

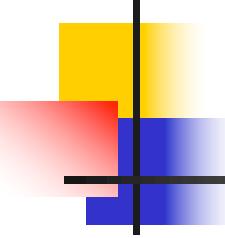
# CSCI 5090/7090- Machine Learning

Spring 2018

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## Bayes Classifier

(slides borrowed from Tom Mitchell, Barnabás Póczos & Aarti Singh



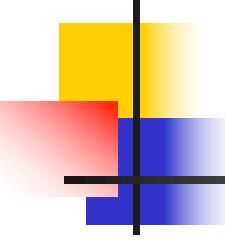
# Joint Distribution:

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sounds like the solution to  
learning  $F: X \rightarrow Y$ ,  
or  $P(Y | X)$ .

Main problem: learning  $P(Y|X)$   
can require more data than we have

consider learning Joint Dist. with 100 attributes  
**# of rows in this table?**  
**# of people on earth?**  
**fraction of rows with 0 training examples?**



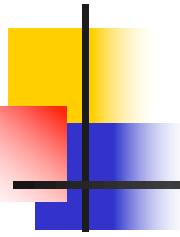
# What to do?

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1. Be smart about how we estimate probabilities from sparse data
  - maximum likelihood estimates
  - maximum a posteriori estimates
  
2. Be smart about how to represent joint distributions
  - Bayes networks, graphical models



1. Be smart about how we estimate probabilities



# Estimating Probability of Heads



- I show you the above coin  $X$ , and hire you to estimate the probability that it will turn up heads ( $X = 1$ ) or tails ( $X = 0$ )
- You flip it repeatedly, observing
  - it turns up heads  $\alpha_1$  times
  - it turns up tails  $\alpha_0$  times
- Your estimate for  $P(X = 1)$  is....?

# Estimating $\theta = P(X=1)$

Test A:

100 flips: 51 Heads ( $X=1$ ), 49 Tails ( $X=0$ )



$X=1$

$X=0$

Test B:

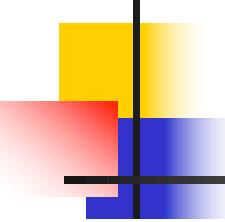
3 flips: 2 Heads ( $X=1$ ), 1 Tails ( $X=0$ )

# Estimating $\theta = P(X=1)$

Case C: (online learning)

- keep flipping, want single learning algorithm that gives reasonable estimate after each flip





# Principles for Estimating Probabilities

Principle 1 (maximum likelihood):

- choose parameters  $\theta$  that maximize  $P(\text{data} | \theta)$

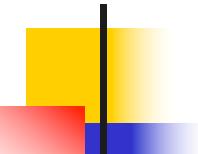
- e.g.,

$$\hat{\theta}^{MLE} = \frac{\alpha_1}{\alpha_1 + \alpha_0}$$

Principle 2 (maximum a posteriori prob.):

- choose parameters  $\theta$  that maximize  $P(\theta | \text{data})$
- e.g.

$$\hat{\theta}^{MAP} = \frac{\alpha_1 + \#\text{hallucinated\_1s}}{(\alpha_1 + \#\text{hallucinated\_1s}) + (\alpha_0 + \#\text{hallucinated\_0s})}$$



## Summary: Maximum Likelihood Estimate



$$X=1 \quad X=0$$

$$P(X=1) = \theta$$

$$P(X=0) = 1-\theta$$

(Bernoulli)

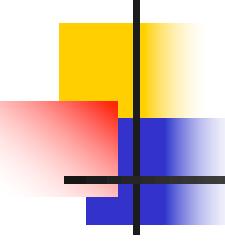
- Each flip yields boolean value for  $X$

$$X \sim \text{Bernoulli}: P(X) = \theta^X(1 - \theta)^{(1-X)}$$

- Data set  $D$  of independent, identically distributed (iid) flips produces  $\alpha_1$  ones,  $\alpha_0$  zeros (Binomial)

$$P(D|\theta) = P(\alpha_1, \alpha_0|\theta) = \theta^{\alpha_1}(1 - \theta)^{\alpha_0}$$

$$\hat{\theta}^{MLE} = \operatorname{argmax}_{\theta} P(D|\theta) = \frac{\alpha_1}{\alpha_1 + \alpha_0}$$



# Principles for Estimating Probabilities

Principle 1 (maximum likelihood):

- choose parameters  $\theta$  that maximize  $P(\text{data} | \theta)$

Principle 2 (maximum a posteriori prob.):

- choose parameters  $\theta$  that maximize

$$P(\theta | \text{data}) = \frac{P(\text{data} | \theta) P(\theta)}{P(\text{data})}$$

# MAP estimation for Binomial distribution

**Coin flip problem:** Likelihood is Binomial

$$P(\mathcal{D} \mid \theta) = \binom{n}{\alpha_H} \theta^{\alpha_H} (1 - \theta)^{\alpha_T}$$

If the prior is Beta distribution,

$$P(\theta) = \frac{\theta^{\beta_H - 1} (1 - \theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

⇒ posterior is Beta distribution

Beta function:  $B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt$

# MAP estimation for Binomial distribution

Likelihood is Binomial:  $P(\mathcal{D} | \theta) = \binom{n}{\alpha_H} \theta^{\alpha_H} (1 - \theta)^{\alpha_T}$

Prior is Beta distribution:  $P(\theta) = \frac{\theta^{\beta_H-1} (1 - \theta)^{\beta_T-1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$

