On decentralized computation of the leader's strategy in bi-level games

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Abstract

Motivated by the omnipresence of hierarchical structures in many real-world applications, this study delves into the intricate realm of bi-level games, with a specific focus on exploring local Stackelberg equilibria as a solution concept. While existing literature offers various methods tailored to specific game structures featuring one leader and multiple followers, a comprehensive framework providing formal convergence guarantees to a local Stackelberg equilibrium appears to be lacking. Drawing inspiration from sensitivity results for nonlinear programs and guided by the imperative to maintain scalability and preserve agent privacy, we propose a decentralized approach based on the projected gradient descent with the Armijo stepsize rule. The main challenge here lies in assuring the existence and well-posedness of Jacobians that describe the leader's decision's influence on the achieved equilibrium of the followers. By meticulous tracking of the Implicit Function Theorem requirements at each iteration, we establish formal convergence guarantees to a local Stackelberg equilibrium for a broad class of bi-level games. Building on our prior work on quadratic aggregative Stackelberg games, we also introduce a decentralized warm-start procedure based on the consensus alternating direction method of multipliers addressing the previously reported initialization issues. Finally, we provide empirical validation through two case studies in smart mobility, showcasing the effectiveness of our general method in handling general convex constraints, and the effectiveness of its extension in tackling initialization issues.

1 Introduction

In the realm of strategic decision-making, games with the inherent leader-follower structure have emerged as one of the fundamental frameworks to model the interplay between agents on multiple levels of hierarchy. These games are characterized by a structure in which a leader, possessing a strategic advantage, makes decisions prior to rational followers who, in return, choose their best response to the leader's action. With the pivotal works on bi-level games formalizing the concepts of Stackelberg (SG) [37] and their broader format, Reverse Stackelberg games (RSG) [16,17], various real-world problems in the domain of energy management [1,29], operational optimization [15,39] and transportation [14,25] gained interest from the perspective of computing a no-regret solution for all participants.

Typically, each lower-level agent competes to minimize a personal objective parametrized by the leader's decision variable and influenced by other followers' decisions. Consequently, the leader aims to minimize a personal objective under the equilibrium constraints imposed by the lower-level game between the followers. If the nature of the application allows for a centralized computation of the solution [9,12,18], the problem can be framed as a bi-level mathematical program with complementarity constraints (MPCC) [34], usually tackled by iterative relaxations of the equilibrium constraints [19] or by recasting it into an instance of a mixed integer program [13,22]. Nevertheless, in the presence of private feasibility constraints, and driven by the essential requirements to preserve privacy and ensure scalability, decentralized systems have become increasingly prevalent in many real-world applications. As a result, several approaches exploiting specific structural assumptions of the analyzed games have been proposed in the literature.

For a specific class of pricing games with a quadratic, aggregative game between the followers [24,25], it has been demonstrated how the concept of Reverse Stackelberg games can be used to incentivize the global optimum of the leader. Formulating dynamic strategies for the leader in the form of functionals, rather than real-valued vectors, facilitated reshaping the lower-level game in a way that gave rise to a Nash Equilibrium corresponding exactly to the minimizer of the leader's objective. A follow-up question naturally imposes - can we establish a connection between the proposed dynamic policies and fixed, real-valued, vector strategies? In general, this procedure is not straightforward. By restricting the leader's

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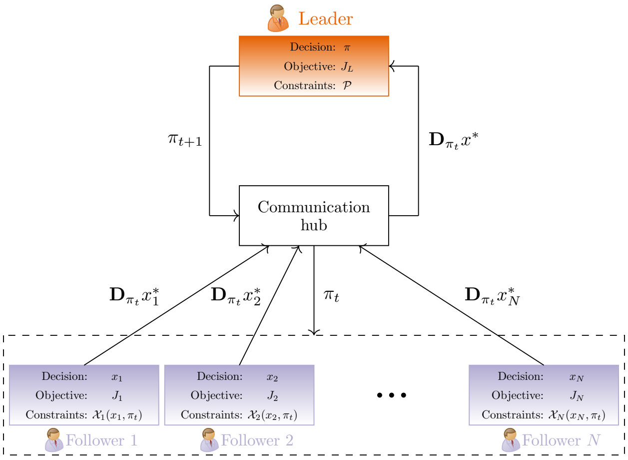

Fig. 1. Schematic sketch of the problem setup. Each of the N followers aims to optimize the personal objective J_i under the parametrized local constraints $x_i \in \mathcal{X}_i(x_i, \pi)$. The followers communicate with the leader through the communication hub that is used as a medium to collect the locally computed Jacobians $\mathbf{D}_{\pi_t} x_i^*$ in every update step of the leader's action.

impact solely to parameterizing the lower-level game, rather than allowing the flexibility to restructure it as in the RSG framework, we enter the realm of Stackelberg games, which are arguably more challenging to solve.

Due to the inherent non-convex nature of the bi-level problems, the existing body of literature predominantly focuses on local Stackelberg equilibria as a viable solution concept [10]. In [21], an iterative method for unconstrained games has been suggested. Conversely, in the context of constrained, quadratic, aggregative games, the authors of [10] propose a two-layer, semi-decentralized algorithm based on iteratively convexifying a regularized version of the underlying MPCC. Looking from a different perspective, the sensitivity results for nonlinear programs [33, 35, 38] hint at the possibility of differentiating the Karush-Kuhn-Tucker (KKT) conditions of the best-response optimization problems in an attempt to estimate how the attained Nash Equilibrium between the followers reacts to a change in the leader's action. In light of the success that gradient-descent-based methods have experienced in many real-world applications and under the paradigm of decentralized equilibrium computation, we aim to design an iterative, first-order-like method for computing a local Stackelberg equilibrium suitable for a broad class of bi-level games.

This paper is a continuation of the preliminary work presented in [27], where the original idea inspired by [38] has been outlined for one specific case of quadratic, aggregative Stackelberg games. In this paper, we extend the analysis to a more general structure illustrated in Figure 1, and show how the computation of Jacobians describing the influence of the leader's strategy on the attained variational Nash Equilibrium [27] of the lower-level game can be performed locally by each of the followers. We start by briefly discussing the connection between the RSG dynamic policies and the static ones used in the SG setup and continue by rigorously tackling the requirements of the Implicit Function Theorem [8] in order to generalize the approach in [27] to also account for non-quadratic, non-aggregative games. On the other hand, in the context of quadratic, aggregative games with polytopic constraints as in [27], we also address the reported initialization issue arising from the fact that in each iteration we differentiate the KKT conditions of an optimization problem equivalent to the standard best-response one. With that in mind, the main contributions of this paper can be summarized as follows:

- We propose a distributed, first-order-like, iterative method based on explicit fulfillment of the Implicit Function Theorem requirements. By ensuring a local improvement of the leader's objective at each iteration, we provide formal convergence guarantees for a broad class of bi-level games.
- For a class of quadratic, aggregative, Stackelberg games linearly parametrized by the decision variable of the leader, we propose a decentralized warm-start procedure based on the alternating direction method of multipliers (ADMM). In line with the distributed nature of the main algorithm, we compute a feasible leader's strategy that yields an interior point variational Nash Equilibrium of the lower-level game, i.e., renders no local inequality constraint active.
- We test the proposed method in an adaptation of the case study from the domain of smart mobility previously analyzed in [27]. Firstly, by introducing the so-called budget constraints parametrized by the leader's decision variable, we demonstrate the effectiveness of the main procedure for the bi-level games with general convex constraints. Then, by going back to the setup in [27], we illustrate the effectiveness of the warm-start procedure in alleviating the initialization issues.

The paper is outlined as follows: the rest of this section is devoted to introducing some basic notation. In Section 2, we introduce the general bi-level setup, discuss the connection between the Stackelberg and Reverse Stackelberg games, postulate the main standing assumptions, and formally introduce the problem. In the following section, Section 3, we revise and generalize the decentralized method for computing the local Stackelberg equilibrium previously outlined in [27]. Section 4 then focuses on the subclass of quadratic, aggregative Stackelberg games and presents the proposed warm-start procedure. Finally, we conclude the paper with Sections 5 and 6 where we present the numerical examples and propose some ideas for future research.

Notation: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative reals, and $\mathbb{Z}_+$ the set of non-negative integers. Let $\mathbf{0}_m$ and $\mathbf{1}_m$ denote the all zero and all one vectors of length m respectively, and $\mathbb{I}_m$ the identity matrix of size $m \times m$. For a finite set $\mathcal{A}$, we let $\mathbb{R}^{\mathcal{A}}_{(+)}$ denote the set of (non-negative) real vectors indexed by the elements of $\mathcal{A}$ and $|\mathcal{A}|$ the cardinality of $\mathcal{A}$. Furthermore, for finite sets $\mathcal{A}$, $\mathcal{B}$ and a set of $|\mathcal{B}|$ vectors $x_i \in \mathbb{R}^{\mathcal{A}}_{(+)}$, we define $x := \operatorname{col}((x_i)_{i\in\mathcal{B}}) \in \mathbb{R}^{|\mathcal{A}||\mathcal{B}|}$ to be their concatenation. For $A \in \mathbb{R}^{n\times n}$, $A \succ 0 (\succeq 0)$ is equivalent to $x^T A x > 0 (\geq 0)$ for all $x \in \mathbb{R}^{n\times n}$. We let $A \otimes B$ denote the Kronecker product between two matrices and for a vector $x \in \mathbb{R}^n$, we let $\mathrm{Dg}(x) \in \mathbb{R}^{n\times n}$ denote a diagonal matrix whose elements on the diagonal correspond to vector x. For a differentiable function $f(x) : \mathbb{R}^n \to \mathbb{R}^m$, we let $\mathbf{D}_x f \in \mathbb{R}^{m\times n}$ denote the Jacobian matrix of f defined as $(\mathbf{D}_x f)_{ij} := \frac{\partial f_i}{\partial x_j}$. If $f(x)$ is a real-valued function, i.e., $m = 1$, we adopt $\nabla_x f := \mathbf{D}_x f \in \mathbb{R}^n$. Finally, for a set-valued mapping $\mathcal{F} : \mathcal{Y} \rightrightarrows \mathcal{X}, \operatorname{gph}(\mathcal{F}) := \{(y, x) \in \mathcal{Y} \times \mathcal{X} \mid x \in \mathcal{F}(y)\}$ denotes its graph.

2 Theoretical preliminaries

Throughout the paper, we consider a bi-level game with a set of $N + 1$ agents $\overline{\mathcal{I}} = \mathcal{I} \cup \{L\}$, where L represents the leading agent and each $i \in \mathcal{I}$ represents one of the N followers. In this setup, the leading agent is the first one to choose an action π from its feasible set $\mathcal{P}$, to which all N followers will respond at once with a personal decision vector x_i from their feasible set $\mathcal{X}_i(\pi)$ that is in their best interest. If $m_F \in \mathbb{N}$ represents the dimension of the follower's decision space, we assume $\mathcal{X}_i(\pi)$ in the form of $\mathcal{X}_i(\pi) := \{x_i \in \mathbb{R}^{m_F} \mid \boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi) \leq \mathbf{0}_{m_{\mathrm{inq},i}} \wedge \boldsymbol{g}_i^{\mathrm{eq}}(x_i, \pi) = \mathbf{0}_{m_{\mathrm{eq},i}}\}$, where $m_{\mathrm{inq},i}, m_{\mathrm{eq},i} \in \mathbb{N}$ denote the number of inequality and equality constraints encompassed in $\mathcal{X}_i(\pi)$. If $m_L \in \mathbb{N}$ represents the dimension of the leader's action, the nature of the leader's strategy can lead to two types of games in general:

- **Reverse Stackelberg Games (RSG)**: where the leader's strategy $\pi \in \mathcal{P}$ is a map $\pi : \mathbb{R}^{Nm_F} \to \mathbb{R}^{m_L}$;
- **Stackelberg Games (SG)**: where the leader's strategy is a fixed real vector, $\pi \in \mathcal{P} \subseteq \mathbb{R}^{m_L}$.

