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OF

GRAPH THEORY

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Schaum's Outline of Theory and Problems of
GRAPH THEORY

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 PRS PRS 9 0 1 0 9 8 7

ISBN 0-07-005489-4

Sponsoring Editor: Barbara Gilson
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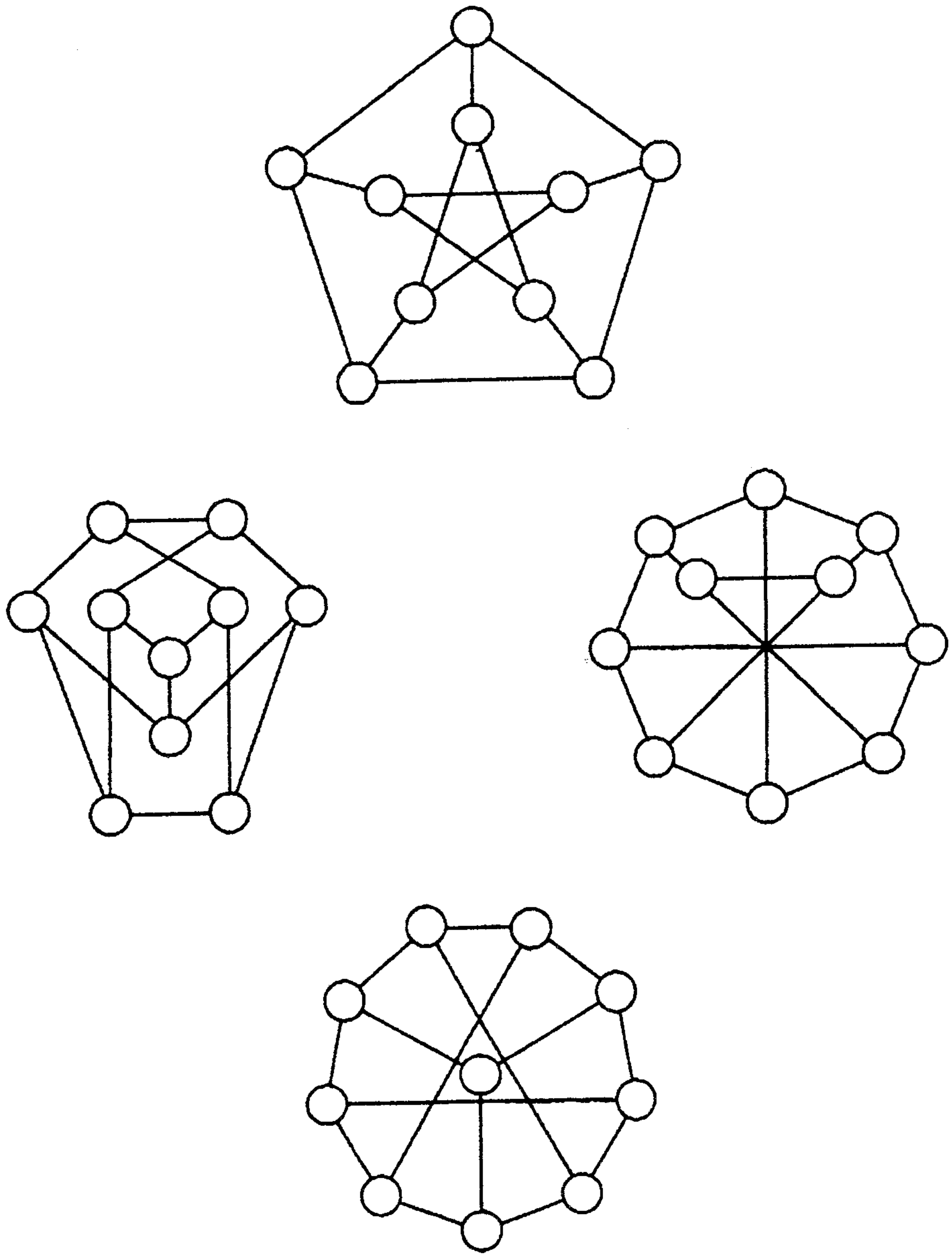

The Petersen Graph and three of its Avatars

Preface

The theory of graphs, with its diverse applications in natural and social sciences in general and in theoretical computer science in particular, is becoming an important component of the mathematics curriculum in colleges and universities all over the world. This book presents the basic concepts of contemporary graph theory in a sequence of nine chapters. It is primarily designed as a supplementary textbook for mathematics and computer science students with a wide range of maturity. At the same time it can also serve as a useful reference book for many academic and industrial professionals who are interested in graph theory.

Graph Theory can be considered a companion volume to *Combinatorics,* which was published as a Schaum Outline in 1995. The style of presentation of the material is the same in both outlines. In each chapter the basic concepts are developed in the first few pages by giving definitions and statements of the major theorems to familiarize the reader with topics that will be fully explored in the selection of solved problems that follow the text. The problems are grouped by topics and are presented in increasing order of maturity and sophistication. In some cases the results established as solutions of problems are some deep theorems and proofs of conjectures that have remained unsettled for several years.

In writing this book I have benefited enormously from the contributions of other mathematicians and scientists. My book brings together the main ideas of graph theory that I learned from the scholarly writings of others distinguished in the field, and no originality is claimed as far as the results presented in the outline are concerned. At the same time, if there are any errors, I accept complete responsibility for their occurrence, and they will be rectified in a subsequent printing of this outline once they are brought to my attention. Any feedback from the reader in this context will be gratefully acknowledged. My e-mail address is vkbal@gauss.umemat.maine.edu and may be used for this purpose.

Since this outline provides basic theory and solved problems, in many cases explicit references are not made to the source of the material. Many people deserve recognition for their specific contributions, and a partial list of books that helped me to prepare this outline is appended as a Select Bibliography for further study.

I am grateful to Dan Archdeacon and Lowell Beineke for the valuable suggestions they made during the course of reviewing parts of the manuscript. In this connection I would also like to thank Kenneth Appel and Douglas West for their helpful hints in the clarification of several results during the preparation of the manuscript. Paul Erdös is no longer with us to show the way. We all miss him dearly. I consider it a singular blessing that I could discuss with him some of the exciting results presented in this outline, and I am forever beholden to him for the kindness and warmth he bestowed on me as well as for his encouragement.

The credit for creating the artwork in this book goes to Dr. Arvind Sharma of the Los Alamos National Laboratory, and it is indeed a great pleasure to acknowledge my indebtedness to him in this regard.

In conclusion I would like to express my immense gratitude to the editorial and production staffs at McGraw-Hill and Progressive Publishing Alternatives for the unfailing cooperation and encouragement extended to me throughout the production of this outline.

V.K. Balakrishnan
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Contents

Chapter 1

Graphs and Digraphs

1.1 INTRODUCTION

Many structures involving real-world situations can be conveniently represented on paper by means of a diagram consisting of a set of points (usually drawn as small circles or dots) together with lines (or curves) joining some or all pairs of these points. For example, the points in a diagram could represent different cities in a country, and a line joining two points that does not pass through a third point may indicate that there is direct air service between the two cities represented by those two points. In some instances, it may happen that there is direct air service from A to B but not from B to A. In such situations, an arrow (directed line or directed curve) is drawn from A to B so that the line joining A and B becomes oriented or directed. The possibility that there can be more than one line joining two points or that there is a line joining a point to itself cannot be ruled out in a more general setting. It is also possible that there can be an arrow from A to B and another arrow from B to A. The particular manner in which these lines and arrows are drawn on a piece of paper is not relevant for our investigations. What really matters is to know whether lines and arrows exist connecting the various points. In some situations, it may be pertinent to ask whether these lines can be drawn such that no two lines intersect except possibly at those points to which they are already joined. A mathematical abstraction of such structures involving points and lines leads us to the concept of graphs and digraphs.

A **graph** G consists of a set V of **vertices** and a collection E (not necessarily a set) of unordered pairs of vertices called **edges.** A graph is symbolically represented as $G = (V, E)$. In this book, unless otherwise specified, both V and E are finite. The **order** of a graph is the number of its vertices, and its **size** is the number of its edges. If u and v are two vertices of a graph and if the *unordered pair* $\{u, v\}$ is an edge denoted by e, we say that e **joins** u and v or that it is an edge between u and v. In this case, the vertices u and v are said to be **incident on** e and e is **incident to** both u and v. Two or more edges that join the same pair of distinct vertices are called **parallel** edges. An edge represented by an unordered pair in which the two elements are not distinct is known as a **loop.** A graph with no loops is a **multigraph.** A graph with at least one loop is a **pseudograph.** A **simple graph** is a graph with no parallel edges and loops. The term *graph* is used in lieu of *simple graph* in many places in this book. The **complete graph** K_n is a graph with n vertices in which there is exactly one edge joining every pair of vertices. The graph K_1 with one vertex and no edge is known as the **trivial graph.** A **bipartite graph** is a simple graph in which the set of vertices can be partitioned into two sets X and Y such that every edge is between a vertex in X and a vertex in Y; it is represented as $G = (X, Y, E)$. The **complete bipartite graph** $K_{m,n}$ is the graph (X, Y, E) with m vertices in X and n vertices in Y in which there is an edge between every vertex in X and every vertex in Y. The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = G_1 \cup G_2 = (V, E)$, where V is the union of V_1 and V_2 and E is the union of E_1 and E_2.

A **directed graph** or **digraph** consists of a finite set V of vertices and *a set A* of *ordered pairs* of distinct vertices called **arcs.** If the ordered pair $\{u, v\}$ is an arc a, we say that the arc a is **directed from** u to v. In this context, arc a is **adjacent from** vertex u and is **adjacent to** vertex v. In a **mixed graph,** there will be at least one edge and at least one arc. If each arc of a digraph is replaced by an edge, the resulting structure is a graph known as the **underlying graph** of the digraph. On the other hand, if each edge of a simple graph is replaced by an arc, the resulting structure is a digraph known as an **orientation** of the simple graph. Any orientation of a complete graph is known as a **tournament.**

Structures thus defined are called graphs because they can be represented graphically on paper. Such graphical representations of structures often enable us to understand and investigate many of their properties. Here are some examples of graphs and digraphs.

Example 1(a). In Fig. 1-1, we have a graph in which the vertex set is $V = \{1, 2, 3, 4, 5, 6, 7\}$. The order is 7 and the size is 8. This is a pseudograph with a loop at vertex 6 and three parallel edges between vertex 2 and vertex 4.

Example 1(b). Suppose each vertex of a graph represents either a recent college graduate or a firm that is hiring college graduates. Join a vertex representing a college graduate and a vertex representing a firm if and only if the firm is interested

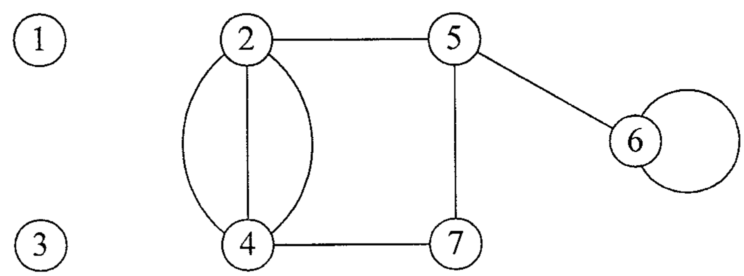


Fig. 1-1

in hiring that graduate. In this situation, we have a bipartite graph $G = (X, Y, E)$, where every vertex in X represents a graduate and every vertex in Y represents a firm.

Example 1(c). Suppose a certain commodity is manufactured at one location denoted by S (the source) and then is transported by trucks to another location denoted by T (the terminal or the sink) along different routes passing through several intermediate locations. This situation can be represented by a digraph in which the intermediate locations, the source, and the terminal are all vertices. We draw a directed edge from a vertex A to another vertex B if it is possible to transport the commodity from the location represented by A to the location represented by B in a truck that does not stop at another location en route.

Example 1(d). Here is an example of a graph in which the number of edges is not finite. Let $V = \{1, 2, 3, \ldots, n\}$ be the set of vertices. Corresponding to each real number in the open interval $(i, i + 1)$, where $i = 1, 2, \ldots, (n - 1)$, we draw an edge joining the vertex i and the vertex $(i + 1)$.

Example 1(e). Here is an example of a graph in which we have an infinite set of vertices. Let $V = \{1, 2, 3, \ldots\}$ be set of vertices. So each positive integer represents a vertex. Join the vertex representing a prime number p and the vertex representing the integer $p + 2$ if and only if $(p + 2)$ is also a prime number. It is not known whether the set of edges is finite or infinite.

Example 1(f). Let V be a finite set of open intervals of real numbers. Join the vertex i representing the open interval I and the vertex j representing another open interval J (which is not equal to I) if and only if the intersection of I and J is nonempty. The graph thus constructed is a simple graph known as an **interval graph.** For example, in the interval graph defined by the open intervals (3, 8), (7, 9), (3, 6), and (5, 10) represented by vertices A, B, C, and D, respectively, it is easy to see that there is an edge between every pair of vertices except between B and C.

Example 1(g). Fig 1-2(a) is the diagram of the complete graph with four vertices. An orientation of this graph describing a tournament with four vertices is shown in Fig. 1-2(b).

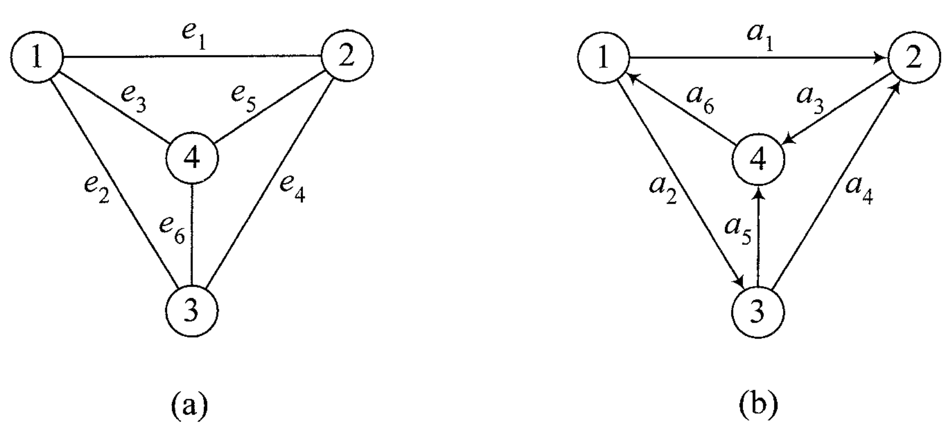


Fig. 1-2

1.2 GRAPH ISOMORPHISM

Two graphs $G = (V, E)$ and $G' = (V', E')$ are **identical** if $V = V'$ and $E = E'$. This rigid approach to the identification and classification of graphs is too restrictive. It is often possible that two graphs have the same structure even though they are not equivalent. In this case, we may consider them to be the same for all practical purposes. For example, any simple graph with vertex set $\{a, b, c\}$ and an edge set consisting of one edge is not structurally different from some other simple graph with vertex set $\{p, q, r\}$ and an edge set consisting of one edge. This structural equivalence between two nonequivalent graphs leads us to the concept of isomorphic graphs.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be **isomorphic** if there exists a one-to-one correspondence f, called an **isomorphism,** from V to V' such that there is an edge between $f(v)$ and $f(w)$ in G' if and only if there is an edge between v and w in G. For all practical purposes, two isomorphic graphs can be considered as one and the same graph. Obviously, two equivalent graphs are isomorphic, but the converse is not true, as pointed out in the previous paragraph. Thus any complete graph with n vertices is isomorphic to any other complete graph with n vertices.

In the **cyclic graph** $\boldsymbol{C_n = (V, E)}$ (where $n > 2$), V is the set $\{1, 2, \ldots, n\}$ and E is $\{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. A **triangle** is a cyclic graph with three vertices.

Example 2. Consider the three graphs in Fig. 1-3. The graphs G_1 and G_2 are isomorphic because of the isomorphism f defined by

$$f(v_1) = w_1,\quad f(v_2) = w_4,\quad f(v_3) = w_3,\quad f(v_4) = w_6,\quad f(v_5) = w_5,\quad \text{and}\quad f(v_6) = w_2$$

G_3 is not isomorphic to either G_1 or G_2.

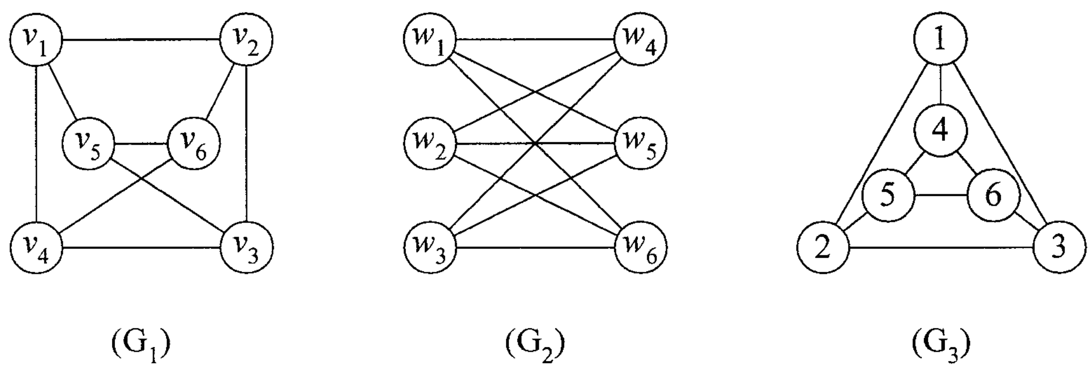


Fig. 1-3

To show that two graphs are isomorphic, one must point out an explicit isomorphism between them. The definition of isomorphism between two digraphs is analogous.

Since we consider two graphs to be the same if they are isomorphic, the trivial graph is the only graph of order one. A simple graph with two vertices may have no edge or one edge. Thus there are two nonisomorphic simple graphs of order two. Any simple graph with three vertices gives rise to four nonisomorphic cases: (1) with no edge at all, (2) with one edge joining a pair, (3) with two edges such that there is no edge joining a pair, and (4) with three edges. In other words, there are four nonisomorphic simple graphs of order 3.

Given two arbitrary simple graphs of the same order and the same size, the problem of determining whether an isomorphism exists between the two is known as the **isomorphism problem** in graph theory. In general, it is not all easy (in other words, there is no "efficient algorithm") to solve an arbitrary instance of the isomorphism problem.

1.3 SUBGRAPHS

The graph $H = (W, F)$ is a **subgraph** of the graph $G = (V, E)$ if W is a subset of V and F is a subset of E. (More generally, an arbitrary graph H can be considered as a subgraph of G if H is isomorphic to a subgraph of G.) If a subgraph H of the graph G is a cyclic graph, H is called a **cycle** in G. A complete subgraph of G is a **clique** in G. Any graph G' for which a given graph G is a subgraph is called a **supergraph** of G.

Any subgraph $H(V, F)$ of $G = (V, E)$ is a **spanning subgraph** of G. A **factor** of a graph is a spanning subgraph with at least one edge. If F is a set of edges in $G = (V, E)$, the spanning subgraph obtained by deleting the edges of F from G is denoted by $G - F$. If F consists of one edge f, we write $G - f$ instead of $G - \{f\}$. If W is a set of vertices in $G = (V, E)$, the graph obtained from G by deleting every vertex in W as well as any edge in E that is adjacent to a vertex in W is denoted by $G - W$. If W consists of a single vertex w, we write $G - w$ instead of $G - \{w\}$.

If $H = (W, F)$ is a subgraph of a graph $G = (V, E)$ such that an edge exists in F between two vertices in W if and only an edge exists in E between those two vertices, the subgraph H is said to be **induced by the set** W and is denoted by $\langle W \rangle$, which is the maximal subgraph of G with respect to the set W.

A subgraph G' of $G = (V, E)$ is a **vertex-induced subgraph** (or simply an **induced subgraph**) of G if there exists a set W of vertices in G such that $\langle W \rangle = G'$. Observe that if W is a subset of vertices of graph G, the subgraph $G - W$ is the induced subgraph $\langle V - W \rangle$, where the set $V - W$ is the relative complement of W in V.

We next consider the edge analog of induced subgraphs. Suppose F is a set of edges in $G = (V, E)$. The subgraph induced by F is the minimal subgraph $\langle F \rangle = (W, F)$, where v is in W if and only if v is adjacent to at least one edge in F. A subgraph G' of G is an **edge-induced subgraph** if there exists a set F of edges in G such that $G' = \langle F \rangle$.

Example 3. In graph G_1 shown in Fig. 1-3, the graph $H = (W, F)$, where $W = \{v_1, v_2, v_5, v_6\}$ and $F = \{\{v_1, v_2\}, \{v_1, v_5\}, \{v_2, v_6\}\}$, is a subgraph of G_1. It becomes an induced subgraph H' if we also adjoin the edge joining v_5 and v_6 to F. Then $H' = \langle W \rangle$. Here $G - W$ is the subgraph consisting of two vertices v_3 and v_4 with an edge joining the two, and this graph is the vertex-induced subgraph $\langle V - W \rangle$.

1.4 DEGREES, INDEGREES, AND OUTDEGREES

If there are p loops at a vertex v that has also q edges (not loops) incident to it, the **degree** (also known as **valence**) of v is $2p + q$. In a graph with no loops, the degree of a vertex is the number of edges adjacent to that vertex. In a graph with no loops, a vertex is said to be an **isolated vertex** if its degree is 0 and an **end-vertex** if its degree is 1. The maximum degree among the vertices of a graph G is denoted by $\Delta(G)$, and the minimum is denoted by $\delta(G)$. A graph is ***k*-regular** if the degree of each vertex is k, and it is **regular** if there exists a nonnegative integer k such that it is k-regular.

When the degrees of the vertices of a graph are added, each edge is counted exactly two times. Thus we have the following result, known as the first theorem of graph theory due to Euler.

Theorem 1.1. The sum of the degrees of a graph is twice the number of edges in it. (See Solved Problem 1.36.)

The result stated above (which implies that the sum of the degrees is even) is also known as the **handshaking lemma** because the number of hands shaken in a party is always even since each handshake involves exactly two hands of two different individuals. A vertex in a graph is an **odd vertex** if its degree is odd. Otherwise, it is an **even vertex.**

Theorem 1.2. Every graph has an even number of odd vertices. (See Solved Problem 1.38.)

Example 4. In graph G shown in Fig. 1-1, let d_i be the degree of vertex i, where $i \in V = \{1, 2, 3, 4, 5, 6, 7\}$. Then $d_1 = d_3 = 0$, $d_2 = 4$, $d_4 = 4$, $d_5 = 3$, $d_6 = 3$, and $d_7 = 2$. The sum of the degrees is 16, which is equal to twice the number of edges. The vertices v_5 and v_6 are odd.

In a digraph, the number of arcs adjacent to a vertex is the **indegree** of that vertex, and the number of arcs adjacent from a vertex is the **outdegree** of that vertex. When the outdegrees (or indegrees) of all the vertices are added, each arc is considered once in the counting process. We thus have the following result.

Theorem 1.3. In a digraph, the sum of the outdegrees of all the vertices is equal to the number of arcs, which is also equal to the sum of all the indegrees of all the vertices. (See Solved Problem 1.42.)

Example 5. The outdegrees of the four vertices 1, 2, 3, and 4 of the digraph in Fig. 1-2(*b*) are 2, 1, 2, and 1. The indegrees are 1, 2, 1, and 2. There are six arcs.

1.5 ADJACENCY MATRICES AND INCIDENCE MATRICES

Let $G = (V, E)$ be a graph where $V = \{1, 2, \ldots, n\}$. The **adjacency matrix of the graph** is the $n \times n$ matrix $A = [a_{ij}]$, where the nondiagonal entry a_{ij} is the number of edges joining vertex i and vertex j and the diagonal entry a_{ii} is twice the number of loops at vertex i. The adjacency matrix of a graph is obviously symmetric, that is, $a_{ij} = a_{ji}$ for every i and every j. The adjacency matrix of a simple graph is a binary matrix (0, 1 matrix) in which each diagonal entry is zero. Notice that in the adjacency matrix of the complete graph K_n, each nondiagonal entry is 1.

Since the n vertices of a graph can be labeled in $n!$ different ways and for each such labeling of vertices we have an adjacency matrix of the graph, by an abuse of notation any of these matrices is considered the adjacency matrix of the graph. At any rate, the adjacency matrix is uniquely determined apart from the ordering of its rows and columns. See Solved Problem 1.61 for more on this.

The **adjacency matrix of a digraph** with vertex set $\{1, 2, \ldots, n\}$ is the $n \times n$ binary matrix $A = [a_{ij}]$ in which $a_{ij} = 1$ if and only if there is an arc from vertex i to vertex j. Each diagonal entry in the adjacency matrix A of a digraph is zero, and A need not be symmetric.

Theorem 1.4. (*i*) In the adjacency matrix of a graph, the sum of the entries in a row (or a column) corresponding to a vertex is its degree, and the sum of all the entries of the matrix is twice the number of edges in the graph. (*ii*) In the adjacency matrix of a digraph, the sum of the entries in a row corresponding to a vertex is its outdegree, the sum of the entries in a column corresponding to a vertex is its indegree, and the sum of all the entries of the matrix is equal to the number of arcs in the digraph. (See Solved Problem 1.59.)

Example 6(a). The adjacency matrix of the graph of Fig. 1-1 is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Example 6(b). The adjacency matrix of the digraph of Fig. 1-2(*b*) is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $G = (V, E)$ be a simple graph where $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The **incidence matrix** $B = [b_{ij}]$ of G is defined as follows. Row i of B corresponds to vertex i for each i. Column k corresponds to edge e_k for each k. If e_k is the edge that joins vertex i and vertex j, the entries b_{ik} and b_{jk} are 1 and all the other entries in column k are zero. If G is a digraph and e_k is the arc from vertex i to vertex j, we define $b_{ik} = -1$ and $b_{jk} = 1$ in column k. Again, all the other entries in column k are zero. Thus the incidence matrix of a simple graph with n vertices and m edges is an $n \times m$ matrix in which each entry is 0 or 1, whereas the **incidence matrix of a digraph** with n vertices and m arcs is an $n \times m$ matrix in which each entry is 0 or 1 or -1. Each column of an incidence matrix has exactly two nonzero entries. We have the following analog of the previous theorem.

Theorem 1.5. (i) The sum of the entries in a row of the incidence matrix of a simple graph corresponding to a vertex is its degree, and the sum of all the entries in the matrix is twice the number of edges. (ii) The sum of the entries in a row of the incidence matrix of a digraph is its outdegree minus its indegree, and the sum of all the entries in the matrix is zero. (See Solved Problem 1.60.)

Example 7(a). The six edges in the simple graph of Fig. 1-2(a) are e_1 joining 1 and 2, e_2 joining 1 and 3, e_3 joining 1 and 4, e_4 joining 2 and 3, e_5 joining 2 and 4, and e_6 joining 3 and 4. The incidence matrix of this simple graph is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Example 7(b). The six arcs in the digraph of Fig. 1-2(b) are a_1 from 1 to 2, a_2 from 1 to 3, a_3 from 2 to 4, a_4 from 3 to 2, a_5 from 3 to 4, and a_6 from 4 to 1. The incidence matrix of this digraph is

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

1.6 DEGREE VECTORS OF SIMPLE GRAPHS

The **degree vector** $d(G)$ of a simple graph G is the sequence of degrees of its vertices arranged in nonincreasing order. If G and G' are isomorphic, $d(G) = d(G')$. But the converse is not true. The three graphs in Fig. 1-3 have all the same degree vector $[3\ \ 3\ \ 3\ \ 3\ \ 3\ \ 3]$, but G_3 is not isomorphic to G_1 or G_2.

Every simple graph has a unique degree vector (which can be easily constructed), and the sum of the terms in the vector is even. Notice that each term in a degree vector with n components is nonnegative and is at most $(n - 1)$, where n is the order of the graph. On the other hand, a finite nonincreasing vector $v = [d_1\ \ d_2\ \ \cdots\ \ d_n]$ of nonnegative integers, where each $d_i \leq (n - 1)$ and the sum of the terms is even, need not be the degree vector of a simple graph. Consider, for example, the vector $v = [3\ \ 3\ \ 3\ \ 1]$. If there were a simple graph with v as a degree vector, the subgraph obtained by deleting the vertex of degree 1 (with three vertices and four edges) would not be simple.

A vector v is called a **graphical vector** if there exists a simple graph such that v is the degree vector of that graph. Thus $[3\ \ 3\ \ 3\ \ 1]$ is not a graphical vector. The following theorem gives a necessary and sufficient condition for a vector to be a graphical vector.

Theorem 1.6 **(Hakimi–Havel).** Let $v = [d_1\ \ d_2\ \ d_3\ \ \cdots\ \ d_k]$ be a nonincreasing vector of k (where k is at least 2) nonnegative integers such that no component d_i exceeds $(k - 1)$. Let v' be the vector obtained from v by deleting d_1 and subtracting 1 from each of the next d_1 components of v. Let v_1 be the nonincreasing vector obtained from v' by rearranging its components if necessary. Then v is a graphical vector if and only if v_1 is a graphical vector. (See Solved Problem 1.68.)

Example 8. Consider $v = [5\ \ 4\ \ 3\ \ 3\ \ 3\ \ 3\ \ 3\ \ 2]$ with eight components in which no component exceeds 7.

Delete the first component 5, and subtract 1 from the next five components of v. We obtain $v' = [3\ \ 2\ \ 2\ \ 2\ \ 2\ \ 3\ \ 2]$. By rearranging the components of v', we get the nonincreasing vector $v_1 = [3\ \ 3\ \ 2\ \ 2\ \ 2\ \ 2\ \ 2]$ with seven components.

According to Theorem 1.6, v is a graphical vector if and only if v_1 is a graphical vector. Proceeding further, we see in the next iteration that v_1 is a graphical vector if and only if $v_2 = [2\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1]$ is a graphical vector. At the next iteration, we see that the vector v_2 is a graphical vector if and only if $v_3 = [2\ \ 1\ \ 1\ \ 1\ \ 1]$ is a graphical vector. Then v_3 is a graphical vector if and only if $v_4 = [1\ \ 1\ \ 0\ \ 0]$ is a graphical vector. At the next stage, we get $v_5 = [0\ \ 0\ \ 0]$. Now v_5, being the degree vector of a simple graph of order 3 with no edges, is a graphical vector. So the given vector v is also a graphical vector.

Using Theorem 1.6, it is possible to test whether an arbitrary vector with integer components is a graphical vector as outlined in the following algorithm.

Algorithm to Test Whether a Given Vector Is Graphical

The input is a nonincreasing vector v with integer components.

Step 0. (Initialization) The current vector is v.

Step 1. If the current vector with k components has a component that exceeds $(k - 1)$, go to step 5. Otherwise, go to step 2.

Step 2. If the current vector has a negative component, go to step 5. Otherwise, go to step 3.

Step 3. If the current vector is the zero vector, go to step 6. Otherwise, go to step 4.

Step 4. (Iteration) Rearrange the components of the current vector so that it becomes nonincreasing with d_1 as the first component. Delete d_1 from the rearranged vector, and subtract 1 from each of the next d_1 components of the rearranged vector. The vector thus constructed is the updated current vector. Go to step 1.

Step 5. The input vector is not graphical. Go to step 7.

Step 6. The input vector is graphical. Go to step 7.

Step 7. Stop.

(Note: The zero vector of step 3 with k components is graphical since it is the degree vector of a simple graph with k vertices and no edges.)

Example 9. Using this algorithm it can be verified that $v = [5\ \ 4\ \ 4\ \ 3\ \ 3\ \ 3\ \ 2]$ is a graphical vector.

Iteration 1:

$$v = [5\ \ 4\ \ 4\ \ 3\ \ 3\ \ 3\ \ 2] \quad \text{and} \quad v_1 = [3\ \ 3\ \ 2\ \ 2\ \ 2\ \ 2]$$

Iteration 2:

$$v = [3\ \ 3\ \ 2\ \ 2\ \ 2\ \ 2] \quad \text{and} \quad v_1 = [2\ \ 2\ \ 2\ \ 1\ \ 1]$$

Iteration 3:

$$v = [2\ \ 2\ \ 2\ \ 1\ \ 1] \quad \text{and} \quad v_1 = [1\ \ 1\ \ 1\ \ 1]$$

Iteration 4:

$$v = [1\ \ 1\ \ 1\ \ 1] \quad \text{and} \quad v_1 = [1\ \ 1\ \ 0]$$

Iteration 5:

$$v = [1\ \ 1\ \ 0] \quad \text{and} \quad v_1 = [0\ \ 0]$$

At the end of the fifth iteration we get the zero vector. So the given vector v is a graphical vector.

A simple graph for which the graphic vector v given above is the degree vector is shown in Fig. 1-4.