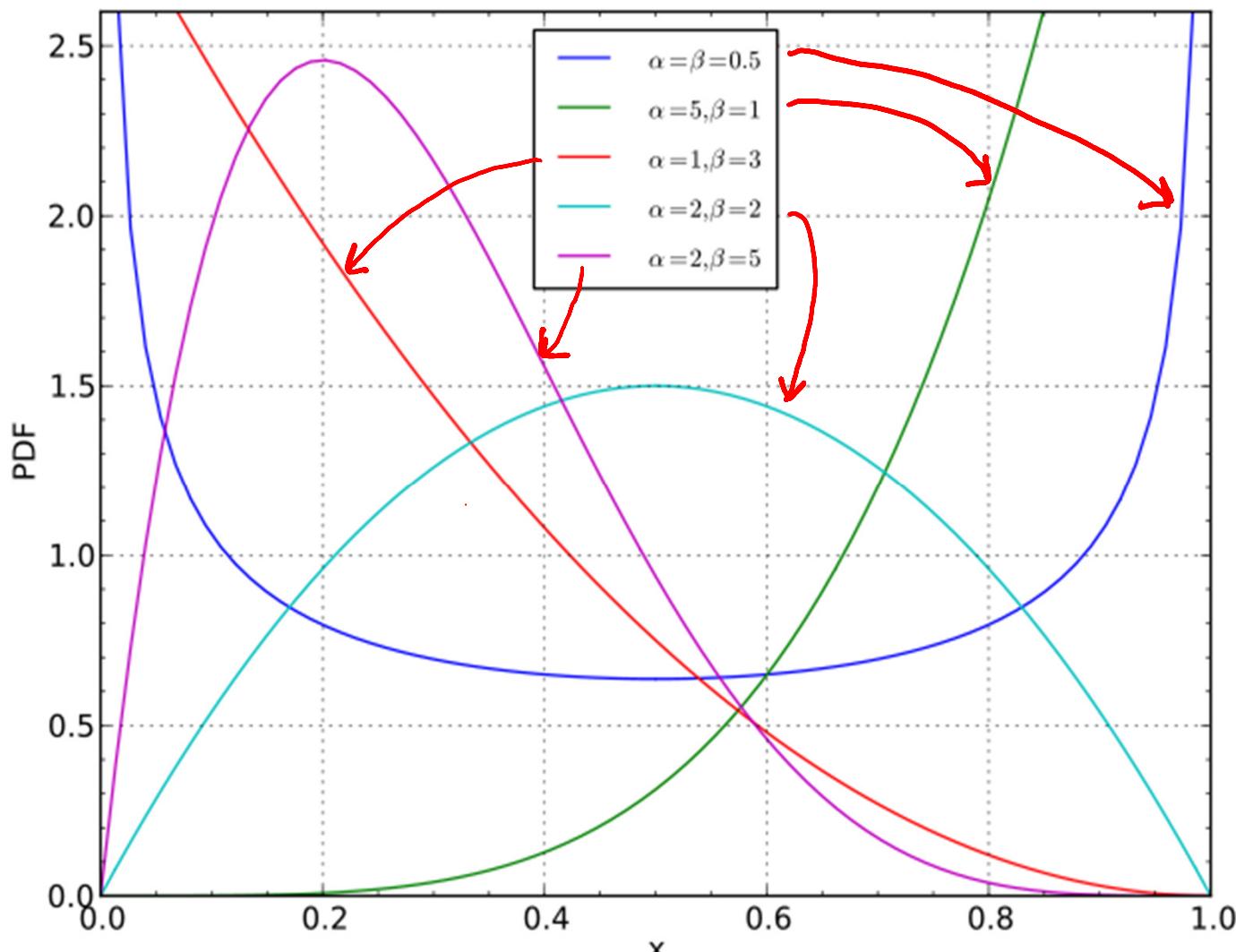
⇒ posterior is Beta distribution

$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

$P(\theta)$  and  $P(\theta|D)$  have the same form! [Conjugate prior]

$$\begin{aligned}\hat{\theta}_{MAP} &= \arg \max_{\theta} P(\theta | D) = \arg \max_{\theta} P(D | \theta)P(\theta) \\ &= \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}\end{aligned}$$

# Beta distribution

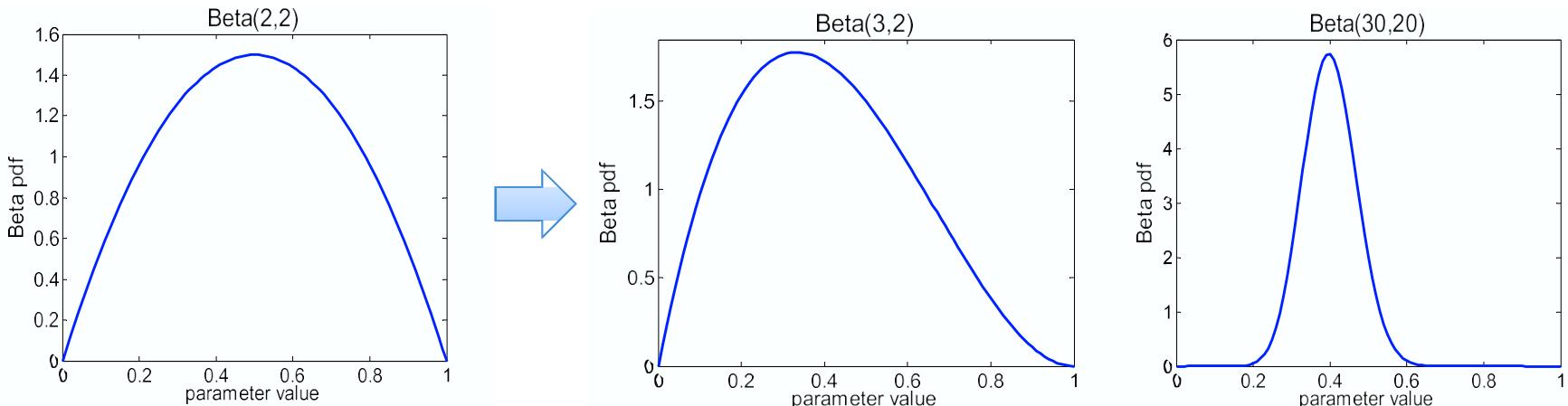


More concentrated as values of  $\alpha$ ,  $\beta$  increase

# Beta conjugate prior

$$P(\theta) \sim Beta(\beta_H, \beta_T)$$

$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$



As  $n = \alpha_H + \alpha_T$   
increases

As we get more samples, effect of prior is “washed out”

- Beta prior equivalent to extra thumbtack flips
- As  $N \rightarrow \infty$ , prior is “forgotten”
- **But, for small sample size, prior is important!**

# From Binomial to Multinomial

**Example:** Dice roll problem (6 outcomes instead of 2)

Likelihood is  $\sim \text{Multinomial}(\theta = \{\theta_1, \theta_2, \dots, \theta_k\})$

$$P(\mathcal{D} | \theta) = \theta_1^{\alpha_1} \theta_2^{\alpha_2} \dots \theta_k^{\alpha_k}$$



If prior is Dirichlet distribution,

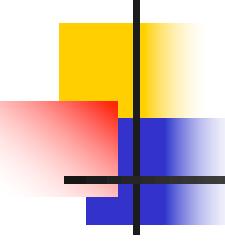
$$P(\theta) = \frac{\prod_{i=1}^k \theta_i^{\beta_i - 1}}{B(\beta_1, \dots, \beta_k)} \sim \text{Dirichlet}(\beta_1, \dots, \beta_k)$$

Then posterior is Dirichlet distribution

$$P(\theta | D) \sim \text{Dirichlet}(\beta_1 + \alpha_1, \dots, \beta_k + \alpha_k)$$

**For Multinomial, conjugate prior is Dirichlet distribution.**

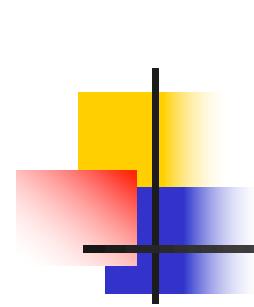
[http://en.wikipedia.org/wiki/Dirichlet\\_distribution](http://en.wikipedia.org/wiki/Dirichlet_distribution)



# Some terminology

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- Likelihood function:  $P(\text{data} | \theta)$
- Prior:  $P(\theta)$
- Posterior:  $P(\theta | \text{data})$
- Conjugate prior:  $P(\theta)$  is the conjugate prior for likelihood function  $P(\text{data} | \theta)$  if the forms of  $P(\theta)$  and  $P(\theta | \text{data})$  are the same.