In any case, we refer to the phase of choosing the optimal $x_i \in \mathcal{X}_i(\pi)$ as the *Lower-level game* and the process of choosing the optimal leader's strategy knowing that the followers will play a best-response as the *Upper-level game*. Furthermore, we define the joint strategy of all followers as $x := \operatorname{col}((x_i)_{i\in\mathcal{I}}) \in \mathcal{X}(\pi)$ and for every $i \in \mathcal{I}$, we define $x_{-i} := \operatorname{col}((x_j)_{j\in\mathcal{I}\setminus i}) \in \mathcal{X}_{-i}(\pi)$, such that $\mathcal{X}(\pi) := \prod_{i\in\mathcal{I}} \mathcal{X}_i(\pi)$ and $\mathcal{X}_{-i}(\pi) := \prod_{j\in\mathcal{I}\setminus i} \mathcal{X}_j(\pi)$.

2.1 Lower-level game

Regardless of the game type, the followers choose their strategies in an attempt to minimize personal objective functions $J_i(x_i, x_{-i}, \pi)$ by playing the best response to other followers' strategies under the umbrella of the Nash Equilibrium concept. The π-parametrized lower-level game $G^0(\mathcal{I}; \pi)$ represents N coupled optimization problems, i.e., $G^0(\mathcal{I}; \pi) := \{G_i^0(\pi, x_{-i}) \mid i \in \mathcal{I}\}$, with

$$G_i^0(\pi, x_{-i}) := \left\{ \begin{array}{c} \min\limits_{x_i\in\mathbb{R}^{m_F}} J_i(x_i, x_{-i}, \pi) \\ \text{s.t. } x_i \in \mathcal{X}_i(\pi) \end{array} \right\}, \qquad (1)$$

and the Nash Equilibrium given in Definition 1.

Definition 1 (Nash Equilibrium) *For any leader's strategy $\pi \in \mathcal{P}$, a joint strategy $x^* \in \mathcal{X}$ is a Nash Equilibrium (NE) of the game $G^0(\mathcal{I}; \pi)$, if for all $i \in \mathcal{I}$ and all $x_i \in \mathcal{X}_i(\pi)$ it holds that $J_i(x_i^*, x_{-i}^*, \pi) \leq J_i(x_i, x_{-i}^*, \pi)$.*

For a particular $\pi \in \mathcal{P}$, it is rarely possible to find a closed-form characterization of the full set of NE in a general setup. Therefore, we postulate standard assumptions about the structure of $G^0(\mathcal{I}; \pi)$ that allow us to focus on the variational Nash Equilibria (v-NE) as a solution concept of the lower-level game.

Standing Assumption 1 *For every $i \in \mathcal{I}$ and any $\pi \in \mathcal{P}$, $x_{-i} \in \mathcal{X}_{-i}$, the cost $J_i(x_i, x_{-i}, \pi)$ is convex and continuously differentiable in x_i. Moreover, it is continuous in $x \in \mathcal{X}(\pi)$ and the sets $\mathcal{X}_i(\pi)$ are nonempty, compact, convex and satisfy Slater's constraint qualification.*

Strictly speaking, under Standing Assumption 1, for every $i \in \mathcal{I}$, the Nash Equilibrium strategy $x_i^* \in \mathcal{X}_i(\pi)$ is the solution of the best-response optimization problem (1) for x_{-i}^*, i.e., $G_i^0(\pi, x_{-i}^*)$. The optimality of $x_i^* \in \mathcal{X}_i(\pi)$ is guaranteed if and only if x_i^* solves the KKT system of equations $l_i(z_i, \pi \mid x_{-i}^*) = 0$, where the vector mapping l_i is defined for every $z_i = (x_i, \lambda_i, \nu_i)$ as

$$l_i(z_i, \pi \mid x_{-i}^*) := \begin{bmatrix} \nabla_{x_i} \mathcal{L}_i(z_i, \pi) \\ \mathrm{Dg}(\lambda_i)\, \boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi) \\ \boldsymbol{g}_i^{\mathrm{eq}}(x_i, \pi) \end{bmatrix}, \qquad (2)$$

with the Lagrangian given by $\mathcal{L}_i(z_i, \pi) = J_i(x_i, x_{-i}^*, \pi) + \lambda_i^T \boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi) + \nu_i^T \boldsymbol{g}_i^{\mathrm{eq}}(x_i, \pi)$ and $\lambda_i \in \mathbb{R}_+^{m_{\mathrm{inq},i}}$ and $\nu_i \in \mathbb{R}^{m_{\mathrm{eq},i}}$ representing the dual variables associated with the inequality and equality constraints. Under Standing Assumption 1, if some $\hat{z}_i = (\hat{x}_i, \hat{\lambda}_i, \hat{\nu}_i)$, with feasible $\hat{x}_i$ and $\hat{\lambda}_i$, satisfies $l_i(\hat{z}_i, \pi \mid x_{-i}^*) = 0$, then $\hat{z}_i$ is the optimizer of (1). On the other hand, based on [11, Prop. 1.4.2], Standing Assumption 1 also ensures that a joint strategy $x \in \mathcal{X}(\pi)$ is a NE if and only if it solves a variational inequality problem, hence providing a closed-form description of the lower-level game's solution set. Namely, if $F(x, \pi) := \operatorname{col}((\nabla_{x_i} J_i(x_i, x_{-i}, \pi))_{i\in\mathcal{I}})$ denotes the pseudo-gradient of $G^0(\mathcal{I}; \pi)$, we can adopt the following assumption about the followers.

Standing Assumption 2 *For any $\pi \in \mathcal{P}$, the agents $i \in \mathcal{I}$ play a joint strategy $x \in \mathcal{N}_0(\pi)$, where $\mathcal{N}_0(\pi)$ is the set of all v-NE of the game $G^0(\mathcal{I};\pi)$, given by $\mathcal{N}_0(\pi) \coloneqq \{x \in \mathcal{X}(\pi) \mid (y-x)^T F(x,\pi) \geq 0, \forall y \in \mathcal{X}(\pi)\}$.*

2.2 Upper-level game

On the upper level, finding the optimal strategy $\pi \in \mathcal{P}$ imposes solving a minimization problem of the leader's objective $J_L : \mathbb{R}^{m_F} \times \mathcal{P} \to \mathbb{R}$. Instances of both SG and RSG can be compactly written as:

$$G_1 \coloneqq \left\{ \begin{array}{c} \min_{\pi \in \mathcal{P}} J_L\left(x^*, \pi\right) \\ \text{s.t. } \left(x^*, \pi\right) \in \operatorname{gph}\left(\mathcal{N}_0\right) \cap \left(\mathbb{R}^{m_F} \times \mathcal{P}\right) \end{array} \right\}. \quad (3)$$

In general, the optimal π in G_1 is a possibly non-unique solution [27, Corr. 1] to a non-convex problem that requires the ability to understand the leader's influence on the position of the lower-level game's v-NE. Moreover, depending on the properties of $F(x,\pi)$, the lower-level game could admit multiple NE for a particular parametrization. In this study, we restrict ourselves to cases where $G^0(\mathcal{I};\pi)$ admits a unique NE for any $\pi \in \mathcal{P}$. Hence, we state the following assumption, common in existing literature [5,30], that ensures the existence and uniqueness of the lower-level game's v-NE [11, Th. 2.3.3]. Moreover, for a particular leader's strategy, it also ensures that the v-NE can be computed as a fixed-point of the projected pseudo-gradient mapping [11,25,31,32].

Standing Assumption 3 *For any $\pi \in \mathcal{P}$, the pseudo-gradient $F(\cdot,\pi)$ is strongly monotone in $x \in \mathcal{X}(\pi)$.*

Concerning the nature of the leader's decision variable, it is evident that SG represents a distinct instance of RSG, wherein the leader's strategy assumes a constant function. This limits the flexibility to incentivize a certain NE of the lower-level game, as the leader's strategies in the form of feedback policies offer a means to directly shape the functional form of the followers' optimization problems. Moreover, with many real-world applications requiring different notions of fairness, solving a SG can be considered arguably more challenging. To elucidate this contrast, we look at the following example for a specific class of games, referred to as quadratic aggregative games in [25,32].

Definition 2 (Quadratic Aggregative Games) *Let the leaader's objective be $J_L = \frac{1}{2}\sigma(x)^T P_L \sigma(x) + q_L^T \sigma(x)$, where $\sigma(x) = \sum_{i \in \mathcal{I}} x_i$. Moreover, let the lower-level game $G^0(\mathcal{I};\pi)$ be defined by*

$$J_i(x_i, x_{-i}, \pi) = \frac{1}{2} x_i^T P_i x_i + x_i^T Q_i \sigma(x_{-i}) + r_i^T x_i + x_i^T S_i \pi_i\,,$$

where $\sigma(x_{-i}) = \sigma(x) - x_i$, $\pi_i \in \mathbb{R}^{m_F}$, $\pi = col((\pi_i)_{i \in \mathcal{I}})$, the matrices P_L, P_i, Q_i, S_i and vectors q_L, r_i are all real valued, $P_L \succ 0$, for every $i \in \mathcal{I}$, $P_i, S_i \succ 0$, and the Standing assumptions 1, 2 and 3 all hold.

If one regards the game in Definition 2 as a Reverse Stackelberg game, it suffices to choose the leader's strategy as a mapping of the form

$$\pi_i(x_i, x_{-i}) = S_i^{-1} \left[\frac{1}{2} \overline{\mathbf{P}}_i x_i + \overline{\mathbf{Q}}_i \sigma(x_{-i}) + \overline{\mathbf{r}}_i \right], \quad (4)$$

where $\overline{\mathbf{P}}_i = P_L - P_i$, $\overline{\mathbf{Q}}_i = P_L - Q_i$ and $\overline{\mathbf{r}}_i = q_L - r_i$, so that the leader's objective and the objectives of the followers satisfy $J_L(x_i, x_{-i}) - J_L(\tilde{x}_i, x_{-i}) = J_i(x_i, x_{-i}) - J_i(\tilde{x}_i, x_{-i})$ for any fixed $x_{-i} \in \mathcal{X}_{-i}$ and any two $x_i, \tilde{x}_i \in \mathcal{X}_i$. This implies that due to (4), the leader's objective J_L becomes the exact potential [28] of the lower-level game by definition. Consequently, this guarantees that the minimizer of J_L aligns with the v-NE of $G^0(\mathcal{I};\pi)$, which can, in this case, be computed using a decentralized, iterative, fixed-point method as in [25,27,32]. Conversely, if one regards the game in Definition 2 as an instance of Stackelberg games, such manipulation is no longer possible. If x^R represents the NE obtained when applying (4) in the RSG setup, one might naively try to plug back x^R into (4) to obtain a static pricing vector $\pi_i^R = \pi_i(x_i^R, x_{-i}^R)$ and use it in the setup of a SG. However, in a general case, this does not yield a viable solution, as illustrated in the following proposition.

