Week 14 : Counting

pending tasks	
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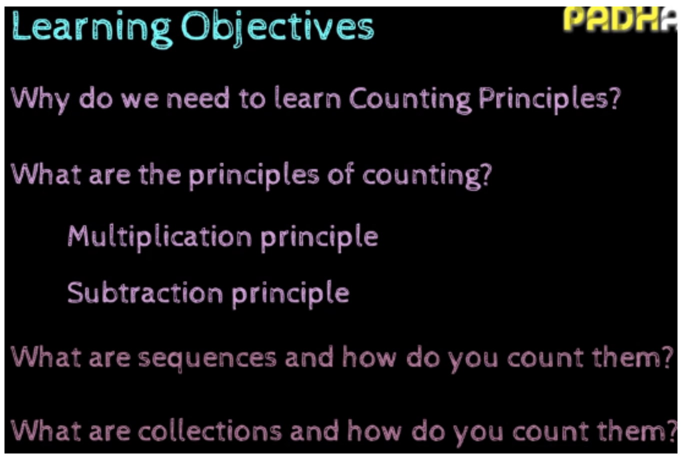


Why do we need Counting and Probability Theory?

- To study a population, statistics are computed on a sample. Thus, it is important to find the probability that a statistic computed from a sample is close to that of the computed from a population.
- Classification problems in ML require probability estimation that a certain element belongs to a particular class i.e. predicting distribution over classes.

Question : What is the probability of getting a head?

Assuming that the coin is fair, w.k.t there are two possible outcomes with equal likelihood. Therefore the probability is 50%.

Question : While rolling a fair dice, what is the probability of getting a 6?

For a fair dice, there are 6 possible outcomes, thus, the probability of 6 is $1/6$.

Question : In a hand of 4 cards, what is the probability of getting 4 aces?

Here the total number of possible outcomes(n) is all possible combinations of 4 cards. Finding n is done using counting.

Very Simple Counting

Question : How many numbers are there between 1 and 358?

358

This gives the principle that the number of numbers between 1 and n is n.

Question : how many numbers are there between 73 and 358?

Subtracting 72 from all numbers converts the problem into numbers between 1 and 286. Using the principle stated the answer is 286. The alternate way is to do n-k+1.

The number of numbers between k and n is n-k+1.

Question : how many numbers are there between 73 and 358 divisible by 7?

Take the first three numbers and the last three numbers and divide by 7. Counting the number of elements in the resulting sequence gives 41.

Question : How many elements are there in the sequence -21,-17,-13,...,391,395,399?

All the elements have a difference of 4. Thus, adding 1 to all elements makes it divisible by 4 and doing so gives the sequence -5,-4,-3,...98,99,100. This gives 100-(-5)+1 = 106 numbers.

Question : How many numbers are there in the sequence 9 5/12, 9 ⅚,10 ¼,..., 21 ½ , 21 11/12 , 22 ⅓ .

Convert all to common denominators and then to proper fractions. It can be observed that the difference between every element is 5, adding 2 to make everything divisible by 5 gives the sequence 23,24,25,...,52,53,54. Thus the number of elements is 54 - 23 +1 = 32.

The Multiplication Principle

def multiplication principle - (t = 3:55)

Question : Can you have a different combo each day of the month?

This involves finding if the number of combos > 30. There are 5*3*3 = 45 combinations. Therefore the answer is yes.

Multiplication principle : The number of ways of making independent choices is the product of the number of choices at each step.

Question : How many ways are there of forming a committee of a boy and a girl from 8 boys and 12 girls?

There are 8*12 = 96 combinations .

Multiplication Principle Special Case : Sequences with Repetition

- The objective is to find the way to make a sequence of k objects from given n objects with repetition.

Question : How many weekly exercise plans can be made from 10 exercises if an exercise can be repeated.

It can be reframed as making a sequence of 7 items from 10 items with repetition. There are 7 independent decisions and 10 choices. Thus there are 10^7 ways.

The number of sequences of k objects made from given n objects when any object in the sequence can be repeated any number of times is n^k .

Question : How many 5 letter words can be formed using alphabets of english language?

26^5.

Multiplication Principle Special Case : Sequences without Repetition

- The objective is to find the way to make a sequence of k objects from given n objects without repetition. The multiplication principle can be used when the number of choices at each step are independent.
- The same exercise regime problem without repetition would result in 10*9*8*...*4. As irrespective of the choice made on the previous day, the number of choices on the next day is independent of the previous choices.

The number of sequences of k objects made from given n objects such that no object in the sequence can be repeated is n(n-1)(n-2)..(n-k+1).

Question : How many 3 digit numbers are there that contain no zero and no repeated digits?