# You should know

---

- Probability basics
  - random variables, events, sample space, conditional probs, ...
  - independence of random variables
  - Bayes rule
  - Joint probability distributions
  - calculating probabilities from the joint distribution
- Point estimation
  - maximum likelihood estimates
  - maximum a posteriori estimates
  - distributions – binomial, Beta, Dirichlet, ...

# Chain Rule & Bayes Rule

Chain rule:

$$P(X, Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

Bayes rule:

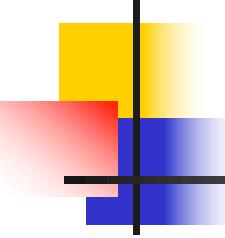
$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Which is shorthand for:

$$(\forall i, j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i)P(Y = y_i)}{P(X = x_j)}$$

Equivalently:

$$(\forall i, j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i)P(Y = y_i)}{\sum_k P(X = x_j | Y = y_k)P(Y = y_k)}$$



# Bayesian Learning

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$\mathcal{D}$  is the measured data.

Our goal is to estimate parameter  $\theta$ .

- Use Bayes rule:

$$P(\theta | \mathcal{D}) = \frac{P(\mathcal{D} | \theta)P(\theta)}{P(\mathcal{D})}$$

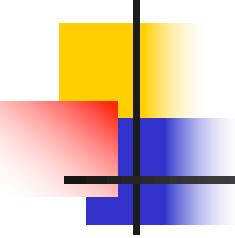
- Or equivalently:

$$P(\theta | \mathcal{D}) \propto P(\mathcal{D} | \theta)P(\theta)$$

posterior      likelihood      prior



# Application of Bayes Rule



# AIDS test (Bayes rule)

## Data

- **Approximately 0.1% are infected**
- **Test detects all infections**
- **Test reports positive for 1% healthy people**

Probability of having AIDS if test is positive:

$$\begin{aligned} P(a = 1|t = 1) &= \frac{P(t = 1|a = 1)P(a = 1)}{P(t = 1)} \\ &= \frac{P(t = 1|a = 1)P(a = 1)}{P(t = 1|a = 1)P(a = 1) + P(t = 1|a = 0)P(a = 0)} \\ &= \frac{1 \cdot 0.001}{1 \cdot 0.001 + 0.01 \cdot 0.999} = 0.091 \end{aligned}$$

Only 9%!

# Improving the diagnosis

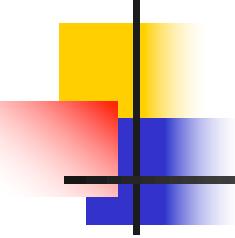
## Use a weaker follow-up test!

- Approximately 0.1% are infected
- Test 2 reports positive for 90% infections
- Test 2 reports positive for 5% healthy people

$$P(a = 0|t_1 = 1, t_2 = 1) = \frac{P(t_1 = 1, t_2 = 1|a = 0)P(a = 0)}{P(t_1 = 1, t_2 = 1|a = 1)P(a = 1) + P(t_1 = 1, t_2 = 1|a = 0)P(a = 0)}$$
$$= \frac{0.01 \cdot 0.05 \cdot 0.999}{1 \cdot 0.9 \cdot 0.001 + 0.01 \cdot 0.05 \cdot 0.999} = 0.357$$

$$P(a = 1|t_1 = 1, t_2 = 1) = 0.643$$

64%!...



# Improving the diagnosis

## Why can't we use Test 1 twice?

- Outcomes are **not** independent,
- but tests 1 and 2 are **conditionally independent (by assumption)**:

$$p(t_1, t_2|a) = p(t_1|a) \cdot p(t_2|a)$$

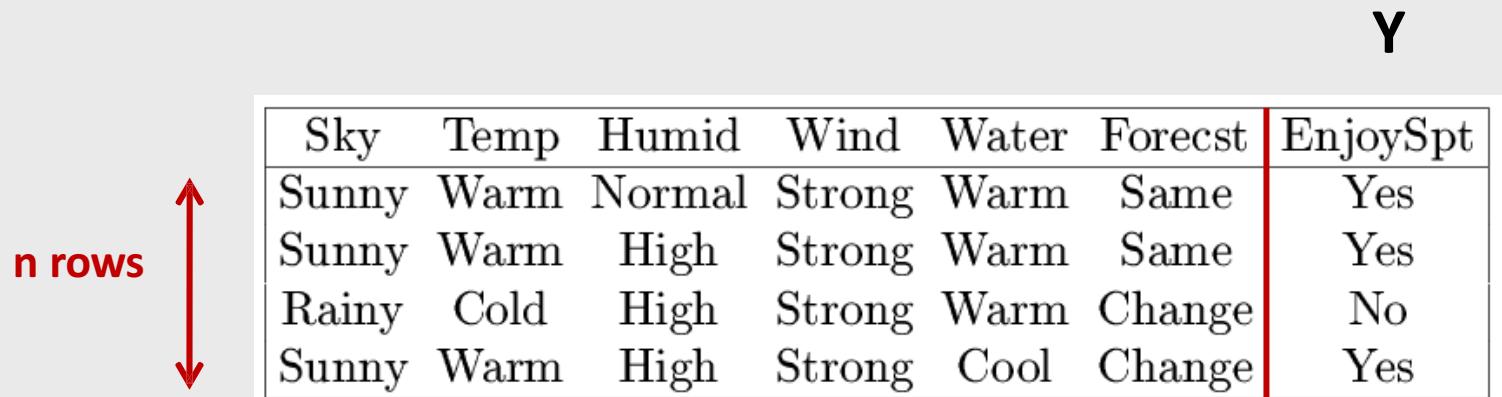


# The Naïve Bayes Classifier

# How many parameters must we estimate?

Suppose  $X = \langle X_1, \dots, X_d \rangle$

where  $X_i$  and  $Y$  are boolean RV's



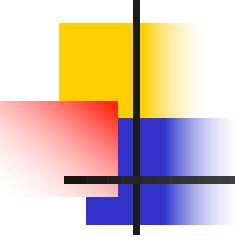
Sky	Temp	Humid	Wind	Water	Forecast	EnjoySpt
Sunny	Warm	Normal	Strong	Warm	Same	Yes
Sunny	Warm	High	Strong	Warm	Same	Yes
Rainy	Cold	High	Strong	Warm	Change	No
Sunny	Warm	High	Strong	Cool	Change	Yes

To estimate  $P(Y|X_1, X_2, \dots, X_n)$

$$(2^{n-1})^2$$

If we have 30  $X_i$ 's instead of 2:  $P(Y|X_1, X_2, \dots, X_{30})$

$$2^{30} \cong 1 \text{ Billion}$$

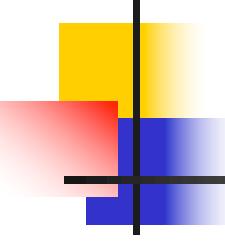


# Naïve Bayes Assumption

**Naïve Bayes assumption:** Features  $X_1$  and  $X_2$  are conditionally independent given the class label  $Y$ :

$$P(X_1, X_2|Y) = P(X_1|Y)P(X_2|Y)$$

More generally:  $P(X_1 \dots X_d|Y) = \prod_{i=1}^d P(X_i|Y)$



# Conditional Independence

Definition:  $X$  is conditionally independent of  $Y$  given  $Z$ , if the probability distribution governing  $X$  is independent of the value of  $Y$ , given the value of  $Z$

$$(\forall i, j, k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write

$$P(X|Y, Z) = P(X|Z)$$

E.g.,

$$P(Thunder | Rain, Lightning) = P(Thunder | Lightning)$$

# Naïve Bayes Assumption

Naïve Bayes uses assumption that the  $X_i$  are conditionally independent, given  $Y$ .