Proposition 1 *Let a bi-level game be defined as in Definition 2 such that $\mathcal{X}_i(\pi) = \mathbb{R}^{m_F}$ for all $i \in \mathcal{I}$. Moreover, let the mapping $\pi : \mathbb{R}^{Nm_F} \to \mathbb{R}^{Nm_F}$ be given by $\pi_i : \mathbb{R}^{Nm_F} \to \mathbb{R}^{m_F}$ and let (4) yield $x^R \in \mathcal{X}(\pi)$. If $\pi_i^R = \pi_i(x_i^R, x_{-i}^R)$, then utilizing π_i^R in a SG gives rise to a NE equal to x^R if and only if $P_L x_i^R = P_i x_i^R$.*

PROOF. It suffices to look at the KKT systems $l_i(z_i, \pi_i \mid x_{-i}^R) = 0$, given by (2), for the SG and RSG scenarios. Namely, after applying (4), the derivative of the i-th follower's Lagrangian evaluated at x_i^R satisfies $P_L x_i^R + P_L \sigma(x_{-i}^R) + q_L = 0$. On the other hand, if π_i^R is applied in the context of SG, x_i^R will remain the NE of $G^0(\mathcal{I};\pi)$ if and only if $\frac{1}{2}(P_L + P_i)x_i^R + P_L\sigma(x_{-i}^R) + q_L = 0$ holds. Hence, x^R remains the NE of $G^0(\mathcal{I};\pi)$ if and only if $\frac{1}{2}P_L x_i^R = \frac{1}{2}P_i x_i^R$, which does not always hold. □

Therefore, even the Stackelberg games that exhibit favorable mathematical properties such as the one in Definition 2 pose significant difficulties in computing the leader's strategy. As previously mentioned in the introduction, if the nature of the application allows centralized computation, one can formulate an MPCC that can be recast into an instance of mixed-integer linear or quadratic problems [22] using the big-M reformulation [13] as demonstrated in [12,18,26]. Unfortunately, such computation could breach the privacy of the lower-level agents in many real-world applications, particularly in terms of sharing information about personal constraint sets $\mathcal{X}_i$. With that in mind, the focus of this paper is entirely redirected towards the decentralized computation of the leader's strategy in Stackelberg games.

2.3 Problem formulation

Owing to the problem's overall non-convex character, with possibly multiple solutions, in this work, we focus on finding the leader's strategy $\pi \in \mathcal{P} \subseteq \mathbb{R}^{m_F}$ based on the concept of local Stackelbrg equilibria (l-SE) previously explored in [10, 20, 23]. For the sake of completeness, we repeat it in Definition 3.

Definition 3 (Local Stackelberg Equilibrium) *Let G_1 be a Stackelberg game as in* (3). *A pair of vectors $(\hat{x}^*, \hat{\pi}) \in \operatorname{gph}(\mathcal{N}_0) \cap (\mathbb{R}^{m_F} \times \mathcal{P})$ is a local Stackelberg equilibrium of G_1 if there exist open neighborhoods $\Omega_{\hat{x}^*}$ and $\Omega_{\hat{\pi}}$ of $\hat{x}^*$ and $\hat{\pi}$ respectively, such that*

$$J_L(\hat{x}^*, \hat{\pi}) \leq \inf_{(x^*,\pi)\in\operatorname{gph}(\mathcal{N}_0)\cap\Omega} J_L(x^*, \pi) \ , \qquad (5)$$

where $\Omega := \Omega_{\hat{x}^*} \times (\mathcal{P} \cap \Omega_{\hat{\pi}})$.

Interestingly, restricting ourselves to the framework of Stackelberg games implies that finding the leader's strategy for the bi-level game (3) in the context of l-SE reduces to finding the local optimum of J_L as a function of π. Namely, under Standing Assumption 1, for any $\pi \in \mathcal{P}$, we have that $|\mathcal{N}_0(\pi)| = 1$. This means that to find the l-SE, we need to find $\hat{\pi}$ and its neighborhood $\Omega_{\hat{\pi}}$, since the condition (5) will always be fulfilled for $\hat{x}^* = \mathcal{N}_0(\hat{\pi})$ and the open ball of radius R given by $\Omega_{\hat{x}^*} := \{x \in \mathcal{X} \mid \|x - \hat{x}^*\| < R\}$, where $R > \max_{x\in\hat{\mathcal{N}}(\Omega_{\hat{\pi}})} \|x - \hat{x}^*\|$ and $\hat{\mathcal{N}}(\Omega_{\hat{\pi}}) = \bigcup_{\pi\in\Omega_{\hat{\pi}}} \mathcal{N}_0(\pi)$.

To summarize, in the following sections we will focus on designing an iterative, decentralized, gradient descent-based algorithm that leverages the guarantees provided by the Implicit Function Theorem [8] concerning the continuous differentiability of $J_L(x^*(\pi), \pi)$ at the current π value. However, before delving deeper into the details, we establish the regularity of the leader's optimization problem through Standing Assumption 4.

Standing Assumption 4 *The leader's constraint set $\mathcal{P} \subseteq \mathbb{R}^{m_L}$ is nonempty, compact and convex. Moreover, $J_L : \mathbb{R}^{Nm_F} \times \mathcal{P} \to \mathbb{R}$ is continuously differentiable in both x^* and π, and for every $i \in \mathcal{I}$, each element of $\boldsymbol{g}_i^{\text{inq}}(x_i, \pi)$ and $\boldsymbol{g}_i^{\text{eq}}(x_i, \pi)$ is continuously differentiable in both x_i and π. Finally, for every $x_{-i}^* \in \mathcal{X}_{-i}(\pi)$, every component of the derivative of the Lagrangian associated with the KKT system $l_i(z_i, \pi \mid x_{-i}^*) = 0$, i.e., $\nabla_{x_i}\mathcal{L}(z_i, \pi)$, is continuously differentiable at both x_i and π.*

3 Decentralized computation of the local Stackelberg equilibrium

To tackle the problem of computing the local Stackelberg equilibrium, the initial step involves introducing the idea of Projected Gradient descent incorporating the Armijo rule. This concept, along with the Implicit Function Theorem, will form the foundation of our method.

3.1 Projected Gradient descent with Armijo rule

We start by adopting the projected gradient descent method with 'Armijo step-size rule along the projection arc' explored in [4]. To update π at iteration $t \in \mathbb{N}$, we first define the mapping $\pi^+ : \mathcal{P} \times \mathbb{R}_+ \to \mathcal{P}$ as

$$\pi^+(\pi_t, s) := \Pi_{\mathcal{P}} \left[\pi_t - s \left. \frac{\mathrm{d}J^L(x_\pi^*, \pi)}{\mathrm{d}\pi} \right|_{\pi=\pi_t} \right] ,$$

where $\Pi_{\mathcal{P}}$ is the projection operator on the leader's constraint set for some particular step size $s \in \mathbb{R}_+$ and x_π^* emphasizes the dependence of the Nash Equilibrium on π. Let β, $\overline{s}$ and δ be fixed scalars such that $\beta, \delta \in (0, 1)$ and $\overline{s} > 0$. Moreover, let $l_t \in \mathbb{Z}_{\geq 0}$ be the smallest non-negative integer such that for $s_t = \beta^{l_t}\overline{s}$ it holds that

$$\begin{aligned} &J^L\left(x_{\pi_t}^*, \pi_t\right) - J^L\left(x_{\pi^+(\pi_t,s_t)}^*, \pi^+(\pi_t, s_t)\right) \geq \\ &\geq \delta \left(\left. \frac{\mathrm{d}J^L(x_\pi^*, \pi)}{\mathrm{d}\pi} \right|_{\pi=\pi_t} \right)^T \left(\pi_t - \pi^+(\pi_t, s_t)\right) . \end{aligned} \qquad (6)$$

Then, the leader's strategy is updated as

$$\pi_{t+1} = \pi^+(\pi_t, s_t) \, . \qquad (7)$$

Under Standing Assumption 4, to observe that l_t is well defined, i.e., a stepsize s_t will be found after a finite number of trials based on the test given by (6), it suffices to invoke the following adaptation of [4, Prop. 2.3.3].

Lemma 1 (Proposition 2.3.3 of [4]) *Let the set $\mathcal{P}$ satisfy Standing Assumption 4, $J_L(x_\pi^*, \pi)$ be continuously differentiable on $\mathcal{P}$ and $\delta \in (0, 1)$. Then, for every $\pi \in \mathcal{P}$, there exists $s_\pi > 0$ such that $J_L(x_\pi^*, \pi) - J_L(x_{\pi^+(\pi,s)}^*, \pi^+(\pi, s)) \geq \delta\nabla_\pi J_L(x_\pi^*, \pi)^T(\pi - \pi^+(\pi, s)))$ holds for every $s \in [0, s_\pi]$.*

Therefore, the complexity of each update step boils down to ensuring that $J_L(x_\pi^*, \pi)$ is continuously differentiable, i.e., showing that the gradient of the leader's objective with respect to the current strategy given by

$$\frac{\mathrm{d}J_L(x_\pi^*, \pi)}{\mathrm{d}\pi} = \frac{\partial J_L(x_\pi^*, \pi)}{\partial \pi} + \mathbf{D}_\pi^T x_\pi^* \frac{\partial J_L(x_\pi^*, \pi)}{\partial x_\pi^*} \ , \qquad (8)$$

is well-defined. In that case, if $\overline{s} > s_{\pi_t}$ we have $l_t = 0$, otherwise the testing procedure (6) terminates after $l_t = \lceil \log_\beta(\frac{s_\pi}{\overline{s}}) \rceil$ iterations.