9*8*7

Question : How many of the above numbers are odd?

This would result in restriction on the last number. It causes the last choice to be dependent on the previous two if it is selected third and the multiplication rule cannot be used. Instead if the units digit were selected first, the principle can be applied and results in 5*8*7.

- Number of 5 letter words without repetition would be 26*25*24*23*22. The number that end with a consonant would be 21*25*24*23*22*21

If the problem specifies a constraint or restriction, then always start by addressing the restriction first.

Example : A Different Kind of Sequence

Question : Given a class of 15 students, on how many ways can a committee consisting of a president, vice-president , treasurer and secretary be formed?

The order of elements matters in a sequence. Here if the order changes the roles change, thus, the order matters. Thus there are 15*14*13*12 ways.

The number of ways of filling k named or numbered slots using a collection of n objects is the same as the number of ways of creating a sequence of k elements such that no object in the sequence can be repeated: n(n-1)(n-2)...(n-k+1).

Multiplication Principle Special Case : Sequence Equals Number of objects

- The objective is to make a sequence of n objects from given n objects.

Question : Suppose there are 9 flower pots that have to be arranged in a line at the entrance of the house. In how many ways can this be done?

9!

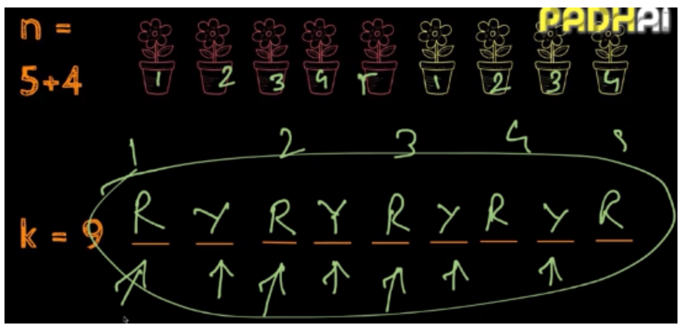
The number of sequences of length n that can be formed using n objects so that no object in the sequence is repeated in n!.

The number of ways in which n objects can be arranged amongst themselves is n!.

The number of permutations of n objects is n.

Also, **The number of ways of filling k named or numbered slots using a collection of n objects is the same as the number of ways of creating a sequence of k elements such that no object in the sequence can be repeated: n(n-1)(n-2)...(n-k+1) is equal to n!/(n-k)!**

Question : There are 5 red and 4 yellow pots, in how many ways can they be arranged such that no two red pots are adjacent to each other?



The pots have to be kept alternatively, giving 5 slots for red pots and 4 for yellow (both alternating). In these slots the pots can be kept in n! number of ways each. Therefore the total number of ways of doing this is 5!*4!.

The Subtraction Principle

A = B+C.

The number of objects that satisfy some condition is equal to the total number of objects in the collection minus the ones which do not satisfy the condition.

Question : How many 3 letter words containing at least one vowel can be formed if no letter is repeated?

The multiplication principle cannot easily be applied here. Subtraction principle is used if calculating the elements of the complement set is easier.

A = set of all 3 letter words with no letters repeated

B = set of all 3 letter words with no letters repeated and at least one vowel

C = 3 letter words with no letters repeated and no vowel

A = B+C , B = A - C

count(B) = 26*25*24 - 21*20*19

Question : How many 5 letter words can be formed that contain at least two consecutive letters which are the same?

A = set of all 5 letter words

B = set of all 5 letter words containing at least two consecutive letters which are the same

C = set of all 5 letter words with no two consecutive letters which are the same

count(B) = count(A) - count(C)

count(B) = 265 - 26*254

Collections

- In a collection the order does not matter.
- In general, formation of a sequence involves two steps : making a collection and re-arranging elements in the collection. The number of elements in a collection (N) is equal to the number of ways of selecting k elements. There are k! Ways of rearranging the k terms. Therefore, the number of sequences is N*k! = n!/(n-k)!. This implies that $N = \frac{n!}{(n-k)!k!}$.

Question : What is the number of ways of selecting 3 vowels from 5 vowels?