Given this assumption, then:

$$\begin{aligned} P(X_1, X_2|Y) &= P(X_1|X_2, Y)P(X_2|Y) \\ &= P(X_1|Y)P(X_2|Y) \end{aligned}$$

in general:  $P(X_1\dots X_n|Y) = \prod_i P(X_i|Y)$

How many parameters to describe  $P(X_1\dots X_n|Y)$ ?  $P(Y)$ ?

Without conditional indep assumption? **2 (2<sup>n</sup> - 1) + 1**

With conditional indep assumption? **2n + 1**

# Naïve Bayes in a Nutshell

Bayes rule:

$$P(Y = y_k | X_1 \dots X_n) = \frac{P(Y = y_k) P(X_1 \dots X_n | Y = y_k)}{\sum_j P(Y = y_j) P(X_1 \dots X_n | Y = y_j)}$$

Assuming conditional independence among  $X_i$ 's:

$$P(Y = y_k | X_1 \dots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

So, classification rule for  $X^{new} = < X_1, \dots, X_n >$  is:

$$Y^{new} \leftarrow \arg \max_{y_k} P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

# Naïve Bayes Algorithm – discrete $X_i$

- Train Naïve Bayes (examples) for each\* value  $y_k$   
estimate  $\pi_k \equiv P(Y = y_k)$   
for each\* value  $x_{ij}$  of each attribute  $X_i$   
estimate  $\theta_{ijk} \equiv P(X_i = x_{ij}|Y = y_k)$
- Classify ( $X^{new}$ )

$$Y^{new} \leftarrow \arg \max_{y_k} P(Y = y_k) \prod_i P(X_i^{new}|Y = y_k)$$

$$Y^{new} \leftarrow \arg \max_{y_k} \pi_k \prod_i \theta_{ijk}$$

\* probabilities must sum to 1, so need estimate only n-1 of these...

# Estimating Parameters: $Y, X_i$ discrete-valued

Maximum likelihood estimates (MLE's): (**Relative Frequencies**)

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$$

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_{ij}|Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

Number of items in  
dataset D for which  $Y=y_k$

# Subtlety: Insufficient training data

What if you never see a training instance where  $X_1 = a$  when  $Y = b$ ?

For example,

there is no  $X_1 = \text{'Earn'}$  when  $Y = \text{'SpamEmail'}$  in our dataset.

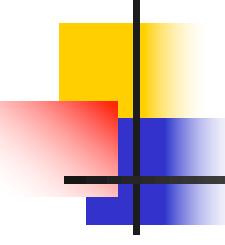
$$\Rightarrow P(X_1 = a, Y = b) = 0 \Rightarrow P(X_1 = a | Y = b) = 0$$

$$\Rightarrow P(X_1 = a, X_2 \dots X_n | Y) = P(X_1 = a | Y) \prod_{i=2}^d P(X_i | Y) = 0$$

Thus, no matter what the values  $X_2, \dots, X_d$  take:

$$P(Y = b | X_1 = a, X_2, \dots, X_d) = 0$$

What now??? What can be done to avoid this?



# Estimating Parameters

- Maximum Likelihood Estimate (MLE): choose  $\theta$  that maximizes probability of observed data  $\mathcal{D}$

$$\hat{\theta} = \arg \max_{\theta} P(\mathcal{D} | \theta)$$

- Maximum a Posteriori (MAP) estimate: choose  $\theta$  that is most probable given prior probability and the data

$$\hat{\theta} = \arg \max_{\theta} P(\theta | \mathcal{D})$$

$$= \arg \max_{\theta} = \frac{P(\mathcal{D} | \theta)P(\theta)}{P(\mathcal{D})}$$

# Conjugate priors

- $P(\theta)$  and  $P(\theta | D)$  have the same form

Eg. 1 Coin flip problem

Likelihood is  $\sim$  Binomial

$$P(D | \theta) = \theta^{\alpha_H} (1 - \theta)^{\alpha_T}$$

If prior is Beta distribution,

$$P(\theta) = \frac{\theta^{\beta_H-1} (1 - \theta)^{\beta_T-1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

Then posterior is Beta distribution

$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

**For Binomial, conjugate prior is Beta distribution.**

[A. Singh]



# Conjugate priors

- $P(\theta)$  and  $P(\theta | D)$  have the same form

Eg. 2 Dice roll problem (6 outcomes instead of 2)

Likelihood is  $\sim \text{Multinomial}(\theta = \{\theta_1, \theta_2, \dots, \theta_k\})$

$$P(D | \theta) = \theta_1^{\alpha_1} \theta_2^{\alpha_2} \dots \theta_k^{\alpha_k}$$

If prior is Dirichlet distribution,

$$P(\theta) = \frac{\prod_{i=1}^k \theta_i^{\beta_i - 1}}{B(\beta_1, \dots, \beta_k)} \sim \text{Dirichlet}(\beta_1, \dots, \beta_k)$$

Then posterior is Dirichlet distribution

$$P(\theta | D) \sim \text{Dirichlet}(\beta_1 + \alpha_1, \dots, \beta_k + \alpha_k)$$

**For Multinomial, conjugate prior is Dirichlet distribution.**



# Estimating Parameters: $Y, X_i$ discrete-valued

**Training data:**  $\{(X^{(j)}, Y^{(j)})\}_{j=1}^n \quad X^{(j)} = (X_1^{(j)}, \dots, X_d^{(j)})$

**Use your expert knowledge & apply prior distributions:**

- Add  $m$  “virtual” examples
- Same as assuming conjugate priors

**Assume priors:**  $Q(Y = b) \quad Q(X_i = a, Y = b)$

**MAP Estimate:**

$$\hat{P}(X_i = a | Y = b) = \frac{\{\#j : X_i^{(j)} = a, Y^{(j)} = b\} + mQ(X_i = a, Y = b)}{\{\#j : Y^{(j)} = b\} + mQ(Y = b)}$$

  
# virtual examples  
with  $Y = b$

# Estimating Parameters: $Y, X_i$ discrete-valued

Maximum likelihood estimates:

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$$

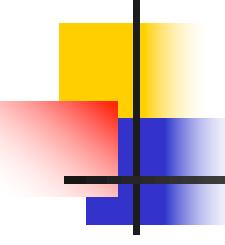
$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

MAP estimates (Beta, Dirichlet priors):

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\} + \alpha_k}{|D| + \sum_m \alpha_m}$$

Only difference:  
"imaginary" examples

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\} + \alpha'_k}{\#D\{Y = y_k\} + \sum_m \alpha'_m}$$



# Case Study: Text Classification

- Classify e-mails
  - $Y = \{\text{Spam}, \text{NotSpam}\}$
- Classify news articles
  - $Y = \{\text{what is the topic of the article?}\}$

What are the features  $X$ ?