The challenging aspect of computing (8) stems from the requirement to compute the Jacobian $\mathbf{D}_\pi x^*_\pi$, i.e., from having to estimate how the NE of $G^0(\mathcal{I};\pi)$ reacts to variations in π. This is particularly challenging as in general there exists no closed-form functional description of the connection between π and the obtained NE x^*_π. Therefore, we aim to achieve this by virtue of the Implicit Function Theorem. Namely, with the constraint sets $\mathcal{X}_i(\pi)$ being local, and knowing that

$$\mathbf{D}^T_\pi x^*_\pi \frac{\partial J_L\left(x^*_\pi,\pi\right)}{\partial x^*_\pi} = \sum_{i\in\mathcal{I}} \mathbf{D}^T_\pi x^*_{\pi,i} \frac{\partial J_L\left(x^*_\pi,\pi\right)}{\partial x^*_{\pi,i}} \,,$$

we can compute $\mathbf{D}_\pi x^*_\pi$ in a distributed manner such that each follower remains in charge of only computing the personal Jacobian $\mathbf{D}_\pi x^*_{\pi,i}$. The Jacobians are then communicated to the leader as illustrated in Figure 1, who, in return, calculates

$$\frac{\mathrm{d}J_L\left(x^*_\pi,\pi\right)}{\mathrm{d}\pi} = \frac{\partial J_L\left(x^*_\pi,\pi\right)}{\partial \pi} + \sum_{i\in\mathcal{I}} \mathbf{D}^T_\pi x^*_{\pi,i} \frac{\partial J_L\left(x^*_\pi,\pi\right)}{\partial x^*_{\pi,i}}$$

before updating its decision via (7). To obtain individual $\mathbf{D}_\pi x^*_{\pi,i}$, we leverage the fact that the computed lower-level NE has to solve the best-response optimization problem of the corresponding follower. Namely, to tackle the requirements of the Implicit Function Theorem, for every $i \in \mathcal{I}$, we formulate an optimization problem equivalent to (1) and directly apply the theorem on the problem's KKT mapping $l_i\left(z_i,\pi \mid x^*_{-i}\right)$. To ease the notation in the following sections, we will suppress the subscript denoting dependence on π when it is clear from the context and refer to the Jacobian of follower $i \in \mathcal{I}$ as $\mathbf{D}_\pi x^*_i$.

3.2 Differentiating the KKT conditions

For a given $\pi \in \mathcal{P}$ and the corresponding unique v-NE $x^* \in \mathcal{X}(\pi)$ of the lower-level game, the Implicit Function Theorem allows us to locally compute Jacobians $\mathbf{D}_\pi x^*_i$ by applying the theorem on the KKT mapping $l_i\left(z_i,\pi \mid x^*_{-i}\right)$. For every $i \in \mathcal{I}$, let the set-valued map $\Xi^*_i : \mathcal{P} \rightrightarrows \mathcal{Z}_i$, with $\mathcal{Z}_i := \mathbb{R}^{m_F} \times \mathbb{R}^{m_{\mathrm{inq},i}}_{\geq 0} \times \mathbb{R}^{m_{\mathrm{eq},i}}$, be

$$\Xi^*_i(\pi) := \left\{ z_i \in \mathcal{Z}_i \,\middle|\, l_i\left(z_i,\pi \mid x^*_{-i}\right) = 0 \right\} \,. \qquad (9)$$

Moreover, let $\Theta = \left[1, m_{\mathrm{inq},i}\right] \cap \mathbb{N}$, and the set of non-strongly active inequality constraints $\Gamma^\pi_i(x^*_i,\lambda^*_i)$ be

$$\Gamma^\pi_i(x^*_i,\lambda^*_i) := \left\{ j \in \Theta \mid \lambda^{j*}_i = 0 \wedge \boldsymbol{g}^{\mathrm{inq}}_i(x^*_i,\pi)_j = 0 \right\} \,,$$

where $\boldsymbol{g}^{\mathrm{inq}}_i(x_i,\pi)_j$ is the j-th inequality constraint and λ^{j*}_i represents the corresponding dual variable of the best-response optimization problem. With a slight abuse of notation, the Implicit Function Theorem from [8] adapted to our problem reads as the following theorem.

Theorem 1 (Theorem 1.B1 of [8]) *Let Standing Assumptions 1– 4 hold and $x^* \in \mathcal{X}(\pi)$ be the unique NE of the game $G^0(\mathcal{I};\pi)$ for some $\pi \in \mathcal{P}$. Furthermore, let the best-response optimization problem of each agent $i \in \mathcal{I}$ be defined via (1), its KKT mapping $l_i(z_i,\pi \mid x^*_{-i})$ via (2), and $\Xi^*_i(\pi)$ be defined via (9). If $l_i(\hat{z}_i,\pi \mid x^*_{-i}) = 0$, $\Gamma^\pi_i(\hat{x}_i,\hat{\lambda}_i)$ is empty and $\mathbf{D}_{z_i} l_i(\hat{z}_i,\pi \mid x^*_{-i})$ is non-singular for some $\hat{z}_i$, then the solution mapping $\Xi^*_i(\pi)$ has a single-valued localization z^*_i around $\hat{z}_i = (\hat{x}_i,\hat{\lambda}_i,\hat{\nu}_i)$, that is continuously differentiable in a neighbourhood Ω_π of π, with the Jacobian satisfying for every $\pi \in \Omega_\pi$*

$$\mathbf{D}_\pi \, z^*_i\left(\pi\right) = -\,\mathbf{D}^{-1}_{z_i} \, l_i\left(\hat{z}_i,\pi \mid x^*_{-i}\right) \mathbf{D}_\pi \, l_i\left(\hat{z}_i,\pi \mid x^*_{-i}\right) \,,$$

where $\mathbf{D}_{z_i} \, l_i\left(\hat{z}_i,\pi \mid x^*_{-i}\right)$ and $\mathbf{D}_\pi \, l_i\left(\hat{z}_i,\pi \mid x^*_{-i}\right)$ satisfy

$$\mathbf{D}_{z_i} \, l_i := \begin{bmatrix} \mathbf{D}_{x_i} \nabla_{x_i}\mathcal{L}_i, & \mathbf{D}^T_{x_i} \boldsymbol{g}^{\mathrm{inq}}_i, & \mathbf{D}^T_{x_i} \boldsymbol{g}^{\mathrm{eq}}_i \\ \mathrm{Dg}\big(\hat{\lambda}_i\big) \, \mathbf{D}_{x_i} \boldsymbol{g}^{\mathrm{inq}}_i, & \mathrm{Dg}\big(\boldsymbol{g}^{\mathrm{inq}}_i\big), & \mathbf{0} \\ \mathbf{D}_{x_i} \boldsymbol{g}^{\mathrm{eq}}_i, & \mathbf{0}, & \mathbf{0} \end{bmatrix}$$

$$\mathbf{D}_\pi \, l_i := \begin{bmatrix} \mathbf{D}_\pi \nabla_{x_i}\mathcal{L}_i \\ \mathrm{Dg}\big(\hat{\lambda}_i\big) \, \mathbf{D}_\pi \boldsymbol{g}^{\mathrm{inq}}_i \\ \mathbf{D}_\pi \boldsymbol{g}^{\mathrm{eq}}_i \end{bmatrix} \,.$$

The triplet $\hat{z}_i = (x^*_i,\lambda^*_i,\nu^*_i)$, with x^*_i, λ^*_i and ν^*_i being the solution of $G^0_i(\pi,x^*_{-i}) = \min_{x_i\in\mathcal{X}_i(\pi)} J_i(x_i,x^*_{-i},\pi)$, satisfies $l_i(\hat{z}_i,\pi \mid x^*_{-i}) = 0$. However, based on the Implicit Function Theorem, extracting the derivative $\mathbf{D}_\pi \, x^*_i$ from $\mathbf{D}_\pi \, z^*_i$ requires that the matrix $\mathbf{D}_{z_i} \, l_i(\hat{z}_i,\pi \mid x^*_{-i})$ be invertible. The necessary condition for this to hold is that the set $\Gamma^\pi_i(x^*_i,\lambda^*_i)$ be empty. Namely, observe that $\Gamma^\pi_i(x^*_i,\lambda^*_i) \neq \emptyset$ implies that there would exist a zero row in $\mathbf{D}_{z_i} \, l_i(\hat{z}_i,\pi \mid x^*_{-i})$, hence making it singular. On the other hand, the sufficient condition for the Implicit Function Theorem to hold directly depends on the structure of the game resulting from the nature of the application and the NE computed prior to the leader's strategy update step. To ensure this, in Section 3.3, we reorganize the constraints of the original best-response optimization problem to form an equivalent one whose KKT map $l_i(z_i,\pi \mid x^*_{-i})$ yields invertible $\mathbf{D}_{z_i} \, l_i(z_i,\pi \mid x^*_{-i})$.

3.3 Equivalent best-response optimization problem

For a particular $\pi \in \mathcal{P}$, let the unique v-NE of the lower-level game $G^0(\mathcal{I};\pi)$ be $x^* \in \mathcal{X}(\pi)$. For every follower $i \in \mathcal{I}$, its NE strategy $x^*_i \in \mathcal{X}_i(\pi)$ explicitly provides information on what inequality constraints are active for a particular leader's strategy. Let $\mathcal{A}_i(x_i^*)$ represent the set of all active inequality constraints at x_i^*, i.e.,

$$\mathcal{A}_i\left(x_i^*\right) \coloneqq \left\{j \in \left[1, m_{\mathrm{inq},i}\right] \cap \mathbb{N} \mid \boldsymbol{g}_i^{\mathrm{inq}}(x_i^*, \pi)_j = 0\right\}. \tag{10}$$

Consequently, let the complement of $\mathcal{A}_i\left(x_i^*\right)$ be

$$\mathcal{A}_i^{\dagger}\left(x_i^*\right) = \left(\left[1, m_{\mathrm{inq},i}\right] \cap \mathbb{N}\right) \setminus \mathcal{A}_i\left(x_i^*\right). \tag{11}$$

If $\left|\mathcal{A}_i\left(x_i^*\right)\right| = m_{\mathrm{act},i} > 0$, then we can define

$$\overline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi) = \mathrm{col}\Big(\left(\boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi)_j\right)_{j \in \mathcal{A}_i\left(x_i^*\right)}\Big), \tag{12}$$

$$\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi) = \mathrm{col}\Big(\left(\boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi)_j\right)_{j \in \mathcal{A}_i^{\dagger}\left(x_i^*\right)}\Big), \tag{13}$$

that effectively split the inequality constraints into a set of active and inactive ones. This allows us to formulate an auxiliary best-response optimization problem equivalent to (1) whose corresponding set $\Gamma_i^{\pi}(x_i^*, \lambda_i^*)$ is empty.

Lemma 2 *Let Standing Assumptions 1–4 hold and $x^* \in \mathcal{X}(\pi)$ be the unique NE of the game $G^0(\mathcal{I};\pi)$ for some $\pi \in \mathcal{P}$. Moreover, let $\mathcal{A}_i(x_i^*)$ and $A_i^{\dagger}(x_i^*)$ be defined as (10) and (11) and $|\mathcal{A}_i(x_i^*)| = m_{\mathrm{act},i} \neq 0$. If $\overline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi)$ and $\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi)$ are given by (12) and (13), then $x_i^* \in \mathcal{X}_i(\pi)$ solves the best-response problem $G_i^0(\pi, x_{-i}^*)$ given by (1) if and only if it solves the surrogate problem*

$$\overline{G}_i^0\left(\pi, x_{-i}^*\right) \coloneqq \left\{\begin{array}{c} \min\limits_{x_i \in \mathbb{R}^{m_F}} J_i\left(x_i, x_{-i}^*, \pi\right) \\ \mathrm{s.t.}\, \underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi) \leq \mathbf{0}_{m_{\mathrm{inq},i} - m_{\mathrm{act},i}} \\ \overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi) = \mathbf{0}_{m_{\mathrm{eq},i} + m_{\mathrm{act},i}} \end{array}\right\}, \tag{14}$$

where $\overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi) = \left[\boldsymbol{g}_i^{\mathrm{eq}T}(x_i, \pi),\ \overline{\boldsymbol{g}}_i^{\mathrm{inq}T}(x_i, \pi)\right]^T$.