Collections	Sequences
(a,e,i)	{(a,e,i), (a,i,e), (e,a,i), (e,i,a), (i,a,e),(i,e,a)}
(a,e,o)	{(a,e,o), (a,o,e), (e,a,o), (e,o,a), (o,a,e), (o,e,a)}
(a,e,u)	{(a,e,u), (a,u,e), (e,a,u), (e,u,a), (u,a,e), (u,e,a)}
(a,i,o)	{(a,i,o), (a,o,i), (i,a,o), (i,o,a), (o,a,i), (o,i,a)}
(a,i,u)	{(a,i,u), (a,u,i), (i,a,u), (i,u,a), (u,a,i), (u,i,a)}
(a,o,u)	{(a,o,u), (a,u,o), (o,a,u), (o,u,a), (u,a,o), (u,o,a)}
(e,i,o)	{(e,i,o), (e,o,i), (i,e,o), (i,o,e), (o,e,i), (o,i,e)}
(e,i,u)	{(e,i,u), (e,u,i), (i,e,u), (i,u,e), (u,e,i), (u,i,e)}
(e,o,u)	{(e,o,u), (e,u,o), (o,e,u), (o,u,e), (u,e,o), (u,o,e)}
(i,o,u)	{(i,o,u), (i,u,o), (o,i,u), (o,u,i), (u,i,o), (u,o,i)}

It can be seen that the number of sequences = number of collections * k!.

Thus, n!(n-k)! = N * k!

N = 5!/2!3! = 10

Question : Given a class of 15 students, in how many ways can a committee of 4 members be formed?

Unlike the sequence example, here the order does not matter. Therefore, it is a collection. The answer is 15!/11!4!

The number of ways of selecting k objects from a given n object is $\frac{n!}{(n-k)!k!}$ And is denoted as $\binom{k}{n}$.

Collections (Some Examples)

Question : consider 10 people in a meeting. If each person shakes hands with every other person in the room what is the total number of handshakes?

This entails counting the number of ways two people can be selected from a group when the order does not matter. Thus the answer is 10!/8!2! = 45

Question : You are going on a vacation and your suitcase has space for only 3 shirts out of 10. In how many ways can you fill the suitcase?

$\binom{10}{3}$ = 120

Question : There are 6 points on a 2-dimensional plane such that no 3 points are collinear. How many line segments can be drawn from these 6 points.

$\binom{6}{3}$ = 15

Question : How many triangles can be drawn from the vertices of a polygon of n sides?

$\binom{3}{n}$.

Collections with Repetitions

Question : How many breakfast combos containing 5 items out of 10 can be formed if multiple servings of the same dish is allowed?

From n+k-1 counters, k dishes have to be selected.

The number of ways of selecting k objects from given n objects with repetitions is $\binom{n+k-1}{k}$.

Collection (+ multiplication principle)

Question : Given a class of 7 boys and 8 girls, in how many ways can a committee of 4 members with 2 boys and 2 girls be formed?

The number of ways of choosing girls and boys are independent steps. Thus , the total number of ways = 8C2* 7C2.

Given : n items of i different types

m1+m2+....+mi = n

Form : collection of k items

k1+k2+....+ki = k

$\binom{k1}{m1} * \binom{k2}{m2} * .... * \binom{ki}{mi}$

Collection (+ subtraction principle)

- Subtraction principle is used when counting the complement is easier.

Question : given 3 cardiologists, 2 neurologists, 4 diabetologists, 5 gynecologists and 7 physicians in how many ways can a 4-member committee be formed containing at least one gynaecologist? A = set of all possible committees of 4 members

B = all possible committees containing at least one gynaecologist

C = all possible committees containing no gynaecologist

B = A - C = 21C4 - 16C4.

Summary

Sequences	
without repetitions	$\frac{n!}{(n-k)!}$
with repetitions	n^k
Collections	
without repetitions	$\frac{n!}{(n-k)!k!}$
with repetitions	$\binom{n+k-1}{k}$

- In a sequence the order matters whereas in a collection the order does not matter.
- To use the multiplication rule, the number of choices at each step must be independent of the choices made at previous steps. The restrictions should be addressed first.
- The subtraction principle can be used when it is easier to calculate the complement of the set.

MCQ

1. 7 of puri, chapathi, idly, dosa, poha, dhokla, pongal and khichadi are to be arranged on the table. How many different arrangements are possible if dosa cannot be next to poha?
    1. **3,600**
    2. 1,440
    3. 2,400
    4. 5,040
2. In a house of the probability that a person is not infected by the flu is 1/3. What is the probability that at least one person in the house is not infected?
    1. 1/81
    2. **65/81**
    3. 2/3
    4. 1/3
3. Using the numbers 2 and 3 how many 10 digit numbers can be formed?
    1. 2^{10}
    2. (22)(5!)
    3. 5! * 5!
    4. 10!
4. Two fair dice are rolled. What is the probability that sum of exposed faces is not a prime number?
    1. 5/12
    2. **7/12**
    3. 0.5
    4. 1/6
5. To make a sandwich there are 3 options of bread and 7 condiments: cheese, tomatoes,onion, ketchup,cucumber, basil and lettuce. As many/few condiments can be included. How many different sandwiches can be made?
    1. 8!
    2. 3*7!
    3. 8*8!
    4. **$3*2^7$**