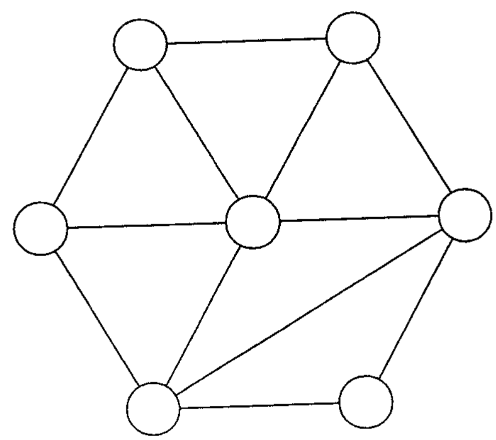


Fig. 1-4

Example 10. Using the algorithm, we now show that the vector $v = [3 \quad 3 \quad 3 \quad 1]$ is not a graphical vector. At the end of the first iteration, we get $[2 \quad 2 \quad 0]$ as the current vector. At the end of the next iteration, we get $[1 \quad -1]$ as the current vector, which has a negative component. The conclusion is that v is not a graphical vector.

Even if a vector is graphical, it is not the case that it is the degree vector of a unique (up to isomorphism) simple graph. The vector $v = [3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3]$ is graphical, but both G_1 and G_3 in Fig. 1-3 have v as the degree vector.

Solved Problems

INTRODUCTION

1.1 Draw the diagram of each of the following graphs $G = (V, E)$:

(*a*) $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 4\}\}$

(*b*) $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{3, 5\}\}$

Solution. See Fig. 1-5.

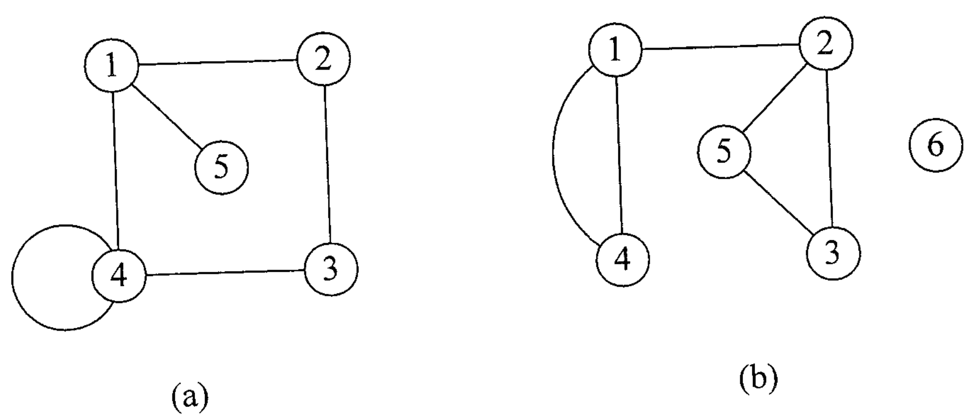


Fig. 1-5

1.2 Draw the diagram of each of the following graphs $G = (V, E)$:

(*a*) $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}\}$.

(*b*) $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$

Solution. See Fig. 1-6.

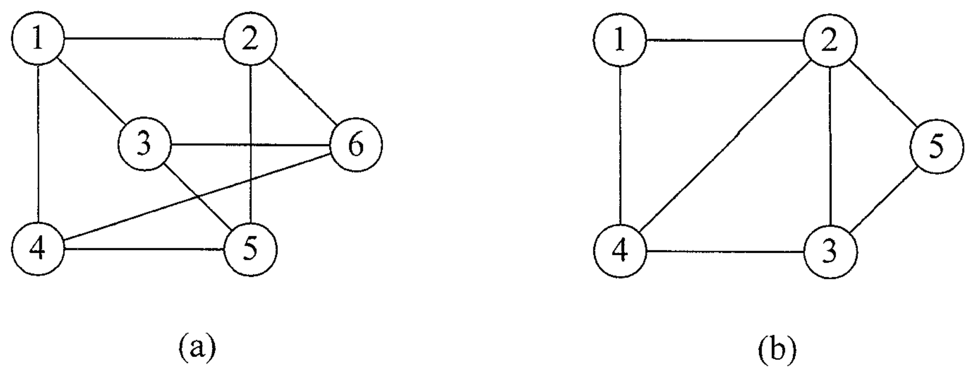


Fig. 1-6

1.3 Identify the simple graphs in the previous two problems. If a simple graph is identified, determine whether it is (*i*) a bipartite graph, (*ii*) a complete graph, (*iii*) a complete bipartite graph, or (*iv*) a complete nonbipartite graph.

Solution. The two graphs in Fig. 1-5 are not simple. The graph in Fig. 1-6(*a*) is a complete bipartite graph (X, Y, E), where $X = \{1, 5, 6\}$ and $Y = \{2, 3, 4\}$. The graph in Fig. 1-6(*b*) is simple and nonbipartite.

1.4 The **complement** of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, F)$ in which there is an edge between two vertices v and w if and only if there is no edge between v and w in G. Obviously, the complement of the complement of $\overline{G}$ is G. Draw the diagrams of the complements of the simple graphs identified in Problems 1.1 and 1.2.

Solution. The complement of Fig. 1-6(*a*) is Fig. 1-7(*a*), and the complement of Fig. 1-6(*b*) is Fig. 1-7(*b*), as shown in Fig. 1-7.

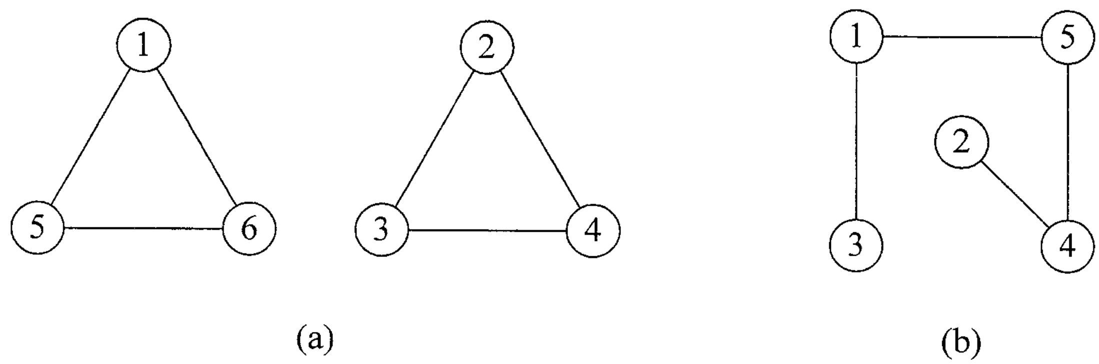


Fig. 1-7

1.5 Show that the complement of a bipartite graph need not be a bipartite graph.

Solution. Figure 1-6(*a*) shows a bipartite graph whose complement, shown in Fig. 1-7(*a*), is not bipartite.

1.6 Draw the diagram of an orientation of the simple nonbipartite graph that is not complete identified from Problems 1.1 and 1.2.

Solution. An orientation of the simple graph in Problem 1.2(b) is as shown in Fig. 1-8.

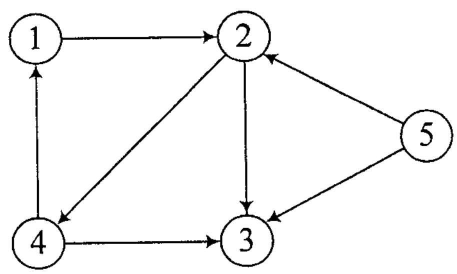


Fig. 1-8

1.7 Any orientation of the complete graph with vertex set $\{1, 2, \ldots, n\}$ is a tournament, and it is a **transitive** tournament if there is an arc from i to k whenever there is an arc from i to j and an arc from j to k for all choices of i, j, and k. Construct both a transitive tournament with four vertices and one that is not transitive.

Solution. A transitive tournament with four vertices is shown in Fig. 1-9. If we replace the arc (1, 3) by the arc (3, 1), the resulting tournament is not transitive.

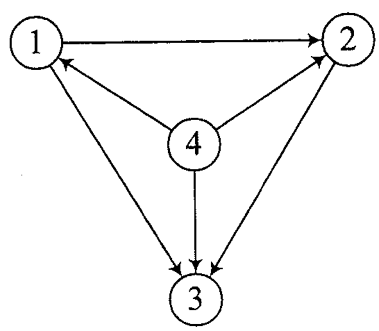


Fig. 1-9

GRAPH ISOMORPHISM

1.8 Show that every simple graph of order n is isomorphic to a subgraph of the complete graph with n vertices.

Solution. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of the simple graph G. Label the vertices of K_n as 1, 2, . . . , n. Define a one-to-one mapping $f(v_i) = i$ from V to the vertex set of K_n. Let H be the subgraph of K_n in which there is an edge between i and j if and only if there is an edge between v_i and v_j in G. Then G is isomorphic to H. Thus every simple graph with n vertices is (isomorphic to) a subgraph of the complete graph K_n.

1.9 If two graphs G and G' are isomorphic, the order of G is equal to the order of G' and the size of G is equal to the size of G'.

Solution. Let $G = (V, E)$ and $G' = (V', E')$. If G and G' are isomorphic, there is a bijection f between V and V'. So both G and G' have the same order. Furthermore, the bijection preserves adjacency and nonadjacency: There is an edge between vertex x and vertex y in G if and only if there is an edge between $f(x)$ and $f(y)$ in G' and there is a loop at x if and only if there is a loop at $f(x)$. So both G and G' have the same size.

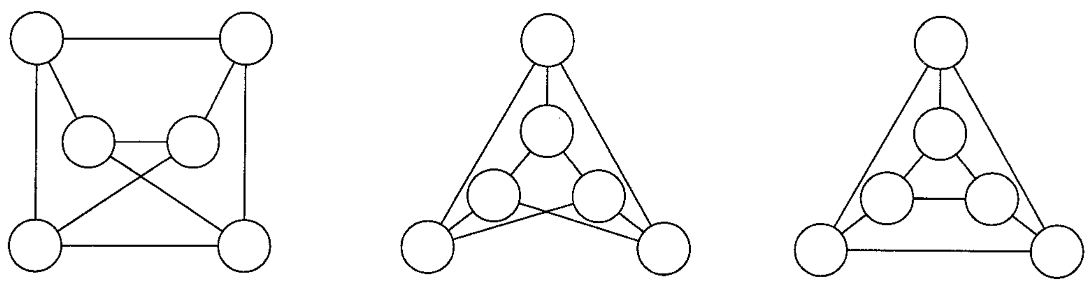
1.10 Show that two graphs need not be isomorphic even when they both have the same order and same size.

Solution. Let $G = (V, E')$ and $G' = (V', E')$, where $V = V' = \{a, b, c, d\}$, $E = \{\{a, b\}, \{b, c\}, \{c, d\}\}$, and $E' = \{\{a, b\}, \{b, c\}, \{b, d\}\}$, be two graphs. It is impossible to define a bijection between V and V' that will preserve adjacency and nonadjacency even though both G and G' have the same order and same size.

1.11 Show that two simple graphs are isomorphic if and only if their complements are isomorphic.

Solution. Suppose $G = (V, E)$ and $H = (W, F)$ are two simple isomorphic graphs with $|V| = |W| = n$ and $|E| = |F| = m$. Then their complements are also of order n. An isomorphism from V to W that preserves adjacency and nonadjacency between G and H is an isomorphism from V to W that preserves nonadjacency and adjacency between their complements.

1.12 Determine whether the three graphs given in Fig. 1-10 are isomorphic.



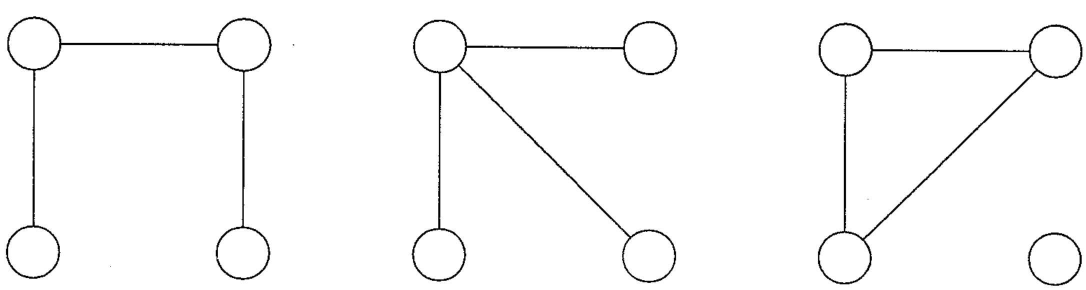
(a) (b) (c)

Fig. 1-10

Solution. Both Figure 1-10(a) and (b) are isomorphic to $K_{3,3}$ and so they are isomorphic to each other. But Figure 1-10(c) is not isomorphic to $K_{3,3}$.

1.13 Suppose $N(n, k)$ is the number of nonisomorphic simple graphs with n vertices and k edges. Find $N(4, 3)$.

Solution. There are exactly three graphs in this category, as shown in Fig. 1-11. So $N(4, 3) = 3$.



(G_1) (G_2) (G_3)

Fig. 1-11

1.14 Find all nonisomorphic simple graphs of order 4.

Solution. The maximum number of edges in a simple graph with four vertices is 6. The complete graph K_4 is the only graph of order 4 and size 6. There is one nonisomorphic graph of order 4 and size 5, and there are two nonisomorphic graphs of order 4 and size 4, as shown in Fig. 1-12. So $N(4, 6) = 1$, $N(4, 5) = 1$, and $N(4, 4) = 2$. In Problem 1.13 it was shown that $N(4, 3) = 3$. It is easy to see that $N(4, 2) = 2$ and $N(4, 1) = N(4, 0) = 1$. So the total number of nonisomorphic simple graphs of order 4 is $N(4, 0) + N(4, 1) + N(4, 2) + N(4, 3) + N(4, 4) + N(4, 5) + N(4, 6) = 1 + 1 + 2 + 3 + 2 + 1 + 1 = 11$.

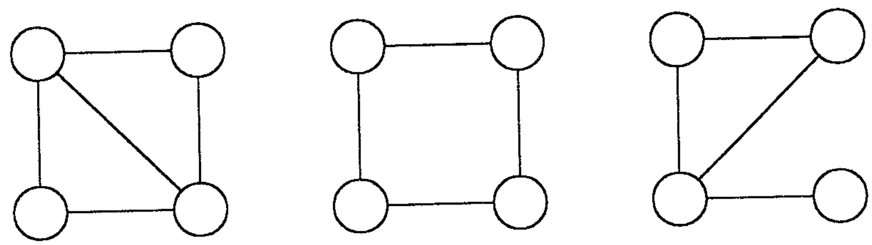


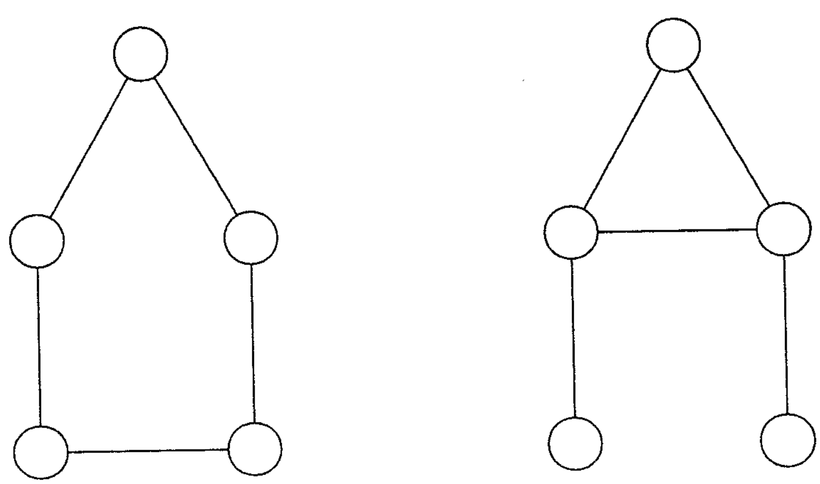
Fig. 1-12

1.15 A simple graph that is isomorphic to its complement is called a **self-complementary** graph. Find a self-complementary graph of order 4.

Solution. The number of edges in the complete graph with four vertices is 6. So if G is a self-complementary graph with four vertices, it should have three edges. Among the three nonisomorphic graphs of order 4 and size 3 shown in Fig. 1-11, the complement of G_1 is isomorphic to G_1, the complement of G_2 is isomorphic to G_3, and the complement of G_3 is isomorphic to G_2. Thus G_1 is the only self-complementary graph of order 4.

1.16 Find two self-complementary graphs of order 5.

Solution. It is easily verified that the graphs G and H shown in Fig. 1-13 are self-complementary.



(G) (H)

Fig. 1-13

SUBGRAPHS

1.17 Find a vertex-induced bipartite subgraph of the graph in Fig. 1-6(a).

Solution. The subgraph induced by the set $W = \{1, 2, 3, 4, 5\}$ is $K_{2,3}$.

1.18 A set I of vertices in a simple graph $G = (V, E)$ is an **independent set** (also known as an **internally stable set**) in G if no two vertices in I are adjacent. A set K of vertices in G is called a **vertex cover** if every edge in the graph is incident to at least one vertex in K. Show that a set K of vertices is a vertex cover if and only if its complement $(V - K)$ is an independent set.

Solution. If K is a vertex cover, no two vertices in $(V - K)$ can be adjacent, so $(V - K)$ is an independent set. On the other hand, if I is any independent set in G, out of the two vertices joined by an edge at least one should be in $(V - I)$. In other words, every edge is adjacent to some vertex in $(V - I)$, so $(V - I)$ is a vertex cover.

1.19 An independent set I in a simple graph G is a **maximum independent set** if there is no independent set I' in G such that $|I'| > |I|$. The number of vertices in a maximum independent set in G is the **independence number** $\alpha(G)$ (also known as the **internal stability number**) of the graph G. A vertex cover K in a graph G is a **minimum vertex cover** if there is no vertex cover K' such that $|K'| < |K|$. The number of vertices in a minimum vertex cover is called the **vertex-covering number** $\beta(G)$ of the graph G. Find the vertex-covering number and the independence number of the graph in Fig. 1-14.

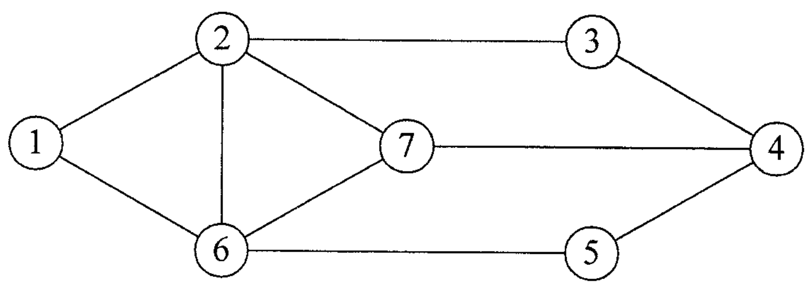


Fig. 1-14

Solution. The set $\{1, 3, 7, 5\}$ is a maximum independent set, so the vertex-independence number of the graph is 4. The set $\{2, 4, 6\}$ is a minimum vertex cover, so the vertex-covering number is 3.

1.20 Show that in a simple graph G of order n, $\alpha(G) + \beta(G) = n$.

Solution. Let I be a maximum independent set in G. Hence $|I| = \alpha(G)$ and $|(V - I)| = n - \alpha(G)$. But $(V - I)$ is a vertex cover. So $|(V - I)| \geq \beta(G)$, and consequently, $n \geq \alpha(G) + \beta(G)$. On the other hand, suppose K is a minimum vertex cover in G. Then $|K| = \beta(G)$ and $|(V - K)| = n - \beta(G)$. But $(V - K)$ is an independent vertex set. So $|(V - K)| \leq \alpha(G)$, and consequently, $n \leq \alpha(G) + \beta(G)$. Thus the equality is established.

1.21 A subset D of vertices in a simple graph G is a **dominating vertex set** (also known as an **external dominating set**) if every vertex not in D is adjacent to at least one vertex in D. Find (*a*) a dominating vertex set that is not independent, (*b*) an independent set that is not a dominating vertex set, and (*c*) a set that is an independent set as well as a dominating vertex set in the graph of Fig. 1-14.

Solution. (*a*) $\{2, 6, 7\}$. (*b*) $\{1, 3\}$. (*c*) $\{1, 4\}$.

1.22 A dominating vertex set D is a **minimum dominating vertex set** if there is no dominating set D' such that $|D'| < |D|$. The number of vectors in a minimum dominating vertex set is the **vertex domination number** $\sigma(G)$ (also known as the **external stability number**) of the graph. Show that the vertex domination number of a simple graph cannot exceed its independence number.

Solution. Let X be any minimum dominating set, Y any set that is both dominating and independent, and Z any maximum independent set. Then $\sigma(G) = |X| \leq |Y| \leq |Z| = \alpha(G)$.

1.23 An independent set is a **maximal independent set** if it is not a proper subset of another independent set. Show that an independent set is a dominating vertex set if and only if it is a maximal independent set. (A *maximal* independent set is not the same as a *maximum* independent set defined in Problem 1.19.)

Solution. Let I be a maximal independent set. If this set is not a dominating set, there will be a vertex v (not in I) that is not adjacent to any vertex in I. In that case, $I \cup \{v\}$ is an independent set violating the maximality of I. Conversely, suppose the independent set I is also a dominating set. If I is not a maximal independent set, there exists an independent set J such that I is a proper subset of J. So there is a vertex in $(J - I)$ that is not adjacent to any vertex in I contradicting the assumption that I is a dominating set.

1.24 Show that if there are at least two people who do not know each other among a set of individuals, it is possible to choose people from that set to form a **committee** such that no two individuals in the committee know each other and that every individual in the set not included in the committee is known to at least one person in the committee.

Solution. Construct a graph in which each vertex represents an individual in the set V of people. Join two vertices by an edge if the two individuals represented by these vertices know each other. The simple graph G thus constructed is known as the **acquaintance graph** of the set V. By hypothesis, G has an independent set consisting of at least two people. Now any maximal independent set in G will be an independent dominating set as shown in Problem 1.23. And a committee is nothing but a dominating independent set. (If G is complete, the problem is trivial. The singleton set consisting of any individual is a committee.)

1.25 A committee S as described in Problem 1.24 is a **minimum committee** if there is no committee S' such that $|S'| < |S|$. The **committee number** of a set is the cardinality of a minimum committee in the set. Find the committee number of the acquaintance graph in Fig. 1-14.

Solution. The set $W = \{1, 4\}$ is both dominating and independent. There is no set with only one vertex that is dominating. So the committee number is 2. (The committee number of any complete graph is 1.)

1.26 A set M of edges in a graph is called a **matching** (also known as an **independent edge set**) if no two edges in M have a vertex in common. A set of edges L is an **edge cover** if every vertex of positive degree is a vertex of at least one edge in L. Show that the complement of a matching need not be an edge cover. (Compare this result with that of Problem 1.18.)

Solution. Consider any simple graph G with three vertices and two edges. The set consisting of one of these edges is matching in G. But the complement of that set is not an edge cover in G.

1.27 A matching M in a simple graph is a **maximum matching** (also known as **maximum cardinality matching**) if there is no matching M' such that $|M'| > |M|$. The **edge independence number** $\alpha_1(G)$ of a graph G is the number of edges in a maximum matching. An edge cover L of a simple graph G is a **minimum edge cover** if there is no edge cover L' of G such that $|L'| < |L|$. The **edge-covering number** $\beta_1(G)$ of the graph is the sum of the number of edges in a minimum edge cover and the number of isolated vertices. Find the edge independence number and the edge-covering number of the graph of Fig. 1-14.

Solution. The set $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is maximum matching, so the edge independence number is 3. The set $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{6, 7\}\}$ is a minimum edge cover, so the edge-covering number is 4.

1.28 (*Gallai's Theorem*) Show that in a simple graph, $\alpha_1 + \beta_1 = n$.

Solution. Suppose M is a maximum matching. If M is an edge cover, $\beta_1 \le \alpha_1$, which implies that $\alpha_1 + \beta_1 \le 2\alpha_1 \le n$. If M is not an edge cover, $\beta_1 \le \alpha_1 + (n - 2\alpha_1)$. Thus in any case, the sum of the two numbers cannot exceed n. Next we establish the reverse inequality. Suppose L is a minimum edge cover in G. Let H be the subgraph of G induced by L, and let M be a maximum matching in H. If W is the set of unmatched vertices in this subgraph, the subgraph of H induced by W has no edges since M is a maximum matching. Thus $|L| - |M| = |(L - M)|$, where $(L - M)$ is the relative complement of M in L. Now $|(L - M)| \geq |W| = n - 2|M|$. Hence $|L| + |M| \geq n$. Since M is matching in G, $\alpha_1 \geq |M|$. Thus $\alpha_1 + \beta_1 \geq |M| + |L| \geq n$. This completes the proof.

1.29 A set F of edges in a graph $G = (V, E)$ is a **dominating edge set** if every edge not in F has a vertex in common with an edge in F. The **edge domination number** $\sigma_1(G)$ is the number of edges in a minimum edge domination set. Find the edge domination number of the graph of Fig. 1-14.

Solution. The edge domination number is 3 since the set $\{\{1, 2\}, \{5, 6\}, \{4, 7\}\}$ is a minimum dominating edge set.

1.30 Show that the edge domination number cannot exceed the edge independence number.

Solution. The proof is as in Problem 1.22.

1.31 Find a necessary and sufficient condition to be satisfied by a matching so that it is a dominating edge set.

Solution. A matching is a dominating edge set if and only if it is a maximal (not necessarily a maximum) matching.

DEGREES, INDEGREES, AND OUTDEGREES

1.32 Find the number of edges in the complete graph with n vertices.

Solution. Suppose the vertex set is $V = \{1, 2, \ldots, n\}$. A vertex i can be selected in n ways. There are exactly $(n - 1)$ edges between a selected vertex i and the remaining $(n - 1)$ vertices. The edge joining i and another vertex j is the same as the edge joining j and i. Thus the number of edges in K_n is $n(n - 1)/2$. Equivalently, an edge in K_n is constructed by choosing any two vertices out of a set of n vertices and joining them. The number of ways of choosing any two elements out of a set of n elements is $n(n - 1)/2$.

1.33 Using techniques from graph theory, show that $1 + 2 + \cdots + n = n(n + 1)/2$.

Solution. Consider the complete graph with $(n + 1)$ vertices. By Problem 1.32, it has $n(n + 1)/2$ edges. Now the total number of edges in the graph can be computed as follows. Suppose the vertices are labeled $1, 2, \ldots, (n + 1)$. Joining vertex 1 and the remaining vertices are n edges. Delete these edges. Then joining vertex 2 and the remaining vertices are $(n - 1)$ edges, which are also deleted. Continue the process until all the edges are deleted. The total number of edges deleted is $n + (n - 1) + \cdots + 2 + 1$, which is equal to the total number of edges in the graph.

1.34 Show that the number of vertices in a self-complementary graph is either $4k$ or $4k + 1$, where k is a positive integer.

Solution. Consider a self-complementary graph $G = (V, E)$ with n vertices and m edges. Since G is isomorphic to its complement, both G and its complement have the same number of edges. Now every edge in the complete graph with V as the set of vertices is either an edge in G or an edge in its complement. Thus $m + m = n(n - 1)/2$, showing that $n(n - 1) = 4k$, where k is a positive integer. So $n = 4k$ or $4k + 1$.

1.35 Find the number of edges in the complete bipartite graph $K_{m,n}$.

Solution. Suppose $K_{m,n} = G(X, Y, E)$. There are n edges adjacent to a vertex in X. No vertex in X is joined to a vertex in X. There are m vertices in X. So the total number of edges is mn.

1.36 Prove Theorem 1.1: The sum of the degrees of a graph is twice the number of edges in it.

Solution. An edge that is not a loop contributes to the degrees of two distinct vertices. A loop at a vertex by definition contributes twice to the degree of that vertex. Thus when the degrees of the vertices are added, each edge (whether it is a loop or not) is counted exactly two times. Thus the sum of the degrees is twice the number of edges.

1.37 Use Theorem 1.1 to find the size of K_n and $K_{m,n}$.

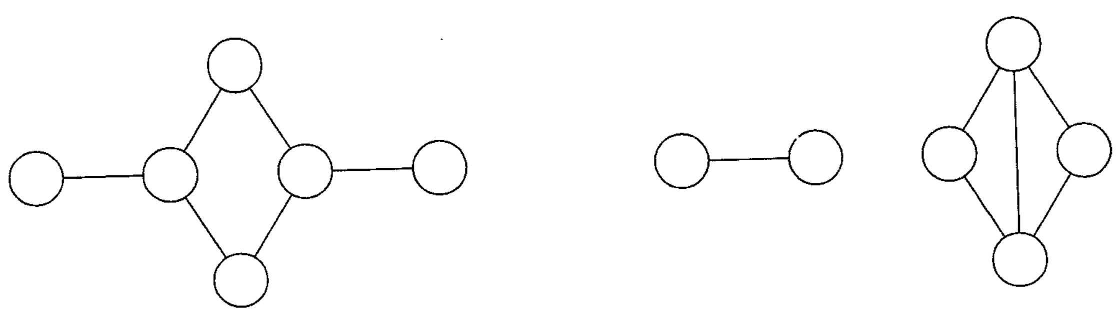
Solution. (*i*) Suppose the size of K_n is m. The degree of each vertex is $(n - 1)$. There are n vertices. Thus the sum of the degrees of the n vertices is $n(n - 1)$, which is $2m$ by the theorem. Hence $m = n(n - 1)/2$. (*ii*) Let the number of edges in $K_{m,n} = G(X, Y, E)$ be p. The degree of each vertex in X is n, and the degree of each vertex in Y is m. The sum of the degrees of the m vertices in X is mn. The sum of the degrees of the n vertices in Y is also mn. The sum of the degrees of all the vertices is $2mn$, which is $2p$. Hence $p = mn$.

1.38 Prove Theorem 1.2: Every graph has an even number of odd vertices.

Solution. Suppose the sum of the degrees of the odd vertices is x and the sum of the degrees of the even vertices is y. The number y is even, and the number $x + y$, being twice the number of edges, is also even. So x is necessarily even. If there are p odd vertices, the even number x is the sum of p odd numbers. So p is even.

1.39 Construct two nonisomorphic simple graphs with six vertices with degrees 1, 1, 2, 2, 3, and 3. Find the size of the graph thus constructed.

Solution. Since the sum of the degrees is 12, the size is 6. Two nonisomorphic graphs G and G' with six vertices and six edges are shown in Fig. 1-15.



(a) (b)

Fig. 1-15

1.40 Show that if G and G' are isomorphic graphs, the degree of each vertex is preserved under the isomorphism.

Solution. Suppose $G = (V, E)$ and $G' = (V', E')$ are two isomorphic graphs with f as the bijection from V to V' that preserves adjacency and nonadjacency. Suppose the degree of v in G is k. Then there are k vertices adjacent to v in G. Let these vertices be v_i $(i = 1, 2, \ldots, k)$. Then the only edges adjacent to $f(v)$ are those edges joining $f(v)$ and $f(v_i)$ for $i = 1, 2, \ldots, k$. So the degree of $f(v)$ is also k.

1.41 Show that two graphs G and G' with the same set $V = \{1, 2, \ldots, n\}$ of vertices such that the degree of vertex i is the same for both the graphs for every i need not be isomorphic.

Solution. The two graphs in Fig. 1-16 are not isomorphic.

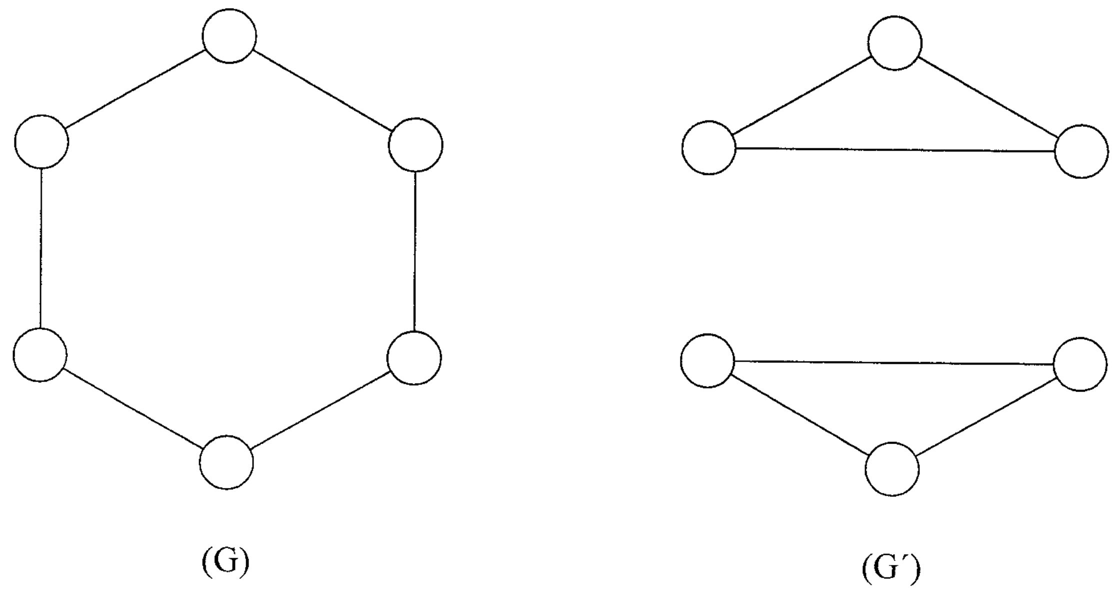


Fig. 1-16

1.42 Prove Theorem 1.3: In a digraph, the sum of the outdegrees of all the vertices is equal to the number of arcs, which is also equal to the sum of the indegrees of all the vertices.