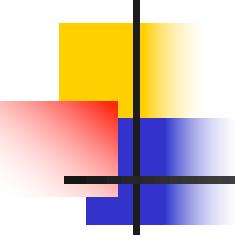
The text!

Let  $X_i$  represent  $i^{\text{th}}$  word in the document

# Data for spam filtering

- date
- time
- recipient path
- IP number
- sender
- encoding
- many more features

Delivered-To: [alex.smola@gmail.com](mailto:alex.smola@gmail.com)  
Received: by 10.216.47.73 with SMTP id s51cs361171web;  
Tue, 3 Jan 2012 14:17:53 -0800 (PST)  
Received: by 10.213.17.145 with SMTP id s17mr2519891eba.147.1325629071725;  
Tue, 03 Jan 2012 14:17:51 -0800 (PST)  
Return-Path: <[alex+caf\\_alex.smola@gmail.com@smola.org](mailto:alex+caf_alex.smola@gmail.com@smola.org)>  
Received: from mail-ey0-f175.google.com (mail-ey0-f175.google.com [209.85.215.175])  
by mx.google.com with ESMTPS id n4si29264232ee57.2012.01.03.14.17.51  
(version=TLSv1/SSLv3 cipher=OTHER);  
Tue, 03 Jan 2012 14:17:51 -0800 (PST)  
Received-SPF: neutral (google.com: 209.85.215.175 is neither permitted nor denied by best guess record for domain of [alex+caf\\_alex.smola@gmail.com@smola.org](mailto:alex+caf_alex.smola@gmail.com@smola.org)) client-ip=209.85.215.175;  
**Authentication-Results:** mx.google.com; spf=neutral (google.com: 209.85.215.175 is neither permitted nor denied by best guess record for domain of [alex+caf\\_alex.smola@gmail.com@smola.org](mailto:alex+caf_alex.smola@gmail.com@smola.org))  
smtp.mail=[alex+caf\\_alex.smola@gmail.com@smola.org](mailto:alex+caf_alex.smola@gmail.com@smola.org); **dkim**=pass (test mode) [header.i=@googlemail.com](mailto:header.i=@googlemail.com)  
Received: by eaal1 with SMTP id l1so15092746eaa.6  
for <[alex.smola@gmail.com](mailto:alex.smola@gmail.com)>; Tue, 03 Jan 2012 14:17:51 -0800 (PST)  
Received: by 10.205.135.18 with SMTP id ie18mr5325064bc.72.1325629071362;  
Tue, 03 Jan 2012 14:17:51 -0800 (PST)  
X-Forwarded-To: [alex.smola@gmail.com](mailto:alex.smola@gmail.com)  
X-Forwarded-For: [alex@smola.org](mailto:alex@smola.org) [alex.smola@gmail.com](mailto:alex.smola@gmail.com)  
Delivered-To: [alex@smola.org](mailto:alex@smola.org)  
Received: by [10.204.65.198](mailto:10.204.65.198) with SMTP id k6cs206093bk1;  
Tue, 3 Jan 2012 14:17:50 -0800 (PST)  
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Tue, 03 Jan 2012 14:17:48 -0800 (PST)  
Return-Path: <[althoff.tim@googlemail.com](mailto:althoff.tim@googlemail.com)>  
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by mx.google.com with ESMTPS id dt4si11767074vdb.93.2012.01.03.14.17.48  
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client-ip=209.85.220.179;  
Received: by vcbf13 with SMTP id f13so11295098vcb.10  
for <[alex@smola.org](mailto:alex@smola.org)>; Tue, 03 Jan 2012 14:17:48 -0800 (PST)  
**DKIM-Signature:** v=1; a=rsa-sha256; c=relaxed/relaxed;  
d=googlemail.com; s=gamma;  
h=mime-version:sender:date:x-google-sender-auth:message-id:subject  
:from:to:**content-type**;  
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Tue, 03 Jan 2012 14:17:47 -0800 (PST)  
Sender: [althoff.tim@googlemail.com](mailto:althoff.tim@googlemail.com)  
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**Subject:** CS 281B. Advanced Topics in Learning and Decision Making  
From: Tim Althoff <[althoff@eecs.berkeley.edu](mailto:althoff@eecs.berkeley.edu)>  
To: [alex@smola.org](mailto:alex@smola.org)  
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$X_i$  represents  $i^{\text{th}}$  word in document

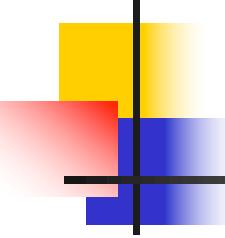
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## Article from rec.sport.hockey

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Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e  
From: xxx@yyy.zzz.edu (John Doe)  
Subject: Re: This year's biggest and worst (opinic  
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most obvious candidate for pleasant surprise is Alex Zhitnik. He came highly touted as a defensive defenseman, but he's clearly much more than that. Great skater and hard shot (though wish he were more accurate). In fact, he pretty much allowed the Kings to trade away that huge defensive liability Paul Coffey. Kelly Hrudey is only the biggest disappointment if you thought he was any good to begin with. But, at best, he's only a mediocre goaltender. A better choice would be Tomas Sandstrom, though not through any fault of his own, but because some thugs in Toronto decided



# NB for Text Classification

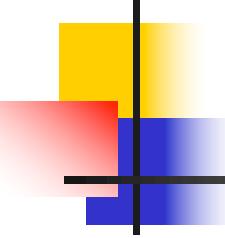
**A problem:** The support of  $P(\mathbf{X}|Y)$  is huge!

- Article at least 1000 words,  $\mathbf{X} = \{X_1, \dots, X_{1000}\}$
- $X_i$  represents  $i^{\text{th}}$  word in document, i.e., the domain of  $X_i$  is the entire vocabulary, e.g., Webster Dictionary (or more).


$$X_i \in \{1, \dots, 50000\} \quad K(1000^{50000} - 1)$$

parameters to estimate without the NB assumption....

$$h_{MAP}(\mathbf{x}) = \arg \max_{1 \leq k \leq K} P(Y = k)P(X_1 = x_1, \dots, X_{1000} = x_{1000} | Y = k)$$



# NB for Text Classification

$X_i \in \{1, \dots, 50000\}$  )  $K(1000^{50000} - 1)$  parameters to estimate....