PROOF. Observe that both problems are convex, so it suffices to look at their KKT optimality conditions. If $x_i^* \in \mathcal{X}_i(\pi)$ solves (1) for some π, then $\nabla_{x_i}[J_i(x_i, x_{-i}^*, \pi) + \lambda_i^T \boldsymbol{g}_i^{\mathrm{inq}}(x_i, \pi) + \nu_i^T \boldsymbol{g}_i^{\mathrm{eq}}(x_i, \pi)] = 0$ is satisfied for x_i^* and some feasible λ_i^* and ν_i^*. We can partition λ^* into $\overline{\lambda}_i$ and $\underline{\lambda}_i$ and rewrite $\nabla_{x_i}[J_i(x_i, x_{-i}^*, \pi) + \underline{\lambda}_i^T \underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi) + \nu_i^T \boldsymbol{g}_i^{\mathrm{eq}}(x_i, \pi) + \overline{\lambda}_i^T \overline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi)] = 0$. However, this is exactly the KKT stationarity condition of the surrogate best-response problem (14) for $\overline{\nu}_i^T = [\nu_i^T,\ \overline{\lambda}_i^T]$. Since the primal and dual feasibility conditions are equivalent, the proof is completed. □

We can now postulate the following results regarding the diferentiability of the KKT mapping $l_i\left(z_i, \pi \mid x_{-i}^*\right)$.

Theorem 2 *Let Standing Assumptions 1– 4 hold and $x^* \in \mathcal{X}(\pi)$ be the unique NE of the game $G^0(\mathcal{I};\pi)$ for some $\pi \in \mathcal{P}$. Let the auxiliary best-response optimization problem $\overline{G}_i^0(\pi, x_{-i}^*)$ be defined as in Lemma 2, $\hat{z}_i = (x_i^*, \lambda_i^*, \nu_i^*)$ be its solution and $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$ be defined as in Theorem 1. If $\mathbf{D}_{x_i} \nabla_{x_i} \mathcal{L}_i(\hat{z}_i, \pi) \succ 0$ and $\mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi)$ has full row rank for $\hat{x}_i$, then the matrix $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$ associated with $\overline{G}_i^0(\pi, x_{-i}^*)$ is invertible and the Jacobian $\mathbf{D}_\pi x_i^*$ is given by*

$$\mathbf{D}_\pi x_i^* = -\Sigma_1^{-1}[\Sigma_3 - \Sigma_2^T(\Sigma_2\Sigma_1^{-1}\Sigma_2^T)^{-1}(\Sigma_2\Sigma_1^{-1}\Sigma_3 - \Sigma_4)],$$

where $\Sigma_1 = \mathbf{D}_{x_i} \nabla_{x_i} \mathcal{L}_i$, $\Sigma_2 = \mathbf{D}_{x_i} \boldsymbol{g}_i^{\mathrm{eq}}$, $\Sigma_3 = \mathbf{D}_\pi \nabla_{x_i} \mathcal{L}_i$ and $\Sigma_4 = \mathbf{D}_\pi \boldsymbol{g}_i^{\mathrm{eq}}$ are all evaluated at $\hat{z}_i, \pi, x_{-i}^*$.

PROOF. We start by noting that $\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i^*, \pi) < 0$ guarantees that $\underline{\lambda}_i^* = \mathbf{0}$ due to complementary slackness, and hence $\overline{\Gamma}_i^{\pi} = \emptyset$. In order to prove invertibility of $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$, we invoke Lemma 3 listed in Appendix. Namely, we can partition $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$ into blocks M_1, M_2, M_3 and M_4, evaluated at $\hat{z}_i, \pi$ and x_{-i}^*, such that $M_1 = \mathbf{D}_{x_i} \nabla_{x_i} \mathcal{L}_i(z_i, \pi)$,

$$M_2^T = \begin{bmatrix} \mathbf{D}_{x_i} \underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi) \\ \mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi) \end{bmatrix}, \quad M_3 = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi) \end{bmatrix},$$

$$M_4 = \begin{bmatrix} \mathrm{Dg}(\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i, \pi)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

For $\hat{z}_i, \pi, x_{-i}^*$, the Shur complement of M_1 is given by

$$\mathrm{Sh}\left(M_1\right) \coloneqq \begin{bmatrix} \mathrm{Dg}(\underline{\boldsymbol{g}}_i^{\mathrm{inq}}) & \mathbf{0} \\ \star & -\mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}} M_1^{-1} \mathbf{D}_{x_i}^T \overline{\boldsymbol{g}}_i^{\mathrm{eq}} \end{bmatrix}.$$

Since $\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i^*, \pi)$ encompasses inactive inequality constraints, we have that $\mathrm{Dg}(\underline{\boldsymbol{g}}_i^{\mathrm{inq}}(x_i^*, \pi)) \prec 0$. Similarly, because $\mathbf{D}_{x_i} \nabla_{x_i} \mathcal{L}_i(\hat{z}_i, \pi) \succ 0$ and $\mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}}(x_i, \pi)$ has full row rank, we have that $\mathbf{D}_{x_i} \overline{\boldsymbol{g}}_i^{\mathrm{eq}} M_1^{-1} \mathbf{D}_{x_i}^T \overline{\boldsymbol{g}}_i^{\mathrm{eq}} \succ 0$, making $\mathrm{Sh}(M_1)$, and hence $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$, nonsingular. Moreover, based on Theorem 1, we have $\mathbf{D}_\pi x_i^* = -\overline{M}_1 \Sigma_3 - \overline{M}_2[\mathbf{0}^T, \Sigma_4^T]^T$, where Lemma 3 gives $\overline{M}_1 = \Sigma_1^{-1}(\mathbb{I} - \Sigma_2^T(\Sigma_2\Sigma_1^{-1}\Sigma_2^T)^{-1}\Sigma_2\Sigma_1^{-1})$ and $\overline{M}_2 = -\Sigma_1^{-1}\left[\star,\ -\Sigma_2^T(\Sigma_2\Sigma_1^{-1}\Sigma_2^T)^{-1}\right]$. Direct computation of the right-hand side completes the proof. □

Theorem 2 offers two general conditions that can be used to assess the invertibility of $\mathbf{D}_{z_i} l_i(\hat{z}_i, \pi \mid x_{-i}^*)$ and, consequently, establish the well-posedness of the Jacobian $\mathbf{D}_\pi x_i^*$. This essentially involves confirming the typical structural characteristics of the followers' cost functions and constraint sets for commonly encountered instances

Algorithm 1 Finding leader's optimal strategy

Input: γ, β, $\overline{s}$, δ, ε, T
Output: π
$\pi_0 = \text{Initialize}()$;
for $t \leftarrow 0$ **to** T **do**
  $x^*_{\pi_t} = \text{ComputeVariationalNE}(\pi_t)$;
  for $i \in \mathcal{I}$ **do** ▷ In parallel
    Define $\underline{\boldsymbol{g}}_i^{\text{inq}}(x_{\pi_t,i}, \pi_t)$, $\overline{\boldsymbol{g}}_i^{\text{eq}}(x_{\pi_t,i}, \pi_t)$;
    Obtain $\mathbf{D}_\pi x^*_{\pi_t,i}$ using Theorem 1 on $\overline{G}_i^0$;
  end for
  Leader:
    $\frac{\mathrm{d}J_L(\cdot)}{\mathrm{d}\pi} = \frac{\partial J_L(\cdot)}{\partial \pi} + \sum_{i\in\mathcal{I}} \mathbf{D}_\pi^T x_i^* \frac{\partial J^L(\cdot)}{\partial x_i^*}$;
    $s_t = \text{ArmijoStep}\left(\beta, \overline{s}, \delta, \pi_t, \frac{\mathrm{d}J^L(\cdot)}{\mathrm{d}\pi}\right)$;
    $\pi_{t+1} = \pi^+(\pi_t, s_t)$;
end for

of Stackelberg games. On the other hand, the closed form of the Jacobian is a direct consequence of Lemma 3 and shows that the Jacobian retains constant functional form during the segments of the leader's update procedure with the same sets of active inequality constraints. In the following section, we will further discuss the applicability of Theorem 2 in particular cases. However, before we proceed, we will first present the formal convergence guarantees for the more general case.

Theorem 3 *Let the Stackelberg game be defined as* (3) *under Standing Assumptions 1– 4. At every update step $t \in \mathbb{N}$ of the leader, let $x_t^* \in \mathcal{X}(\pi_t)$ be the unique v-NE of the lower level game and the surrogate best-response optimization problem of the i-th follower be defined as in Lemma 2. If the sequence $\{\pi_t\}$ generated by the projected gradient descent method defined by* (6) *and* (7) *fulfills the conditions of Theorem 2, then it holds that*

$$\lim_{t\to+\infty} \left[J_L\big(x^*_{\pi_{t+1}}, \pi_{t+1}\big) - J_L\big(x^*_{\pi_t}, \pi_t\big) \right] = 0\,,$$

and every limit point of $\{\pi_t\}$ is stationary.

PROOF. First, note that based on the Armijo rule, the sequence $\{J_L(x^*_{\pi_t}, \pi_t)\}_{t=1}^{\infty}$ is monotonically nonincreasing. Because $J_L\left(x^*_\pi, \pi\right)$ is continuous in $z^T = [(x^*_\pi)^T, \pi^T]$ and $\cup_{\pi\in\mathcal{P}}\mathcal{X}(\pi)\times\mathcal{P}$ is compact, there exists $J_L^{\min} \in \mathbb{R}$ such that $J_L\left(x^*_\pi, \pi\right) \geq J_L^{\min}$ for all $z \in \cup_{\pi\in\mathcal{P}}\mathcal{X}(\pi)\times\mathcal{P}$. Since the sequence $\left\{J_L\left(x^*_{\pi_t}, \pi_t\right)\right\}_{t=1}^{\infty}$ is monotonically nonincreasing and bounded, it converges to a finite value implying $\lim_{t\to+\infty}[J_L(x^*_{\pi_{t+1}}, \pi_{t+1}) - J_L\left(x^*_{\pi_t}, \pi_t\right)] = 0$. Since Theorem 2 guarantees that $J_L\left(x^*_\pi, \pi\right)$ is continuously differentiable at every $\pi \in \mathcal{P}$, every limit point of $\{\pi_t\}$ is stationary based on [4, P2.3.3]. □

The complete iterative procedure for finding a local Stackelberg equilibrium is outlined in Algorithm 1. As previously shown, the Jacobian computations required for performing the update step on the upper level can be entirely parallelized regardless of the type of game being played among the followers. Therefore, the overall distributed nature of the complete procedure is entirely dictated by the subproblem of computing the v-NE for a particular leader's strategy. In the forthcoming Section 4, we narrow our focus to a special case of games presented in Defintion 2 with polytopic constraints of the followers. As demonstrated in [25, 27, 31, 32], these games enable the computation of the followers' v-NE in a semi-decentralized manner, involving only the exchange of aggregated follower decisions facilitated by a central aggregator entity, i.e., the communication hub. When $\mathcal{P}$ is a polytope, we demonstrate how this communication hub can also be used to design a decentralized warm-start procedure that may help mitigate issues arising from an unfavorable initial value of the leader's decision variable $\pi_0 \in \mathcal{P}$.