Solution. The outdegree of a vertex is the number of arcs adjacent from that vertex. So when we add all the outdegrees, each arc is counted exactly once. Likewise, when the indegrees are summed, each arc is counted exactly once. Thus the sum of the outdegrees and the sum of the indegrees are both equal to the total number of arcs in the digraph.

1.43 Show that there is no simple graph with 12 vertices and 28 edges in which (*i*) the degree of each vertex is either 3 or 4, and (*ii*) the degree of each vertex is either 3 or 6.

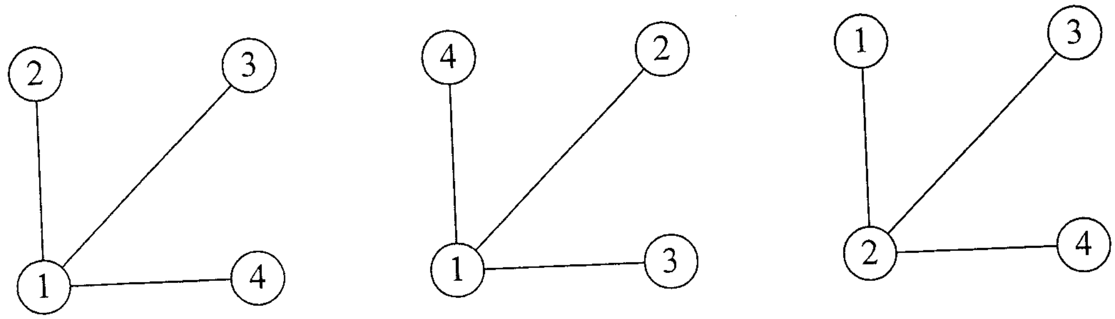
Solution. Suppose there is a graph with p vertices of degree 3 in the graph. (*i*) If the remaining $(12 - p)$ vertices have all degree 4, the equation $3p + 4(12 - p) = 56$ gives a negative value for p. (*ii*) If the remaining $(12 - p)$ vertices have all degree 6, the equation $3p + 6(12 - p) = 56$ gives a noninteger value for p.

1.44 Show that there is no simple graph with four vertices such that three vertices have degree 3 and one vertex has degree 1.

Solution. Suppose there exists a graph G as stipulated. The size of this graph is 5 since the sum of the degrees is 10. So G is a graph with four vertices and five edges. There is only one (up to isomorphism) graph with four vertices and five edges, as shown in Fig. 1.12 with degrees 3, 3, 2, and 2. So there is no simple graph that satisfies the given requirement.

1.45 A **labeled graph with n vertices** is obtained by assigning the labels $1, 2, \ldots, n$ to the vertices of a given graph G with n vertices and m edges. Two labeled graphs thus obtained from a given graph G are necessarily isomorphic, but they need not be equivalent. Label the vertices of a simple graph with four vertices of degrees 1, 1, 1, and 3, creating three isomorphic graphs G_1, G_2, and G_3 such that (*i*) G_1 and G_2 are equivalent, and (*ii*) G_1 and G_3 are not equivalent.

Solution. See Fig. 1-17.



(G_1) (G_2) (G_3)

Fig. 1-17

1.46 Find the number of nonequivalent labeled graphs with n vertices.

Solution. Let $L(n, k)$ be the number of nonequivalent labeled graphs with n vertices and k edges. Then the number of nonequivalent graphs of order n is $L(n, 0) + L(n, 1) + \cdots + L(n, r)$, where r is the number of edges in a complete graph with n vertices. Now we can choose k edges out of r edges in $C(r, k)$ ways, where $C(r, k)$ is the binomial coefficient representing the number of ways of choosing k elements from a set of r elements. Thus $L(n, k)$ is equal to $C(r, k)$. So the total number of labeled graphs with n vertices is $C(r, 0) + C(r, 1) + \cdots + C(r, r)$, which is the binomial expansion of $(1 + 1)^r$. Thus the answer is 2^r, where $r = n(n - 1)/2$.

1.47 Show that the number of vertices in a k-regular graph is even if k is odd.

Solution. If the number of vertices n, the product kn is twice the number of edges. Thus n is even if k is odd.

1.48 Show that it is not possible to have a group of seven people such that each person in the group knows exactly three other people in the group.

Solution. Construct an acquaintance graph with seven vertices such that each vertex represents a person. Join two vertices by an edge if the two individuals know each other. If each knows exactly three people, we should have a 3-regular graph with seven vertices, which is a contradiction by Problem 1.47.

1.49 Prove that in any group of six people, there will be either three people who know one another or three people who do not know one another.

Solution. Suppose G is the acquaintance graph involving six people. Then its complementary graph can be considered as the *nonacquaintance* graph in the sense that there will be an edge (in that graph) between two vertices representing two people if and only if they do not know each other. Suppose v is a vertex. Then either in G or in its complement there will be at least three edges incident to v since the degree of every vertex in the complete graph with six vertices is 5. Assume without loss of generality that v is adjacent to vertices p, q, and r in G. If there is an edge joining any two of these three vertices, v and these two vertices form a set of three mutual acquaintances. Otherwise, the three individuals p, q, and r form a set of three nonacquaintances.

1.50 The positive integer n has the **(p, q)-Ramsey property** if for every graph G with n vertices, either K_p is a subgraph of G or K_q is a subgraph of the complement of G. Show that the positive integer 6 has the (3, 3)-Ramsey property whereas the positive integer 5 does not.

Solution. It follows from Problem 1.49 that for any graph G with six vertices, there is a triangle either in G or in its complement. So the number 6 has the (3, 3)-Ramsey property. The cyclic graph with five vertices has no triangle and is isomorphic to its complement. So the number 5 does not have the (3, 3)-Ramsey property.

1.51 Show that the following properties are equivalent: (*i*) the positive integer n has the (p, q)-Ramsey property, (*ii*) every simple graph with n vertices has a clique of p vertices or an independent set of q vertices, and (*iii*) the edges of K_n can be colored using two colors such that there will be either a clique K_p in which all edges are of one color or a clique K_q in which all edges are of the other color.

Solution. These statements are reformulations of the definition given in Problem 1.50.

1.52 The smallest integer n that has the (p, q)-Ramsey property is called a **Ramsey number,** denoted $\boldsymbol{R(p, q)}$. Show that (*a*) $R(p, q) = R(q, p)$, (*b*) $R(p, 2) = p$, and (*c*) $R(3, 3) = 6$.

Solution. (*a*) This is an immediate consequence of the definition. (*b*) For any graph G with n vertices, either G or its complement has an edge. (*c*) This follows from Problems 1.49 through 1.51.

1.53 Show that if a bipartite graph $G = (X, Y, E)$ is regular, both X and Y have the same number of elements.

Solution. Suppose there are x vertices in X and y vertices in Y. If the degree of each vertex is r, the total number of edges is rx, which is also equal to ry. Thus $x = y$.

1.54 A **cubic graph** is simple graph in which the degree of each vertex is 3. Construct two nonisomorphic cubic graphs each with six vertices.

Solution. The bipartite graph $K_{3,3}$ is a cubic graph with six vertices. A nonbipartite cubic graph of order 6 is the graph in Fig. 1-10(*c*).

1.55 Find the maximum number of edges in a bipartite graph.

Solution. Suppose in $G = (X, Y, E)$ there are m vertices in X and n vertices in Y. The number of edges in G cannot exceed mn, which is a maximum when $m = n$. So the maximum number of edges is n^2 when G has $2n$ vertices.

1.56 The ***k*-cube** (also known as the **hypercube**) is the graph $\boldsymbol{Q_k}$ whose vertices are the ordered k-tuples of binary numbers, two vertices being joined by an edge if and only if they differ exactly in one component. Show that a k-cube is a k-regular bipartite graph, and find the number of vertices and edges in a k-cube.

Solution. Let X denote the set of k-tuples consisting of the zero k-tuple and all the k-tuples that differ from the zero k-tuple in an even number of components, and let Y be the set of k-tuples that differ from the zero k-tuple in an odd number of components. Then every edge in the cube is between a vertex in X and a vertex in Y. Moreover, both X and Y have the same number of vertices. Since each component of the k-tuple is either 0 or 1, there are 2^k vertices. For any vertex that corresponds to a fixed k-tuple, there are k vertices with k-tuples that differ from the given k-tuple in exactly one component. Thus the k-cube is a k-regular bipartite graph with $(k)(2^k)/2$ edges.

1.57 Find the fewest vertices needed to construct a complete graph with at least 1000 edges.

Solution. If the number of vertices is n, we have the inequality $n(n - 1) \geq 2000$. So $n \geq 46$.

ADJACENCY MATRICES AND INCIDENCE MATRICES

1.58 Show that the vertices of the bipartite graph $G = (X, Y, E)$ with m vertices in X and n vertices in Y can be enumerated so that the adjacency matrix has the form

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

where A is an $m \times n$ matrix in which each entry is 0 or 1, A^T is the transpose of A, and 0 is a matrix in which each entry is zero.

Solution. Label the vertices in X from 1 to m, and label the vertices in Y from $(m + 1)$ to $(m + n)$. Then the adjacency matrix is of the form given above.

1.59 Prove Theorem 1.4: (*i*) In the adjacency matrix of a graph, the sum of the entries in a row (or a column) corresponding to a vertex is its degree, and the sum of all the entries of the matrix is twice the number of edges in the graph. (*ii*) In the adjacency matrix of a digraph, the sum of the entries in a row corresponding to a vertex is its outdegree, the sum of the entries in a column corresponding to a vertex is its indegree, and the sum of all the entries of the matrix is equal to the number of arcs in the digraph.

Solution. Suppose the vertex set of G is $\{1, 2, \ldots, n\}$. (*i*) In the adjacency matrix, the (i, j) entry is equal to the number of edges joining vertex i and vertex j. When we add all the entries of the ith row (or the ith column), we count the number of edges adjacent to vertex i and add them. So the sum of the ith row (column) is equal to the degree of vertex i. When all the entries in the matrix are added, we obtain the sum of the degrees of all vertices, which is twice the sum of edges by Theorem 1.1. (*ii*) If the (i, j) entry is positive, there is an arc from i to j. So the sum of the entries in the ith row is the outdegree of i, whereas the sum of the entries in the jth column is the indegree of j. When we add all the entries of the matrix, we get the sum of all outdegrees (which is also the sum of all indegrees), and by Theorem 1.3, this sum is equal to the number of arcs.

1.60 Prove Theorem 1.5: (*i*) The sum of the entries in a row of the incidence matrix of a simple graph corresponding to a vertex is its degree, and the sum of all the entries in the matrix is twice the number of edges. (*ii*) The sum of the entries in a row of the incidence matrix of a digraph is its outdegree minus its indegree, and the sum of all the entries in the matrix is zero.

Solution. Let the vertices be $1, 2, \ldots, n$. (*i*) Suppose the edges are e_i $(i = 1, 2, \ldots, m)$, and let the jth column correspond to e_j. The sum of each column is 2, so the sum of all the entries in the matrix is $2m$, which is twice the number of edges. Moreover, if the (i, j) entry is positive, there is an edge joining i and j. So when all the entries in the ith row are added, all the edges adjacent to i are accounted for. Thus the sum of the ith row is the degree of vertex i. (*ii*) Suppose the arcs are a_i $(i = 1, 2, \ldots, m)$. Each column has two nonzero entries ($+1$ and -1), and therefore each column sum is zero. So the sum of all the entries in the matrix is zero. If the (i, j) entry is -1, it indicates that there is an arc from i, and if it is $+1$, it indicates that there is an arc directed to vertex i. So the sum of all the entries in the ith row is equal to its outdegree minus its indegree.

1.61 A **permutation matrix** is a square binary matrix that has exactly one 1 in each row and column. Two matrices A and A' are **isomorphic** if there is a permutation matrix P such that $A'P = PA$. Show that two graphs are isomorphic if and only if their adjacency matrices are isomorphic.

Solution. Suppose A and A' are the adjacency matrices of two isomorphic graphs. Then one of these matrices can be obtained from the other by rearranging rows and then rearranging the corresponding columns. Now rearranging rows of A is equivalent to premultiplying by a permutation matrix P obtaining the product matrix PA. The subsequent rearrangement of corresponding columns is equivalent to postmultiplying PA by P^{-1}. (Recall that any permutation matrix P is nonsingular.) Thus $A' = PAP^{-1}$. Conversely, if $A'P = PA$, A' can be obtained from A by rearranging columns and then rows, showing that the two graphs are isomorphic.

1.62 Find the adjacency matrices A and A' of the two isomorphic graphs given in Fig. 1-18, and obtain a permutation matrix P such that $A'P = PA$.

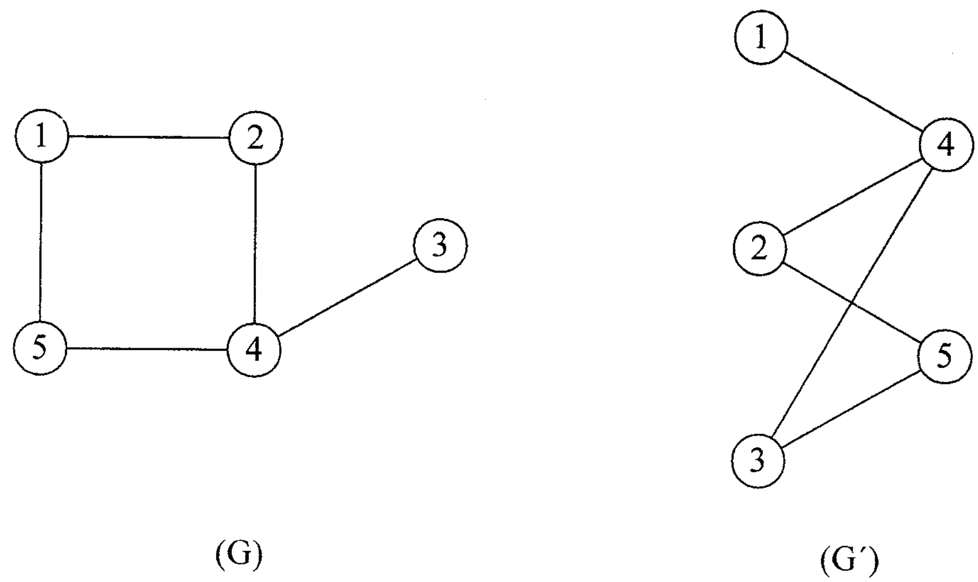


Fig. 1-18

Solution. The adjacency matrices are A and A', where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

An isomorphism f from G to G' is $f(1) = 5, f(2) = 2, f(3) = 1, f(4) = 4$, and $f(5) = 3$. To obtain A' from A, we replace the fifth row of A by its first row, the first row of A by its third row, and the third row of A by its fifth row. Suppose the resulting matrix is denoted B. Then replace the fifth column of B by its first column, the first column of B by its third column, and the third column of B by its fifth column. The resulting matrix is A'. To obtain the following permutation matrix P from the 5×5 identity matrix I, we perform the same sequence of operations on I that was performed on A to obtain B:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to verify that $PA = B = A'P$. (The permutation matrix P depends on the isomorphism f.)

1.63 The **characteristic polynomial** of a simple graph with n vertices is the determinant of the matrix $(A - \lambda I)$, where A is the adjacency matrix and I is the $n \times n$ identity matrix. Show that if two graphs are isomorphic, their characteristic polynomials are the same. (Note: The determinant of A is written $\det A$.)

Solution. Suppose A and A' are the characteristic polynomials of two isomorphic graphs G and G'. Then $A' = PAP^{-1}$. So $(A' - \lambda I) = (PAP^{-1} - \lambda I) = (PAP^{-1} - \lambda PIP^{-1}) = P(A - \lambda I)P^{-1}$. Thus

$$\begin{aligned} \det(A' - \lambda I) &= \det P(A - \lambda I)P^{-1} \\ &= \det P \det(A - \lambda I) \det P^{-1} \\ &= \det P \det(A - \lambda I)(1/\det P) \\ &= \det(A - \lambda I) \end{aligned}$$

1.64 By computing the characteristic polynomials of the two graphs shown in Fig. 1-19, show that two nonisomorphic graphs can have the same characteristic polynomial.

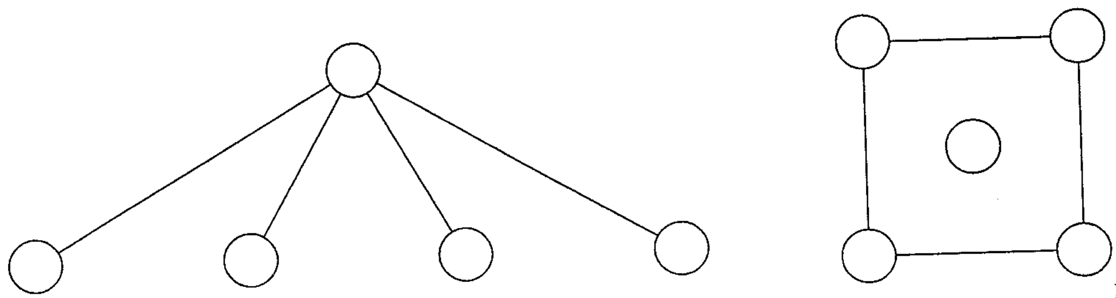


Fig. 1-19

Solution. By expanding the appropriate determinants and simplifying, it is verified that both graphs have the same characteristic polynomial $4\lambda^3 - \lambda^5$. But obviously the two graphs are not isomorphic.

1.65 If $A = [a_{ij}]$ is the adjacency matrix of a simple graph G with n vertices, the **binary code of G with respect to A** is the nonnegative integer $a_{12}2^0 + a_{13}2^1 + \cdots + a_{1n}2^{n-1} + a_{23}2^n + \cdots + a_{2n}2^{2n-3} + \cdots + a_{n-1,n}2^{k-1}$, where $k = n(n-1)/2$. Find the binary code of the adjacency matrix of the graph $G = (V, E)$, where $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$.

Solution. The adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The binary code of A is $1.2^0 + 0.2^1 + 1.2^2 + 0.2^3 + 1.2^4 + 1.2^5 = 53$.

1.66 Show that it is possible to construct a simple graph G if the binary code of one of its adjacency matrices and the order of G are known.

Solution. The given binary code can be uniquely expressed as the sum of terms like $a_i.2^i$, where the coefficient a_i is either 0 or 1. This information gives all the entries above the diagonal of the adjacency matrix provided we know the order of the graph. For example, suppose the binary code is 573, which is equal to $512 + 32 + 16 + 8 + 4 + 1 = 1.2^0 + 0.2^1 + 1.2^2 + 1.2^3 + 1.2^4 + 1.2^5 + 0.2^6 + 0.2^7 + 0.2^8 + 1.2^9$.

(*i*) Suppose A and A' are the adjacency matrices of graphs with 5 and 6 vertices, respectively. Then these two matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(*ii*) If the order of the graph is 7, it can be easily verified that vertex 3 is an isolated vertex since the third row (and the third column) of the 7×7 adjacency matrix is the zero vector.

1.67 The **minicode of a graph** is its smallest binary code, and the largest binary code is the **maxicode** of the graph. Find the minicode and maxicode of a simple graph with three vertices and two edges.

Solution. The three vertices give rise to 3! adjacency matrices, out of which three are distinct. These matrices give the binary codes 3, 5, and 6. Thus the minicode is 3 and maxicode is 6.

DEGREE VECTORS AND GRAPHICAL VECTORS

1.68 Prove Theorem 1.6: Let $v = [d_1 \quad d_2 \quad d_3 \quad \cdots \quad d_k]$ be a nonincreasing vector of k (where k is at least 2) nonnegative integers such that no component d_i exceeds $(k - 1)$. Let v' be the vector obtained from v by deleting d_1 and subtracting 1 from each of the next d_1 components of v. Let v_1 be the nonincreasing vector obtained from v' by rearranging its components if necessary. Then v is a graphical vector if and only if v_1 is a graphical vector.

Solution. (*i*) Suppose v_1 is graphical. So there exists a graph G_1 of order $(k - 1)$ for which the degree vector is v_1. Now relabel the vertices of G_1 as $x_2, x_3, \ldots, x_k$ such that the degree of x_i is the ith component of v'. Construct a new vertex x_1 and join x_1 to each of the first d_1 vertices in the ordered set $\{x_2, x_3, \ldots, x_k\}$. The first component of the degree vector of the new graph G thus constructed is d_1, and the next d_1 components are d_j. Thus the first $d_1 + 1$ components of the degree vector of G are the same as the first $d_1 + 1$ components of v. The remaining components of the degree vector are the same as the last $k - (d_1 + 1)$ components of v. Thus the degree vector of G is v, and v is graphical.

(*ii*) Suppose $v = [d_1 \quad d_2 \quad \cdots \quad d_k]$ is a graphical vector. There can be more than one graph with (the ordered) vertex set $V = \{x_1, x_2, \ldots, x_k\}$ such that degree $x_i = d_i$. Choose a graph G with degree vector v such that the sum of the degrees of those vertices adjacent to the first vertex x_1 is as large as possible. Then we prove that x_1 is adjacent to the next d_1 vertices, starting from x_2 in the ordered set V. Suppose this is not the case. So there exist vertices x_j and x_k such that (1) x_j is not adjacent to x_1, (2) x_k is adjacent to x_1, and (3) $d_j > d_k$. Let the sum of the degrees of all vertices adjacent to x_1 in the graph G be $d_k + t$, where $t \geq 0$. Since $d_j > d_k$, there should be a vertex x_i that is adjacent to x_j but not adjacent to x_k. In the graph G, we now delete the edge joining x_1 and x_k and the edge joining x_i and x_j. Then construct an edge joining the nonadjacent vertices x_1 and x_k and another edge joining the nonadjacent vertices x_i and x_j. The degree vector of the newly constructed graph G' is also the same vector v. In G', the sum of the degrees of the vertices adjacent to x_1 is $d_j + t$, which is more than $d_k + t$. This contradicts the assumption that the sum of the degrees of the vertices adjacent to x_1 in G is a maximum. So x_1 is adjacent to the next d_1 vertices in the ordered set V starting from x_2. Thus the vector v_1 constructed from the vector v as in the hypothesis of the theorem is the degree vector of the graph $G - x_1$. So v_1 is also a graphical vector.

1.69 Prove that the algorithm in Section 1.6 determines whether a given vector of nonnegative integers is a graphical vector.

Solution. It is enough if we show that the repetitive process in step 4 of the algorithm eventually results in a zero vector or a vector with at least one negative component. Suppose we start with a vector with n components. Each component is at most $(n - 1)$. At the end of the next iteration, we have a vector with $(n - 1)$ components. Each component is at most $(n - 2)$. At the end of the kth iteration, we have a vector with $(n - k)$ components. Each component is at most $n - k - 1$. If step 4 were applied $(n - 2)$ times, we have a vector with two components. Each component is at most 1. At this stage we have either the vector $v = [1 \quad 1]$ or $w = [1 \quad 0]$. By iterating once more, we have either the zero vector with one component or the vector with -1 as the only component.

1.70 Test whether $[5 \quad 4 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 2]$ is a graphical vector. If it is graphical, draw a simple graph with this vector as the degree vector.

Solution.

Iteration 1: $v = [5 \quad 4 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 2]$ and $v_1 = [3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2]$

Iteration 2: $v = [3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2]$ and $v_1 = [2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1]$

Iteration 3: $v = [2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1]$ and $v_1 = [2 \quad 1 \quad 1 \quad 1 \quad 1]$

Iteration 4: $v = [2 \quad 1 \quad 1 \quad 1 \quad 1]$ and $v_1 = [1 \quad 1 \quad 0 \quad 0]$

Iteration 5: $v = [1 \quad 1 \quad 0 \quad 0]$ and $v_1 = [0 \quad 0 \quad 0]$

So the given vector is graphical. A simple graph with v as degree vector is shown in Fig. 1-20.

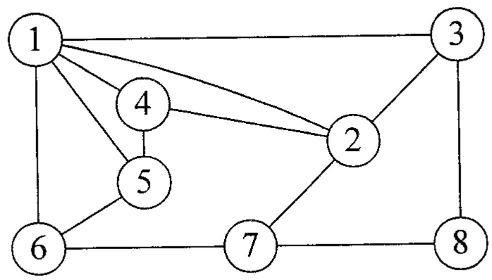


Fig. 1-20

1.71 Test whether $[6 \quad 6 \quad 5 \quad 4 \quad 3 \quad 3 \quad 1]$ is graphical.

Solution.

Iteration 1: $v = [6 \quad 6 \quad 5 \quad 4 \quad 3 \quad 3 \quad 1]$ and $v_1 = [5 \quad 4 \quad 3 \quad 2 \quad 2 \quad 0]$

Iteration 2: $v = [5 \quad 4 \quad 3 \quad 2 \quad 2 \quad 0]$ and $v_1 = [3 \quad 2 \quad 1 \quad 1 \quad -1]$

Since we obtain a vector with a negative component, we conclude that the given vector is not graphical.

1.72 Let $v = [d_1 \quad d_2 \quad \cdots \quad d_n]$ and $w = [w_n \quad w_{n-1} \quad \cdots \quad w_2 \quad w_1]$, where $w_i = n - 1 - d_i$. Show that v is graphical if and only if w is graphical.

Solution. Suppose v is the degree vector of $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. It is easy to see that w is the degree vector of the complement of G. Thus v is graphical if and only if w is graphical.

1.73 Show that there is no simple graph with six vertices of which the degrees of five vertices are 5, 5, 3, 2, and 1.

Solution. Suppose there is a simple graph, and let x be the degree of the sixth vertex. The sum of the 6 degrees has to be even, and x cannot exceed 5. So $x = 0$ or 2 or 4.

$x = 0$ implies $v = [5 \quad 5 \quad 3 \quad 2 \quad 1 \quad 0]$. At the end of the first iteration we get $[4 \quad 2 \quad 1 \quad 0 \quad -1]$. So x is not 0.

$x = 2$ implies $v = [5 \quad 5 \quad 3 \quad 2 \quad 2 \quad 1]$. At the end of the second iteration we get $[1 \quad 0 \quad 0 \quad -1]$. So x is not 2.

$x = 4$ implies $v = [5 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1]$. At the end of the second iteration we get $[2 \quad 1 \quad 0 \quad -1]$. So x is not 4.

1.74 Find x if $[8 \quad x \quad 7 \quad 6 \quad 6 \quad 5 \quad 4 \quad 3 \quad 3 \quad 1 \quad 1 \quad 1]$ is a graphical vector.

Solution. Obviously x is either 8 or 7.

$x = 8$ eventually leads to $[-1 \quad 0 \quad 0 \quad 0 \quad 0]$. So x is not 8.

$x = 7$ eventually leads us to $v = [0 \quad 0 \quad 0 \quad 0 \quad 0]$. So $x = 7$.

1.75 Show that a finite nonincreasing vector in which no two components are equal cannot be a graphical vector.

Solution. Consider the nonincreasing vector v with k components in which each component is a nonnegative integer. If no two components are equal, $v = [k-1 \quad k-2 \quad \cdots \quad 2 \quad 1 \quad 0]$. If we delete the first component and subtract 1 from each of the remaining components, we get a vector v_1 in which the last component is negative. So v_1 is not graphical, and therefore v is not graphical.

1.76 Show that in a simple graph, there are at least two vertices with equal degrees.

Solution. Suppose no two degrees in a simple graph G are equal. Then no two components of the degree vector of G are equal. This is a contradiction, as established in Problem 1.76.

1.77 Show that there exists a simple graph with 12 vertices and 28 edges such that the degree of each vertex is either 3 or 5. Draw this graph.

Solution. Suppose there are p vertices of degree 3. Then the equation $3p + 5(12 - p) = (2)(28)$ gives the unique solution $p = 2$. So if there exists a graph with two vertices of degree 3 and 10 vertices of degree 5, its degree vector is $v = [5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 3 \quad 3]$. To show that there exists a graph with the desired property, it is enough if we demonstrate that v is a graphical vector. After nine iterations we get the 0 vector:

$$v_1 = [5 \quad 5 \quad 5 \quad 5 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3]$$

$$v_2 = [4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3 \quad 3 \quad 3]$$

$$v_3 = [4 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3]$$

$$v_4 = [3 \quad 3 \quad 3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2]$$

$$v_5 = [2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2]$$

$$v_6 = [2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1]$$

$$v_7 = [2 \quad 1 \quad 1 \quad 1 \quad 1]$$

$$v_8 = [1 \quad 1 \quad 0 \quad 0]$$

$$v_9 = [0 \quad 0 \quad 0]$$

So v is indeed a graphical vector. A simple graph with v as the degree vector is shown in Fig. 1-21.

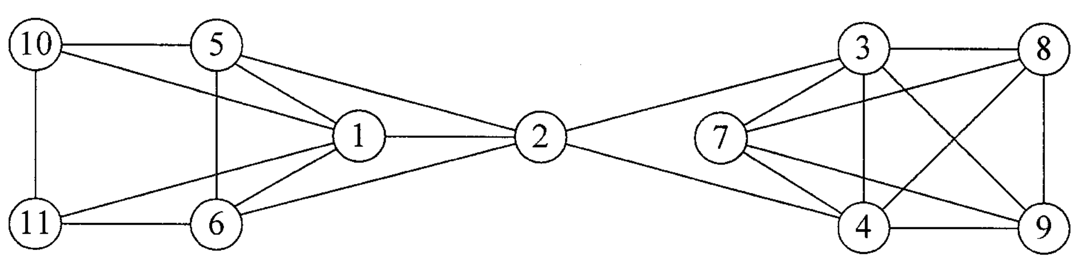


Fig. 1-21

1.78 Show that there exists a simple graph with seven vertices and 12 edges such that the degree of each vertex is 2 or 3 or 4.

Solution. Suppose there are p vectors of degree 2 and q vectors of degree 3. Then the only solution in positive integers of the equation $2p + 3q + 4(7 - p - q) = 24$ is $p = 1$ and $q = 2$. Thus if there is a graph with the desired property, it should have one vertex of degree 2, two vertices of degree 3, and four vertices of degree 4, giving a unique degree vector $v = [4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3 \quad 2]$. It is easily verified that this is indeed a graphical vector.

Supplementary Problems

1.79 Find the complements of (*a*) K_n and (*b*) $K_{m,n}$.
Ans. (*a*) The graph with n vertices and no edges. (*b*) The graph with $(m + n)$ vertices consisting of two parts (components): a complete graph with m vertices and a complete graph with n vertices.

1.80 Find the number of nonisomorphic graphs with four vertices and at most 3 edges.
Ans. $N(4, 0) + N(4, 1) + N(4, 2) + N(4, 3) = 1 + 1 + 2 + 3 = 7$

1.81 Find the vertex covering number and the independence number of K_n and $K_{m,n}$.
Ans. $\alpha(K_n) = 1$, $\alpha(K_{m,n}) = \max\{m, n\}$, $\beta(K_n) = n - 1$ and $\beta(K_{m,n}) = (m + n) - \max\{m, n\}$

1.82 Find the vertex dominating number of K_n and $K_{m,n}$. *Ans.* $\sigma(K_n) = 1$ and $\sigma(K_{m,n}) = \min\{m, n\}$

1.83 Find the committee number of (*a*) K_n and (*b*) $K_{m,n}$. *Ans.* (*a*) 1; (*b*) minimum of $\{m, n\}$.

1.84 If I is an independent set in a graph, find the subgraph induced by I.
Ans. The graph with I as vertex set and no edges.

1.85 Find the maximum number of edges in (*a*) a simple graph with n vertices and (*b*) a bipartite graph (X, Y, E), where the cardinalities of X and Y are m and n, respectively. [*Hint:* (*a*) Since K_n has exactly $n(n - 1)/2$ edges, the number of edges in a simple graph with n vertices cannot exceed $n(n - 1)/2$.] (*b*) Since $K_{m,n}$ has exactly mn edges, the number of edges in the bipartite graph cannot exceed mn.

1.86 It is known that there exists a simple graph with 12 vertices and 28 edges in which the degree of each vertex is either 3 or 5. Find the number of vertices of degree 3.
Ans. There are two vertices of degree 3 and 10 vertices of degree 5.

1.87 Find the number of nonequivalent graphs with four vertices and three edges. [*Hint:* The complete graph K_4 has six edges, out of which any three can be chosen in $C(6, 3) = 20$ ways. Thus there are 20 nonequivalent graphs with four vertices and three edges, out of which there are exactly three nonisomorphic graphs, as established earlier.]

1.88 Find the number of nonequivalent graphs with five vertices and three edges. [*Hint:* The complete graph K_5 has 10 edges. Out of these 10 edges, any three can be chosen in $C(10, 3) = 120$ ways. So there are 120 nonequivalent graphs.]