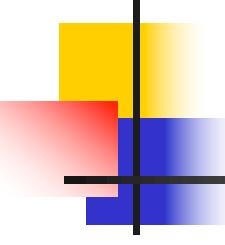
**NB assumption helps a lot!!!**

If  $P(X_i=x_i | Y=y)$  is the probability of observing word  $x_i$  at the  $i^{\text{th}}$  position in a document on topic  $y$

$1000K(50000-1)$  parameters to estimate with NB assumption

NB assumption helps, but still lots of parameters to estimate.

$$h_{NB}(x) = \arg \max_y P(y) \prod_{i=1}^{LengthDoc} P(X_i = x_i | y)$$



# Bag of words model

Typical additional assumption:

**Position in document doesn't matter:**

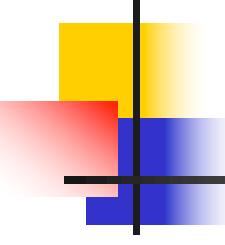
$$P(X_i=x_i | Y=y) = P(X_k=x_i | Y=y)$$

- “Bag of words” model – order of words on the page ignored  
The document is just a bag of words: i.i.d. words
- Sounds really silly, but often works very well!

$K(50000-1)$  parameters to estimate

The probability of a document with words  $x_1, x_2, \dots$

$$\prod_{i=1}^{LengthDoc} P(x_i|y) = \prod_{w=1}^W P(w|y)^{count_w}$$



# Bag of words model

**in is lecture lecture next over person  
remember room sitting the the  
the to to up wake when you**

**When the lecture is over, remember to  
wake up the person sitting next to you  
in the lecture room.**

# Bag of words approach

the world of **TOTAL**



**all about the company**

Our energy exploration, production, and distribution operations span the globe, with activities in more than 100 countries.

At TOTAL, we draw our greatest strength from our fast-growing oil and gas reserves. Our strategic emphasis on natural gas provides a strong position in a rapidly expanding market.

Our expanding refining and marketing operations in Asia and the Mediterranean Rim complement already solid positions in Europe, Africa, and the U.S.

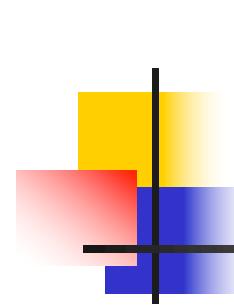
Our growing specialty chemicals sector adds balance and profit to the core energy business.

► All About The Company

- Global Activities
- Corporate Structure
- TOTAL's Story
- Upstream Strategy
- Downstream Strategy
- Chemicals Strategy
- TOTAL Foundation
- Homepage



aardvark	0
about	2
all	2
Africa.	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
...	
Zaire	0



# Learning to classify document: $P(Y | X)$ the “Bag of Words” model

- $Y$  discrete valued. e.g., Spam or not
- $X = \langle X_1, X_2, \dots X_n \rangle$  = document
- $X_i$  is a random variable describing the word at position  $i$  in the document
- possible values for  $X_i$  : any word  $w_k$  in English
- Document = bag of words: the vector of counts for all  $w_k$ 's
- This vector of counts follows a ?? Distribution

# Naïve Bayes Algorithm – discrete $X_i$

- Train Naïve Bayes
  - (examples) for each value  $y_k$  estimate  $\pi_k \equiv P(Y = y_k)$
  - for each value  $x_{ij}$  of each attribute  $X_i$  estimate  $\theta_{ijk} \equiv P(X_i = x_{ij}|Y = y_k)$

prob that word  $x_{ij}$  appears in position i, given  $Y=y_k$

- Classify ( $X^{new}$ )

$$Y^{new} \leftarrow \arg \max_{y_k} P(Y = y_k) \prod_i P(X_i^{new}|Y = y_k)$$

$$Y^{new} \leftarrow \arg \max_{y_k} \pi_k \prod_i \theta_{ijk}$$

\* Additional assumption: word probabilities are position independent

$$\theta_{ijk} = \theta_{mjk} \text{ for } i \neq m$$

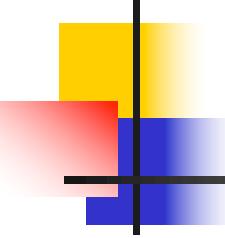
# MAP estimates for bag of words

Map estimate for multinomial

$$\theta_i = \frac{\alpha_i + \beta_i - 1}{\sum_{m=1}^k \alpha_m + \sum_{m=1}^k (\beta_m - 1)}$$

$$\theta_{aardvark} = P(X_i = \text{aardvark}) = \frac{\# \text{ observed 'aardvark'} + \# \text{ hallucinated 'aardvark'} - 1}{\# \text{ observed words} + \# \text{ hallucinated words} - k}$$

What  $\beta$ 's should we choose?



# Twenty news groups results

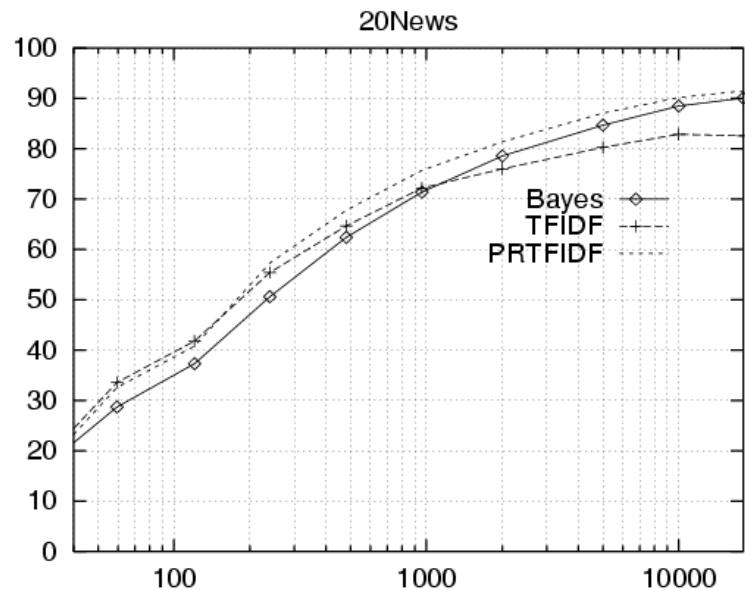
Given 1000 training documents from each group  
Learn to classify new documents according to  
which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

**Naïve Bayes: 89% accuracy**

# Twenty news groups results

## Learning Curve for 20 Newsgroups

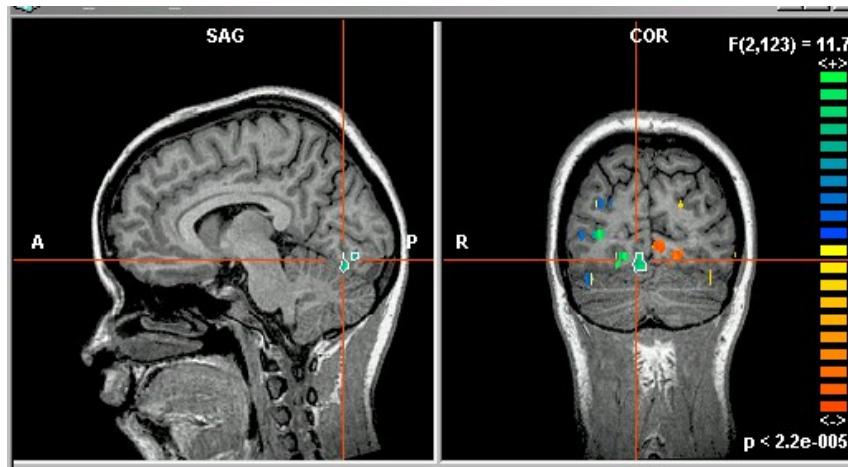


Accuracy vs. Training set size (1/3 withheld for test)

