SANDPILE MODEL

The concept of self-organized criticality (SOC) was introduced by Bak et al. (1987) using the example of a sandpile. If a sandpile is formed on a horizontal circular base with any arbitrary initial distribution of sand grains, a sandpile of fixed conical shape (steady state) is formed by slowly adding sand grains one after another (external drive). In the steady state, the surface of the sandpile makes on average a constant angle with the horizontal plane, known as the angle of repose. The addition of each sand grain results in some activity on the surface of the pile: an avalanche of sand mass follows, which propagates on the surface of the sandpile. In the stationary regime, avalanches are of many different sizes and Bak et al. (1987) argued that they would have a power law distribution. If one starts with an initial uncritical state, initially most of the avalanches are small, but the range of sizes of avalanches grows with time. After a long time, the system arrives at a critical state, in which the avalanches extend over all length and time scales (Bak, 1996; Jensen, 1998; Dhar, 1999; Sornette, 2004).

Laboratory experiments on sandpiles, however, have not in general shown evidence of criticality in sandpiles due to the effects of inertia and dilatation (moving grains require more space) (Nagel, 1992), except for small avalanches (Held et al., 1990) or with elongated rice grains (Malte-Sørensen et al., 1999) where these effects are minimized. Small avalanches have small velocities (and thus negligible kinetic energy), and they activate only the first surface layer of the pile. Elongated rice grains slip at or near the surface as a result of their anisotropy (thus minimizing dilatational effects), and they also build up scaffold-like structures, which enhance the threshold nature of the dynamics.

On the theoretical front, a large number of discrete and continuous sandpile models have been studied. Among them, the so-called abelian sandpile model is the simplest and most popular (Dhar, 1999). Other variants include Zhang's model, which has modified rules for sandpile evolution (Zhang, 1989), a model for abelian distributed processors and other stochastic rule models (Dhar, 1999), and the Eulerian Walkers model (Priezzhev et al., 1996).

In the abelian sandpile model, each lattice site is characterized by its height h. Starting from an arbitrary initial distribution of heights, grains are added one at a time at randomly selected sites n: $h_n \to h_n + 1$. The sand column at any arbitrary site i becomes unstable when h_i exceeds a threshold value h_c and topples to reduce its height to $h_i \to h_i - 2d$, where d is the space dimension of the lattice. The $2d$ grains lost from the site i are redistributed on the $2d$ neighboring sites $\{j\}$, which gain a unit sand grain each: $h_j \to h_j + 1$. This toppling may make some of the neighboring sites unstable. Consequently, these sites will topple themselves, possibly making further neighbors unstable. In this way, a cascade of topplings propagate, which finally terminates when all sites in the system become stable. When this avalanche has stopped, the next grain is added on a site chosen randomly. This condition is equivalent to assuming that the rate of adding sand is much slower than the natural rate of relaxation of the system. The large separation of the driving and of the relaxation time scales is usually considered to be a defining characteristic of SOC. Finally, the system must be open to the outside, that is, it must dissipate energy or matter, for instance. An outcoming flux of grains must balance the incoming flux of grains, for a stationary state to occur. Usually, the outcoming flux occurs on the boundary of the system: even if the number of grains is conserved inside the box, it loses some grains at the boundaries. Even in a very large box, the effect of the dissipating boundaries are essential: increasing the box size will have the effect of lengthening the transient regime over which the SOC establishes itself; the SOC state is built from the long-range correlations that establish a delicate balance between internal avalanches and avalanches that touch the boundaries (Middleton & Tang, 1995).

The simplicity of the abelian model is that the final stable height configuration of the system is independent of the sequence in which sand grains are added to the system to reach this stable configuration (hence the name "abelian" referring to the mathematical property of commutativity). On a stable configuration $\mathcal{C}$, if two grains are added, first at i and then at j, the resulting stable configuration $\mathcal{C}'$ is exactly the same as in the case where the grains were added first at j and then at i. In other sandpile models, where the stability of a sand column depends on the local slope or the local curvature, the dynamics is not abelian, since toppling of one unstable site may convert another unstable site to a stable site. Many such rules have been studied in the literature (Manna, 1991; Kadanoff et al., 1989).