1.89 If G is a k-regular graph with n vertices, find the number of triangles in G and its complement.
Ans. $C(n, 3) - nk(n - k - 1)/2$

1.90 Show that if every edge in a graph joins an odd vertex and an even vertex, the graph is bipartite. Is the converse true?
Ans. Let X be the set of all odd vertices and Y be the set of all even vertices in the graph G. Then $G = (X, Y, E)$ is a bipartite graph where E is the set of edges in G. The converse is not true since it is possible to have a bipartite graph in which there is a left vertex (in X) that is odd (or even) and a right vertex (in Y) that is even (or odd).

1.91 Any root of the characteristic polynomial of a graph is an **eigenvalue** of the graph. The **spectrum** of a graph is the collection of all its eigenvalues. Find the spectrum of (*a*) K_3, (*b*) K_4, and (*c*) $K_{2,2}$.
Ans. Since the adjacency matrix is symmetric, every eigenvalue is a real number. (*a*) -1, -1, and 2; (*b*) -1, -1, -1, and 3; (*c*) 0, 0, 2, -2

1.92 Find the spectrum of K_n. *Ans.* The number -1 repeated $(n - 1)$ times and the number $(n - 1)$

1.93 Find the minicode and the maxicode of the simple graph with four vertices such that the degrees of the vertices are 1, 2, 2, and 3.
Ans. The minicode is 15, and the maxicode is 60.

1.94 Find the minicode and maxicode of K_n.
Ans. Since every nondiagonal element of any adjacency matrix is 1, both the minicode and the maxicode are equal to $1 + 2 + 2^2 + \cdots + 2^{k-1}$, where $k = n(n - 1)/2$.

Chapter 2

Connectivity

2.1 PATHS, CIRCUITS, AND CYCLES

Let v and w be two vertices in a graph. A **walk between v and w** in the graph is a finite alternating sequence $v = v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n = w$ of vertices and edges of the graph such that each edge e_i in the sequence joins vertex v_{i-1} and vertex v_i. The vertices and edges in a walk need not be distinct. Two walks $v_0, e_1, v_1, \ldots, e_n, v_n$ and $u_0, f_1, u_1, \ldots, f_m, u_m$ are **equal** if $n = m$, $v_i = u_i$, and $e_i = f_i$ for $0 \le i \le n$. Two walks are said to be **different** if they are not equal. The number of edges in a walk is the **length** of the walk. If the graph is simple, the edges in the sequence defining a walk between v and w need not be explicitly listed; the walk $v = v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n = w$, can be expressed as $v_0 — v_1 — v_2 — \cdots — v_n$ unambiguously. A walk in which no edge is repeated is a **trail.** The walk $v = v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{n-1}, e_n, v_n = w$, in which the vertices v_i $(0 < i < n)$ are all distinct is a **path between v and w;** the $(n - 1)$ vertices $v_i (0 < i < n)$ are called the **intermediate vertices** of the path. Obviously, every path is a trail.

If v and w are vertices in a directed graph, a **directed walk from v to w** is a finite sequence $v = v_0, a_1, v_1, a_2, v_2, a_3, \ldots, a_n, v_n = w$, of vertices and arcs of the digraph such that each arc a_i in the sequence is an arc from v_{i-1} to v_i. This is written $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$. A directed walk is a **directed trail** if the arcs are distinct. A **directed path from v to w** in a digraph is a directed walk from v to w in which no vertices repeat.

Example 1. In graph G in Fig. 2-1(a), the sequence 2, e_4, 1, e_1, 4, e_2, 1, e_1, 4 is a walk between vertex 2 and vertex 4. The sequence 2, e_3, 1, e_1, 4, e_2, 1, e_4, 2, e_5, 3 is a trail between 2 and 3. The sequence 2, e_5, 3, e_6, 4, e_1, 1 is a path between 2 and 1. In the directed graph shown in Fig. 2-1(b), $2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 6$ is a directed walk from 1 to 6, $1 \rightarrow 2 \rightarrow 4 \rightarrow 1 \rightarrow 3$ is a directed trail from 1 to 3, and $1 \rightarrow 2 \rightarrow 5 \rightarrow 7$ is a directed path from 1 to 7.

Theorem 2.1. Every walk in a graph between v and w contains a path between v and w, and every directed walk from v to w in a digraph contains a directed path from v to w. (See Solved Problem 2.1.)

Theorem 2.2. If A is the adjacency matrix of a simple graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, the $(i - j)$ entry in the kth power of A is the number of *different* walks of length k between the vertices i and j. In particular, the $(i - i)$ diagonal entry in A^2 is the degree of vertex i for each i. (See Solved Problem 2.2.)

Example 2. The adjacency matrix A of the simple graph shown in Fig. 2-2, the matrix A^2, and the matrix A^4 are

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}, \quad \text{and} \quad A^4 = \begin{bmatrix} 9 & 3 & 11 & 1 & 6 \\ 3 & 15 & 7 & 11 & 8 \\ 11 & 6 & 15 & 3 & 8 \\ 1 & 11 & 3 & 9 & 6 \\ 6 & 8 & 8 & 6 & 8 \end{bmatrix}$$

The degrees of the vertices 1, 2, 3, 4, 5 are 2, 3, 3, 2, 2, respectively, in agreement with the diagonal entries of A^2. The (1, 5) entry in A^4 is 6, indicating that there are six *different* walks of length 4 between 1 and 5. These walks are $1 — 4 — 1 — 2 — 5$, $1 — 2 — 1 — 2 — 5$, $1 — 4 — 3 — 2 — 5$, $1 — 2 — 5 — 2 — 5$, $1 — 2 — 3 — 2 — 5$, and $1 — 2 — 5 — 3 — 5$.

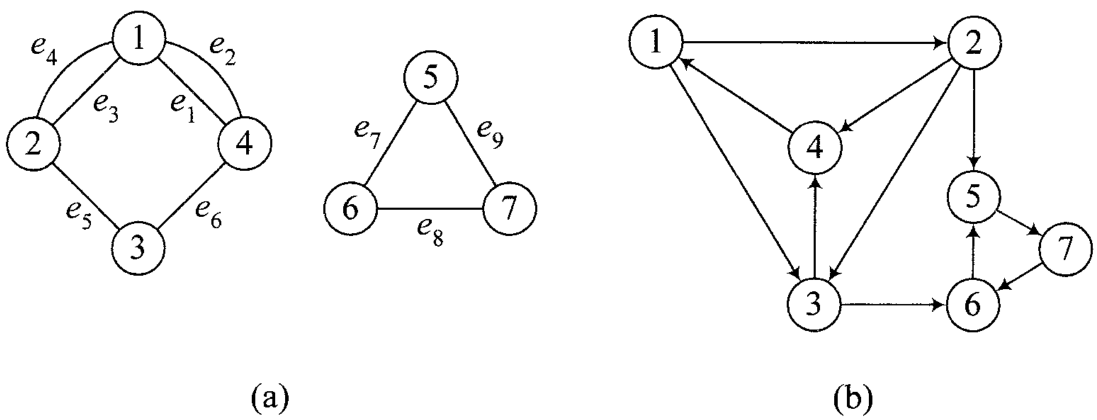


Fig. 2-1

A **closed walk** in a graph is a walk between a vertex and itself. A closed walk in which no edges repeat is a **circuit.** A **cycle** is a circuit with no repeated vertices. Notice that the closed walk v, e_1, w, e_2, v is a cycle, but the closed walk v, e_1, w, e_1, v with no repeated intermediate vertices is not a cycle since it is not a circuit. The subgraph C of a simple graph G is a **cycle in *G*** if and only if C is a cyclic graph. In a simple graph G, any cycle consisting of k vertices (that is, passing through k vertices) is a ***k*-cycle** in G; it is an **odd cycle** if k is odd and an **even cycle** if k is even. The terms **directed circuits** and **directed cycles** in the case of digraphs are defined analogously.

Example 3. In the simple graph G shown in Fig. 2-3, the closed walk 1, e_1, 2, e_5, 5, e_6, 3, e_3, 4, e_7, 5, e_8, 1 is a circuit, and the closed walk 1 — 2 — 3 — 4 — 1 is an even cycle.

One way to ascertain whether or not a given graph is bipartite is by identifying its cycles. If there is an odd cycle in a graph G, G is not bipartite. In this context, we have the following theorem, which characterizes bipartite graphs.

Theorem 2.3. A simple graph with three or more vertices is bipartite if and only if it has no odd cycles (See Solved Problem 2.10.)

2.2 CONNECTED GRAPHS AND DIGRAPHS

A pair of vertices in a graph is a **connected pair** if there is a path between them. A graph G is a **connected graph** if every pair of vertices in G is a connected pair; otherwise, it is a **disconnected** graph. A connected subgraph H of a graph G is a **component** of G if $H = H'$ whenever H' is a connected subgraph (of G) that contains H. In other words, a component of a graph is a maximal connected subgraph. A graph is connected if and only if the number of its components is one.

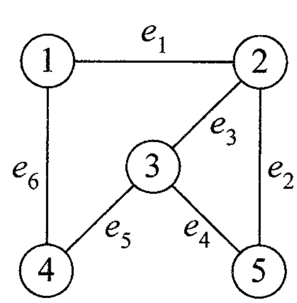


Fig. 2-2

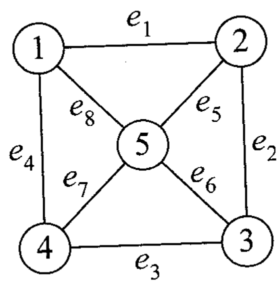


Fig. 2-3

If F is a set of edges in a graph $G = (V, E)$, the graph obtained from G by deleting all the edges belonging to F is denoted by $G - F$. If F consists of a single edge f, $G - F$ is denoted by $G - f$. A set F of edges in G is called a **disconnecting set** in G if $G - F$ has more than one component. If a disconnecting set F consists of a single edge f, that edge is called a **bridge** (also known as a **cut edge** or an **isthmus**). A graph is said to be ***k* edge connected** if every disconnecting set in it has at least k edges. The **edge-connectivity number $\lambda(G)$ of a graph G** is the minimum size of a disconnecting set in G; *by definition*, it is zero when G is the trivial graph K_1. Thus $\lambda(G)$ is zero if and only if G is a disconnected graph or the trivial graph, and it is k edge connected if and only if $\lambda(G)$ is at least k. A disconnecting set F is said to be a **cut set** (also known as a **bond**) if no proper subset of F is a disconnecting set.

Example 4. The graph G of Fig. 2-4 with 13 vertices is not connected since there are several pairs of vertices that are not connected. For example, vertex 5 and vertex 12 do not form a connected pair. The components of G are G^1, G^2, and G^3, as shown in the figure. The set {{1, 6}, {2, 6}} is a disconnecting set but not a cut set. The set {{10, 12}, {10, 13}} is a cut set. The edge {2, 6} is a bridge. Since G is disconnected, its edge-connectivity number is zero. The edge-connectivity number of the component G^2 is 2, and it is thus two edge connected.

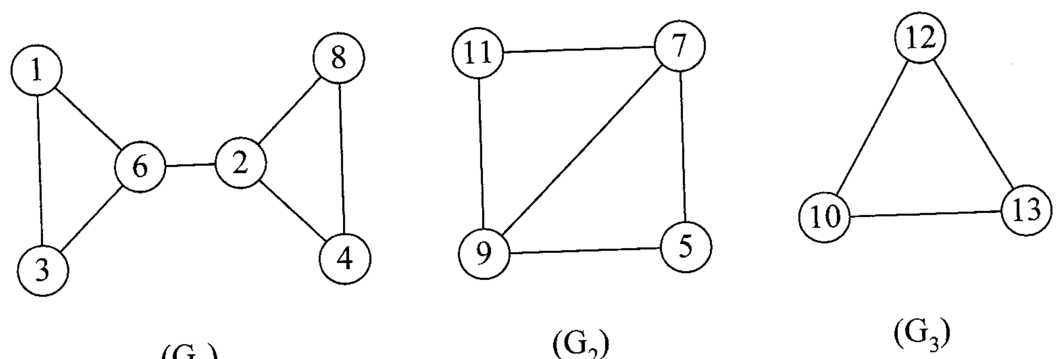


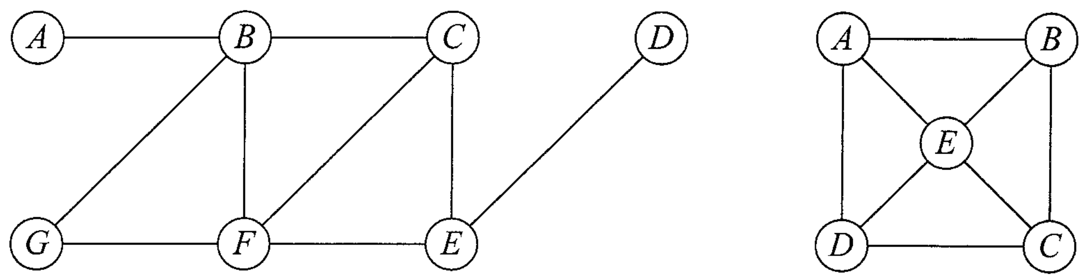
Fig. 2-4

We now define analogous concepts regarding the deletion of vertices. If W is a set of vertices in $G = (V, E)$, the graph obtained from G by deleting all the vertices belonging to W as well as the edges incident to the vertices in W is denoted by $G - W$. If W consists of a single vertex w, the graph $G - W$ is denoted by $G - w$. A set W of vertices in G is called a **separating set** (also known as a **vertex cut**) in G if $G - W$ has more than one component. Observe that neither V nor $V - w$ are separating sets. If a separating set consists of a single vertex w, w is known as a **cut vertex** (or **articulation vertex**). The **connectivity number $\kappa(G)$** of a graph G is the minimum size of a separating set in it. Since a complete graph has no separating set, we adopt the convention that the connectivity number of the complete graph of order n is $(n - 1)$ for all n. A graph G is said to be ***k*-connected** if $\kappa(G) \geq k$. Thus K_n is $(n - 1)$-connected for all n, and a graph that is not complete is k-connected if and only if every separating set in it has at least k vertices. The connectivity number of a graph G is zero if and only if G is either the trivial graph K_1 or is a disconnected graph. A cyclic graph is 2-connected. In Fig. 2-4, vertex 6 is a cut vertex and the subgraph G^1 is 1-connected, whereas both the subgraphs G^2 and G^3 are 2-connected.

Theorem 2.4 **(Whitney's Inequality).** For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. (See Solved Problem 2.11.) (Edge connectivity and connectivity are discussed in more detail in Chapter 6 in the context of Ford–Fulkerson theorem and Menger's theorem.)

The number of edges in a path with as few edges as possible between two vertices v and w of a connected graph G is denoted by $d(v, w)$. The **eccentricity $e(v)$** of the vertex v is the maximum value of $d(v, w)$, where w varies through all the vertices of G. The **radius $r(G)$** of G is the eccentricity of the vertex of minimum eccentricity. A vertex v is a **central vertex** if its eccentricity is equal to the radius of the graph. The **center $C(G)$ of a graph** is the set of all its central vertices.

Example 5. In Fig. 2-5(a), the eccentricities of the vertices A, B, C, D, E, F, and G are 4, 3, 2, 4, 3, 2, and 3, respectively. The central vertices are C and F. The center is the set $\{C, F\}$. In Fig. 2-5(b), the eccentricity of vertex E is 1 and the eccentricities of the other vertices are 2. E is the only central vertex, and the center is the singleton set $\{E\}$.



(a) (b)

Fig. 2-5

In the case of digraphs, the concept of connectivity becomes more variegated. A digraph G is **strongly connected** if there is a directed path from each vertex to every other vertex. A **strong component of a digraph** is a maximal strongly connected subgraph. A digraph is **unilaterally connected** if for every pair of vertices v and w in G, there is either a path from v to w or from w to v. A digraph is **weakly connected** if its underlying graph is connected.

Example 6. The digraph G' of Fig. 2-1(b) is not strongly connected since there is no directed path from vertex 5 to vertex 3. It is weakly connected because the underlying graph is connected. It is easy to verify that this digraph is unilaterally connected.

A **mixed graph** is a structure $G = (V, E)$, where V is a finite set of vertices and E is a finite set of pairs of vertices in which some pairs are ordered (defining arcs of G) and some pairs are unordered (defining edges of G). The undirected graph $U(G)$ obtained by converting each arc of a mixed graph G into an edge is the underlying graph of G, and G is connected (by definition) if $U(G)$ is connected. The digraph $D(G)$ obtained from G after replacing each edge between v and w by two arcs (v, w) and (w, v) is called the directed graph of G, and G is strongly connected (by definition) if $D(G)$ is strongly connected. An undirected edge e in a connected mixed graph G is a bridge if the deletion of e makes G disconnected.

2.3 TREES AND SPANNING TREES

An **acyclic graph** (also known as a **forest**) is a graph with no cycles. A **tree** is a connected acyclic graph. Thus each component of a forest is a tree, and any tree is a connected forest.

Example 7. The graph in Fig. 2-6 with 13 vertices is a forest consisting of three trees.

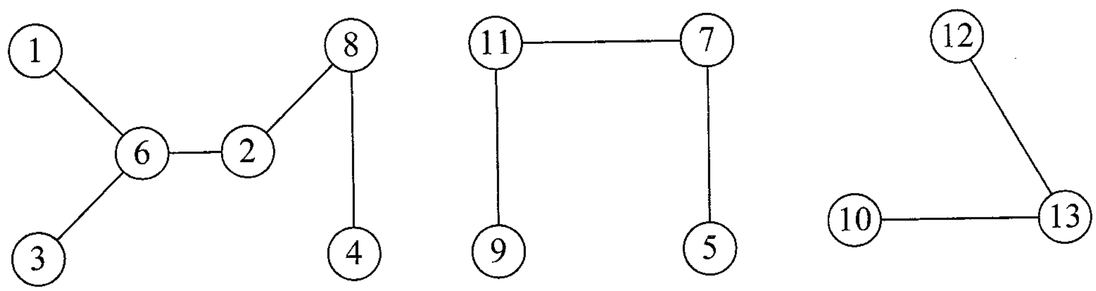


Fig. 2-6

Theorem 2.5. The following are equivalent in a graph G with n vertices.

(*i*) G is a tree.

(*ii*) There is a unique path between every pair of vertices in G.

(*iii*) G is connected, and every edge in G is a bridge.

(*iv*) G is connected, and it has $(n - 1)$ edges.

(*v*) G is acyclic, and it has $(n - 1)$ edges.

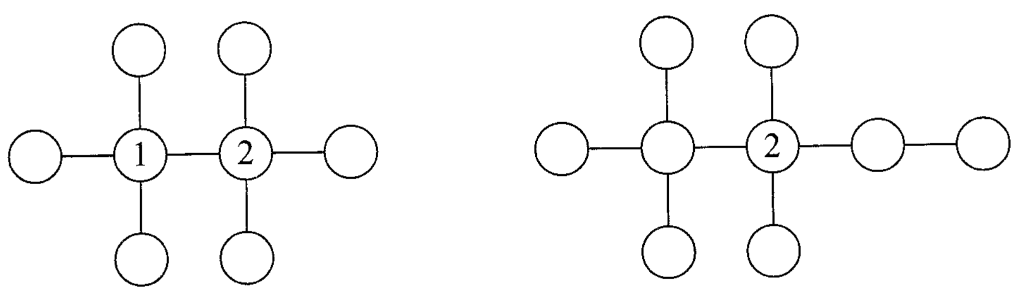
(*vi*) G is acyclic, and whenever any two arbitrary nonadjacent vertices in G are joined by an edge, the resulting enlarged graph G' has a unique cycle.

(*vii*) G is connected, and whenever any two arbitrary nonadjacent vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle.

(See Solved Problem 2.56.)

Theorem 2.6. The center of a tree is either a singleton set consisting of a unique vertex or a set consisting of two adjacent vertices. (The converse is not true. See Example 5.) (See Solved Problem 2.62.)

Example 8. In the tree shown in Fig. 2-7(a), the center is the set $\{1, 2\}$. In the tree shown in Fig. 2-7(b), the center is the singleton set $\{2\}$.



(a) (b)

Fig. 2-7

An acyclic spanning subgraph of a graph G is a **spanning forest** in G. A spanning forest in the graph of Fig. 2-4 is the forest shown in Fig. 2-6. An acyclic connected spanning subgraph (if it exists) of G is called a **spanning tree** in G. In the connected graph G of Fig. 2-3, the set $\{e_1, e_2, e_6, e_7\}$ of edges constitutes the edges of a spanning tree in G.

Theorem 2.7. A graph is connected if and only if it has a spanning tree. (See Solved Problem 2.57.)

Theorem 2.8. Let G be a simple graph with n vertices. If a spanning subgraph H satisfies any two of the following three properties, it will satisfy the third property also. (i) H is connected. (ii) H has $(n - 1)$ edges. (iii) H is acyclic. (See Solved Problem 2.58.)

A tree with n vertices is a **labeled tree** if each vertex is assigned a unique positive integer between 1 and n such that no two vertices get the same label. Two labeled trees $T = (V, E)$ and $T' = (V, E')$ are **distinct** if E and E' are not the same set. For example, the labeled tree $T = (V, E)$ with edge set $\{\{1, 2\}$ and $\{2, 3\}\}$ and the labeled tree $T' = (V, E')$ with edge set $\{\{1, 3\}, \{2, 3\}\}$ are distinct even though they are isomorphic. Two nonisomorphic labeled trees are, of course, distinct labeled trees.

Theorem 2.9 **(Cayley's Theorem).** The number of distinct labeled trees with n vertices is n^{n-2}, which is also equal to the number of spanning trees in K_n. (See Solved Problem 2.73.)

Given a graph with n vertices, it is natural to ask whether it is connected. One method to test the connectivity of a graph is by using a recursive procedure known as the **depth first search (DFS) technique** in which we relabel the vertices as follows.

Suppose the vertices of G are $v_1, v_2, \ldots, v_n$. Start the search from any vertex, and relabel that vertex 1. If this vertex has no adjacent vertices, the graph is disconnected. Otherwise, select any vertex adjacent to vertex 1 and relabel it vertex 2, marking the edge $\{1, 2\}$ joining vertex 1 and vertex 2 as a used edge. If vertex 2 has an adjacent vertex other than vertex 1, relabel that vertex 3. If vertex 2 has no other adjacent vertex, revert back to vertex 1 and see whether vertex 1 has an adjacent vertex other than vertex 2. If the answer is no and if $n > 2$, the graph is disconnected. If the answer is yes, relabel the newly located vertex as vertex 3. In either case, mark the edge $\{2, 3\}$ as a used edge. After relabeling a vertex as vertex i, select an arbitrary vertex that is not yet relabeled and that is adjacent to i, relabel that vertex $(i + 1)$. Mark the edge joining i and $(i + 1)$ as a used edge. If a newly relabeled vertex v has no adjacent vertex, go back to vertex w, which is adjacent to v with used edge $\{v, w\}$, and continue the search from w. The procedure continues until all the n vertices are relabeled $1, 2, \ldots, n$, indicating that the graph is connected; or, we are back at vertex 1 with the number of relabeled vertices less than n, showing that the graph is not connected.

If we find that a graph G of order n is a connected graph by using the depth first search, the set of $(n - 1)$ used edges in G constitutes the edges of a spanning tree (in G) known as **DFS spanning tree.**

Example 9. The DFS technique described above is used to test the connectivity of the simple graph with vertex set v_i $(i = 1, 2, \ldots, 8)$, the adjacency matrix of which is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Take v_2 and relabel it vertex 1. A vertex adjacent to v_2 is v_7, as can be seen from the adjacency matrix. So v_7 is labeled vertex 2, and edge e_1 joining v_2 and v_7 is marked as a used edge. Then v_6 is labeled vertex 3, and edge e_2 joining v_7 and v_6 is marked as a used edge. At the next stage, v_3 is labeled vertex 4 with used edge $e_3 = \{v_6, v_3\}$, and v_4 is labeled vertex 5 with used edge $e_4 = \{v_3, v_4\}$. Now vertex v_4 (labeled 5) has no adjacent vertex other than v_3 (which is labeled 4). So we revert to vertex 4 and start the search from there. A vertex adjacent to v_3 is v_5, which now gets the label 6, and $e_5 = \{v_3, v_5\}$ becomes a used edge. The search now reverts to vertex v_7 (labeled 2), and v_1 is an adjacent unlabeled vertex that now becomes vertex 7 with $e_6 = \{v_7, v_1\}$ as a used edge. Finally, label vertex v_8 as vertex 8 with used edge $e_7 = \{v_8, v_1\}$. Since all the eight vertices are relabeled, we conclude that G is connected. The seven used edges obtained in this search constitute the edges of a spanning tree, as shown in Fig. 2-8.

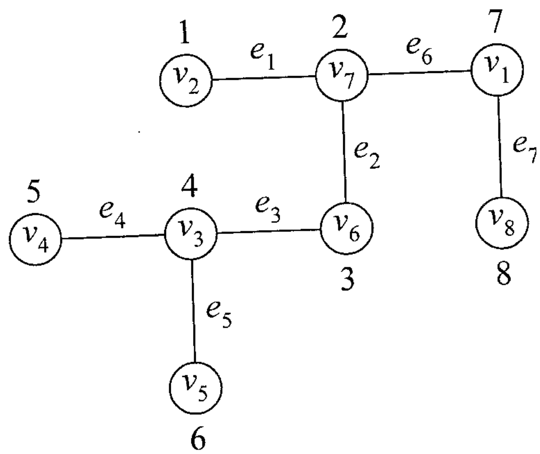


Fig. 2-8

2.4 STRONG ORIENTATIONS OF GRAPHS

If each edge of a simple graph G is replaced by an arc and if the resulting digraph G' is strongly connected, the digraph G' is called a **strong orientation** of G. A simple graph is **strongly orientable** if it has a strong orientation. If each street in a certain region of a city is to be converted into a one-way street, after the conversion one should be able to drive from any street corner to any other street corner in the region. So a conversion of this type is possible only if the streets in the region constitute a strongly orientable graph with street corners as vertices. For a graph to be strongly orientable, it has to be a connected graph in the first place. Furthermore, it should not become disconnected if an edge is deleted from the graph. In other words, a strongly orientable graph is necessarily a connected graph in which no edge is a bridge. The converse of this assertion is also true. Thus we have the following theorem.

Theorem 2.10 **(Robbins's Theorem).** A graph is strongly orientable if and only if it is connected and has no bridges. (See Solved Problem 2.87.)

Example 10. The edge joining vertex 3 and vertex 4 in the connected graph of Fig. 2-9 is a bridge. This graph has no strong orientation.

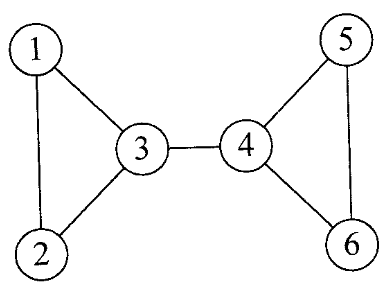


Fig. 2-9

Given a strongly orientable graph with n vertices and m edges, one algorithm mimics the depth first search, enabling us to orient the edges such that the resulting digraph is strongly connected. The procedure is as follows. Suppose the n vertices are relabeled $1, 2, \ldots, n$ with $(n - 1)$ of the m edges specially designated as used edges. A used edge between i and j is converted into an arc from i to j if $i < j$, and an unused edge between i and j is converted into an arc from i to j if $i > j$.

Theorem 2.11 **(Roberts's Theorem).** The orientation procedure using the depth first search in a connected graph with no bridges yields a strongly connected digraph. (See Solved Problem 2.93.)

Example 11. The graph G in Fig. 2-10(a) is connected in which no edge is a bridge. A spanning tree in G obtained by the depth first search is shown in Fig. 2-10(b), with the eight vertices labeled 1, 2, . . . , 8, respectively, and with seven used edges. After orienting the edges according to the rule stipulated in Theorem 2.11, the strongly connected digraph is as shown in Fig. 2-10(c).

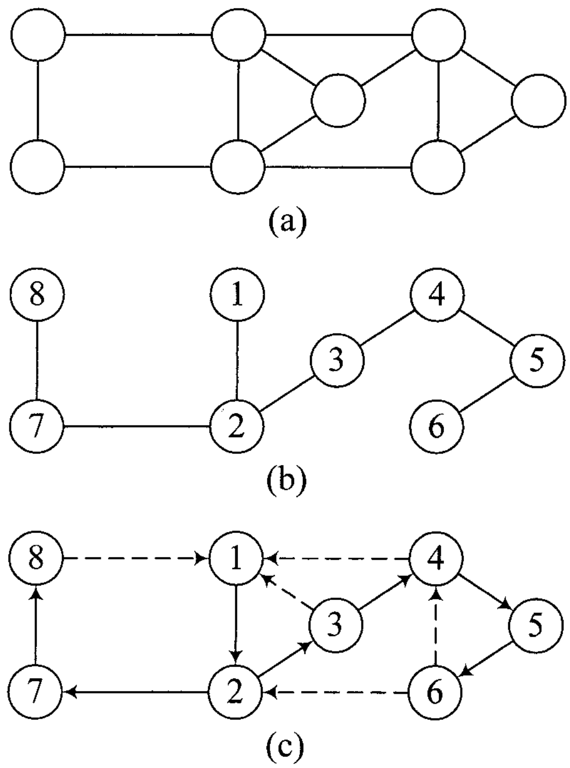


Fig. 2-10

Solved Problems

PATHS, CIRCUITS AND CYCLES

2.1 Prove Theorem 2.1: Every walk in a graph between v and w contains a path between v and w, and every directed walk from v to w in a digraph contains a directed path from v to w.

Solution. Let W be a walk between v and w. If $v = w$, there is the trivial path with no edges. Therefore, assume that v and w are not the same vertex. Suppose W is the walk $v = v_0 - v_1 - \cdots - v_n = w$. It is possible that the same vertex has more than one label in this sequence. If no vertex of the graph appears more than once in the sequence, we have a path between v and w. Otherwise, there will be at least one vertex that appears as v_i and v_j in the sequence with $i < j$. If we remove the terms $v_{i+1}, v_{i+2}, \ldots, v_j$ from the sequence, we still have a walk between v and w that contains fewer edges. We continue this process until each repeated vertex appears only once in the walk; at that stage, we have a path between v and w. The proof in the case of directed walks is similar.

2.2 Prove Theorem 2.2: If A is the adjacency matrix of a simple graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, the $(i - j)$ entry in the kth power of A is the number of *different* walks of length k between the vertices i and j. In particular, the $(i - i)$ diagonal entry in A^2 is the degree of vertex i for each i. (See Solved Problem 2.2.)

Solution. The proof is by induction on k. This is true when $k = 1$. Assume that the result is true for $(k - 1)$. By the induction hypothesis, the (i, j) entry in A^{k-1} is the number of walks between i and j of length $(k - 1)$. The (i, j) entry in A is 1 if and only if i and j are adjacent. Now $A^k = A^{k-1} \cdot A$. So the (i, j) entry in A^k is the number of walks between i and j of length k. Thus it is true for k.

2.3 If the edges in a graph are labeled $e_i (i = 1, 2, \ldots, m)$ and the set of paths between two vertices v and w are labeled p_i $(i = 1, 2, \ldots, k)$, the **v, w path matrix** is the $k \times m$ binary matrix P_{uv} in which the (i, j) entry corresponding to the path p_i is 1 if p_i contains the edge e_j and 0 otherwise. Obtain a path matrix between vertices 1 and 3 in Fig. 2-2.

Solution. In the graph shown in Fig. 2-2, there are three paths, p_1: 1 — 2 — 3, p_2: 1 — 4 — 3, and p_3: 1 — 2 — 5 — 3, between vertex 1 and vertex 3. The edges are labeled as shown. The 1, 3 path matrix of this graph with respect to this labeling is

$$P = P_{13} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(The columns are labeled in the order of the edges: $e_1, \ldots, e_6$.)

2.4 Let B be the incidence matrix of the simple graph $G = (V, E)$ in which $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, and the paths between vertex i and vertex j are labeled $p_1, p_2, \ldots, p_k$. If P is the (i, j) path matrix, the (binary) product BP^T is the binary matrix S in which the only nonzero entries are those in the ith and jth rows. (Multiplication of binary matrices here is mod 2.)

Solution. The product of the ith row of B and any column of P^T is 1. Likewise, the product of the jth row of B and any column of P^T is 1. The product (modulo 2) of any other row of B with every column of P^T is 0.

2.5 Verify the assertion in Problem 2.4 by considering a path matrix in Fig. 2-2.

Solution. In Fig. 2-2, the binary product BP^T, where B is the incidence matrix and P is the (1, 3) path matrix (obtained in Problem 2.3), is equal to the 5×3 binary matrix in which row 1 and row 3 are nonzero vectors and the other rows are zero vectors:

$$BP^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.6 If the edge set in a simple graph is labeled $E = \{e_1, e_2, \ldots, e_m\}$ and the set of cycles is labeled $\{C_1, C_2, \ldots, C_k\}$, the **cycle matrix** of the graph is the $k \times m$ binary matrix C defined as follows. In the row corresponding to the ith cycle C_i, the (i, j) entry is 1 if and only if e_j is an edge in C_i. Obtain a cycle matrix of the graph of Fig. 2-3.