For code and data, see  
[www.cs.cmu.edu/~tom/mlbook.html](http://www.cs.cmu.edu/~tom/mlbook.html)  
click on "Software and Data"

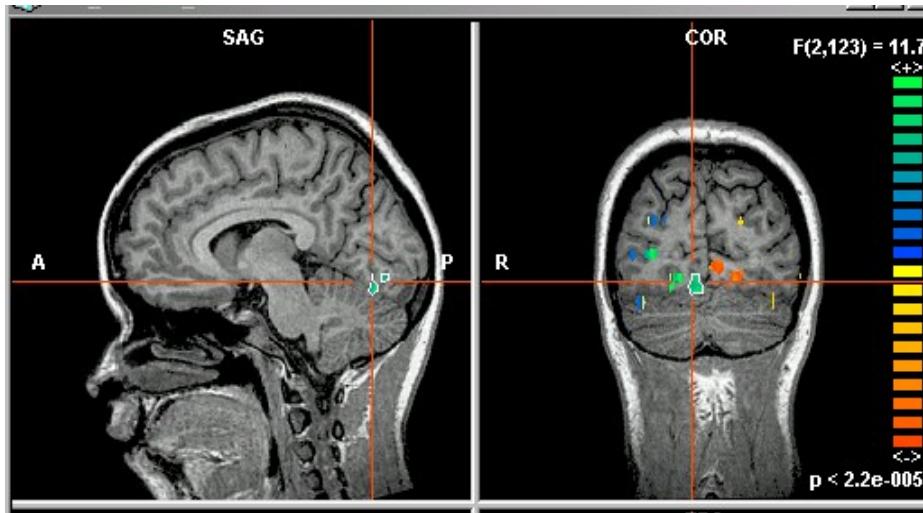
# What if we have continuous $X_i$ ?

Eg., image classification:  $X_i$  is  $i^{\text{th}}$  pixel



# What if we have continuous $X_i$ ?

image classification:  $X_i$  is  $i^{\text{th}}$  pixel,  $Y$  = mental state



Still have:

$$P(Y = y_k | X_1 \dots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

Just need to decide how to represent  $P(X_i | Y)$

# What if features are continuous?

Eg., image classification:  $X_i$  is i<sup>th</sup> pixel

Gaussian Naïve Bayes (GNB): assume

$$P(X_i = x \mid Y = y_k) = \frac{1}{\sigma_{ik}\sqrt{2\pi}} e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$

Sometimes assume  $\sigma_{ik}$

- is independent of Y (i.e.,  $\sigma_i$ ),
- or independent of  $X_i$  (i.e.,  $\sigma_k$ )
- or both (i.e.,  $\sigma$ )

# Gaussian Naïve Bayes Algorithm – continuous $X_i$

(but still discrete  $Y$ )

- Train Naïve Bayes

(examples) for each value  $y_k$

estimate\*  $\pi_k \equiv P(Y = y_k)$

for each attribute  $X_i$  estimate

class conditional mean  $\mu_{ik}$ , variance  $\sigma_{ik}$

- Classify ( $X^{new}$ )

$$Y^{new} \leftarrow \arg \max_{y_k} P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg \max_{y_k} \pi_k \prod_i Normal(X_i^{new}, \mu_{ik}, \sigma_{ik})$$

\* probabilities must sum to 1, so need estimate only n-1 parameters...

# Estimating parameters: Y discrete, Xi continuous

$$\begin{aligned} h_{NB}(x) &= \arg \max_y P(y) \prod_i P(X_i = x_i | y) \\ &\approx \arg \max_k \hat{P}(Y = k) \prod_i \mathcal{N}(\hat{\mu}_{ik}, \hat{\sigma}_{ik}) \end{aligned}$$

$$\hat{\mu}_{MLE} = \frac{1}{N} \sum_{j=1}^N x_j$$

$$\hat{\sigma}_{unbiased}^2 = \frac{1}{N-1} \sum_{j=1}^N (x_j - \hat{\mu})^2$$

# Estimating parameters: Y discrete, Xi continuous

Maximum likelihood estimates:

$$\hat{\mu}_{MLE} = \frac{1}{N} \sum_{j=1}^N x_j$$

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

i<sup>th</sup> pixel in j<sup>th</sup> training image    δ(z)=1 if z true, else 0

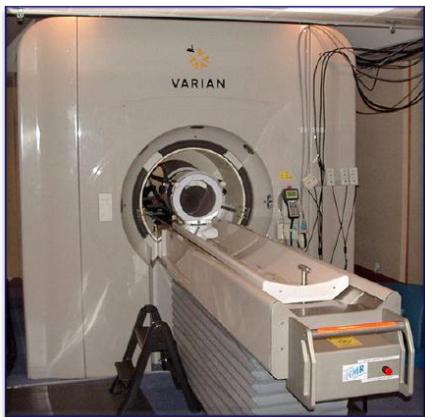
k<sup>th</sup> class    j<sup>th</sup> training image

$$\hat{\sigma}_{unbiased}^2 = \frac{1}{N-1} \sum_{j=1}^N (x_j - \hat{\mu})^2$$

$$\hat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k) - 1} \sum_j (X_i^j - \hat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