4 Quadratic aggregative Stackelberg games

We focus on a particular instance of games in Definition 2 with $\pi_i = \pi_j = \pi$ for all $i, j \in \mathcal{I}$. In that case, the lower-level agents minimize a quadratic cost of the form

$$J_i = \frac{1}{2} x_i^T P_i x_i + x_i^T Q_i \sigma(x_{-i}) + r_i^T x_i + x_i^T S_i \pi\,, \quad (15)$$

under π-parametrized local polytopic constraints $x_i \in \mathcal{X}_i(\pi)$ given by $\boldsymbol{g}_i^{\text{inq}}(x_i, \pi) = G_i(\pi)x_i - b_i(\pi)$ and $\boldsymbol{g}_i^{\text{eq}}(x_i, \pi) = A_i(\pi)x_i - b_i(\pi)$. In light of Theorem 2, if $x^* \in \mathcal{X}(\pi)$ denotes the v-NE of the lower level game and $|\mathcal{A}_i(x_i^*)| \neq 0$, we proceed to formulate the surrogate best-response optimization problem by letting $\overline{G}_i(\pi) \in \mathbb{R}^{m_{\text{act},i}\times m_F}$ be a matrix whose rows are the rows of $G_i(\pi)$ listed in $\mathcal{A}_i(x_i^*)$. Moreover, we let $\underline{G}_i(\pi)$ encompass all the remaining rows of $G_i(\pi)$ and decompose the vector $h_i(\pi)$ into $\overline{h}_i(\pi)$ and $\underline{h}_i(\pi)$ such that $\overline{G}_i(\pi)x_i^* = \overline{h}_i(\pi)$ and $\underline{G}_i(\pi)x_i^* < \underline{h}_i(\pi)$. For the cost (15), the conditions of Theorem 2 ensuring the invertibility of

$$\mathbf{D}_{z_i} l_i = \left[\begin{array}{c|cc} P_i & \underline{G}_i^T(\pi) & \overline{A}_i^T(\pi) \\ \hline \mathbf{0} & \mathrm{Dg}\left(\underline{G}_i(\pi)x_i^* - \underline{h}_i(\pi)\right) & \mathbf{0} \\ \overline{A}_i(\pi) & \mathbf{0} & \mathbf{0} \end{array}\right],$$

related to $\overline{G}_i^0(\pi, x^*_{-i})$ reduce to $\mathbf{D}_{x_i}\nabla_{x_i} L(\hat{z}_i, \pi) = P_i \succ 0$ and ensuring that $\overline{A}_i(\pi) = [A_i^T(\pi),\ \overline{G}_i^T(\pi)]^T$ is full row rank. Note that the latter can be easily accounted for during the construction step of the surrogate best-response problem. When designing $\overline{A}_i(\pi)$, it suffices to exclude the active inequality constraints for which there already exists a linearly dependant equality constraint.

As mentioned earlier, quadratic aggregative games allow us to tackle the initialization problem to a certain ex-

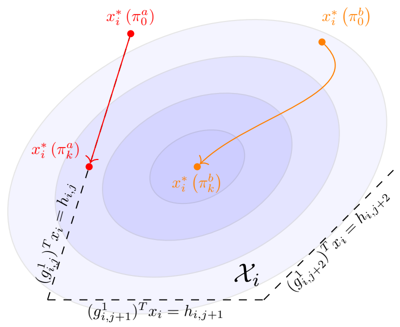

Fig. 2. Ilustrative example of the lower-level NE evolution when the projected gradient descent algorithm is initialized by π_0^a, i.e., such that the NE is on the boundary of the feasible set $\mathcal{X}_i$ (red), and by π_0^b, i.e., such that the NE is in the interior (orange).

tent. Our previous work [27] empirically demonstrated that by adjusting the $\pi_0 \in \mathcal{P}$ value used to initiate the iterative update procedure of the leader we can give rise to local Stackelberg equilibria of varying quality with respect to the value of the leader's objective. While it is possible to sample the leader's action space and repeat the complete iterative process multiple times to find a better solution, opting for initial values π_0 that immediately lead to a v-NE that causes certain inequality constraints to become active can unnecessarily hinder the subsequent steps of the procedure. As illustrated in Figure 2, since the Jacobian is calculated with respect to the surrogate best-response problem, poorly choosing the initial value π_0 means that we effectively start with a lower-level game where each follower has more equality constraints than originally postulated. Although later on we do not have precise control over the trajectory of the leader's iterative procedure, if $\mathcal{P}$ and $\mathcal{X}_i(\pi)$ are polytopes, with $\mathcal{X}_i(\pi)$ being implicitly governed by a linear map of π and x_i, the structure of the analyzed games allows us to increase the flexibility of the algorithm by at least providing a feasible point π_0 that would yield an interior point v-NE in the first iteration of the algorithm. In other words, in the following section we will present a decentralized method for computing a π_0, should such a point exist, that results in a v-NE of the lower-level game that renders no inequality constraint active.

4.1 Decentralized initialization procedure for agents with polytopic actions spaces

As mentioned earlier, a decentralized initialization procedure can be devised for specific configurations of the agents' constraint sets. Before we go deeper into details about the procedure, we summarize all structural requirements on $\mathcal{P}$ and $\mathcal{X}_i(\pi)$ in the following assumption.

Assumption 1 *The constraint sets $\mathcal{P} \subset \mathbb{R}^{m_L}$ and $\mathcal{X}_i(\pi) \subset \mathbb{R}^{m_F}$ are given by bounded polytopes*

$$\mathcal{P} := \{\pi \mid \mathbf{A}_\pi \pi = \mathbf{b}_\pi \wedge \mathbf{G}_\pi \pi \leq \mathbf{h}_\pi\}, \tag{16}$$

$$\mathcal{X}_i(\pi) := \{x_i \mid A_i x_i + A_i^\pi \pi = b_i \wedge G_i x_i + G_i^\pi \pi \leq h_i\}. \tag{17}$$

To find a $\pi_0 \in \mathcal{P}$ that yields an interior point v-NE $x^* \in \mathcal{X}(\pi_0)$, we are essentially interested in ensuring that such a π_0 entails existence of a positive slack vector δ_i such that $G_i x_i^* + G_i^\pi \pi_0 + \delta_i \leq h_i$. However, as we hope to avoid imposing information exchange regarding the personal constraint sets of the followers, we anticipate a consensus-based mechanism to compute π_0.

For every $i \in \mathcal{I}$, we look at the KKT conditions of the best-response optimization problem and define $\psi_i \in \mathbb{R}^{m_{\psi_i}}$, with $m_{\psi_i} = N m_F + m_L + m_{\mathrm{eq},i} + m_{\mathrm{inq},i}$ and

$$\psi_i := [\tilde{\mathbf{x}}_i^T,\ \tilde{\mathbf{p}}_i^T,\ \nu_i^T,\ \delta_i^T]^T,$$

where $\tilde{\mathbf{x}}_i = \mathrm{col}((\tilde{x}_i^j)_{j\in\mathcal{I}}) \in \mathcal{X}(\pi_0)$ denotes the i-th follower's local copy of the complete v-NE x^*, $\tilde{\mathbf{p}}_i \in \mathcal{P}$ being the local copy of the vector π_0, ν_i being the Lagrangian multiplier associated with equality constraints of the best response optimization problem and δ_i is the slack vector that we are looking for. Furthermore, let $\Lambda_{\tilde{x}_i}$, $\Lambda_{\tilde{p}_i}$, Λ_{δ_i} and Λ_i be selection matrices such that $\Lambda_{\tilde{x}_i}\psi_i = \tilde{x}_i^i$, $\Lambda_{\tilde{p}_i}\psi_i = \mathbf{p}_i$, $\Lambda_{\delta_i}\psi_i = \delta_i$ and $\Lambda_i\psi_i = [\tilde{\mathbf{x}}_i^T, \tilde{\mathbf{p}}_i^T]^T$. We can now postulate a necessary and sufficient feasibility test based on linear programming.

Theorem 4 (Internal v-NE feasibility check) *Let the Stackelberg game be defined as* (3) *under Standing Assumptions 1– 4, the structural Assumption 1 and objective functions given by* (15)*. There exists a $\pi_0 \in \mathcal{P}$ such that the corresponding v-NE, $x^* \in \mathcal{X}(\pi_0)$, of the lower-level game $G^0(\mathcal{I};\pi_0)$ renders no inequality constraint active if and only if there exists $\varepsilon > 0$ such that the following linear optimization problem has a solution*

$$\begin{aligned}
\underset{\{\psi_i\}_{i\in\mathcal{I}},\,\beta}{\text{minimize}} \quad & -\sum_{i\in\mathcal{I}} \mathbf{1}^T \Lambda_{\delta_i}\psi_i \\
\text{subject to} \quad & \Lambda_i\psi_i - \beta = \mathbf{0} && \text{, (18a)} \\
& [W_i \ \ S_i^T \ \ A_i^T \ \ \mathbf{0}]\psi_i + r_i = \mathbf{0} && \text{, (18b)} \\
& (A_i\Lambda_{\tilde{x}_i} + A_i^\pi \Lambda_{\tilde{p}_i})\psi_i = b_i && \text{, (18c)} \\
& (G_i\Lambda_{\tilde{x}_i} + G_i^\pi \Lambda_{\tilde{p}_i} + \Lambda_{\delta_i})\psi_i \leq h_i && \text{, (18d)} \\
& -\Lambda_{\delta_i}\psi_i \leq -\varepsilon\mathbf{1} && \text{, (18e)} \\
& \mathbf{A}_\pi \Lambda_{\tilde{p}_i}\psi_i = \mathbf{b}_\pi && , \\
& \mathbf{G}_\pi \Lambda_{\tilde{p}_i}\psi_i \leq \mathbf{h}_\pi
\end{aligned}$$

where $W_i = [P_i, \quad \mathbf{1}_{N-1}^T \otimes \mathbb{I}]$.

