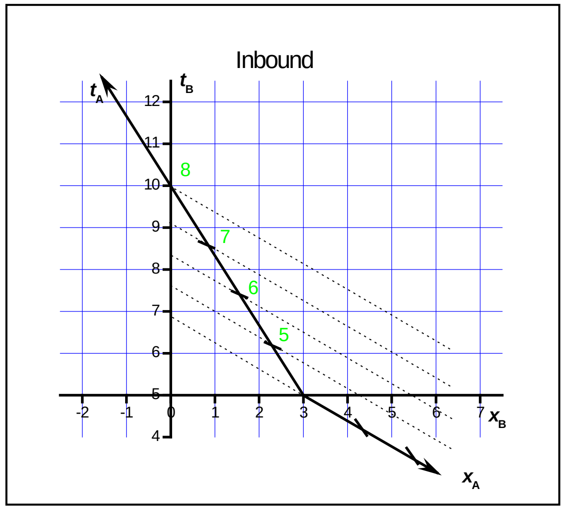
1. Alice is traveling with speed v = 0.6c to the planet that is 3 light-years away from the Earth and back to the Earth. Bob is waiting on the Earth. The world line of Alice is shown on the Bob's spacetime map. How much time one needs to Alice to reach the destination, according to her clocks? Bob's clock? How much slow the Alice's clocks are running relative to Bob's ones? What is the total travel time on her clocks? On his clocks?
   Answer: Alice's wristwatch time is $= \sqrt{5^2 - 3^2} = 4$ years. On Bob's clock 3 light-years/0.6 *c* = 5 years. The Alice's clocks are running slower by 5/4 = 1.25 times slower. The total travel time for Alice is 4 + 4 = 8 years. For Bob, the time is 5 + 5 = 10 years.



2. Spacetime maps for Alice and Bob are shown on one plot for the outbound trip. According to Alice, how long did she travel to the planet? On her wristwatch? What is the reading of the Bob's clock in perspective of Alice at the instance when she reached the planet?
   Answer: According to her wristwatch, she traveled 4 years. In perspective of Alice Bod spent 3.2 years on Earth.



3. Spacetime maps for Alice and Bob are shown on one plot for the inbound trip. According to Alice, how long did she travel back home? On her wristwatch? What is the reading of the Bob's clock in perspective of Alice at the instance when she reached the Earth? How long was Alice away from home in perspective of Bob? In perspective of Alice? Explain.
Answer: She traveled the same 4 years on her inbound trip according to her wristwatch. She has traveled 10 years in Bob's perspective. The entire trip for Alice is 8 years.



4. Resolution of the Twin Paradox could be given using the properties of the spacetime geometry. A straight line between two events always represents the longest interval. How can you demonstrate it?
Answer: She must travel with acceleration while flying to the planet and back. Her world line must be a curve (variable slope). Bob remains in the inertial frame of reference, while Alice must move (at least partially) in an noninertial frame of reference. The situation is not symmetrical. Splitting both paths into segments it is clearly seen that Alice's time reading is less than the Bob's reading. The straight line has the longest wristwatch time.