Solution. The five cycles in Fig. 2-3 are $C_1 = \{e_1, e_5, e_8\}$, $C_2 = \{e_2, e_5, e_6\}$, $C_3 = \{e_3, e_6, e_7\}$, $C_4 = \{e_4, e_7, e_8\}$, and $C_5 = \{e_1, e_2, e_3, e_4\}$. The cycle matrix C is the following 5×8 matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(The columns are labeled in the order of the edges: $e_1, \ldots, e_8$.)

2.7 Let B be the incidence matrix of the simple graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, and the cycles in G are labeled $C_1, C_2, \ldots, C_k$. If C is the $k \times m$ cycle matrix, the (binary) matrix product BC^T and the (binary) matrix product CB^T are zero matrices. (Multiplication of binary matrices here is mod 2.)

Solution. The product of any row of the incidence matrix and any column of the transpose of the cycle matrix is 0 (mod 2). Similarly, the product of any row of the cycle matrix with any column of the transpose of the incidence matrix is 0 (mod 2).

2.8 Verify the assertion of Problem 2.7 by considering the cycle matrix in Fig. 2-3.

Solution. In Fig. 2-3, the cycle matrix C is as in Problem 2.6, and the incidence matrix B is

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is easily verified that the binary matrix product BC^T and the binary matrix product CB^T are both zero matrices.

CONNECTED GRAPHS AND DIGRAPHS

2.9 Show that a graph G is bipartite if and only if every component of G is bipartite.

Solution. Every subgraph of a bipartite graph G is bipartite. In particular, every component of G is bipartite. Conversely, suppose that the components of $G = (V, E)$ are $G_i = (X_i, Y_i, F_i)$, where $i = 1, 2, \ldots, k$. Let $X = \cup X_i$, $Y = \cup Y_i$, and $F = \cup F_i$. Then $G = (X, Y, F)$.

2.10 Prove Theorem 2.3: A simple graph with three or more vertices is bipartite if and only if it has no odd cycles.

Solution. Let $G = (X, Y, E)$ be a bipartite graph. It is easy to see that if there is a cycle C in G, C should have an even number of vertices (and therefore an even number of edges) since the vertices of C are alternatively from X and from Y. Thus C is an even cycle.

On the other hand, suppose $G = (V, E)$ has no odd cycles. Assume without loss of generality that G is connected as in Problem 2.9. Let $d(u, v)$ be the length of a path between u and v of minimum length. A u, v path of length $d(u, v)$ is called a shortest path between u and v. Let u be any vertex in G.

Define $X = \{x \in V\colon d(u, x) \text{ is even}\}$ and $Y = V - X$. We have to establish that whenever v and w are two vertices in X (or in Y), there is no edge joining v and w.

Case (i): Let v be any vertex in X other than u. There is a path of even length between u and v. If there is an edge between u and v, we will get an odd cycle. So u is not adjacent to any vertex in X.

Case (ii): Let v and w be two vertices in X other than u. Suppose there is an edge e joining v and w. Let P be a shortest u, v path of length $2m$, and let Q be a shortest u, w path of length $2n$. If these two shortest paths have no common vertex other than u, these two paths and the edge together will form an odd cycle. If the two paths have common vertices, let u' be that common vertex such that the subpath P' between u' and v and the subpath Q' between u' and w have no vertex in common. Since P and Q are shortest paths, the subpath of P between u and u' is a shortest (u, u') path. The subpath of Q between u and u' is also a shortest (u, u') path. Thus both these subpaths have equal number of edges. Let the length of the shortest path between u and u' be k. Then the length of the subpath of P between u' and v is $2m - k$, and the length of the subpath of Q between u' and w is $2n - k$. If there is an edge between v and w, we will have a cycle of length $(2m - k) + (2n - k) + 1$, which will be an odd cycle. So no two vertices in X are adjacent.

Case (iii): Suppose v and w are in Y. Then, as in (*ii*), the subpath from u' to v of length $(2m - 1) - k$, the subpath from u' to w of length $(2n - 1)$, and the edge joining v and w will form an odd cycle. Thus no two vertices in Y are adjacent.

Case (iv): Suppose u is the only vertex in X. The remaining vertices are all in Y. Every edge is from u to some vertex in Y.

This completes the proof.

2.11 Prove Theorem 2.4: For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Solution. The set of edges incident to a vertex of minimum degree is a disconnecting set, so $\lambda(G) \leq \delta(G)$. If $\lambda(G)$ is zero, the graph is either trivial or disconnected, implying that $\kappa(G)$ is also zero. If $\lambda(G) = 1$, the graph G has a bridge, implying that the graph has a cut vertex; therefore, $\kappa(G) = 1$. Let us assume that $\lambda(G) > 1$. If we delete $\lambda(G) - 1$ edges from the graph, we get a connected subgraph with a bridge joining two vertices v and w. For each deleted edge, we can choose a vertex incident to it other than v and w. Let W be the set of vertices thus chosen. Suppose the deletion of all the vertices in W results in a graph G'. If G' is not connected, it follows that $\kappa(G) < \lambda(G)$. If G' is connected, it has a vertex u whose deletion results in a trivial graph or a disconnected graph. Thus the set of vertices consisting of u and the chosen vertices [at most $\lambda(G) - 1$ in number] constitutes a separating set for G, implying that $\kappa(G) \leq \lambda(G)$.

2.12 Exhibit a graph for which the inequality established in Problem 2.11 is strict.

Solution. For graph G in Fig. 2-11, vertex 3 with degree 4 is a vertex of minimum degree. So $\delta(G) = 4$. Edges $\{3, 6\}$, $\{5, 6\}$, and $\{5, 8\}$ constitute a cut set. Thus $\lambda(G) = 3$. Vertices 3 and 5 together constitute a separating set of minimum cardinality. Hence $\kappa(G) = 2$.

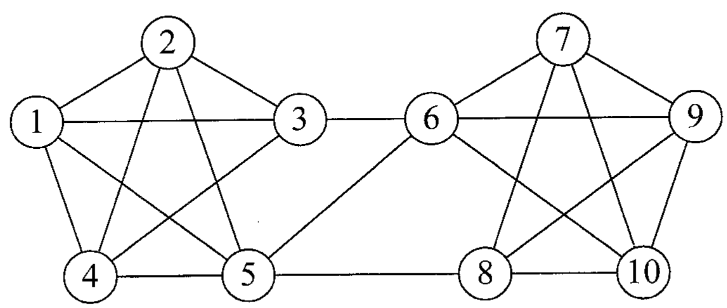


Fig. 2-11

2.13 A matrix is called a **totally unimodular (TU) matrix** if the determinant of every square submatrix is either -1 or 0 or 1. Show that each entry in a TU matrix is -1 or 0 or 1 but that the converse is not true.

Solution. Each entry of a TU matrix is the determinant of a 1×1 matrix; therefore, it is -1 or 0 or 1. On the other hand, the matrix with first row $[1 \quad -1]$ and second row $[1 \quad 1]$ is not a TU matrix since its determinant is 2.

2.14 Show that the matrix $A = [a_{ij}]$ in which each element is -1 or 0 or 1 is a TU matrix if it satisfies the following two conditions: (i) no column can have more than two nonzero elements, and (ii) it is possible to partition the set I of rows of the matrix into sets I_1 and I_2 such that if a_{ij} and a_{kj} are the two nonzero elements in column j, row i and row k belong to the same subset of the partition if and only if they are of opposite sign.

Solution. Let C be any $k \times k$ submatrix of A. The proof is by induction on k. If $k = 1$, the theorem is true. Suppose it holds for $(k - 1)$. Let C' be any $(k - 1) \times (k - 1)$ submatrix of A. By hypothesis, the determinant of C' is -1 or 0 or 1. There are three different possibilities:

1. C has a column in which every entry is 0. Then the determinant of C is 0.

2. C has a column with exactly one nonzero entry that could be -1 or 1. By expanding the determinant of C along this column, we find that that determinant of C is -1 or 0 or 1.
3. Every column of C has exactly two nonzero entries.

Suppose $E = \{r_1, r_2, \ldots, r_k\}$ is the set of rows of C. By hypothesis, the set of k rows is partitioned into two subsets I_1 and I_2. Without loss of generality, let us assume that I_1 is the set of the first p rows and I_2 is the set of the remaining $(k - p)$ rows. It is possible that $p = 0$.

By the way these two subsets are constructed, it is easy to see that $r_1 + r_2 + \cdots + r_p = r_{p+1} + \cdots + r_k$, showing that E is a linearly dependent set. Thus the determinant of C is 0 in this case.

2.15 Show that (*a*) the incidence matrix of a digraph and (*b*) the incidence matrix of a bipartite graph are both TU matrices.

Solution. A matrix in which each element is -1 or 0 or 1 is totally unimodular if in each column there is at most one $+1$ and at most one -1 as a consequence of the result established in Problem 2.14. So the incidence matrix of a digraph and the incidence matrix of a bipartite graph are totally unimodular.

2.16 Show that a graph is bipartite if and only if its incidence matrix is a totally unimodular matrix.

Solution. If G is bipartite, its incidence matrix is a TU matrix, as shown in Problem 2.15. Suppose the incidence matrix of $G = (V, E)$ is a TU matrix. If G is not bipartite, there is at least one odd cycle in G. Let $V = \{1, 2, 3, \ldots, n\}$, and assume that the first $(2k + 1)$ vertices constitute an odd cycle as $1 — 2 — 3 — \cdots — (2k + 1) — 1$.

Let A be the incidence matrix such that the first $(2k + 1)$ rows correspond to the first $(2k + 1)$ vertices and the first $(2k + 1)$ columns correspond to edges $\{1, 2\}, \{2, 3\}, \ldots, \{2k, 2k + 1\}$, and $\{2k + 1, 1\}$. Then the determinant of the submatrix formed by the first $(2k + 1)$ rows and the first $(2k + 1)$ columns is 2. (In fact, the determinant of the incidence matrix of an odd cycle is always -2 or 2.) This contradicts that A is a TU matrix.

2.17 The smallest number of colors needed to color the vertices of a graph such that each vertex gets a unique color and no two adjacent vertices get the same color is called the **chromatic number** (or **vertex chromatic number**) of the graph. Show that a graph is bipartite if and only if its chromatic number is two.

Solution. In the bipartite graph $G = (X, Y, E)$, assign the same color (say red) to each vertex in X. Then assign a unique color other than red (say blue) to each vertex in Y. Thus the chromatic number of G is 2. On the other hand, suppose the chromatic number of $G = (V, E)$ is two. Let X be the set of vertices such that each vertex in X has the same color. Let $Y = (V - X)$. Then every edge in G is between a vertex in X and a vertex in Y. So $G = (X, Y, E)$.

2.18 Show that the following are equivalent in a simple graph G: (*a*) G is bipartite, (*b*) G has no odd cycles, (*c*) the incidence matrix of G is a totally unimodular matrix, and (*d*) the chromatic number of G is two.

Solution. This is a consequence of Problems 2.10, 2.16, and 2.17.

2.19 Suppose each set in a family of subsets of a finite set is represented as a vertex. Two vertices representing two distinct subsets belonging to the family are joined by an edge if they have at least one element in common. The simple graph thus constructed is called the **intersection graph** of the family of subsets of the given set. Construct the intersection graph of the family of subsets of the set $X = \{1, 2, \ldots, 10\}$ with the family $\{A, B, C, D, E, F\}$, where $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 2, 3\}$, $D = \{4, 5, 6, 8, 9\}$, $E = \{5, 6, 7, 9\}$, and $F = \{4, 6, 10\}$.

Solution. The only nonempty intersections between pairs of distinct sets are $A \cap C$, $A \cap D$, $A \cap E$, $B \cap C$, $B \cap D$, $B \cap F$, $D \cap E$, $D \cap F$, and $E \cap F$. Thus we join A and C, A and D, A and E, B and C, B and D, B and F, D and E, D and F, and, finally, E and F by edges. The intersection graph thus constructed has six vertices and nine edges, as shown in Fig. 2-12.

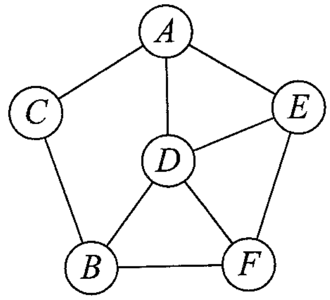


Fig. 2-12

2.20 Show that every simple graph is (isomorphic to) the intersection graph of a family of subsets of a finite set.

Solution. Let $G = (V, E)$, where V is the set $\{1, 2, \ldots, n\}$, be a simple graph. We construct a unique intersection graph corresponding to this graph as follows. For each vertex i in G, we define the set $X(i)$ as the union of the set $\{i\}$ and the set of all edges in G adjacent to i. Thus we have a family of n sets, each representing a vertex. It is easy to see that the intersection of $X(i)$ and $X(j)$ is nonempty if and only if there is an edge between i and j in G. The graph G is isomorphic to the intersection graph of the family $\{X(1), X(2), \ldots, X(n)\}$.

2.21 The **intersection number** $\omega(G)$ of a graph G is the minimum number of elements in a set X such that G is an intersection graph of a family of subsets of X. Show that the intersection number of a connected graph cannot exceed its size.

Solution. Suppose the connected graph is $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. Let $X(i)$ be the set of edges adjacent to vertex i. Then the union of the family $\{X(1), X(2), \ldots, X(n)\}$ is the set E. Thus G is the intersection graph of the family. So the intersection number cannot exceed the size of the graph.

2.22 Construct (*a*) a connected graph such that its intersection number is equal to its size and (*b*) a connected graph such that its intersection number is less than its size.

Solution.

(*a*) Consider the cyclic graph C_4 with edges e_i $(i = 1, 2, 3, 4)$. The intersection number of this graph is four since it is isomorphic to the intersection graph defined by the family $\{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\}$.

(*b*) Construct the graph G of size 5 by joining two nonadjacent vertices of the graph C_4 by an edge denoted by e_5. The intersection number of this graph of size 5 is four since it is isomorphic to the intersection graph defined by the family $\{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_1, e_4\}\}$.

2.23 Show that the intersection number of a connected graph with at least four vertices is equal to its size if and only if it has no triangles.

Solution. Consider any connected graph $G = (V, E)$ with n vertices and m edges, where n is at least four with no triangles. If $\omega(G) = k$, there exists a set X of cardinality k such that G is isomorphic to an intersection graph of a family of subsets of X. Specifically, there exists a family $\{X_1, X_2, \ldots, X_n\}$ of nonempty subsets of X such that (1) each vertex of G corresponds to a unique set in this family and (2) each edge of G corresponds to a pair of unique sets in the family with a common element that does not appear in any other subset in the family. So the cardinality of X should not be less than the number of edges of G. In other words, $k \geq m$. But $k \leq m$, as established in Problem 2.21. Thus $k = m$.

On the other hand, consider a connected graph $G = (V, E)$ with n vertices, m edges, and $\omega(G) = m$. We now prove that G has no triangle. Suppose G has a triangle. Let $G_1 = (V, E_1)$ be a maximal triangle-free spanning subgraph of G with m_1 edges. Then $\omega(G) = m_1$. Thus there exists a set X of cardinality m_1 such that G_1 is the intersection graph of a family of subsets X. Specifically, there exists a family $\{X_1, X_2, \ldots, X_n\}$ of subsets of X

such that vertex v_i of G_1 corresponds to the subset X_i for each i. Let e be any edge of G that is not in G_1, and let $G_2 = (V, E_2)$ be the spanning subgraph obtained by adding e to the graph G_1. This addition creates a unique triangle in G_2. Assume that this triangle is formed by vertices v_1, v_2, and v_3, and assume that edge e is the edge that joins v_1 and v_2. Notice that the intersection of X_1 and X_2 is not empty since $\{v_1, v_2\}$ is an edge in the graph G_1. Let $t \in (X_1 \cap X_2)$.

Two mutually exclusive cases need to be examined. (*i*) In G_1, the degree of $v_2 = 2$. In this case, we replace X_2 by $\{t\}$ and X_3 by $X_3 \cup \{t\}$ in the family $\{X_1, X_2, X_3, \ldots, X_n\}$ of subsets of X. (*ii*) In G_1, the degree of $v_2 > 2$. In this case, we replace X_3 by $X_3 \cup \{t\}$ in the family. In either case, we have a family of subsets of X defining the intersection graph of G_2. If G is isomorphic to G_2, graph G will have a triangle that contradicts the hypothesis. So assume that this is not the case. In that case, let $m - (m_1 + 1) = m_0$, where $m_0 > 0$. So the intersection number of G is $m_1 + m_0 = m - 1 < m$, which is a contradiction.

2.24 Find the intersection number of K_n, where $n > 1$.

Solution. The complete graph with two vertices is the intersection graph of the family $\{\{1\}, \{1, 2\}\}$, so its intersection number is two. The complete graph with three vertices is the intersection graph of the family $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, so its intersection number is three. The intersection number of any complete graph with n vertices (where $n > 3$) is less than its size as established in Problem 2.23. It is easy to see that K_4 is the intersection graph of the family $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, so its intersection number is three. To obtain a family of subsets for K_5, we introduce a new element 4 and adjoin it to each of the sets in the family for K_4. The family consisting of the four enlarged sets and the singleton set $\{4\}$ defines the intersection graph corresponding to K_5. Thus the intersection number for K_5 is four. By a simple inductive argument, it follows that the intersection number of a complete graph with n vertices (where $n > 3$) is $n - 1$.

2.25 The intersection graph of a finite family of open intervals of the real line is called an **interval graph.** Show that the cyclic graph with n vertices is (isomorphic to) an interval graph only when $n = 3$.

Solution.

(*i*) Suppose the vertices of a cyclic graph with three vertices are A, B, and C. Give an assignment of open intervals to these three vertices as $I(A) = (a, a')$, $I(B) = (b, b')$, and $I(C) = (c, c')$, where $a < b < c < a' < b' < c'$. The vertices of C_3 correspond to these three open intervals.

(*ii*) We must show that such an interval assignment is not possible for a cyclic graph when $n > 3$. It is enough if we show this when $n = 4$. Suppose the vertices are A, B, C, and D such that there is no edge between A and C and no edge between B and D. So any interval assignment $\{I(A), I(B), I(C), \text{and } I(D)\}$ should satisfy the requirement that the intervals $I(B)$ and $I(D)$ are disjoint and the intervals $I(A) \cap I(B)$ and $I(A) \cap I(D)$ are nonempty. Once these assignments are made for A, B, and D, we have to make an assignment for $I(C)$ such that $I(C) \cap I(A)$ is empty and, at the same time, both sets $I(C) \cap I(B)$ and $I(C) \cap I(D)$ are nonempty. It is simply impossible to make an assignment $I(C)$ without violating the earlier assignments.

Thus every interval graph is an intersection graph, but an intersection graph need not be an interval graph in general.

2.26 Show that (*a*) any induced subgraph of an interval graph is an interval graph, and (*b*) an arbitrary subgraph of an interval graph need not be an interval graph.

Solution.

(*a*) Let $H = (W, F)$ be an induced subgraph of the interval graph $G = (V, E)$. The interval assignments in G for the vertices in W will serve as the interval assignments for the vertices in W for the graph H as well.

(*b*) The graph G obtained by joining any two nonadjacent vertices of the cyclic graph C_4 is an interval graph in which the subgraph C_4 is not an interval graph.

2.27 A graph G is called a **chordal graph** if in every cycle C in G there is an edge (belonging to G) joining two nonadjacent (in C) vertices. Show that every interval graph is a chordal graph.

Solution. Suppose an interval graph G is not a chordal graph. Thus it has a cycle C with four or more vertices such that there is no edge in G between any pair of nonadjacent (in C) vertices. In other words, G has a cyclic subgraph C_n as an induced subgraph, where n is more than three, which is not an interval graph; this contradicts that an induced subgraph of an interval graph is an interval graph.

2.28 Show that a graph G is chordal if and only if C_n is not an induced subgraph of G for any $n > 3$.

Solution. This is a consequence of the definition and that C_n is not chordal when $n > 3$.

2.29 Give an example of a chordal graph that is not an interval graph.

Solution. Each of the two graphs in Fig. 2-13 is a chordal graph but not an interval graph.

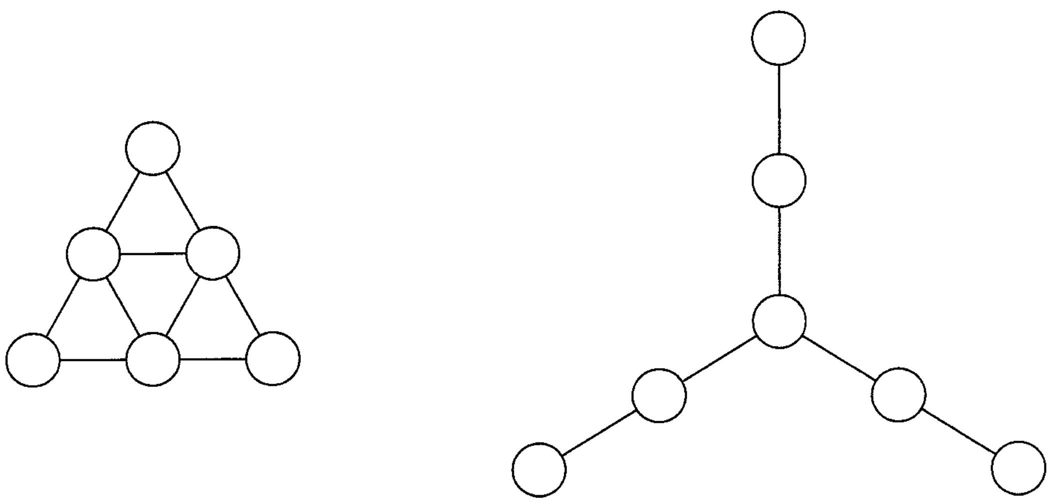


Fig. 2-13

2.30 The graph $G = (V, E)$ is called an **indifference graph** if for every positive number δ, there exists a mapping f from V to the set of real numbers such that $|f(v) - f(w)| < \delta$ if and only if v and w are adjacent. Show that the graph in Fig. 2-14 is an indifference graph.

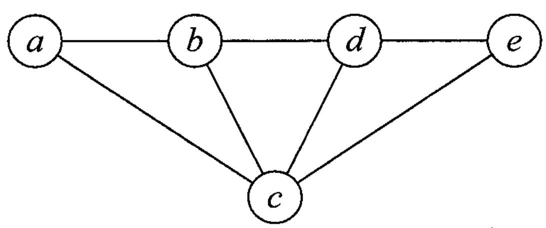


Fig. 2-14

Solution. First notice that without loss of generality, we may take $\delta = 1$. By assigning the values $f(a) = 0.2, f(b) = 0.7, f(c) = 0.9, f(d) = 1.3$, and $f(e) = 1.8$, we see that the graph is indeed an indifference graph.

2.31 Show that every indifference graph is an interval graph.

Solution. Let $G = (V, E)$ be an indifference graph that assigns the value $f(v)$ for each vertex v in the graph. Corresponding to each vertex v, we define the open interval $I(v) = (f(v) - \delta/2, f(v) + \delta/2)$. Then G is an interval graph.

2.32 By constructing an example, show that an interval graph need not be an indifference graph.

Solution. In graph $K_{1,3}$ shown in Fig. 2-15, let $I(a) = (2, 6)$, $I(b) = (8, 10)$, $I(c) = (13, 17)$, and $I(d) =$ 4, 15). Thus G is an interval graph. Suppose G is an indifference graph with weight function f. Assume without loss of generality that $f(a) < f(b) < f(c)$. If $f(c) - f(a) < 2\delta$, either $f(b) - f(c) < \delta$ or $f(a) - f(b) < \delta$. Thus $f(c) - f(a) \geq 2\delta$. Then $f(d)$ cannot be within δ units of both $f(a)$ and $f(c)$. Thus $K_{1,3}$ is an interval graph but not an indifference graph.

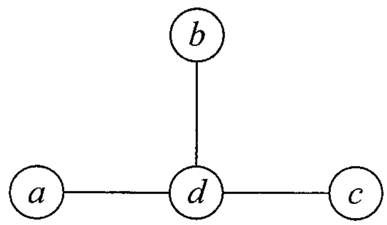


Fig. 2-15

2.33 Show that every indifference graph is an interval graph, every interval graph is a chordal graph, and every chordal graph is an intersection graph.

Solution. This follows from Problems 2.20, 2.27, and 2.31.

2.34 An interval graph is called a **unit interval graph** if the length of the open interval that corresponds to a vertex is the same for every vertex. Show that a graph is an indifference graph if and only if it is a unit interval graph.

Solution. Suppose $G = (V, E)$ is an indifference graph with a weight function f defined on V. Let $\delta = 1$. For each vertex v of G, define the open interval $I(v) = (f(v) - \frac{1}{2}f(v) + \frac{1}{2})$. Now v and w are adjacent in the indifference graph G if and only if $|f(v) - f(w)| < 1$. This is true if and only if the intervals $I(v)$ and $I(w)$ have a nonempty intersection. Thus G is a unit interval graph. Conversely, suppose G is a unit interval graph with $\delta =$ 1. If vertex v corresponds to the unit interval (a, b), define $f(v) = (a + b)/2$. Thus G is an indifference graph.

2.35 (*Ghouila–Houri Theorem*) A digraph D is a **transitive digraph** if there is an arc from u to v whenever there is an arc from u to w and there is an arc from w to v for any set of three distinct vertices u, v, and w in D. A simple graph G is a **transitively orientable graph** (or a **comparability graph**) if G has an orientation D that is a transitive digraph. Show that the complement of an interval graph is a transitively orientable graph.

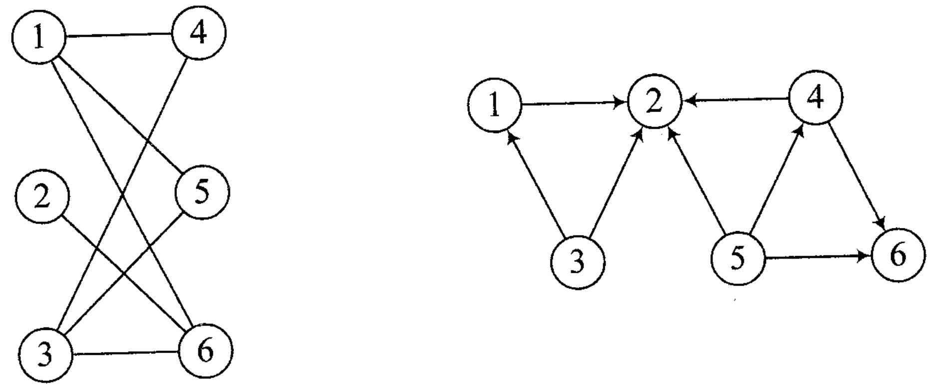
Solution. Let $\{I(v): v \in V\}$ be the interval assignment for the vertices in the interval graph $G = (V, E)$. Suppose v and w are two nonadjacent vertices in G. Then $I(v)$ and $I(w)$ are disjoint. So either $I(v)$ is completely on the left of $I(w)$ or completely on the right. Let us write $I(v) < I(w)$ if $I(v)$ is on the left of $I(w)$. Since v and w are not adjacent in G, there is an edge between v and w in the complement of G. If $I(v) < I(w)$, let the edge between v and w become an arc from v to w. Otherwise, the edge is an arc from w to v. Thus we have an orientation of the complement of G. If $I(u) < I(v)$ and $I(v) < I(w)$, $I(u) < I(w)$. Thus the digraph thus constructed is transitively orientable.

2.36 Obtain a transitive orientation for the complement of the interval graph $K_{1,3}$.

Solution. The complement of $K_{1,3}$ is the graph consisting of an isolated vertex and a triangle, the edges of which can be oriented such that it becomes transitively orientable.

2.37 Show that if the complement of G is transitively orientable, it is not necessary that G is an interval graph by constructing an example.

Solution. The bipartite graph G in Fig. 2-16(a) has a cycle 1 — 5 — 3 — 4 — 1 with no edge joining 1 and 3 or 4 and 5. So G is not chordal; therefore, it is not an interval graph. But its complement is transitively orientable, as shown in Fig. 2-16(b).



(a) (b)

Fig. 2-16

2.38 Suppose $G = (V, E)$ is a simple graph with at least one edge. The **line graph $L(G)$** (also known as the **interchange graph, adjoint graph, derived graph,** or **edge graph**) of G is the graph (W, F), where there is a one-to-one correspondence ϕ from E to W such that there is an edge between $\phi(e)$ and $\phi(e')$ if and only if the edges e and e' have a vertex in common. Construct the line graph of the graph of K_4.

Solution. The line graph of K_4 is as shown in Fig. 2-17.

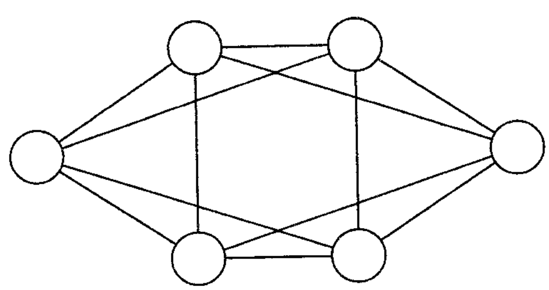
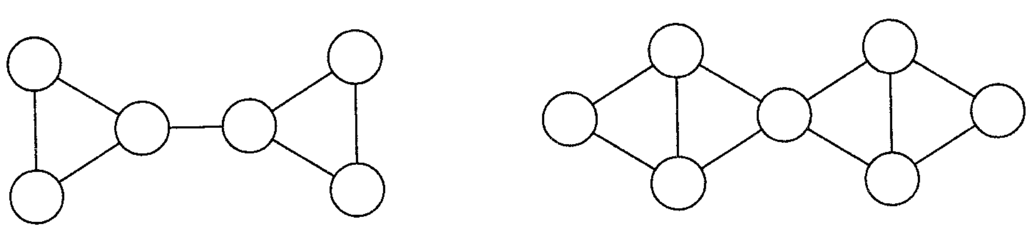


Fig. 2-17

2.39 Find the line graph of the graph shown in Fig. 2-18(a).

Solution. The line graph is shown in Fig. 2-18(b).



(a) (b)

Fig. 2-18

2.40 Show that the line graph of $G = (V, E)$ is the intersection graph of a family of subsets of the product set $V \times V$.

Solution. Each edge in G defines a two-element set in the product set. Then $L(G)$ is the intersection graph of the set S of all two-element sets that correspond to the edges of G.

2.41 If v is the vertex in the line graph $L(G)$ that corresponds to the edge joining vertex x and vertex y in G, find the degree of v in $L(G)$.

Solution. The degree of v in $L(G)$ = (degree of x in G + degree of y in G) $- 2$.

2.42 Find the order and size of $L(K_n)$.

Solution. The order of $L(K_n)$ is $n(n-1)/2$, which is the size of K_n. An edge joining two vertices in K_n is adjacent to $2(n-2)$ edges. So the degree of each vertex in $L(K_n)$ is $2(n-2)$. Hence, the sum of the degrees of all the vertices in $L(K_n)$ is $n(n-1)(n-2)$, which is twice the size of $L(K_n)$.

2.43 Find the order and size of the line graph $L(G)$ of a simple graph G with n vertices and m edges.

Solution. By definition, the order of $L(G)$ is m. In counting the number of edges in $L(G)$, we have to examine only those vertices in G with degree more than 1. Each edge of $L(G)$ corresponds to a pair of vertices. Let $V = \{1, 2, \ldots, n\}$ be the vertex set of G, and let d_i be the degree of vertex i. If $d_i > 1$, any two of the d_i edges that are incident at vertex i can be chosen in $C(d_i, 2)$ ways. Two edges e and e' of G that are incident at vertex i correspond to two adjacent vertices in $L(G)$ joined by an edge. Let the degrees of the vertices of G be $d_1, d_2, \ldots, d_n$. Thus the total number of edges in $L(G)$ is

$$r = C(d_1, 2) + C(d_2, 2) + \cdots + C(d_n, 2)$$

Then

$$r = \frac{d_1(d_1-1)}{2} + \cdots + \frac{d_n(d_n-1)}{2} =$$

$$\frac{\{[(d_1)^2 + \cdots + (d_n)^2] - [d_1 + \cdots + d_n)]\}}{2} = \frac{\{[(d_1)^2 + \cdots + (d_n)^2] - 2m\}}{2}$$

2.44 Suppose G is a simple graph with five vertices with degrees 1, 2, 3, 3, and 3. Find the number of vertices and edges in $L(G)$.

Solution. The sum of degrees is 12, so G has six edges. Thus $L(G)$ has six vertices. The sum of the squares of the degrees is 32. Hence, the number of edges in $L(G)$ is $(32 - 12)/2 = 10$.

2.45 Find the line graph of (a) a simple path with k edges, where $k > 1$, and (b) the cyclic graph C_n with n edges, where $n > 2$.

Solution.

(a) The line graph of a path with k edges is a path with $(k-1)$ edges.

(b) The line graph of C_n is C_n.

2.46 Show that there is no graph G such that $L(G) = K_{1,3}$.

Solution. Since $K_{1,3}$ has four vertices, if there is a graph G it should have four edges. Suppose these four edges are a, b, c, and d. Assume that the left vertex of $K_{1,3}$ corresponds to the side a and that the other three correspond to the right vertices. There is no graph G with four sides a, b, c, and d such that a has a vertex in common with the three edges and at the same time no two of these three edges have a vertex in common.