PROOF. Firstly, observe that the dummy variable $\beta \in \mathbb{R}^{m_\beta}$, where $m_\beta = N m_F + m_L$, ensures through (18a) that all the followers have equal local copies of the pricing vector and the v-NE. Moreover, since $\varepsilon > 0$, (18d) and (18e) ensure that any optimal solution of this problem renders no inequality constraint active. Due to complementarity slackness, we search for solutions where the dual variables satisfy $\lambda_i = \mathbf{0}$. Hence, (18b) represents the stationarity condition of the KKT system for the convex best-response optimization problem. By adding (18c) and (18d), we form a complete set of KKT optimality conditions, so any optimal β corresponds to a π_0 for which the v-NE is an interior point. $\square$

Firstly, we note that (18) could have been posed as a feasibility problem for simplicity. However, we opt for the proposed functional form as it places the attained v-NE further away from the boundaries. Clearly, the optimal solution β^* encodes a viable π_0 and its corresponding v-NE, i.e., $\beta^* = [(x^*_{\pi_0})^T,\ \pi_0^T]$. Thanks to the separable objective function and constraints of the optimization problem (18), we can preserve the privacy of the followers and solve (18) in a decentralized fashion through the consensus alternating direction method of multipliers (ADMM) [6]. Here, β acts as a global variable to be shared among all the followers and is the only one that needs to be updated in a centralized manner, e.g., in case studies presented in [25, 27, 31, 32], this could be served by the same central entity required for computing the v-NE. If we let the polytope

$$\Omega_{\psi_i} \coloneqq \{\psi_i \in \mathbb{R}^{m_{\psi_i}} \mid M_i^1 \psi_i = v_i^1 \wedge M_i^2 \psi_i \leq v_i^2\}$$

encode all the constraints of (18) except (18a), then the augmented Lagrangian of (18) is

$$\begin{aligned}\mathcal{L}_\rho(\{\psi_i\}, \beta, \{y_i\}) = \sum_{i\in\mathcal{I}} &-\mathbf{1}^T \Lambda_{\delta_i} \psi_i + I_{\Omega_{\psi_i}}(\psi_i) \\ &+ y_i^T(\Lambda_i \psi_i - \beta) + \frac{\rho}{2}\left\|\Lambda_i \psi_i - \beta\right\|_2^2 ,\end{aligned}$$

where $I_{\Omega_{\psi_i}}(\psi_i)$ denotes the indicator function and $\rho > 0$ is an a priori chosen parameter. The consensus ADMM consists of repeating the following three steps

$$\begin{aligned}\psi_i^{k+1} &= \underset{\psi_i \in \Omega_{\psi_i}}{\arg\min}\, \mathcal{L}_\rho(\psi_i, \{\psi_j^k\}_{j\in\mathcal{I}\setminus\{i\}}, \beta^k, \{y_i^k\}), \\ \beta^{k+1} &= \underset{\beta \in \mathbb{R}^{m_\beta}}{\arg\min}\, \mathcal{L}_\rho(\{\psi_i^{k+1}\}, \beta, \{y_i^k\}), \\ y_i^{k+1} &= y_i^k + \rho(\Lambda_i \psi_i^{k+1} - \beta^{k+1}).\end{aligned} \tag{19}$$

Due to the separability of the augmented Lagrangian, solving the N convex quadratic optimization problems for updating individual ψ_i can be done in parallel. The same holds for updating the dual variables y_i. On the other hand, the unconstrained quadratic minimization problem to be solved to obtain β^{k+1} yields

$$\beta^{k+1} = \frac{1}{N}\left[\frac{1}{\rho}\sum_{i\in\mathcal{I}} y_i^k + \sum_{i\in\mathcal{I}} \Lambda_i \psi_i^{k+1}\right],$$

and requires that the followers communicate their updated local estimate of π_0 and the corresponding v-NE, both encoded in ψ_i^{k+1}, to the central aggregator who will then update the consensus variable β. Formal convergence guarantees are given in the following theorem.

Theorem 5 *Let $\Delta_i^k = \Lambda_i \psi_i^k - \beta^k$ denote the residual at each iteration of the consensus ADMM given by* (19). *If A_i is full row rank then $\Delta_i^k \to 0$ when $k \to \infty$.*

PROOF. We aim to directly invoke [7, Th 4.1]. Namely, for [7, Th 4.1] to hold, we first observe that the extended, real-valued function $\overline{f} = \sum_{i\in\mathcal{I}} -\mathbf{1}^T \Lambda_{\delta_i}\psi_i + I_{\Omega_{\psi_i}}(\psi_i)$ is closed, proper and convex. Secondly, we need to make sure that the solution set of (18) is bounded. For this, it suffices to prove that Ω_{ψ_i} is bounded as then $\beta = \Lambda_i \psi_i$ is bounded as well. Under Standing Assumption 1 $\tilde{\mathbf{x}}_i \in \mathcal{X}(\pi)$ is bounded and $\tilde{\mathbf{p}}_i \in \mathcal{P}$ is bounded because of (16). From (18b), we have $A_i^T \nu_i = \gamma_i$ for $\gamma_i \coloneqq -r_i - W_i \tilde{\mathbf{x}}_i - S_i^T \tilde{\mathbf{p}}_i$. If γ_i^j denotes the j-th element of the vector, then for every $j \in [1, m_F] \cap \mathbb{N}$ there exist $\underline{\gamma}_i^j, \overline{\gamma}_i^j \in \mathbb{R}$ such that $\underline{\gamma}_i^j \leq \gamma_i^j \leq \overline{\gamma}_i^j$ since both $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{p}}_i$ are bounded. If we set $\gamma_i^{\min} = \min_j \underline{\gamma}_i^j$ and $\gamma_i^{\max} = \max_j \overline{\gamma}_i^j$, then $\gamma_i^{\min}\mathbf{1} \leq A_i^T \nu_i \leq \gamma_i^{\max}\mathbf{1}$. If A_i is full row rank, then the polytope $\gamma_i^{\min}\mathbf{1} \leq A_i^T \nu_i \leq \gamma_i^{\max}\mathbf{1}$ is bounded. Similarly, we can establish that δ_i is bounded based on (18d) and (18e), which completes the proof. $\square$

In the following section, we will introduce in detail the two numerical case studies showcasing the performance of the main decentralized algorithm and its corresponding warm-start procedure.

5 Numerical examples

We consider two scenarios of a case study in the smart mobility domain previously introduced in [25,27]. In particular, we analyze a market model depicted in Figure 3, where ride-hailing companies $\mathcal{I} = \{I_1, I_2, I_3\}$ compete to meet demand requests that are distributed heterogeneously across the city of Shenzhen [2]. The city is inherently partitioned into four Voronoi-based regions by the available charging infrastructure that consists of stations $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ and is controlled by the central authority L through adjustable electricity prices $\pi \in [p_{\min}, p_{\max}]^4$. At a particular point in time, we assume that each company $i \in \mathcal{I}$ wants to recharge its N_i vehicles by distributing them among charging stations $\mathcal{M}$. Namely, we let the vector $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^4$ denote the strategic decision of company i that describes the fleet split among charging stations, i.e., $\|x_i\|_1 = N_i$ and $x_i^j \geq 0$ represents what fraction of the fleet is to be directed to a particular station $j \in \mathcal{M}$.

The central authority, which can for example be the power-providing company or the government, may have

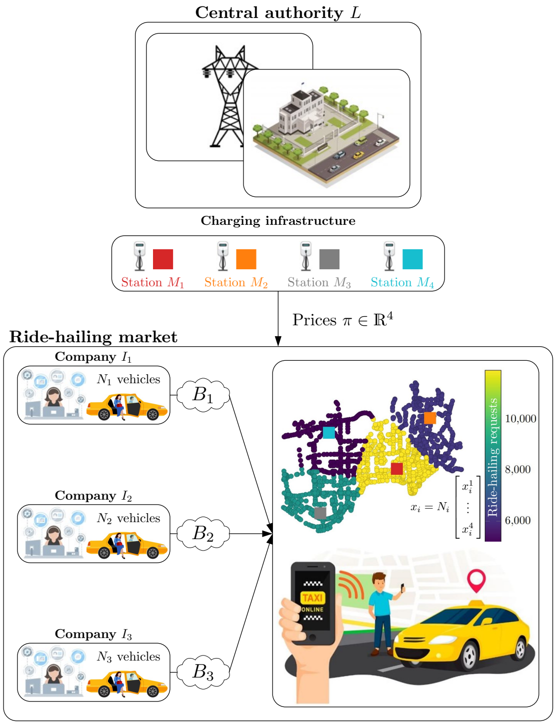

Fig. 3. Illustration of the problem setup with ride-hailing companies $\mathcal{I} = \{I_1, I_2, I_3\}$ operating in a region with charging stations $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$. The central authority L chooses the electricity price $\pi \in \mathcal{P} \subseteq \mathbb{R}^4$ so as to respect the discount budget B_i of each company $i \in \mathcal{I}$.

an interest in balancing the demand on the power grid or it might aim to design pricing incentives to enhance coverage and encourage idle taxi drivers to avoid flocking to the more demand-attractive areas. We assume that the nominal prices of charging are encoded in $\pi_{\text{base}} \in \mathbb{R}^4$ and that the central authority is then interested in determining the optimal discount $\Delta\pi \in \mathbb{R}^4$, all while adhering to the total monetary discount budgets $B_i \in \mathbb{R}$ assigned to companies $\mathcal{I}$ based on external subsidies that they receive. Upon the announcement of the pricing vector $\pi := \pi_{\text{base}} - \Delta\pi$, every company operator is interested in minimizing its operational cost under the feasibility constraints imposed by the battery status of its vehicles. Similar to the objectives analyzed in [24,27,36,40], the operator's cost is assumed in the form of a sum of three terms, i.e., $J_i(x_i, \sigma(x_{-i}), \pi) = J_i^1(x_i, \sigma(x_{-i})) + J_i^2(x_i) + J_i^3(x_i, \pi)$, where $J_i^1(x_i, \sigma(x_{-i}))$ denotes the expected queuing cost at different charging stations due to their limited capacities, $J_i^2(x_i)$ denotes the negative expected revenue in the regions around charging stations, $J_i^3(x_i, \pi))$ denotes the charging cost and $\sigma(\cdot)$ is defined as in Definition 2. The resulting form is quadratic and given by $J_i(x_i, \sigma(x_{-i}), \pi) = \frac{1}{2} x_i^T P_i x_i + x_i^T Q_i \sigma(x_{-i}) + r_i^T x_i + \pi^T S_i x_i$. On the other hand, we assume that the central authority chooses a desired vehicle distribution vector $\mathcal{Z} \in [0,1]^4$ satisfying $\|\mathcal{Z}\|_1 = 1$ and plays the game with the ride-hailing companies in an attempt to minimize the cost $J_G(\sigma(x)) = \frac{1}{2}\|\sigma(x) - \mathbf{1}^T n \mathcal{Z}\|_2^2$, with $n = \text{col}((N_i)_{i \in \mathcal{I}})$ being the vector containing the number of vehicles per company that need to be recharged.

Concerning the constraint sets of ride-hailing companies, they encompass information about the number of vehicles that can reach a certain station under a linear battery discharge model and given the current battery level after the rush-hour period simulation. It has been shown in [25] that a specifically designed polytopic constraint allows for the consistent matching of each ride-hailing vehicle with precisely one charging station in an attempt to respect the allocation given by the split x. For every $i \in \mathcal{I}$, the matching constraints in accordance with [25] are given by $\mathcal{X}_i^m := \{x_i \in \mathbb{R}^4 \mid A_i x_i = b_i \wedge G_i x_i \leq h_i\}$, for some properly chosen A_i, b_i, G_i, h_i. Apart from them, we also account for the limited discount budget B_i through the constraint $\mathcal{X}_i^b(\pi) := \{x_i \in \mathbb{R}^4 \mid (\pi_{\text{base}} - \pi)^T S_i x_i \leq B_i\}$. Hence, for any pricing strategy $\pi \in \mathcal{P}$ and for every $i \in \mathcal{I}$, the resulting constraint set is given by $\mathcal{X}_i(\pi) := \mathcal{X}_i^m \cap \mathcal{X}_i^b(\pi)$. Generally speaking, $\mathcal{X}_i(\pi)$ is a polytopic constraint in x_i but does not comply with the structure proposed in Assumption 1 of Section 4. Therefore, we test two scenarios:

(1) To illustrate the performance of the algorithm in a more general scenario, we shift away from the original setup in [27] and assume that the discount budgets are finite, i.e., $B_i < \infty$ for all $i \in \mathcal{I}$;
(2) To demonstrate the effects of the warm-start procedure, we set $B_i = \infty$ for all $i \in \mathcal{I}$, which yields $\mathcal{X}_i^b(\pi) = \mathbb{R}^4$ and gives rise to an identical problem setup as the one analyzed in [27].