An avalanche is a cascade of topplings of a number of sites created by the addition of a sand grain. The strength of an avalanche can be quantified in several ways:

- size (s): the total number of topplings in the avalanche,
- area (a): the number of distinct sites that toppled,
- lifetime (t): the duration of the avalanche, and
- radius (r): the maximum distance of a toppled site from the origin.

These four different quantities are not independent and are related to each other by scaling laws. Between any two such measures x, y belonging to the set $\{s, a, t, r\}$, one can define a mutual dependence by the scaling of the expectation of one quantity y as a function of the other x:

$$\langle y \rangle \sim x^{\gamma_{xy}}, \tag{1}$$

where γ_{xy} is called a critical exponent, index, or dimension. This equation quantifies a nonlinear generalized proportionality between the two observables x and y ($\ln\langle y \rangle$ is proportional to $\ln x$). The exponents γ_{xy} are related to one another, for example,

$$\gamma_{ts} = \gamma_{tr}\gamma_{rs}. \tag{2}$$

For the abelian sandpile model, it can be shown that the avalanche clusters cannot have any holes and in addition that $\gamma_{rs} = 2$ in two dimensions, that is,

$$\langle s \rangle \sim r^2. \tag{3}$$

In words, the size (number of toppling grains) of an avalanche is proportional to its surface. It has also been shown that $\gamma_{rt} = \frac{5}{4}$:

$$\langle t \rangle \sim r^{\frac{5}{4}}, \tag{4}$$

that is, the average duration $\langle t \rangle$ of an avalanche grows with its typical radius r faster than linearly. However, averages reflect only a part of the rich behavior of sandpile models. A significant information is provided by the full distribution function $P(x)$ for any measure $x \in \{s, a, t, r\}$. Associated with the above scaling laws (1) and (2), one often finds the finite size scaling form for $P(x)$:

$$P(x) \sim x^{-\tau_x} f_x\left(\frac{x}{L^{\sigma_x}}\right). \tag{5}$$

The exponent σ_x determines the variation of the cut-off of the tail of the distribution of the quantity x with the system size L. As long as $x < L^{\sigma_x}$, expression (5) describes a power law distribution of x, reflecting a self-similar structure of the set of avalanches. When x becomes comparable with L^{σ_x}, the function f_x ensures a fast fall-off of $P(x)$ describing the impact of the finite size L of the system on the statistics of the fluctuations of x. Scaling relations like $\gamma_{xy} = (\tau_x - 1)/(\tau_y - 1)$ connect any two measures. Scaling assumptions (5) for the avalanche sizes have not been demonstrated and may be open to doubt (Kadanoff et al., 1989). This seems to be due to the effect of rare large avalanches dissipating at the border, which strongly influence the statistics.

Many different sandpile models have been studied. However, the precise classification of various models into different universality classes in terms of their critical exponents is not yet available. Exact values of all the critical exponents of the most widely studied abelian model are still not known in two dimensions. Some effort has also been made towards the analytical calculation of avalanche size exponents (Ktitarev & Priezzhev, 1998). Blanchard et al. (2000) have developed a dynamical system theory for a certain class of SOC models (like Zhang's model, 1989), for which the whole SOC dynamics can either be described in terms of iterated function systems, or as a piecewise hyperbolic dynamical system of skew-product type where one coordinate encodes the sequence of activations. The product involves activation (corresponding to a kind of Bernouilli shift map) and relaxation (leading to to contractive maps).

In summary, the sandpile model of Bak et al. (1987) and its many extensions have helped found the new concept of self-organized criticality, which is now a useful item in the toolbox and set of concepts used to study complex systems involving triggered activities.

Didier Sornette

See also **Avalanches; Critical phenomena; Fractals; Nonequilibrium statistical mechanics**

Further Reading

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