2.47 Construct an example to show that if $L(G)$ and $L(H)$ are isomorphic, it is not necessary that G and H are isomorphic.

Solution. It is easy to see that $L(K_3) = K_3 = L(K_{1,3})$.

2.48 Show that (*a*) a graph G is isomorphic to its line graph if and only if the degree of each vertex is 2, (*b*) a connected graph is isomorphic to its line graph if and only if it is a cyclic graph, and (*c*) the line graph of a connected graph G is (isomorphic to) K_n if and only if G is (isomorphic to) $K_{1,n}$ when $n > 3$.

Solution.

(*a*) If the degree of each vertex of G is 2, the degree of each vertex of $L(G)$ is also 2, and G and $L(G)$ both have the same number of vertices and the same number edges. So G and $L(G)$ are isomorphic. Conversely, if both G and $L(G)$ are isomorphic, they both have the same number of vertices and same number of edges, and the degree of each vertex is 2.

(*b*) This is a special case of (*a*).

(*c*) If $n > 3$, $L(K_{1,n}) = K_n$. Conversely, if $L(G) = K_n$, G has n edges, and all these edges have exactly one vertex in common since the degree of each vertex in K_n is $(n - 1)$.

2.49 Let X be the set of all the vertices and edges of a graph G with n vertices and m edges. Construct a graph $T(G) = (X, F)$ in which two elements x and y are joined by an edge if (*a*) x and y are adjacent vertices in G, (*b*) x is a vertex and y is an edge such that one of the vertices of y is x, or (*c*) if both x and y are edges, they have a vertex in common. The graph thus constructed is called the **total graph** of the graph. Construct the total graph of the complete graph with three vertices.

Solution. The total graph is a four-regular graph with six vertices.

TREES AND SPANNING TREES

2.50 Show that a graph is a tree if and only if there is a unique path between every pair of vertices in the graph.

Solution. Suppose the graph G is a tree. Let v and w be any two vertices in G. Since G is connected, there is a path P between v and w. If Q is another path between these two vertices, let $e = \{v_i, v_{i+1}\}$ be the first edge in P that is not in Q as we go from v to w along P. Let W and W' be the set of intermediate vertices between v_i and w in P and Q, respectively. If W and W' have no vertices in common, we have a cycle in the graph that is acyclic by assumption. If the intersection of W and W' is not empty, let u be the first common vertex as we go from v_i to w either along P or along Q. In this case, we also locate a cycle in the graph. Hence, there is a unique path between every pair of vertices in the tree. Conversely, let G be a graph in which there is a unique path between every pair of vertices. Then G is connected. Suppose G is not a tree. Then there is a cycle C in G. Obviously, there are two paths between any pair of vertices in C, which contradicts the hypothesis.

2.51 Show that a graph is a tree if and only if it is connected and every edge in it is a bridge.

Solution. If G is a tree, it is connected by definition. By Problem 2.50, there is a unique path between every pair of vertices in G. In particular, edge e joining two vertices v and w is the path P: v, e, w. If e is deleted, there is no path between v and w. Thus every edge in a tree is a bridge.

To establish the converse, suppose the connected graph in which every edge is a bridge is not a tree. Let G' be the subgraph of G obtained from G after deleting edge $e = \{v, w\}$ belonging to some cycle C in G. This graph G' is not connected since e is a bridge in G. Let p and q be any two arbitrary vertices. Since G is connected, there is a path P between p and q in G. If e is not an edge in this path P, P is a path in G' between p and q. Suppose e is an edge in P. Let P_1 be the subpath of P between p and v, and let P_2 be the subpath of P between w and q.

Moreover, let P' be the unique path between v and w in the cycle that does not contain e. Suppose Q is the union of these three paths. Then Q is a path in G' between p and q. Thus there is a path between every pair of vertices in G'. But G' is not a connected graph. This is a contradiction.

2.52 Show that a graph with n vertices is a tree if and only if it is connected and has $(n - 1)$ edges.

Solution. Suppose G is a tree with n vertices. It is a connected graph. We prove that it has $(n - 1)$ edges by induction on n. If $n = 1$, it is certainly true. Assume that it is true for $n = 1, 2, \ldots, (n - 1)$. Since every edge is a bridge (as established in Problem 2.51), the subgraph G' obtained from G after deleting an edge will have two components G_1 and G_2 with n_1 and n_2 vertices, respectively, where $n_1 + n_2 = n$. By the induction hypothesis, the number of edges in both the components together is $(n_1 - 1) + (n_2 - 1) = (n - 2)$. Thus the number of edges in G will be $(n - 2) + 1 = (n - 1)$.

Suppose the connected graph G with n vertices and $(n - 1)$ edges is not a tree. Then it has an edge e that is not a bridge. If e is deleted, the resulting subgraph is still a connected graph with n edges and $(n - 2)$ edges. We continue this process of locating edges that are not bridges and deleting them until we get a connected subgraph G' with n vertices and $(n - k)$ edges (where $k > 1$) in which every edge is a bridge. But G' is a tree, so it should have $(n - 1)$ edges. Thus $n - 1 = n - k$, where $k > 1$. This is a contradiction.

2.53 Show that a graph with n vertices is a tree if and only if it is acyclic and has $(n - 1)$ edges.

Solution. If G is a tree with n edges, it is acyclic by definition and has $(n - 1)$ edges, as established in Problem 2.52. On the other hand, consider an acyclic graph G with n vertices and $(n - 1)$ edges. Suppose G is not connected. Let the components of G be $G_i (i = 1, 2, \ldots, k)$ such that G_i has n_i vertices, where $n_1 + n_2 + \cdots + n_k = n$. Notice that each component G_i is a tree with $n_i - 1$ edges. Thus the total number of edges in G is $n - k$, where $k > 1$. This contradiction establishes that G is connected. Thus G is a tree.

2.54 Show that a graph G is a tree if and only if it is acyclic and whenever any arbitrary two vertices in G are joined by an edge, the resulting enlarged graph G' has exactly one cycle.

Solution. If G is a tree, it is connected and acyclic. Let u and v be any two nonadjacent vertices in G. There is a unique path between u and v. If we join u and v by an edge, this edge and path P create a unique cycle in the enlarged graph G'. On the other hand, suppose G is an acyclic graph in which u and v are two any arbitrary nonadjacent vertices such that the linking of the two by a new edge creates a unique cycle in G'. This implies that there is a path in G between u and v. So G is connected and hence is a tree.

2.55 Show that a graph G is a tree if and only if it is connected and whenever any two arbitrary vertices in G are joined by an edge, the resulting enlarged graph G' has exactly one cycle.

Solution. If G is a tree, it is connected and acyclic. If two nonadjacent vertices are joined by an edge, the unique path G between the two vertices and the edge together form a unique cycle. On the other hand, suppose G is connected. There cannot be a cycle in G since the enlarged graph G' obtained by joining two nonadjacent vertices has a unique cycle. So G is a tree.

2.56 Prove Theorem 2.5.

Solution. The proof follows from Problems 2.50 through 2.55.

2.57 Prove Theorem 2.7: A graph is connected if and only if it has a spanning tree.

Solution. Let G be a connected graph. Delete edges from G that are not bridges until we get a connected subgraph H in which each edge is a bridge. Then H is a spanning tree. On the other hand, if there is a spanning tree in G, there is a path between every pair of vertices in G: thus G is connected.

2.58 Prove Theorem 2.8: Let G be a simple graph with n vertices. If a spanning subgraph H satisfies any two of the following three properties, it will satisfy the third property also. (i) H is connected. (ii) H has $(n - 1)$ edges. (iii) H is acyclic.

Solution. The proof follows from Problems 2.54 and 2.55.

2.59 Show that if a graph is disconnected, its complement is connected.

Solution. If a graph G is not connected, it will have at least two components. Suppose u and v are two vertices belonging to two different components of G. Then these two vertices are adjacent in the complement of the graph. In other words, G and its complement cannot both be disconnected graphs. So whenever G is a disconnected graph, its complement is necessarily a connected graph.

2.60 A vertex of degree 1 in a graph is called a **terminal vertex** (or **pendant vertex** or **end-vertex**). Show that every tree of order two or more has at least two terminal vertices.

Solution. Suppose the degrees of the n vertices of a tree are d_i, where $i = 1, 2, \ldots, n$. Then $d_1 + d_2 + \cdots + d_n = 2n - 2$. If each degree is more than 1, the sum of the n degrees is at least $2n$. So there is at least one vertex (say vertex 1) with degree 1. Then $d_2 + d_3 + \cdots + d_n = 2n - 1$. At least one of these $(n - 1)$ positive numbers is necessarily 1. So there is one more vertex of degree 1. Thus at least two of the degrees must be 1.

2.61 Show that the vector d $= [d_1 \quad d_2 \quad \ldots \quad d_n]$ of positive integers, where $d_1 \leq d_2 \leq \ldots \leq d_n$ is the degree vector of a tree with n vertices if and only if $d_1 + d_2 + \cdots + d_n = 2(n - 1)$.

Solution. The necessity is obvious. We prove the sufficiency by induction on n. The property holds for $n = 1$ and $n = 2$. Assume that the property holds for $(n - 1)$ integers, where $n \geq 3$. Let $0 < d_1 \leq d_2 \leq \cdots \leq d_n$ and $d_1 + d_2 + \cdots + d_n = 2(n - 1)$. At least one of these numbers is 1. So $d_1 = 1$. Also $d_n > 1$. Let $d' = d_n - 1$. Then $d_2 + \cdots + d_{n-1} + d' = 2(n - 2)$. So by the induction hypothesis, there exists a tree T with $(n - 1)$ vertices and degrees $d_2, d_3, \ldots, d_{n-1}$ and d'. Construct a new vertex x and join that to the vertex of degree d'. Now we have a tree with n vertices with degrees $1, d_2, \ldots, d_n$. Thus the property hold for n. (Notice that an arbitrary vector satisfying the property stipulated in the problem could also be the degree vector of a graph that is not a tree.)

2.62 Prove Theorem 2.6: The center of a tree is either a singleton set consisting of a unique vertex or a set consisting of two adjacent vertices.

Solution. If a tree has two vertices, the center is the set of those two vertices. If there are three vertices in a tree, the center is the set consisting of the nonterminal vertex. A tree with four vertices is either $K_{1,3}$ (with three terminal vertices) or a path with two terminal vertices. In the former case, the cardinality of the center is 1; in the latter case, the center is the set of two adjacent nonterminal vertices. More generally, let T be a tree with five or more vertices, and let T' be the tree obtained from T by deleting all terminal vertices of T simultaneously. Observe that the eccentricity of any vertex in T' is one less than the eccentricity of that vertex in T. Thus the center of T is equal to the center of T'. If the process of deleting terminal vertices is carried out successively, we finally have a tree with four or fewer vertices.

2.63 A path P between two distinct vertices in a connected graph G is a **diametral** path if there is no other path in G whose length is more than the length of P. Show that (a) every diametral path in a tree will pass through its central vertices, and (b) the center of a tree can be located once a diametral path in the tree is discerned.

Solution. Let t be the length of any diametral path in a tree, and let P be a fixed diametral path joining the vertices v and w.

(a) If t is even, there exists a unique vertex c in P that is equidistant from either v or w. In this case, c is a central vertex. Suppose Q is another diametral path. Since the graph is connected, the two diametral paths should have a vertex in common. If c is not a common vertex, it is possible to obtain path whose length is more than t. So if the length of a diametral path is even, there exists a unique central vertex on that path through which every diametral path passes.

(b) If t is odd, there exist two vertices c' and c'' in P such that the number of edges in the path between v and c' is equal to the number of edges between w and c''. In this case, both c' and c'' are central vertices. Suppose Q is another diametral path. Then both P and Q share the edge joining c' and c'' as a common edge. Thus once a diametral path in a tree is located, it is easy to find the center of the tree.

2.64 A tree with exactly one vertex v of degree 2 in which the degree of every nonterminal vertex (other than v) is 3 is called a **binary tree,** and the **root** of the binary tree is the unique vertex of degree 2. Show that the number of vertices in a binary tree is odd.

Solution. Every vertex other than the root is an odd vertex. The number of odd vertices is even. If we now include the root also, the total number of vertices is odd.

2.65 Show that the number of terminal vertices in a binary tree with n vertices is $(n + 1)/2$.

Solution. Suppose there are k terminal vertices. Then the sum of the degrees of the n vertices is $k + 2 + 3(n - k - 1)$, which is equal to $2(n - 1)$ since the graph is a tree. Thus $k = (n + 1)/2$.

2.66 Show that if T is a tree with n vertices and G is a graph with $\delta(G) \geq (n - 1)$, T is isomorphic to a subgraph of G.

Solution. The proof is by induction on n. This is true when the tree has two vertices. The induction hypothesis is that if T' is any tree with $(n - 1)$ vertices and G' is any graph with $\delta(G') \geq (n - 2)$, then T' is isomorphic to a subgraph of G'. Let T be any tree with n vertices, and let G be any graph with $\delta(G) \geq (n - 1)$. Let v be any terminal vertex in T, and let u be the vertex adjacent to v in T. Then $T - v$ is a tree with $(n - 1)$ vertices. Moreover, $\delta(G) \geq (n - 1) > (n - 2)$. So by the induction hypothesis, the tree $T - v$ is isomorphic to a subgraph of G. Let u' be the vertex in G that corresponds (for this isomorphism) to vertex u. Then $\delta(u') \geq (n - 1)$ in G. The graph $T - v$ has only $n - 2$ vertices in addition to vertex u. So there should be a vertex w in G that is adjacent to u' such that w does not correspond to any vertex in $T - v$. By identifying v with vertex w, we see that T is isomorphic to a subgraph of G. Thus the theorem is true for n as well.

2.67 Show that a tree with n vertices is isomorphic to a subgraph of the complement of the cyclic graph with $(n + 2)$ vertices.

Solution. The complement of the cyclic graph with $(n + 2)$ vertices is an r-regular graph G, where $r = n > n - 1$. So by Problem 2.66, any tree with n vertices is isomorphic to a subgraph of G.

2.68 A graph is said to be a **unicyclic graph** if it has exactly one cyclic subgraph. Show that if any two of the following conditions are satisfied in a graph with n vertices and m vertices, the third condition also is satisfied: (a) G is connected, (b) G is unicyclic, and (c) $n = m$.

Solution.

(a) Suppose G is connected and unicyclic. Let the vertices in the cycle be $v_1, v_2, \ldots, v_k$, and let T_i be a tree with root at v_i containing $n_i - 1$ vertices, excluding the root and the two roots adjacent to the root in the cycle. Then the total number of vertices in the graph is $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) + k$, which is equal to the total number of edges in G.

(b) Suppose G is connected and $n = m$. Since $m > (n - 1)$, there is at least one cycle. Suppose there is more than one cycle. Then $m > n$.

(c) Suppose G is unicyclic and $m = n$. If G is not connected, $m < n$.

2.69 Show that each labeled tree with n vertices corresponds to a unique vector $s = [s_1 \quad s_2 \quad \cdots \quad s_{n-2}]$, where $s_i \in N = \{1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, (n-2)$.

Solution. Let T be any labeled tree, and let W be the set of terminal vertices in T. Arrange the vertices in W such that their labels are in increasing order. If w_1 is the first element in W, find the label s_1 of the unique vertex adjacent to w_1. Then delete w_1 from T to obtain a tree T'. Let W' be the set of all terminal vertices in T'. Arrange the vertices in the set W' such that their labels are in increasing order. If w_1' is the first element in W', find the label s_2 of the unique vertex adjacent to w_1' in T'. The operation is repeated until s_{n-3} has been defined, leaving behind a tree with exactly two vertices with labels p and q, where $p < q$. We take $s_{n-2} = p$. Thus each spanning tree defines a unique vector with $(n-2)$ components.

2.70 Find the unique vector corresponding to the labeled tree shown in Fig. 2-19.

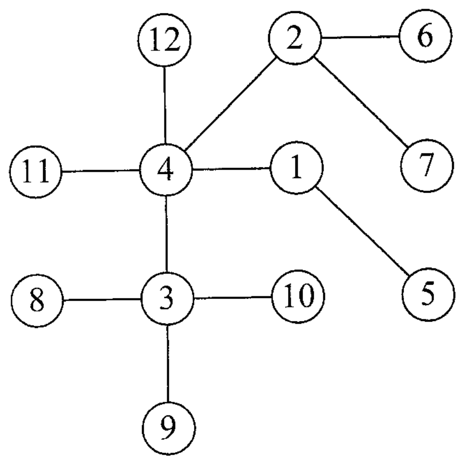


Fig. 2-19

Solution. Since there are 12 vertices in T, we are looking for a vector s with 10 components, each component being an integer between 1 and 12. Here $W = \{5, 6, 7, 8, 9, 10, 11, 12\}$ is the set of all terminal vertices in T, with labels arranged in increasing order. The vertex adjacent to 5 is 1, so $s_1 = 5$. Deleting vertex 5 from T, we get the subtree (the current tree, again denoted by T) in which the set of terminal vertices (the current set, again denoted by W) is $W = \{1, 5, 6, 7, 8, 9, 10, 11, 12\}$. The vertex adjacent to 1 is 4, so $s_2 = 4$.

Deleting 1 from the current tree, we get $W = \{6, 7, 8, 9, 10, 11, 12\}$ and $s_3 = 2$. In the next iteration, $W = \{7, 8, 9, 10, 11, 12\}$ and $s_4 = 2$. In the next iteration, $W = \{2, 8, 9, 10, 11, 12\}$ and $s_5 = 4$. In the next iteration, $W = \{8, 9, 10, 11, 12\}$ and $s_6 = 3$. In the next iteration, $W = \{9, 10, 11, 12\}$ and $s_7 = 3$. In the next iteration, $W = \{10, 11, 12\}$ and $s_8 = 3$. In the next iteration, $W = \{11, 12\}$ and $s_9 = 4$. At the final stage, we have the tree consisting of two vertices 4 and 12, so $s_{10} = 4$. Thus $s = [1 \quad 4 \quad 2 \quad 2 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4]$ is the unique vector defined by the given labeled tree.

2.71 Show that every vector s with $(n-2)$ components, where each component is an element of $N = \{1, 2, \ldots, n\}$, corresponds to a unique labeled tree with n vertices.

Solution. Observe that in Problem 2.69, when we construct the vector s for the given labeled tree, the vertex i (with degree d_i) of T occurs $d_i - 1$ times in s. In particular, terminal vertices do not appear in s. We exploit this idea to prove the reverse implication.

Let v_1 = the first element in N that is not in s, v_2 = the first element in $N - \{v_1\}$ that is not in the subvector $s - s_1$ obtained by deleting s_1 from s, v_3 = the first element in $N - \{v_1, v_2\}$ that is not in the subvector $s - s_1 - s_2$ obtained by deleting s_2 from $s - s_1$, and so on. This process is repeated until we get the set $\{v_1, v_2, \ldots, v_{n-2}\}$. The two remaining vertices are denoted by x and y. Now join s_i and v_i for each i. Also join x and y. The graph thus defined with n vertices has $n - 1$ edges and acyclic-giving a spanning tree that is unique.

2.72 Obtain the unique labeled tree corresponding to the vector $s = [1 \quad 4 \quad 2 \quad 2 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4]$.

Solution. Since there are 10 components in s, we are looking for a labeled tree with 12 vertices. So $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Since the elements in s are 1, 2, 3, and 4, the first element in N that is not in s is 5. So $v_1 = 5$, and we join s_1 and v_1 by an edge. Thus the first edge in T is obtained by joining 1 and 5.

At this stage, $N - \{v_1\} = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12\}$, and the elements in the subvector $s - s_1 = [4 \quad 2 \quad 2 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4]$ are 2, 3, and 4. The first element in $N - \{v_1\}$ that does not appear in the subvector is 1. So $v_2 = 1$. We now join 4 and 1.

At the next iteration, $N - \{v_1, v_2\} = \{2, 3, 4, 6, 7, 8, 9, 10, 11, 12\}$, and the corresponding subvector $s - s_1 - s_2$ is $[2 \quad 2 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4]$. Thus $v_3 = 6$. Join 2 and 6.

Continuing this process, we get $v_4 = 7$, $v_5 = 2$, $v_6 = 8$, $v_7 = 9$, $v_8 = 10$, $v_9 = 3$, and $v_{10} = 11$. We then join each of these vertices to the corresponding components of the subvector $[2 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4]$, which is obtained by deleting the first 3 components of s.

At the end, we have the set $N - \{5, 1, 6, 7, 2, 8, 9, 10, 3, 11\}$ consisting of exactly two vertices, 4 and 12. At this stage we join 4 and 12. The unique labeled tree is the tree in Fig. 2-19, as expected.

2.73 Prove Theorem 2.9 (Cayley's theorem): The number of distinct labeled trees with n vertices is n^{n-2}, which is also equal to the number of spanning trees in K_n. (*a*) Show that the number of distinct labeled trees with n vertices is n^{n-2}. (*b*) Show that the number of spanning trees in K_n is also n^{n-2}.

Solution.

(*a*) A vector $s = [s_1 \quad s_2 \quad \cdots \quad s_{n-2}]$ of $(n - 2)$ components from the set $N = \{1, 2, \ldots, n\}$ can be formed in n^{n-2} ways. From Problems 2.69 and 2.71, we see that there is a bijection between the set of all distinct labeled trees with n vertices and the set of these vectors.

(*b*) Every labeled tree with n vertices corresponds to a unique spanning tree in K_n. On the other hand, every spanning tree in K_n is a uniquely labeled tree. Thus the total number of spanning trees in K_n is n^{n-2}.

2.74 A directed tree T such that there is a unique vertex v of indegree zero and that the indegree of every other vertex is 1 is called an **arborescence** rooted at v. Find the number of distinct labeled arborescences with n vertices.

Solution. If T is a tree with n vertices, we can choose a root v to construct an arborescence in n ways. Once a root v is selected, a unique arborescence rooted at v is obtained by orienting each edge so that any vertex w can be reached from v by a unique directed path. By Cayley's theorem, for each choice of a root there are n^{n-2} labeled arborescences. Thus there are $n.n^{n-2} = n^{n-1}$ labeled arborescences with n vertices.

2.75 (*Palmer's Generalization of Cayley's Theorem*) A tree with n vertices is an **edge-labeled tree** if each edge is assigned a unique positive integer between 1 and $(n - 1)$. Show that the number of distinct edge-labeled trees with $(n - 1)$ labeled edges and n unlabeled vertices is n^{n-3}.

Solution. Suppose v is a fixed vertex in a labeled tree with n vertices. There is a unique path P from any vertex i to the vertex v. Define $f_v(i) = j$ if j is the first vertex in the path P after the vertex i. Thus $f_v(i) = j$ for i = $1, 2, \ldots, (n - 1)$. The function f_v: $\{1, 2, \ldots, (n - 1)\} \to \{1, 2, \ldots, n\}$ is the **tree function with respect to** v of the vertex labeled tree. Assign the label i to the edge joining i and j if $f_v(i) = j$. Thus each edge is assigned a label from the set $\{1, 2, \ldots, (n - 1)\}$.

Let X be the set of all vertex labeled trees, and let Y be the set of all edge labeled trees with unlabeled vertices. The cardinality of X is n^{n-2} by Cayley's theorem.

We have a mapping of X into Y via the tree function. If $n \geq 3$, each edge-labeled tree is the image of n vertex-labeled trees since the vertex v can be chosen in n ways and the labels of the other vertices are uniquely determined by the labels of these edges. Thus there are $n^{n-2}/n = n^{n-3}$ edge-labeled trees with n unlabeled vertices.

2.76 Let G be an undirected graph with n labeled vertices and m labeled edges. Assign each edge an arbitrary orientation, and let A be the incidence matrix of the resulting digraph. Show that (*a*) if G is connected, the rank of A is $(n - 1)$; and (*b*) the determinant of any nonsingular submatrix of A is either -1 or 1.

Solution.

(*a*) Each column has $(n - 2)$ zeros. The other two entries are 1 and -1. So the sum of all n rows is a zero vector with m components. Thus the rank of A is less than n. If $k < n$, the sum of any k rows should not be zero; otherwise, G would not be connected. So any set of k rows is linearly independent if $k < n$. Thus the rank of A is at least $(n - 1)$.

(*b*) If B is any nonsingular $r \times r$ submatrix, no column in B can be the zero vector. If each column of B contains two nonzero entries, B will become nonsingular. So there is at least one column in B with exactly one nonzero entry. We then expand the determinant of B along this column. Thus the determinant of B is the product of the nonzero entry in that column and the determinant of B', where B' is a $(k - 1) \times (k - 1)$ nonsingular submatrix. By a simple inductive argument, we conclude that the determinant of B is either -1 or 1.

2.77 Suppose an arbitrary incidence matrix A of a connected graph $G = (V, E)$ with n vertices is defined as in Problem 2.77. The **reduced incidence matrix A_r** is the matrix obtained from A by deleting a row, say the nth row. Show that any $(n - 1) \times (n - 1)$ submatrix B of matrix A_r is nonsingular if and only if the edges corresponding to the columns of B constitute the edges of a spanning tree in G.

Solution. Let B be an $(n - 1) \times (n - 1)$ submatrix of the reduced incidence matrix. Let $F = (V, E)$ be the spanning subgraph of G defined by the columns of B. Then B is the reduced incidence matrix of F and is nonsingular if and only if its rank is $(n - 1)$. Hence, B is nonsingular if and only if F is connected. But F has n vertices and $(n - 1)$ edges. So B is nonsingular if and only if F is a spanning tree.

2.78 (*Matrix Tree Theorem*) Show that if A_r is the reduced incidence matrix (as defined in Problem 2.77) of a connected graph G and if $(A_r)^T$ is its transpose, the number of spanning trees in G is equal to the determinant of $A_r(A_r)^T$.

Solution. Suppose P is a $p \times q$ matrix and Q is a $q \times p$ matrix, where $p \leq q$. Then it is a known result (known as **Cauchy–Binet formula** in matrix theory) that det $PQ = \Sigma$ (det B)(det C), where the sum is taken over all $p \times p$ matrices B and C of P and Q such that the columns of P in B are numbered the same as the rows of Q in C. Let $P = A_r$ and $Q = (A_r)^T$. Then (det A_r)(det $(A_r)^T$) $= \Sigma$ (det B)(det B^T) $= \Sigma$ (det $B)^2 = \Sigma\, 1$, where the last summation is over all $(n - 1) \times (n - 1)$ nonsingular submatrices of A_r. According to Problem 2.77, each such matrix corresponds to a spanning tree.

2.79 If e is an edge in graph G, $G - e$ is the subgraph of G obtained from G by deleting e from G. After edge e joining the vertices v and w is deleted, suppose the vertices v and w are merged to constitute a single vertex. The resulting graph G' is called the **contracted graph** obtained by contracting edge e and is denoted by $G.e$. If $\tau(G)$ is the number of spanning trees in G, show that $\tau(G) = \tau(G - e) + \tau(G.e)$.

Solution. Every spanning tree in G that does not contain edge e corresponds to a spanning tree in $G - e$. Every spanning tree in G that contains edge e corresponds to a spanning tree in the contracted graph $G.e$.

2.80 Show that if $T_i = (V_i, E_i)$, where $i = 1, 2, \ldots, k$ are subtrees of $T = (V, E)$ such that every pair of subtrees have at least one vertex in common, the entire set of subtrees have a vertex in common.

Solution. Let n be the number of vertices of T. The proof is by induction on n. The desired property holds if $n = 2$. Assume that the property holds for all trees with at most n vertices.

Let T be a tree with $(n + 1)$ vertices in which x is a terminal vertex adjacent to a vertex y. Suppose the subtrees $T_1, T_2, \ldots, T_k$ of T are such that every pair of them has at least one vertex in common. If x is not a vertex in any of these trees, the trees are subtrees of a tree with n vertices; thus the property holds for the graph T with $(n + 1)$ vertices. If one of these trees is the tree with just one vertex x, x is common to all the vertices; thus the property holds in this case.

We now examine the remaining case. Let $T_i(x)$ be the subtree of T_i obtained by deleting x from T_i. If x is a vertex common to T_i and T_j, y is also a vertex common to T_i and T_j. Therefore, y is a common vertex for $T_i(x)$ and $T_j(x)$. Thus by the induction hypothesis, the subtrees $T_i(x)$ have a common vertex. Therefore, the entire collection $\{T_i\}$ has a vertex in common.

2.81 Obtain a DFS spanning tree starting the search from vertex 2 in the graph shown in Fig. 2-20.

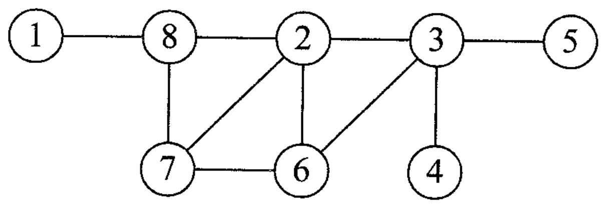


Fig. 2-20

Solution. From 2, we go to 3 and then from 3 to 4. Return to 3 and from 3 to 5. Return to 3 and then 3 to 6, 6 to 7, 7 to 8, and finally 8 to 1. If the search starts from 2, the edges of a DFS spanning tree are {2, 3}, {3, 4}, {3, 5}, {3, 6}, {6, 7}, {7, 8}, and {8, 1}.

STRONG ORIENTATIONS OF GRAPHS

2.82 Show that a digraph $G = (V, E)$ is strongly connected if and only if the following property is satisfied: For every nonempty subset X of vertices, there exists an arc from a vertex x in X to a vertex y in the complement of X.

Solution. Suppose $G = (V, E)$ is strongly connected and X is an arbitrary nonempty subset of V. Let u be an arbitrary vertex in X and y be an arbitrary vertex in $Y = (V - X)$. Then there is at least one directed path P from u to v that will have an arc of the form (x, y), where $x \in X$ and $y \in Y$. So a strongly connected graph satisfies the property. To prove the sufficiency part, assume that G is a digraph that satisfies the property. Suppose G is not strongly connected. Suppose u and v are two vertices such that there is no directed path from u to v in the digraph. Let X be the set of all vertices that are terminal vertices of directed paths that originate from the vertex u. By assumption, X is a proper subset of V. So there exists an arc e from vertex $x \in X$ to vertex $y \in (V - X)$. Now x is the terminal vertex of a directed path P from u, and this path can be enlarged into path P' from u to y using arc e. Thus the vertex y cannot be in $(V - X)$, which is a contradiction.

2.83 Show that if a tournament has a directed circuit, it has a directed triangle.

Solution. If a tournament $G = (V, E)$ has a directed circuit, it will have a directed cycle C. Suppose C: $v_1 \to v_2 \to \cdots \to v_k \to v_1$ is a directed cycle of minimum length in G. We have to show that $k = 3$. Suppose $k > 3$. There are two cases: (*i*) There exists an arc from v_1 to v_{k-1} creating a directed cycle passing through three vertices. This is a contradiction. (*ii*) There is an arc from v_{k-1} to v_1 creating a cycle of length $(k - 1)$. This also is a contradiction. So $k = 3$.

2.84 If G is a tournament with n vertices, the vector whose components are the n outdegrees arranged in a nondecreasing order is called the **score vector** of the tournament. A tournament is a **transitive tournament** if whenever (u, v) and (v, w) are arcs, (u, w) is also an arc. Show that a tournament G with n vertices is transitive if and only if its score vector is $[0 \quad 1 \quad 2 \quad \cdots \quad (n - 1)]$.

Solution. Let G be a transitive tournament. In any tournament, the outdegree is at most $(n - 1)$. Suppose there are two vertices u and v with equal outdegrees. If there is an arc from u to v, the outdegree of u is more than the outdegree of v. If there is an arc from v to u, the outdegree of v is more than that of u. In other words, in a transitive tournament, the outdegrees are all distinct. Now suppose $G = (V, E)$ is a tournament with score vector $s = [0 \quad 1 \quad 2 \quad \cdots \quad (n - 1)]$. If $V = \{v_1, v_2, \ldots, v_n\}$, E will be the set $\{(v_i, v_j)\colon 1 \le j < i \le n\}$ because of the way the outdegrees are defined. Then G is obviously transitive.

2.85 Suppose D' is the digraph obtained from a *strongly connected mixed* graph G (see Section 2.2) by deleting an undirected edge $e = \{x, y\}$ from G and replacing every other edge in G by two arcs in opposite directions. Let X be the set of all vertices v such that there is a directed path in D' from x to v consisting of at least one arc. Likewise, let Y be the set of all vertices v such that there is a direct path in D' from y to v consisting of at least one arc. If x is not in Y and y is not in X, $e = \{x, y\}$ is a bridge in the mixed graph.

Solution. Notice that x is not in X and y is not in Y by definition. Let w be any vertex that is neither x nor y. To go from x to w in G, (*i*) either we have to use a path emanating from x that does not go through y or (*ii*) first go to y using e, and then use a path from y to w. Thus w has to be either in X or in Y.