The number of vehicles per company that want to recharge is given by $n = [194, 181, 157]$ and $\mathcal{Z}$ is chosen to correspond to the total number of requests in each cell. For the analyzed case study, $\mathcal{Z}$ is such that $\mathbf{1}^T n \mathcal{Z} = [198, 103, 144, 87]$ and we set $p_{\min} = 0.0$ and $p_{\max} = 5.0$. For the extensive list of all remaining parameters in the simulation, we refer the reader to [25].

5.1 *Finite discount budgets*

In this case study, the finite discount budgets are given by vector $B = [14000, 13000, 12000]$ and the base price is given by $\pi_{\text{base}} = [5.0, 3.0, 5.0, 3.0]$. Before each update step of the pricing policy, we perform $k_{\text{v-NE}} = 5000$ of the Picard-Banach fixed point iteration procedure to compute the v-NE of the lower-level game [27] for the current value of the pricing vector. For the outer loop, we set the number of iterations to $k_{\text{l-SE}} = 350$ and observe the average duration of one update step of approximately $\tau_{\text{avg}} \approx 0.5$ sec. For the given number of iterations and the pricing vector $\pi_{\text{init}}^1 = [4.0, 2.0, 3.0, 1.0]$ used to initialize the outer loop of the procedure, the system manages to

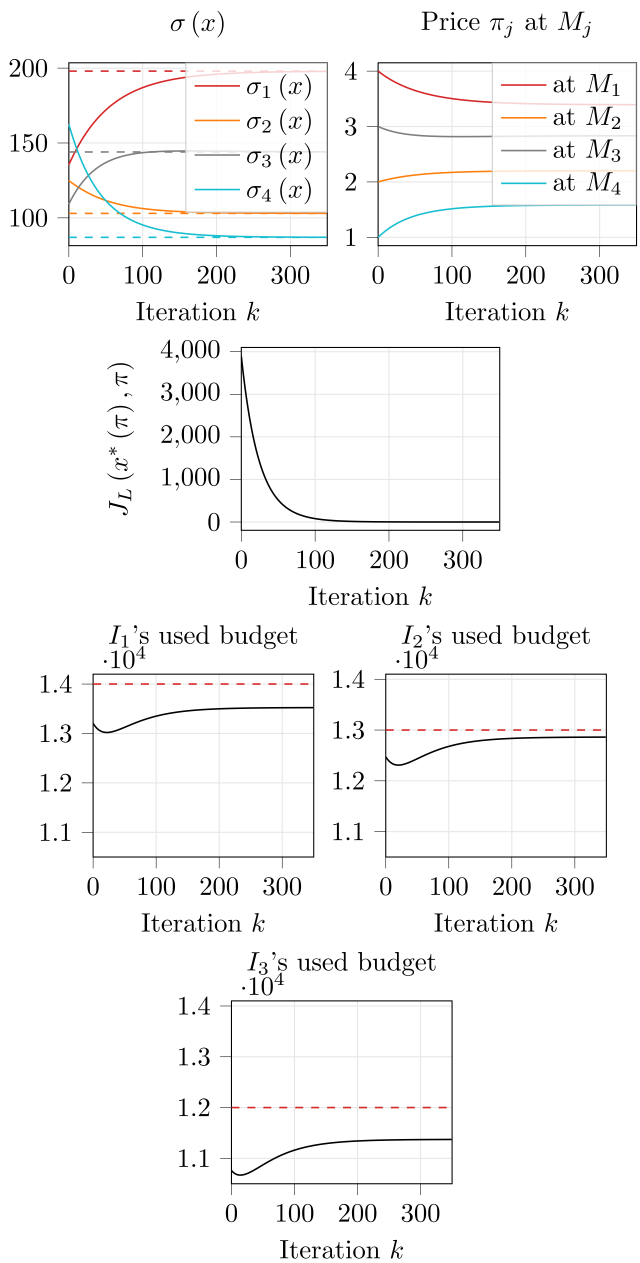

Fig. 4. The plots show the evolution of the total vehicle accumulation at the charging stations $\sigma(x)$, the price of charging π_j at the station M_j, the leader's objective $J_L(x^*(\pi), \pi)$, and the portion of the budgets used at each iteration.

achieve perfect matching with respect to the desired vehicle distribution and attains $J_L(x^*(\pi^*), \pi^*) = 0.022$. This is further supported by plots in Figure 4. The three upper plots show the evolution of the attained vehicle accumulations at different charging stations, the evolution of the pricing vector π, and the corresponding value of the central authority's objective function. The lower three plots demonstrate that no discount budget constraint has been violated during the iterative procedure, i.e., the used discount budget for company I_1 is $B_1^{\text{used}} \approx 13523$, for company I_2 is $B_2^{\text{used}} \approx 12860$, and for company I_3 is $B_3^{\text{used}} \approx 11371$. A full overview of the relevant numerical values is presented in Table 1. It is important to note that the initial value π_{init} has been obtained via sparse grid search as the setup does not comply with the structure of inequality constraints in Assumption 1 of Section 4. Since the complexity of the grid-search procedure grows exponentially in the size of π and polynomially in the granularity of the grid, it is evident that this kind of heuristic is in general not suitable for larger problem sizes. However, for a broad class of bi-level games where the agent's constraints are given by (16) and (17), we can deploy our iterative warm-up procedure. Therefore, in the following subsection, we will shift back our focus to the original problem setup of [27].

Table 1
Vehicle distribution and charging prices

Stations $\mathcal{M}$	Vehicle distribution $\mathbf{1}^T n\mathcal{Z}$	$\sigma(x^*)$	Charging prices π_j^*	$\Delta\pi_j^*$
M_1	198	197.81	3.39	1.61
M_2	103	103.04	2.20	0.80
M_3	144	144.09	2.83	2.17
M_4	87	87.06	1.58	1.42

5.2 *Infinite discount budgets*

When discount budgets are infinite for every ride-hailing company, starting the outer loop with the initial value $\pi_{\text{init}}^2 = [3.0, 3.0, 3.0, 3.0]$ yields that the value of the central authority's objective converges to $J_L(x^*(\pi^*), \pi^*) = 2001$ [27]. As previously discussed, π_{init}^2 already renders certain inequality constraints active which immediately creates a distinction between the original and the surrogate best-response optimization problems. Instead, we let the warm-start procedure with $\rho = 1.0$ run for $k_{\text{w}} = 500$ iterations to obtain the initial pricing vector $\pi_{\text{init}}^{\text{w}} = [4.8, 3.3, 3.9, 2.7]$. Starting the outer loop with $\pi_{\text{init}}^{\text{w}}$ induces an interior v-NE in the first iteration and the complete algorithm is later capable of recovering the perfect matching attained when starting from π_{init}^1. Since the theoretical optimal value for the central authority's objective is zero, the generation of two distinct pricing vectors from the initial states π_{init}^1 and $\pi_{\text{init}}^{\text{w}}$ indicates the general non-uniqueness of the solution in these bi-level games. In Figure 5, we depict the evolution of the complete algorithm for different initial pricing vectors while Table 2 lists all the relevant numerical values. It is interesting to note that the warm-start procedure provides a significantly smaller starting value of the central authority's objective compared to π_{init}^1 and π_{init}^2. However, from the perspective of ride-hailing company operators, starting from π_{init}^1 results in more favorable charging prices in terms of the pricing vector's magnitude and hence, the total charging costs.

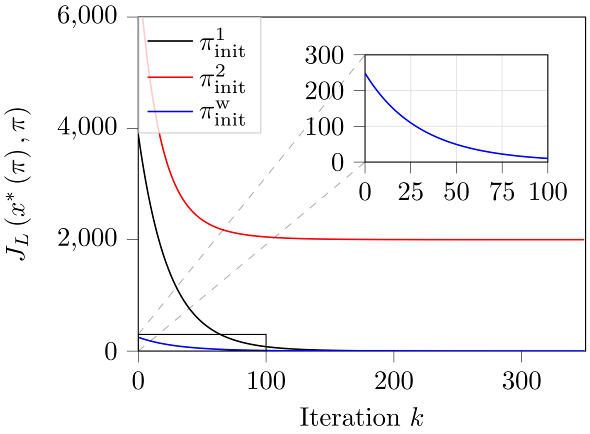

Fig. 5. Evolution of the central authority's objective for different initializations of the upper-level iterative loop.

Table 2
Performance comparison for different initializations

Initial π	Resulting prices at M_j				Attained J_L
	π_1	π_2	π_3	π_4	
π^1_{init}	3.39	2.20	2.83	1.58	0.0222
π^2_{init}	3.57	2.38	3.01	3.00	2000.0
$\pi^{\text{w}}_{\text{init}}$	4.58	3.36	3.99	2.75	0.0039

6 Conclusion

In this work, we formalized an iterative framework based on the Implicit Function Theorem for tackling the problem of computing the local Stackelberg equilibrium in bi-level games. Apart from generalizing the idea introduced in [27] to suit a broader class of games, we address the reported shortcomings of random initialization for a class of quadratic, aggregative games with polytopic constraints. In light of the overall decentralized nature of the approach, we formulate an internal v-NE feasibility test in the form of a linear program that can be efficiently solved by a distributed alternating direction method of multipliers. In addition to theoretical guarantees, we provide an experimental demonstration of the performance improvement in the previously analyzed case study in the smart mobility domain.

In the future, it would be interesting to investigate if other distributed initialization methods could be designed that would be capable of tackling convex constraints beyond polyhedral form. Moreover, the proposed method allows for addressing various real-world problems in the domain of energy management, transportation, etc., hence making it interesting from the practical perspective as well.

A Appendix

Lemma 3 (Chapter 2.17 of [3]) *Let $M_1 \in \mathbb{R}^{p\times p}$, $M_2, M_3^T \in \mathbb{R}^{p\times q}$ and $M_4 \in \mathbb{R}^{q\times q}$. If M_1 is nonsingular, then the inverse*

$$M^{-1} := \begin{bmatrix} \overline{M}_1 & \overline{M}_2 \\ \overline{M}_3 & \overline{M}_4 \end{bmatrix} \quad \text{of } M := \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

exists if and only if Shur complement of M_1 in M, i.e., $\mathrm{Sh}(M_1) = M_4 - M_3 M_1^{-1} M_2$, is nonsingular. The blocks are given by $\overline{M}_1 = M_1^{-1} + M_1^{-1} M_2 \,\mathrm{Sh}(M_1)^{-1} M_3 M_1^{-1}$, $\overline{M}_2 = -M_1^{-1} M_2 \,\mathrm{Sh}(M_1)^{-1}$, $\overline{M}_3 = -\,\mathrm{Sh}(M_1)^{-1} M_3 M_1^{-1}$ and $\overline{M}_4 = \mathrm{Sh}(M_1)^{-1}$.

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