# Example: GNB for classifying mental states

Classify a person's cognitive activity, based on brain image



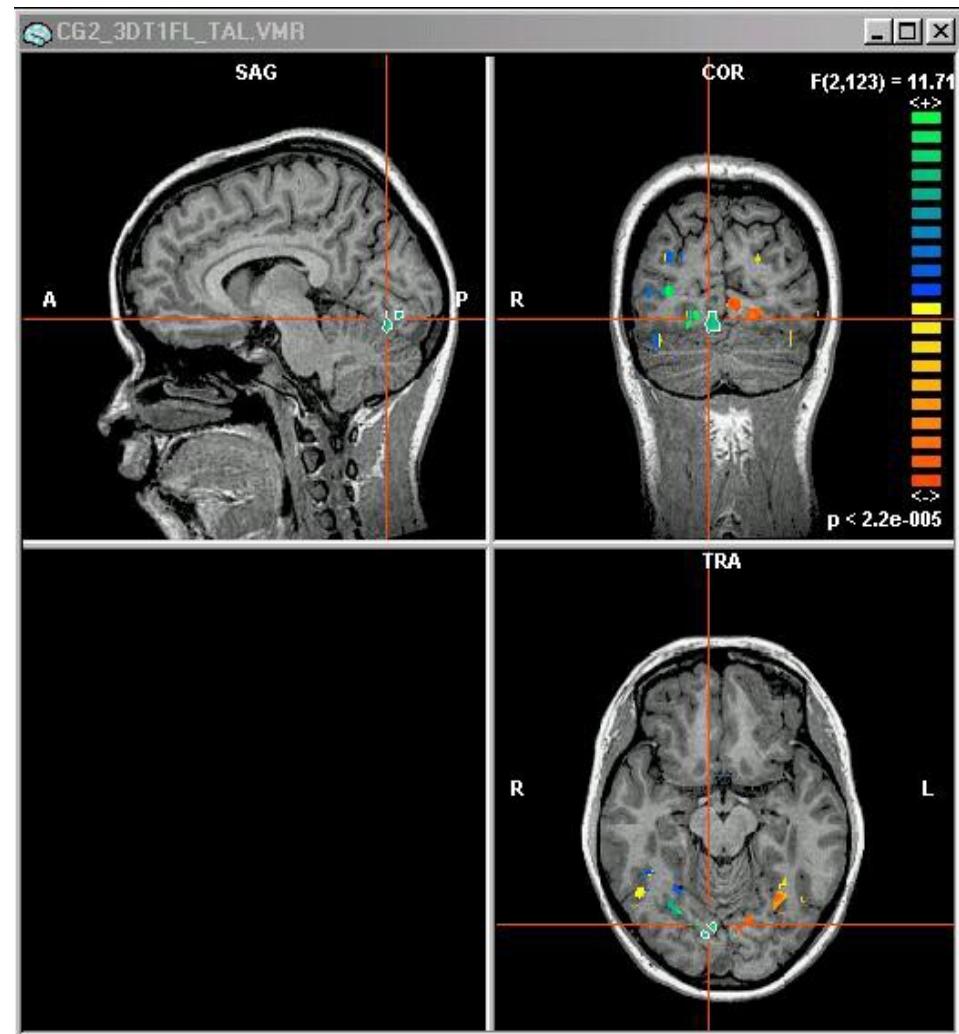
~1 mm resolution

~2 images per sec.

15,000 voxels/image

non-invasive, safe

measures Blood Oxygen Level Dependent (BOLD) response

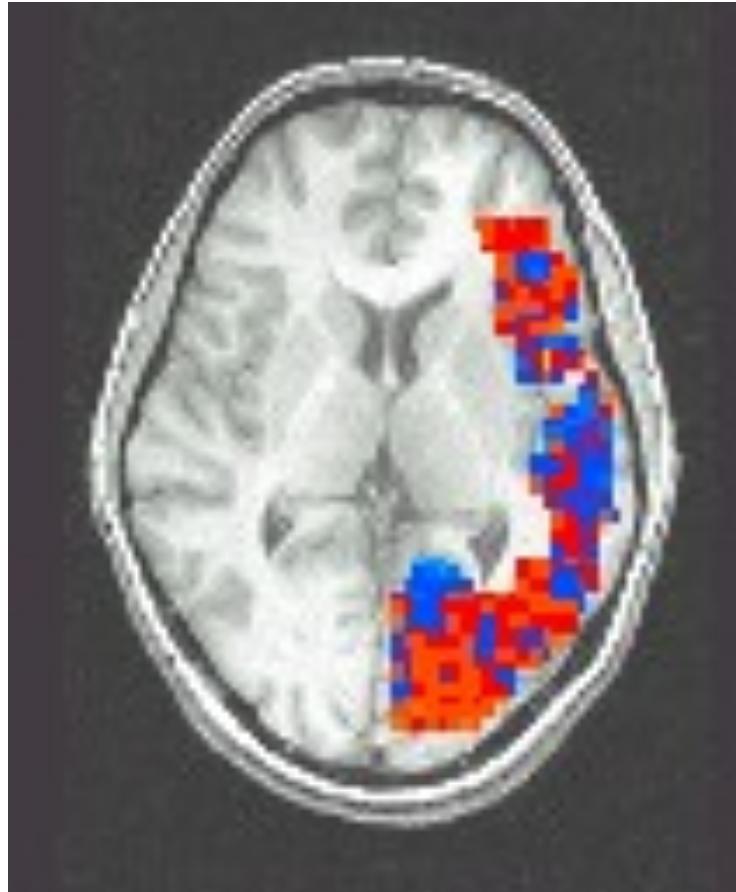


[Mitchell et al.]

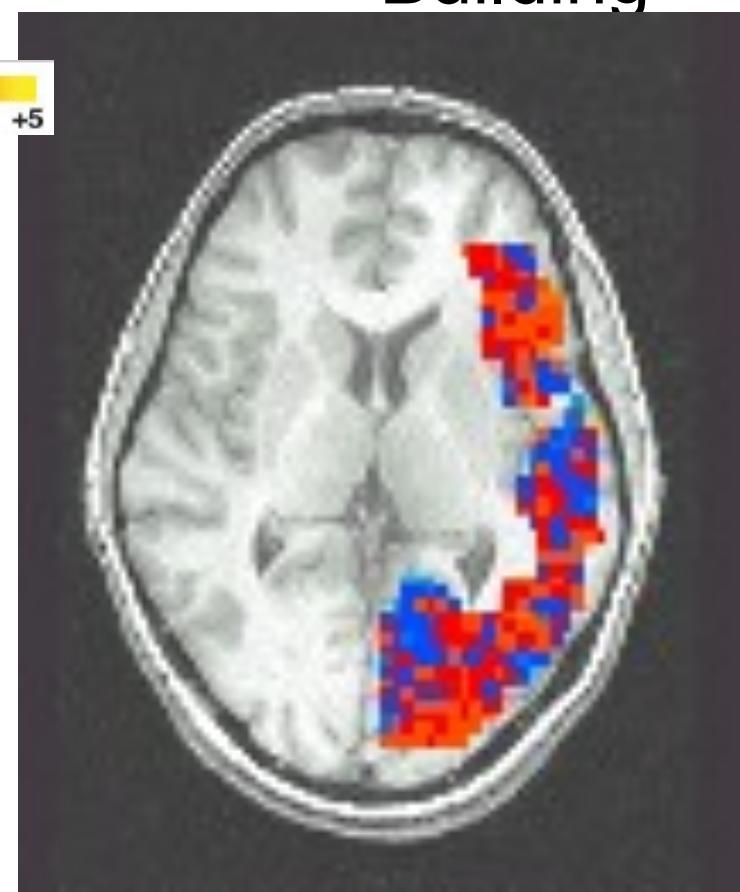
# Learned Naïve Bayes Models – Means for $P(\text{BrainActivity} \mid \text{WordCategory})$

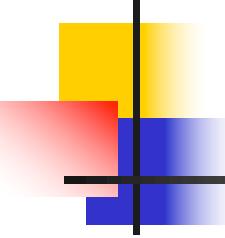
**Pairwise classification accuracy:** [Mitchell et al.]  
**78-99%, 12 participants**

Tool words



Building





# What you should know...

- Training and using classifiers based on Bayes rule
- Conditional independence
  - What it is
  - Why it's important
- Naïve Bayes
  - What it is
  - Why we use it so much
  - Training using MLE, MAP estimates
  - Discrete variables and continuous (Gaussian)