Suppose w is both in X and in Y. Then there is a directed path P_1 (using arcs from D') from x to w, and there is a directed path P_2 (using arcs from D') from y to w. Let $D(G)$ be the digraph obtained from G by converting each edge into two arcs in opposite direction. By definition, G is strongly connected if and only if $D(G)$ is strongly connected. By hypothesis, there is a directed path P_3 [consisting of arcs from $D(G)$] from w to x and a directed path P_4 [consisting of arcs from $D(G)$] from w to y.

There are three cases to be examined:

(*i*) Both P_3 and P_4 did not use e in either direction. In that case, there is a path in D' from x to y that will imply that y is in X, which is a contradiction, and there is a path in D' from y to x that will imply that x is in Y, which also is a contradiction.

(*ii*) Path P_3 is from w to y, using arcs from D' and then using the arc (y, x), which will imply that there is a directed path in D' from x to y. In that case, y will be in X, which is a contradiction.

(*iii*) Path P_4 is from w to y, using edges from D' and then using the arc (x, y). This will imply that x is in Y, which is also a contradiction.

Thus sets X and Y are disjoint, consequently, there is no edge or arc connecting a vertex in X and a vertex in Y. So the only reason that the mixed graph is connected is because of the existence of edge e joining x and y. In other words, e is a bridge.

2.86 Show that if G is a strongly connected mixed graph, it is possible to convert any edge that is not a bridge into an arc such that the resulting mixed graph is also strongly connected.

Solution. Let $e = \{x, y\}$ be any edge in a strongly connected mixed graph G. Construct X and Y as in Problem 2.87. If e is not a bridge, either x is in Y or y is in X. If x is in Y, edge e is converted into an arc from x to y. If y is in X, edge e is converted into an arc from y to x. In either case, the resulting mixed graph is strongly connected.

2.87 Prove Theorem 2.10 (Robbins's theorem): A graph is strongly orientable if and only if it is connected and has no bridges.

Solution. If an undirected graph G is strongly orientable, it is necessarily connected; thus no edge in it can be a bridge. To prove the converse, start from an undirected connected graph with no bridges. By Problem 2.86, we can orient one edge at a time. Notice at each stage that if $e = \{x, y\}$ is not a bridge, it will not become a bridge after the orientation.

2.88 Suppose G is a connected graph in which no edge is a bridge, and let T be the DFS spanning (directed) tree obtained by starting the search from a fixed vertex r called the root of the tree. Show that there is a directed path in T to every vertex from root r.

Solution. As a result of the search, each vertex v gets a label $f(v)$ that is a positive integer between 1 and n. The vertex r with label $f(r) = 1$ is the root of the DFS spanning tree T. We can use an inductive argument (induction on the label) to establish that there is a directed path from the root to every other vertex. If $f(r) = 1$, there is a directed path from 1 to 1. Suppose there is a directed path from the root to any vertex x, where $f(x) < k$. Assume there is a vertex y such that $f(y) = k$. Now there is some vertex z in T [with $f(z) < k$] such that there is an arc from z to y. By the induction hypothesis, there is a directed path from root r to z and an arc from z to y. So there is a directed path from r to y. Thus there is a directed path from the root to every other vertex.

2.89 Let G be a connected simple graph in which no edge is a bridge, and let T be a DFS spanning tree in the graph. Assume that $f(x)$ is the unique label of vertex x assigned to it during the search, as in Problem 2.88. If $\{v, u\}$ is any edge that is not used as an arc in T and if w is any vertex such that $f(v) < f(w) \leq f(u)$, there is a directed path in T from v to w.

Solution. The proof is by induction on $f(w)$. Initially, let $f(w) = f(v) + 1$. Now $\{v, u\}$ is an edge with $f(u) > f(v)$. So arc (v, w) is in T. In this case, there is a directed path from v to w. Suppose the result is true for vertices with labels less than t, and suppose $f(w) = t$. Then there is a vertex x with $f(x) < f(w)$ so that arc (x, w) is in T; consequently, $f(x) \geq f(v)$. Now $f(x) > f(v)$; otherwise, (v, w) will be in T and there will be a path in T from v to w. Thus there is a path in T from v to x. The arc (x, w) is in T. So there is a path from v to w in T.

2.90 Suppose G and T are as in Problem 2.89. If there is a directed path P in T from vertex a to vertex x and a directed path Q in T from another vertex b to x, either there is a directed path from a to b or from b to a in tree T.

Solution. The proof again is by induction on the length of the path P. Observe that the indegree of each vertex (other than the root) is 1. If (a, x) is in T, path Q uses this arc since (a, x) is the only arc pointing to x in T. So there is a path from b to a. By inductive reasoning, there is an arc (w, x) in T for some w. Now both P and Q must use this arc. Thus there are paths in T from both a and b to w. So there is either a path from a to b or from b to a.

2.91 Suppose G and T are as in Problem 2.89 with the same labeling as before. If $\{u, v\}$ is an edge not in T and if $f(u) < f(v)$, the edge is converted into an arc from v to u. Thus each edge in G is now converted into an arc, giving an orientation G' of graph G based on the depth first search. The length of the (unique) path in T from the root to vertex x is denoted by $d(x)$. If $d(x) \geq 1$ and y is any vertex with $d(y) < d(x)$, there is a directed path in G' from x to y.

Solution. Since $d(x) > 0$, there is a vertex w such that there is an arc in T from w to x. Let $R = \{u$: there is a path in T from x to $u\}$. Let $\bar{R}$ be the (complement) set of vertices not in R. Both these sets are nonempty. So there exist u in R and v in the complement such that there is an edge in G between these two vertices. While orienting this edge, there are four possibilities:

(*i*) There is an arc from u to v in T. This will imply that v is in R, which is a contradiction.

(*ii*) There is an arc from v to u in T. Since there is a path from x to u in T, by Problem 2.90, there is a path from v to x or from x to v in T. Since v is not in R, there cannot be a path in T from x to v. Thus there is a path from v to x and then a path from x to u. This implies that $v = w$ and $x = u$.

(*iii*) There is an arc from v to u in G' but not in T. So $f(v) > f(u)$, implying that (Problem 2.89) that there is a path in T from u to v showing that v is in R.

(*iv*) There is an arc from u to v in G' but not in T.

Since the first three alternatives are ruled out, we conclude that there exist u in R and y in $\bar{R}$ such that there is an arc in G' (but not in T) from u to y. So $f(u) > f(y)$, implying that there is a path in T from y to u. Since there is a path in T from x to u, by Problem 2.90, there has to be a path in T from y to x. Hence $d(y) < d(x)$. Thus there is a path in T from x to u and the arc (u, y) in G'. So there is a directed path in G' from x to y. An inductive argument as before completes the proof.

2.92 Let G and G' be as in Problem 2.91. Show that there is a directed path in G' from every vertex to the root.

Solution. This is proved by induction on $d(x)$. If $d(x) = 0$, x is r. The result is true. Suppose the result is true for all vertices v with $d(v) < t$. Let $d(x) = t$. By Problem 2.91, there is a path from x to vertex y with $d(y) < t$. So by the induction hypothesis, there is a path in G' from y to r.

2.93 Prove Theorem 2.11 (Roberts's theorem): The orientation procedure using the depth first search in a connected graph with no bridges yields a strongly connected digraph. (This is another way of establishing Robbins's theorem. Thus the proof given here may be called Roberts's proof of Robbins's theorem.)

Solution. Let G, T, and G' be as in Problem 2.91. There is a directed path in T (and hence in G') from the root to every other vertex, as established in Problem 2.88. There is a directed path from every vertex to the root, as established in Problem 2.92. So G' is strongly connected.

Cut Vertices, Bridges, and Blocks

2.94 Show that a vertex v of a connected graph is a cut vertex if and only if there exist two distinct vertices u and w such that every path between these two vertices passes through v.

Solution. Let v be a cut vertex in a connected graph G. Then $G - v$ has at least two components. If we choose u from one component and w from another component, any path in G between u and w has to pass through v. On the other hand, suppose there are two vertices in a connected graph such that every path between these two vertices passes through vertex v. If this vertex is deleted, there cannot be a path between these two vertices in the resulting graph. In other words, this deletion makes G disconnected. So v is a cut vertex of G.

2.95 Show that any nontrivial graph has at least two vertices that are not cut vertices.

Solution. We may assume without loss of generality that graph G is connected. The distance $d(u, v)$ between two vertices u and v is the number of edges in a path with as few edges as possible between u and v. Let x and y be two vertices such that $d(x, y)$ is a maximum. Suppose x is a cut vertex. Then $G - x$ has at least two components. Let z be a vertex in a component that does not contain y. In G, there is a path between y and z. Since x is a cut vertex, this path has to pass through x, implying that $d(z, y) > d(x, y)$, which contradicts the maximality of $d(x, y)$. So x is not a cut vertex. Similarly, y is also not a cut vertex.

2.96 Show that an edge of a connected graph is a bridge if and only if there exist vertices v and w such that every path between these two vertices contains this edge.

Solution. The deletion of a bridge from a connected graph creates two connected components of the graph, and any path in the original graph joining a vertex in one component and a vertex in the other component definitely contains the bridge. On the other hand, if there are two vertices such that every path between these two vertices contains an edge of the graph, the deletion of this edge will no doubt disconnect the graph.

2.97 Show that an edge is a bridge if and only if no cycle contains that edge.

Solution. Assume without loss of generality that the graph under consideration is connected. If e is a bridge joining two vertices x and y and if there is a cycle that contains this edge, there is a path in the graph between these two vertices other than the edge. Since the graph is connected, every vertex in the graph is connected to every vertex in the cycle. So the deletion of e will not disconnect the graph, which contradicts that e is a bridge. Conversely, let G be a connected graph and e be an edge joining x and y such that no cycle contains this edge. Suppose e is not a bridge. Then $G - e$ is still a connected graph that has a path joining x and y. This path and the edge e together constitute a cycle containing e in G, contradicting the hypothesis.

2.98 A nontrivial connected graph is called a **nonseparable graph** if it has no cut vertices. A subgraph H of a graph G is a **block in graph G** if H is nonseparable and maximal with respect to this property: If there is a nonseparable subgraph H' such that H is a subgraph of H', $H = H'$. Obtain the blocks of the graph in Fig. 2-21.

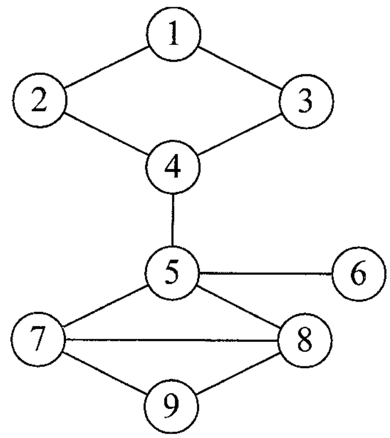


Fig. 2-21

Solution. By definition, each block in a graph is an induced subgraph, and the blocks collectively constitute a partition of the edges of the graph. The graph in Fig. 2-21 has four blocks, as shown in Fig. 2-22.

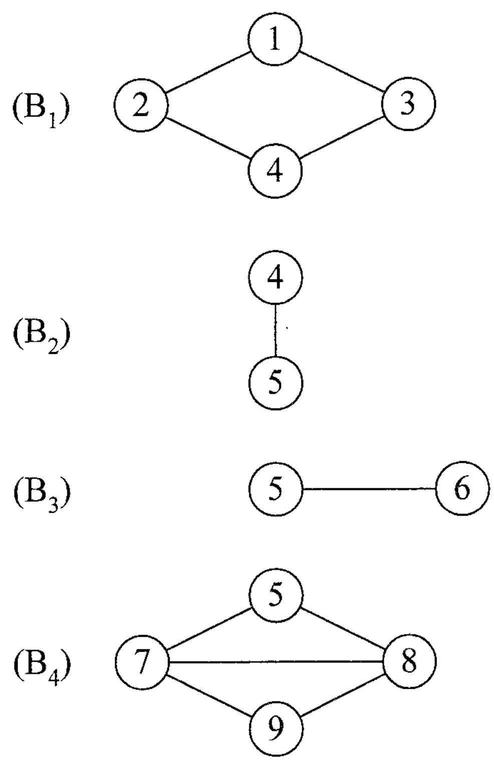


Fig. 2-22

2.99 Show that two blocks have at most one vertex in common. What distinguishing feature does a vertex common to two blocks have?

Solution. We may assume that the graph G under consideration is connected. If a vertex belonging to a block of G is deleted, the graph does not get disconnected. If two blocks B_1 and B_2 have two vertices x and y in common, let $G' = G - x$. In graph B_1, there is a path between every vertex in B_1 to every vertex in $(B_1 \cap B_2) - x$. Similarly, in graph B_2, there is a path between every vertex in B_2 and every vertex in $(B_1 \cap B_2) - x$. Hence, the union of these two blocks is a block violating the maximality of these blocks. If two blocks have a vertex in common, that vertex is necessarily a cut vertex.

2.100 Show that G is 2-connected if and only if G is connected with at least three vertices but no cut vertices.

Solution. If G is 2-connected, it is connected with at least three vertices, and the deletion of a vertex will not disconnect it. So it has no cut vertex. Conversely, any connected graph with at least three vertices and no cut vertices cannot become disconnected by deleting one vertex. Thus the graph is 2-connected.

2.101 Show that the center of a connected graph is a subset of the set of vertices of a block.

Solution. Suppose the vertices belonging to the center of a connected graph are not vertices belonging to the same block. So there exists a cut vertex v and two blocks B and B' such that each block contains vertices from the center. Let x be any vertex such that the distance $d(x, v)$ is equal to the eccentricity $e(v)$, and let P be a path between x and v of length $e(v)$. This path cannot have vertices from both the components. Suppose P has no vertex in B. Let y be a vertex common to both B' and the center, and let P' be a path of minimum length joining y and v. Then the two paths P and P' together form a path between x and y, implying that $e(y) > e(v)$, which contradicts that y is in the center.

Supplementary Problems

2.102 Show that in a graph with n vertices, the length of a path cannot exceed $(n - 1)$ and the length of a cycle cannot exceed n. [*Hint:* If u and v are two vertices, the path between u and v can have at most $(n - 2)$ distinct vertices. So the maximum length of the path is $(n - 1)$. Likewise, the maximum length of a cycle is n.]

2.103 Show that a graph G with n vertices is connected if and only if no entry in $(A + A^2 + \cdots + A^{n-1})$ is zero, where A is its adjacency matrix. [*Hint:* Let $V = \{1, 2, \ldots, n\}$ be the vertex set of G. Suppose G is connected. Let i and j be any two vertices. There is a path of length p between i and j, where $0 < p < n$. So the (i, j) entry in A^p is nonzero. The reverse implication is obvious.]

2.104 Show that the sum of the diagonal elements of the second power of the adjacency matrix is twice the number of edges of the graph. [*Hint:* Each diagonal element is equal to the degree of a vertex. So the sum of the diagonal elements is twice the number of edges.]

2.105 Find the intersection number for K_n when $n > 4$. *Ans.* $(n - 1)$

2.106 Let $T(G)$ be the total graph of the graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. Let d_i be the degree of i in G. (*a*) Find the degree of i in $T(G)$. (*b*) If e is an edge in G joining i and j, find the degree of e in $T(G)$. (*c*) Find the number of edges in $T(G)$ if G has m edges.
Ans. (*a*) 2(deg i), (*b*) deg i + deg j, (*c*) $2m + \frac{1}{2}\Sigma\,(d_i)^2$.

2.107 If both G and its complement are trees, find the order of G. [*Hint:* $(n - 1) + (n - 1) = n(n - 1)/2$. So $n = 4$.]

2.108 Find the number of edges in a forest with n vertices and k trees. [*Hint:* $m = (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = (n - k)$.]

2.109 Show that a tree is a bipartite graph. [*Hint:* A tree has no cycles. Specifically, it has no odd cycles. So it is bipartite.]

2.110 Show that if a tree has exactly two terminal vertices, the degree of every other vertex is 2 and thus it is a path. [*Hint:* $d_1 + d_2 + \cdots + d_{n-2} + 1 + 1 = (2n - 2)$, where each $d_i > 1$. Hence, each $d_i = 2$.]

2.111 Find the number of terminal vertices in a tree with n vertices. [*Hint:* Let $v_1, v_2, \ldots, v_k$ be the vertices of degree 3 or more in a tree with n vertices. Then the number of vertices of degree 1 is $2 + (d_1 + d_2 + \cdots + d_k) - 2k$, where d_i is the degree of v_i.]

2.112 Show that if the degree of every nonterminal vertex in a tree is 3, the number of vertices in the tree is even. [*Hint:* Let k be number of terminal vertices. Solve $k + 3(n - k) = 2(n - 1)$ for k.]

2.113 In a tree with 14 terminal vertices, the degree of every nonterminal vertex is either 4 or 5. Find the number of vertices of degree 4 and of degree 5.
Ans. There are three vertices of degree 4 and two vertices of degree 5.

2.114 Find the number of distinct labeled trees with five vertices. *Ans.* $5^{5-2} = 125$

2.115 If G is the subgraph obtained by deleting an edge from K_n, find the number of spanning trees in G.
Ans. $(n - 2)^{n-3}$

2.116 Show that a bipartite graph is a transitively orientable graph but not conversely. [*Hint:* A bipartite graph has no odd cycle and therefore no triangle. Thus the condition for transitive orientability is vacuously satisfied. On the other hand, consider an orientation C' (of a cycle G) that is not a directed cycle. Here C is not bipartite, but C' is a transitive orientation of C.]

Chapter 3

Eulerian and Hamiltonian Graphs

3.1 EULERIAN GRAPHS AND DIGRAPHS

A trail between two distinct vertices in a connected graph G (it need not be a simple graph) is an **Eulerian trail** if it contains all the edges of G. A circuit that contains all the edges is an **Eulerian circuit.** A graph is said to be **semi-Eulerian** (or **unicursal**) if it has an Eulerian trail. An **Eulerian graph** is a graph that has an Eulerian circuit. In Fig. 3-1(a), the trail $\{e_1, e_2, \ldots, e_{11}\}$ is an Eulerian trail between the vertices 1 and 4, and this graph is unicursal but not Eulerian. The graph in Fig. 3-1(b), however, is an Eulerian graph with an Eulerian circuit $\{e_1, e_2, \ldots, e_{10}\}$.

Theorem 3.1. In graph $G = (V, E)$, the following four statements are equivalent: (i) G is Eulerian, (ii) G is connected and each vertex in it is even, (iii) G is connected and there exists a partition of E such that the edges in each subset of the partition constitutes a cycle in G, and (iv) G is connected and each edge in G is an edge in an odd number of cycles in G. (Since we are considering graphs that are not necessarily simple, the number of edges in a cycle could be less than three.) (See Solved Problems 3.2, 3.5, and 3.6.)

Example 1. In the connected graph of Fig. 3-1(b), the degree of each vertex is even. The set of edges can be partitioned into $E_1 = \{e_1, e_4, e_8\}$, $E_2 = \{e_2, e_3\}$, $E_3 = \{e_5, e_6, e_7\}$, and $E_4 = \{e_9, e_{10}\}$ such that the edges in each subset constitute a cycle in the Eulerian graph. The edge e_1 joining vertices 1 and 2 is an edge that belongs to the following 15 cycles in the graph:

1. 1—2—3—1
2. 1—2—4—3—1 (2—4 is e_2)
3. 1—2—4—3—1 (2—4 is e_3)
4. 1—2—4—5—1 (2—4 is e_2, and 5—1 is e_9)
5. 1—2—4—5—1 (2—4 is e_3, and 5—1 is e_9)
6. 1—2—4—5—1 (2—4 is e_2, and 5—1 is e_{10})
7. 1—2—4—5—1 (2—4 is e_3, and 5—1 is e_{10})
8. 1—2—3—5—1 (5—1 is e_9)
9. 1—2—3—5—1 (5—1 is e_{10})
10. 1—2—3—4—5—1 (5—1 is e_9)
11. 1—2—3—4—5—1 (5—1 is e_{10})
12. 1—2—4—3—5—1 (2—4 is e_2, and 5—1 is e_9)
13. 1—2—4—3—5—1 (2—4 is e_2, and 5—1 is e_{10})
14. 1—2—4—3—5—1 (2—4 is e_3, and 5—1 is e_9)
15. 1—2—4—3—5—1 (2—4 is e_3, and 5—1 is e_{10})

We can obtain an Eulerian circuit in G because the set of edges in an Eulerian graph G is the union of edge-disjoint cycles. The algorithm is as follows:

Step 1. Start from any vertex v and construct a cycle C.

Step 2. If C contains all the edges of the graph, stop. If not, select a vertex w common to C and the subgraph G' obtained from G by deleting all the edges of C from it.

Step 3. Starting from w, construct a cycle in G', say C'.

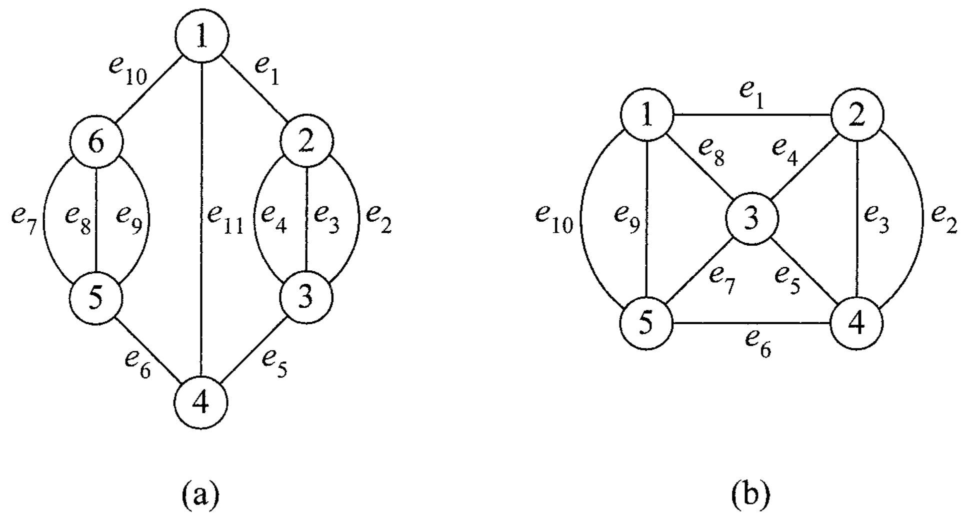


Fig. 3-1

Step 4. Combine the edges of C and C' to obtain a circuit in G. The new circuit is C. Return to Step 2.

(See Solved Problem 3.5.)

We now present another method (known as **Fleury's algorithm**) to obtain an Eulerian circuit starting from any vertex in a connected graph $G = (V, E)$ in which each vertex is even.

Step 1. Initially, $i = 0$. Start from vertex v_0 and define $T_0 : v_0$.

Step 2. Let $T_i = v_0, e_1, v_1, e_2, \ldots, e_i, v_i$ be the trail between v_0 and v_i at stage i. Select an edge e_{i+1} joining v_i and v_{i+1} from the set $E_i = E - \{e_1, e_2, \ldots, e_i\}$. If edge e_{i+1} is a bridge in the subgraph obtained from G after deleting the edges belonging to E_i from E, select it for inclusion in the updated trail $T_{i+1} = v_0, e_1, v_1, e_2, \ldots, e_i, v_i, e_{i+1}, v_{i+1}$ only if there is no other choice. If there is no such edge, stop.

Step 3. Replace i by $i + 1$ and go to step 2.

(See Solved Problem 3.7.)

Example 2. In the connected graph G in Fig. 3-2 (with six vertices and 11 edges), the degree of each vertex is even. Using Fleury's algorithm, an Eulerian circuit in this graph is obtained as follows. There will be 11 iterations, and at the end, we will have the subgraph with six vertices in which the degree of each vertex is zero.

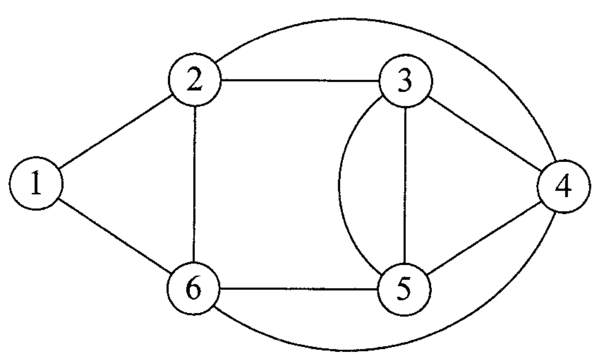


Fig. 3-2

Iteration 1: We start from vertex 2, which is $v(0)$. This is the current vertex. Choose the edge $\{2, 4\}$ as $e(1)$. This is the current edge, which is deleted. Vertex 4 now becomes $v(1)$. The updated graph G (1) is shown in Fig. 3-3.

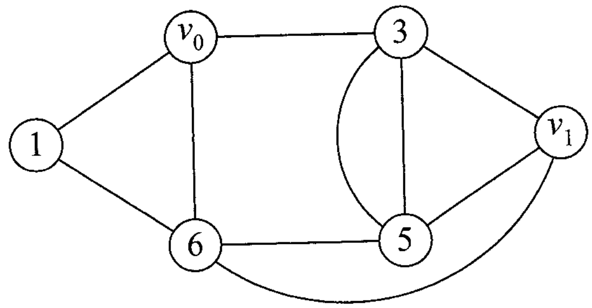


Fig. 3-3

Iteration 2: The current vertex is $v(1)$, and we select $\{v(1), 6\}$ as the current edge $e(2)$, which is deleted. Vertex 6 is now $v(2)$. The updated graph $G(2)$ is shown in Fig. 3-4.

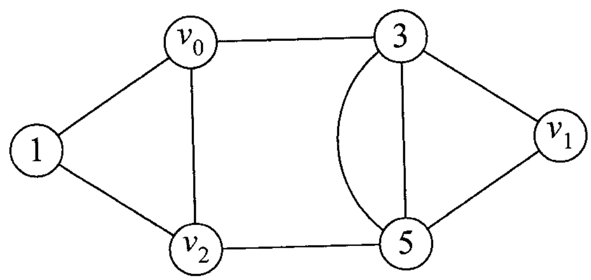


Fig. 3-4

Iteration 3: Select $\{v(2), 5\}$ as $e(3)$, and delete it. Vertex 5 is $v(3)$. The updated graph $G(3)$ is shown in Fig. 3-5.

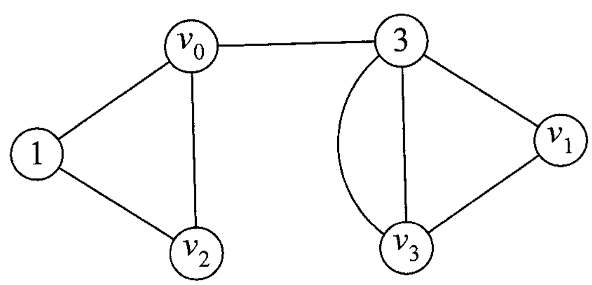


Fig. 3-5

Iteration 4: Select one of the edges between $v(3)$ and vertex 3 as the current edge $e(4)$, and delete it. Vertex 3 is now $v(4)$, and the updated graph $G(4)$ is shown in Fig. 3-6.

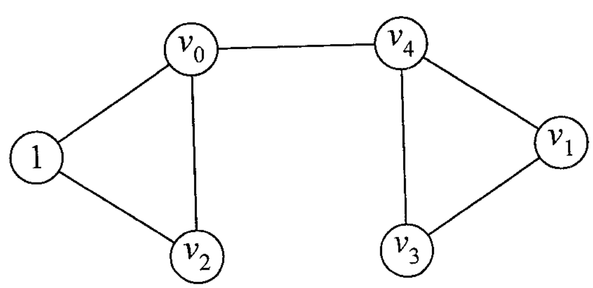


Fig. 3-6

Iteration 5: The current vertex is $v(4)$, and we have to select an edge incident with this vertex. **The edge joining $v(4)$ and $v(0)$ is not selected since it is a bridge.** Instead, we select the edge joining $v(4)$ and $v(1)$ as the edge $e(5)$ to be deleted. At this stage, vertex $v(1)$ gets the updated label $v(5)$. The updated graph $G(5)$ is shown in Fig. 3-7.

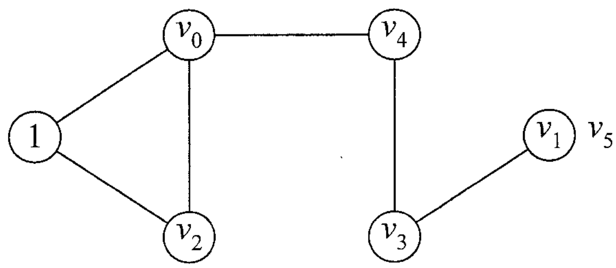


Fig. 3-7

Iteration 6: The current vertex is $v(5)$, and the **only edge** incident to this vertex is the **bridge** that joins this vertex and $v(3)$, which now gets the label $v(6)$. Bridge $e(6)$ is deleted, making $v(5)$ as an isolated vertex, as shown in Fig. 3-8.

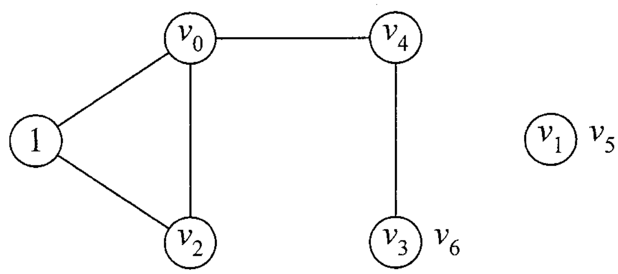


Fig. 3-8

Iteration 7: Select the edge joining $v(6)$ and $v(4)$ as $e(7)$, and delete it. The new label for $v(4)$ is $v(7)$, as shown in Fig. 3-9.

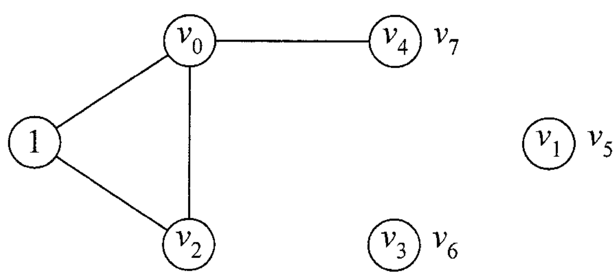


Fig. 3-9

Iteration 8: Select $\{v(4), v(0)\}$ as $e(8)$, and delete it. Now $v(0)$ has the label $v(8)$, as shown in Fig. 3-10.

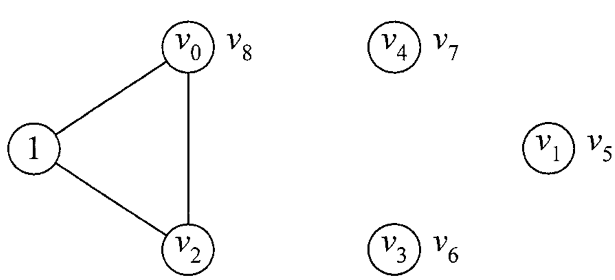


Fig. 3-10

Iteration 9: Select $\{v(8), 1\}$ as $e(9)$, and delete it. Vertex 1 has the label $v(9)$, as shown in Fig. 3-11.

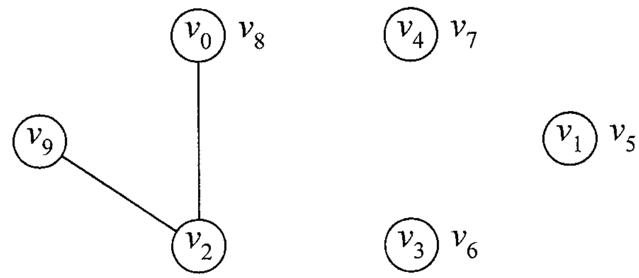


Fig. 3-11

Iteration 10: Select $\{v(9), v(2)\}$ as $e(10)$, and delete it. Vertex $v(2)$ now gets the revised label $v(10)$, as shown in Fig. 3-12.

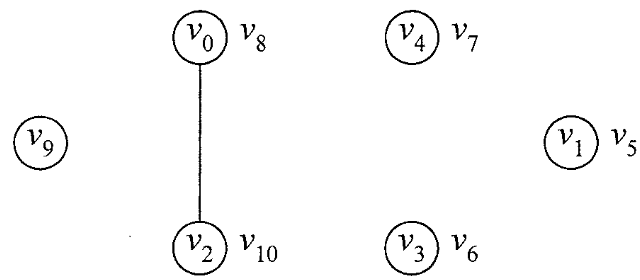


Fig. 3-12

Iteration 11: Select $\{v(10), v(8)\}$ as $e(11)$, and delete it. Vertex $v(8)$ has the revised label $v(11)$, as shown in Fig. 3-13. At this stage, we have a graph with no edges, and the algorithm terminates. The sequence of edges $e(1), e(2), \ldots, e(3)$ gives an Eulerian circuit in the graph.

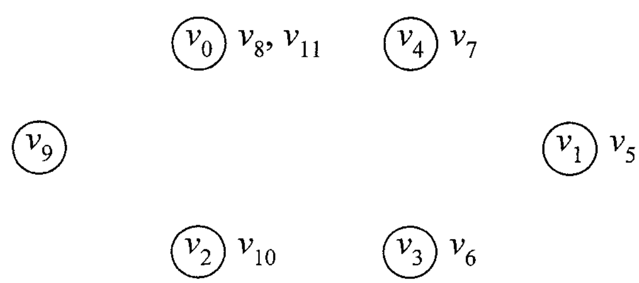


Fig. 3-13

Theorem 3.2. A connected graph is semi-Eulerian if and only if it is connected and the number of odd vertices in it is exactly two. Furthermore, in a semi-Eulerian graph, any Eulerian trail is between its two odd vertices. (See Solved Problem 3.11.)

Example 3. In the semi-Eulerian graph in Fig. 3-1(a), vertices 1 and 4 are odd and the remaining vertices are even. Any Eulerian trail in this graph is a trail between vertex 1 and vertex 4.

We now consider the straightforward extension of some of these concepts to directed graphs. Let D be a weakly connected digraph. A **directed Eulerian trail** in D from vertex v to another vertex w is a directed trail that contains all the arcs of D, and the digraph D is **semi-Eulerian** if it has a directed Eulerian trail. A

directed Eulerian circuit in D is a directed circuit that contains all its arcs, and the digraph is **Eulerian** if it contains all its arcs. Figure 3-14(a) shows an Eulerian trail $1 \to 5 \to 1 \to 4 \to 5 \to 4 \to 3 \to 2 \to 5 \to 2$ from vertex 1 to vertex 2, so this digraph is semi-Eulerian. Figure 3-14(b) shows a digraph with an Eulerian circuit $1 \to 2 \to 3 \to 2 \to 5 \to 3 \to 4 \to 1 \to 4 \to 5 \to 1$, and so this digraph is Eulerian.

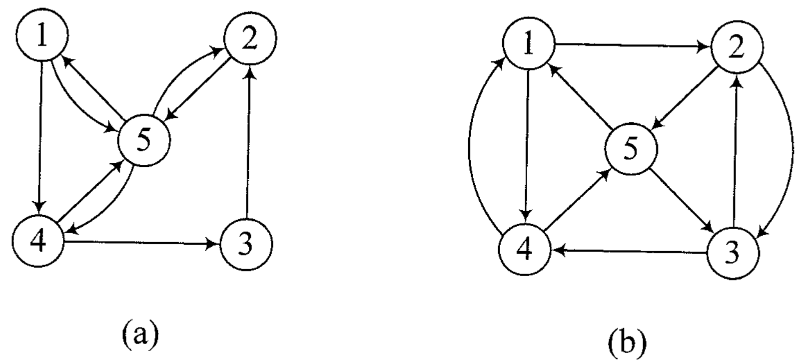


Fig. 3-14

The following theorem characterizes Eulerian and semi-Eulerian digraphs.

Theorem 3.3. (i) A weakly connected digraph is Eulerian if and only if the indegree of each vertex is equal to its outdegree. (ii) A weakly connected digraph is semi-Eulerian if and only if there are two vertices v and w such that (a) (outdegree of v) = (indegree of v) + 1, (b) (outdegree of w) = (indegree of w) − 1, and (c) the outdegree of every other vertex is equal to its indegree. In a weakly connected digraph satisfying these three properties, any directed Eulerian trail is from v to w. (See Solved Problems 3.12 and 3.13.)

Example 4. In the digraph of Fig. 3-14(a), (outdegree of vertex 1) = (its indegree) + 1 and (outdegree of vertex 2) = (its indegree) − 1, and for every other vertex, both the outdegree and indegree are equal. Any directed Eulerian trail is from vertex 1 to vertex 2. In the digraph of Fig. 3-14(b), the outdegree and indegree are the same for every vertex, and it has a directed Eulerian circuit.

3.2 HAMILTONIAN GRAPHS AND DIGRAPHS

A path between two vertices in a graph is a **Hamiltonian path** if it passes through every vertex of the graph. A closed Hamiltonian path is called a **Hamiltonian cycle** in the graph. In other words, a Hamiltonian path is a spanning path, and a Hamiltonian cycle is a spanning cycle. A **Hamiltonian graph** is a graph that has a Hamiltonian cycle. Every Hamiltonian graph has a Hamiltonian path, but the converse is not true; consider the graph that is a path. In our discussion of Hamiltonian graphs, we restrict our attention to simple graphs because the trails under consideration pass through a vertex at most once. In the case of directed graphs, the definitions are analogous.

Unlike Eulerian graphs, there is no elegant (easily testable) characterization of Hamiltonian graphs, even though many necessary conditions and many sufficient conditions are known for the existence of spanning cycles and spanning paths in connected graphs and weakly connected digraphs. Of course, any Hamiltonian graph is necessarily 2-connected since the deletion of a vertex from the graph results in a connected graph that has a Hamiltonian path. Consequently, no graph with a cut vertex is Hamiltonian. Every cyclic graph is Hamiltonian, as is every complete graph with three or more vertices. A complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n$.

The following necessary condition is a straightforward generalization of the fact that no vertex of a Hamiltonian graph is a cut vertex.

Theorem 3.4. **(A necessary condition for a graph to be Hamiltonian).** If $G = (V, E)$ is Hamiltonian and if W is any nonempty proper subset of V, the graph $G - W$ has at most $|W|$ components.

Proof. Let G_i ($i = 1, 2, \ldots, k$) be the distinct components of $G - W$, and let C be any Hamiltonian cycle in graph G. If the last (considered clockwise) vertex in C that belongs to G_i is denoted by u_i, the vertex in the cycle immediately after u_i is necessarily in W for each i. So each component defines a unique vertex in W. Hence, the number of distinct components cannot exceed the cardinality of W.

Example 5. The converse of Theorem 3.4 is false. The complete bipartite graph $K_{2,n}$ (where $n > 2$) has no cut vertex, but it is not Hamiltonian.

We will obtain several sufficient conditions for the existence of spanning cycles and spanning paths in an arbitrary graph in Solved Problems at the end of the chapter. A simple graph with n vertices and as many edges as possible is the complete graph, which is certainly Hamiltonian. In other words, if the degree of each vertex is $(n - 1)$, the graph is Hamiltonian. Is it possible to replace this requirement by a weaker condition that will still guarantee the existence of a Hamiltonian cycle in a connected graph? In this regard, one such sufficiency criterion is the following elegant result.

Theorem 3.5 **(Ore's Theorem: A sufficient condition for a graph to be Hamiltonian).** A simple graph with n vertices (where $n > 2$) is Hamiltonian if the sum of the degrees of every pair of non-adjacent vertices is at least n.

Proof. Suppose a graph G with n vertices satisfying the given inequality condition is not Hamiltonian. So it is a subgraph of the complete graph K_n with fewer edges. We recursively add edges to the graph by joining nonadjacent vertices until we obtain a graph H such that the addition of one more edge joining two nonadjacent vertices in H will produce a Hamiltonian graph with n vertices. Let x and y be two nonadjacent vertices in H. Thus they are nonadjacent in G also. Since the sum of the degree of x and the degree of y is at least n in G, this sum is at least n in H as well. If we join the nonadjacent vertices x and y, the resulting graph is Hamiltonian. Hence, in graph H, there is a Hamiltonian path between the vertices x and y. If we write $x = v_1$ and $y = v_n$, this Hamiltonian path can be written as $v_1—v_2—\cdots—v_{i-1}—v_i—v_{i+1}—\cdots—v_{n-1}—v_n$. Suppose the degree of v_1 is r in graph H. If there is an edge between v_1 and v_i in this graph, the existence of an edge between v_{i-1} and v_n will imply that H is Hamiltonian. So whenever vertices v_1 and v_i are adjacent in H, vertices v_n and v_{i-1} are not adjacent. This is true for $i = 2, 3, \ldots, (n - 1)$. Hence, the degree of v_n cannot exceed $(n - 1) - r$ since the degree of v_1 is r. This implies that the sum of the degrees of the two nonadjacent (in G) vertices is less than n, which contradicts the hypothesis. So any connected graph satisfying the given condition is Hamiltonian.

Theorem 3.6 **(Dirac's Theorem: A sufficient condition for a graph to be Hamiltonian).** A simple graph with n vertices (where $n > 2$) is Hamiltonian if the degree of every vertex is at least $n/2$.

Proof. If each degree is at least $n/2$, the sum of every pair of vertices is at least n. In particular, the sum of every pair of nonadjacent vertices is at least n. So by Ore's theorem, the graph is Hamiltonian.

Example 6. The converses of both Theorem 3.5 and Theorem 3.6 are false. The cyclic graph with five or more vertices is Hamiltonian, but the degree of every vertex in that graph is only 2. The sum of the degrees of every pair of vertices is only 4.

A graph is said to be **Hamilton-connected** if there is a Hamiltonian path in that graph between every pair of vertices. Obviously, every Hamiltonian-connected graph is Hamiltonian. But a Hamiltonian graph need not be Hamiltonian-connected; the cyclic graph with more than three vertices is Hamiltonian but not Hamiltonian-connected. The following sufficient condition for a graph to be Hamilton-connected is also due to O. Ore (1963).

Theorem 3.7. A graph G with n vertices (n is at least three) is Hamilton-connected if the sum of the degrees of any two nonadjacent vertices is more than n. In particular, it is Hamilton-connected if the degree of each vertex is more than $n/2$. (See Solved Problem 3.60.)

Example 7. Figure 3-15 shows a graph that satisfies the sufficiency criterion (first part) stated in Theorem 3.7.

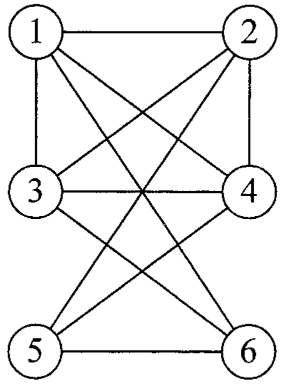


Fig. 3-15

We conclude this section with a brief discussion on Hamiltonian digraphs. Again there is no nice characterization of these graphs. We state two sets of theorems stating sufficient conditions—one set for the existence of directed spanning cycles and the other for the existence of directed spanning paths—in an arbitrary digraph. The interested reader may refer to *Graphs and Digraphs* by Mehdi Behzad, Gary Chartrand, and Linda Lesniak-Foster (1979) for proofs of these assertions.

***Theorem 3.8*(a)** **(Meyniel's Theorem).** A strongly connected digraph with n vertices is Hamiltonian if the sum of the degrees of every pair of nonadjacent vertices is at least $(2n - 1)$.

Theorem 3.8(b) **(Woodall's Theorem).** A nontrivial digraph with n vertices is Hamiltonian if (outdegree of u) + (indegree of v) is at least n whenever u and v are vertices such that there is no arc from u to v.

Theorem 3.8(c) **(Ghoula–Houri Theorem).** A strongly connected digraph with n vertices is Hamiltonian if the degree of each vertex is at least n.

Theorem 3.8(d). A digraph with n vertices is Hamiltonian if both the indegree and outdegree of each vertex are at least $n/2$.

***Theorem 3.9*(a).** A digraph with n vertices has a directed Hamiltonian path if the sum of the degrees of every pair of nonadjacent vertices is at least $(2n - 3)$.

Theorem 3.9(b). A digraph with n vertices has a directed Hamiltonian path if (outdegree of u) + (indegree of v) is at least $(n - 1)$ whenever u and v are vertices such that there is no arc from u to v.

Theorem 3.9(c). A digraph with n vertices has a directed Hamiltonian path if the degree of each vertex is at least $(n - 1)$.

Theorem 3.9(d). A digraph with n vertices has a directed Hamiltonian path if both the indegree and outdegree of each vertex are at least $(n - 1)/2$.

3.3 TOURNAMENTS

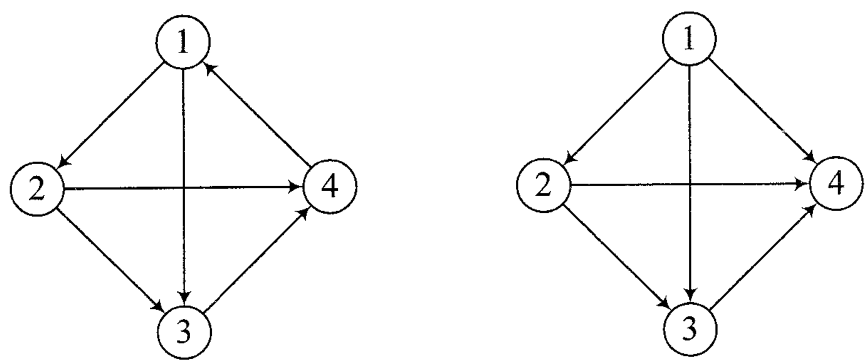
The digraph obtained by converting each edge of the complete graph K_n into an arc is known as a **tournament** with n vertices. In other words, any tournament is an orientation of a complete graph. A digraph of this kind can be used to record the results of games in a tournament in which each team (player) plays against every other team (player) in a match such that no match ends in a draw. If the match between two teams ends in a draw, they continue to play against each other until one of them becomes the winner. A typical vertex v represents a team, and the presence of an arc from v to another vertex w indicates that in the match between v and w, the winner is v. The outdegree of a vertex v is the number of matches in which v is the winner, and its indegree is the number of matches v lost during the tournament. The sum of the outdegree and the indegree of each vertex is the order of the digraph, hence, by Theorem 3.9(c), we notice that every tournament has a directed

Hamiltonian path. We state this as a theorem and present a simple proof by induction on the order of the digraph.

Theorem 3.10 **(Redei's Theorem).** Every tournament has a directed Hamiltonian path.

Proof. The induction hypothesis is that every tournament of order n has a Hamiltonian path. If $n = 2$, the theorem is true. Consider any tournament G with $(n + 1)$ vertices. If v is a vertex of this tournament, graph $G - v$ is a tournament with n vertices, so it has a directed Hamiltonian path: $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$. If there is an arc from v to v_1 or from v_n to v, we are done. If this is not the case, let i be the largest integer such that there is no arc from v to v_i. Thus there is an arc from v_i to v and an arc from v_{i+1} to v. Then we have a directed Hamiltonian path in G from v_1 to v_n in which v is an intermediate vertex. So the theorem is true for any tournament with $(n + 1)$ vertices.

That every tournament of order n has a Hamiltonian path implies that once a Hamiltonian path P is discerned, the vertices can be labeled $v_i (i = 1, 2, \ldots, n)$ and ranked such that v_i defeats v_{i+1}. Then, according to this choice of P, v_1 is the "best" team and v_n is the "worst" team. But a tournament can have more than one Hamiltonian path or a unique Hamiltonian path. Figure 3-16(a) shows four Hamiltonian paths, suggesting that each team could be the "best" as well as the "worst," whereas Fig. 3-16(b) shows a unique Hamiltonian path from vertex 1 to vertex 4.



(a) (b)

Fig. 3-16

Since every tournament has a Hamiltonian path, it is natural to find the conditions under which a tournament will have a unique Hamiltonian path. A tournament is called a **transitive tournament** if u defeats w whenever u defeats v and v defeats w. Observe that this definition implies that a tournament is transitive if and only if it has no directed cycles consisting of three arcs. A transitive tournament can be characterized in more than one way, as can be seen from the following two theorems. Theorem 3.12 settles the uniqueness problem.

Theorem 3.11. A tournament is transitive if and only if it is acyclic.

Proof. Let G be an acyclic tournament. Suppose u, v, and w are three vertices in the graph. If there is an arc from u to v and an arc from v to w, there has to be an arc from u to w since the graph is acyclic. So an acyclic tournament is necessarily transitive. On the other hand, consider a transitive tournament. By definition, it cannot have a cycle consisting of three arcs. If it has a cycle with more than three arcs, it should, again by transitivity, have a cycle with three arcs, which is an impossibility. Thus a transitive tournament is acyclic.

Theorem 3.12. A tournament has a unique Hamiltonian path if and only if it is a transitive tournament.

Proof. If there are two Hamiltonian paths in a transitive tournament G, there will be two vertices u and v such that in one path u precedes v and in the other v precedes u. Then, due to transitivity, there is an arc from u to v as well as an arc from v to u, which is a contradiction. So if a tournament is transitive, it has a unique Hamiltonian path. To prove the converse, consider a tournament G with a unique Hamiltonian path P: $v_1 \rightarrow v_2 \rightarrow \cdots v_n$. If $i < j$ implies that there is an arc from v_i to v_j, the tournament is obviously transitive. Suppose this is not the case. Select the smallest i such that for some j with $i < j$, there is an arc from v_j to v_i. Once i is selected, choose j as large as possible. If $1 < i$ and $j < n$, there is an arc from v_{i-1} to v_{i+1}. There is also an arc from v_i to v_{j+1}. So we have another Hamiltonian path:

$$v_1 \to v_2 \to \cdots \to v_{i-1} \to v_{i+1} \to \cdots \to v_j \to v_i \to v_{j+1} \to \cdots \to v_n$$

which is not the same as the path P. If $i = 1$ and $j < n$, we have the spanning path $v_2 \to \cdots \to v_j \to v_1 \to v_{j+1} \to v_n$. If $1 < i$ and $j = n$, there is the additional spanning path $v_1 \to v_{i-1} \to v_{i+1} \to v_n \to v_i$, which is a contradiction. Finally, if $i = 1$ and $j = n$, we get a Hamiltonian cycle in the graph. Thus, if we assume that the tournament is not transitive, the uniqueness requirement is violated. So any tournament with a unique spanning path is transitive.

We conclude this section with a nice characterization of Hamiltonian tournaments. For a proof of this theorem and many related results in tournaments, see the section on tournaments in the Solved Problems at the end of the chapter.

Theorem 3.13 **(Camion's Theorem).** A tournament is Hamiltonian if and only if it is strongly connected. (See Solved Problem 3.80.)

Solved Problems

EULERIAN GRAPHS AND DIGRAPHS

3.1 If the degree of each vertex in a graph is at least two, show that there is a cycle in the graph.

Solution. If there is a loop at a vertex, that loop can be considered a cycle. If there is more than one edge between two vertices, any two edges joining two vertices will form a cycle. Suppose the graph is simple. Let v_0 be any vertex in the graph, and let e_1 be the edge joining this vertex and vertex v_1. Now there exists a third vertex v_2 and edge e_2 joining v_1 and v_2. This process of finding new vertices and edges is repeated, and at the kth stage, we have edge e_k joining vertices v_{k-1} and v_k and a path from v_0 to v_k consisting of k edges. Since the number of vertices in the graph is finite, we must ultimately choose a vertex that has been chosen before. Suppose v_1 is the first repeated vertex in this process. Then the path between the two occurrences of this repeated vertex is a cycle.

3.2 (*Euler–Hierholzer Theorem*) Show that a connected graph is Eulerian if and only if the degree of each vertex of the graph is even.

Solution. Whenever a circuit passes through an arbitrary vertex of the graph, two distinct edges are used up; as such, each such passing results in a contribution of two to the degree of that vertex. So if there is a circuit that contains all the edges of the graph, the degree of each vertex is necessarily even. To prove the converse, we use induction on the size of a connected graph. If there are no edges, there is only one vertex in the graph since the graph is connected; thus the problem is trivial. Suppose the degree of each vertex is even. Since the graph is connected, each degree is at least 2. So by Problem 3.1, there is a cycle C in the graph. If this cycle contains all the edges, we are done. Suppose this is not the case. Then all the edges belonging to this cycle are deleted from the graph, resulting in a spanning subgraph H (with fewer edges) that need not be connected. But each vertex in H is even. The induction hypothesis is that every graph with fewer edges in which each degree is even is Eulerian. So each component of H is Eulerian. Furthermore, each component has a vertex in common with the cycle C. We can thus obtain an Eulerian circuit in G as follows. Start from any vertex of C and traverse the edges of this cycle sequentially until we reach vertex v_1 that is also a vertex in a (nontrivial) component that is Eulerian. Then we traverse all the edges of this component sequentially starting from v_1 and return to it, and then continue along the edges of C until we find another vertex v_2 that is a vertex in another component. We repeat this process and eventually return to the starting vertex in C, thereby obtaining an Eulerian circuit in the graph.

3.3 (*The Königsberg Bridge Problem*) The two islands (say the East Island and the West Island) in the river Pregel (known as Pregolya these days) that flows from east to west through the city of Königsberg (now known as Kaliningrad) in eastern Prussia (now a part of Russia) were connected by a bridge. Two bridges connected the west island (W) to the north shore (N), and two bridges connected it to the south shore (S). A bridge connected the east island (E) and the north shore, and another bridge connected it to the south shore. Show that the following problem posed to Leonard Euler (in 1736) by the citizens of Königsberg is not solvable: Start from one of these four land masses in the city and return to that point after walking along each bridge exactly once.

Solution. Consider the graph (Fig. 3-17) with vertices denoted by N, S, E, and W. Each bridge is represented by an edge joining the corresponding vertices. We thus have a connected graph with four vertices and seven edges in which the degree of each vertex is odd. So by Euler's theorem, this problem has no solution.

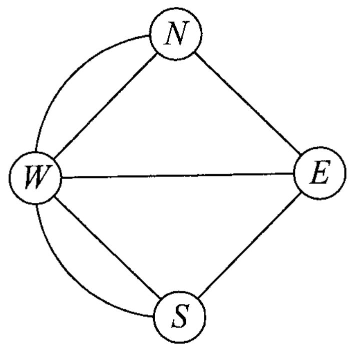


Fig. 3-17

3.4 Solve the modified Königsberg bridge problem by (a) deleting two edges from the graph, (b) constructing two new edges, and (c) deleting an edge and constructing an edge.

Solution.

(a) Delete the edges $\{N, W\}$ and $\{S, E\}$ in Fig. 3-17.

(b) Join N and E by an edge, and then join S and W by another edge.

(c) Delete the edge joining W and E, and then join N and S by an edge. In each case, we end up with a connected graph in which the degree of each vertex is even.

3.5 Show that a graph is Eulerian if and only if it is connected and if the set of its edges can be partitioned into a disjoint union of cycles.

Solution. Suppose G is an Eulerian graph. Then each vertex in G is even and of degree at least 2. So it is not a tree since it does not have a vertex of degree 1. (See Solved Problem 2.60.) Thus there is at least one cycle C_1 in the graph. If G is not this cycle, let G_1 be the subgraph (possibly disconnected) obtained from G by deleting all the edges belonging to the cycle. Since every vertex in a cycle is of degree 2, every vertex in G_1 is also even and, as before, has cycle C_2. Let $G_2 = G_1 - C_2 = G - C_1 - C_2$. We repeat this process of identifying cycles until we get the graph $G_k = G - C_1 - C_2 - \ldots - C_k$ with no edges. Thus the set of edges is the disjoint union of these k cycles. Conversely, suppose the set of edges in a connected graph G is the disjoint union of k cycles. Consider any one of these cycles, say cycle C_1. Since the graph is connected, there is a cycle (say C_2) such that the two cycles have a vertex v_1 in common. Let Q_{12} be the circuit that consists of all the edges in these two cycles. As before, there is a cycle C_3 such that this cycle and circuit Q_{12} have no edge in common but do have vertex v_2 in common. Let Q_{123} be the circuit that contains all the edges of these three edge-disjoint cycles. We repeat this procedure until we get a circuit that contains all the edges of the graph. Thus the graph is Eulerian.

3.6 (*Toida-McKee Theorem*) Show that a connected graph is Eulerian if and only if every edge in it belongs to an odd number of cycles.

Solution. Let e be an edge joining two vertices x and y in an Eulerian graph G. The subgraph $G' = G - e$ is necessarily a connected graph. Let X be the set of trails in G' between x and y in which y is not a repeated vertex. Since the degree of x in G' is odd, the number of ways of choosing the *first* edge in the trail (considering it as a trail *from x to y*) is odd. Once the first edge is chosen, the number of ways of choosing a *second* edge in the trail is also odd. So $|X|$ is odd. Let Y be the set of trails in X with repeated vertices. Suppose v is a repeated vertex in a trail T, and let Q be the circuit consisting of the edges in T between any two occurrences of vertex v. Then, by reversing the order in which the edges of circuit Q appear in T, we can construct a new trail T'. So $|Y|$ is an even number. Thus the number of paths between x and y is $|X| - |Y|$, which is an odd number. So edge e belongs to an odd number of cycles in the graph. Conversely, let G be a connected graph with the property that every edge belongs to an odd number of cycles. Suppose G is not Eulerian. Let v be an odd vertex in G. For each edge e incident with v, let $r(e)$ be the number of cycles that contain edge e. Any cycle passing through v contains two edges incident at v. So the sum $\Sigma r(e)$ (where the summation is over all the edges e incident at v) is even. But by hypothesis, $r(e)$ is odd, so the number of terms in the summation is odd. This contradiction shows that G is Eulerian.

3.7 Show that any trail constructed by Fleury's algorithm in an Eulerian graph is an Eulerian circuit.

Solution. Suppose $T(0)$: $v(0)$, $e(1)$, $v(1)$, $e(2)$, $v(2)$, . . . , $v(i-1)$, $e(i)$, $v(i)$ is the trail at stage i, and let $G(i)$ be the subgraph $G - \{e(1), e(2), \ldots, e(i)\}$. If the trail is not closed, both the *first vertex* $v(0)$ and the *last vertex* $v(i)$ are odd and every other vertex is even.

Case (i): The degree of $v(i)$ is 1. So there is a unique edge e in $G(i)$ joining $v(i)$ and a vertex w, and this e has to be selected for upgrading the trail. Once this choice is made, $v(i)$ becomes an isolated vertex in $G(i+1)$. In other words, e is a bridge in $G(i)$, and the upgrading of the trail using this bridge not only disconnects $G(i)$ but makes the last vertex become an isolated vertex in $G(i+1)$.

Case (ii): If the degree of $v(i)$ is more than 1, we claim that there is at most one bridge incident at $v(i)$. Suppose $e = \{v(i), p\}$ and $f = \{v(i), q\}$ are two bridges in $G(i)$. If we delete these two bridges from $G(i)$, we get a disconnected graph G' with three components: one component with $v(i)$, another one with p, and a third one with q. In G', we see that the degree of $v(i)$ is still odd since two edges incident with it are deleted. Since p and q are even vertices in $G(i)$, they now become odd vertices in G'. The deletion of these two bridges does not affect the degrees of the other vertices. Thus G' is a disconnected graph with three components and exactly four odd vertices. So there are at least two components in which the number of odd vertices is odd. But the number of odd vertices in any graph is even. Thus there is at most one bridge incident at $v(i)$. So whenever there are three or more edges incident at $v(i)$, we can always select one of them (which is not a bridge) and call it $e(i+1)$ in upgrading the trail.

On the other hand, if the current trail $T(i)$ is a closed trail, the degree of each of its vertices is even in $G(i)$. If the degree of $v(i)$ is positive, no edge incident with $v(i)$ is a bridge; otherwise, we will have a component with one odd vertex. These edges are used in upgrading the trail, and we return to this vertex later.

Thus we conclude that once we start constructing a trail using Fleury's algorithm in an Eulerian graph, the only stage at which we find it impossible to execute step 2 of the algorithm is when we reach a stage in which the current trail is a circuit containing all the edges of the graph, since any selection of a bridge (for inclusion in the trail in the absence of any other edge incident to the current last vertex) always results in an isolated vertex.

3.8 Using Fleury's algorithm, obtain an Eulerian circuit in the graph of Fig. 3-18.

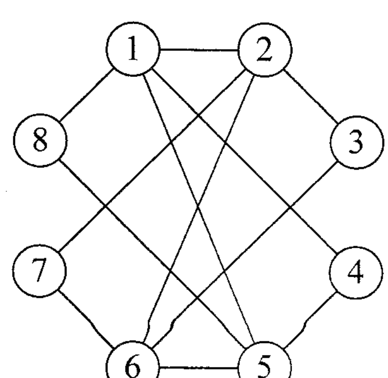


Fig. 3-18

Solution. The starting vertex is 1, and we let $T(0) = 1$. (For convenience, the edge joining i and j is written as ij.)

$T(1)$: $T(0)$, 12, 2
$T(2)$: $T(1)$, 23, 3
$T(3)$: $T(2)$, 36, 6
$T(4)$: $T(3)$, 67, 7
$T(5)$: $T(4)$, 72, 2
$T(6)$: $T(5)$, 26, 6
$T(7)$: $T(6)$, 65, 5
$T(8)$: $T(7)$, 58, 8
$T(9)$: $T(8)$, 81, 1
$T(10)$: $T(9)$, 15, 5
$T(11)$: $T(10)$, 54, 4
$T(12)$: $T(11)$, 41, 1

An Eulerian circuit obtained by applying Fleury's algorithm in this simple graph (Fig. 3-18) is 1—2 — 3 — 6 — 7 — 2 — 6 — 5 — 8 — 1 — 5 — 4 — 1.

3.9 If the number of odd vertices in a connected graph $G = (V, E)$ is $2k$, show that the set E can be partitioned into k subsets such that the edges in each subset constitute a trail between two odd vertices.

Solution. Suppose the odd vertices are v_i $(i = 1, 2, \ldots, k)$ and w_i $(i = 1, 2, \ldots, k)$. Construct k new vertices x_i $(i = 1, 2, \ldots, k)$ and $2k$ new edges $\{x_i, v_i\}$ and $\{x_i, w_i\}$ for $i = 1, 2, \ldots, k$. In the graph G' thus constructed, each vertex is even, so G' is Eulerian. Construct an Eulerian circuit Q in G'. Observe that in this circuit, the new edges adjacent to the new vertex x_i appear consecutively for each i. Now delete from this circuit all the new vertices (and, of course, all the new edges). The remaining edges in Q precisely constitute k pairwise disjoint sets, forming a partition of E such that the edges in each subset of the partition constitute a trail between two distinct odd vertices.

3.10 Locate the odd vertices in the graph of Fig. 3-19 and then partition the set of edges in the graph into subsets such that the edges in each subset constitute a trail.

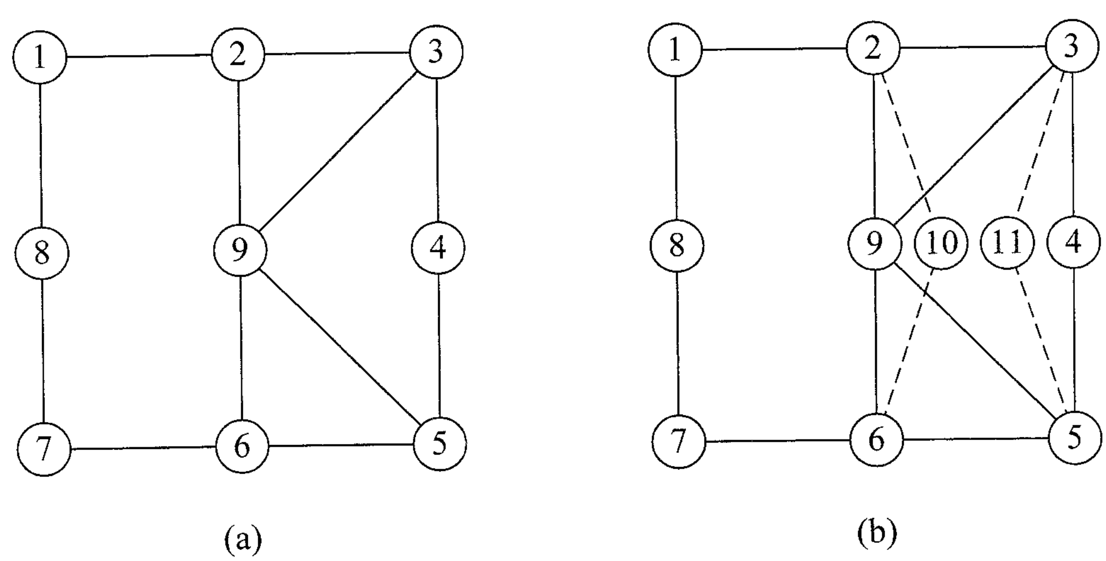


Fig. 3-19

Solution. The odd vertices are 2, 3, 5, and 6. So the set of edges can be partitioned into a family consisting of two subsets. We construct two new vertices 10 and 11 and new edges {2, 10}, {6, 10}, {3, 11}, and {5, 11} to obtain an Eulerian graph as in Figure 3-19(*b*). An Eulerian circuit in the new graph is constructed, and the four new edges are deleted from this circuit. We then obtain a trail 3 — 9 — 5 — 6 between the odd vertices 3 and 6. We also obtain the trail 5 — 4 — 3 — 2 — 1 — 8 — 7 — 6 — 9 — 2 between the odd vertices 5 and 2.

3.11 Prove Theorem 3.2: A connected graph is semi-Eulerian if and only if the number of odd vertices in it is exactly two. Furthermore, in a semi-Eulerian graph, any Eulerian trail is between its two odd vertices.

Solution. Suppose there is a trail between two distinct vertices u and v that contains all the edges of the graph. Any vertex in the trail other than these two is an intermediate vertex, the degree of which in the graph (and in the circuit) is necessarily even because whenever the trail passes through an intermediate vertex, it uses two edges: one to enter and one to exit. By the same argument, both the starting vertex and the terminal vertex are odd. So if the graph is semi-Eulerian, there exist two unique odd vertices and an Eulerian trail between these two vertices such that the degree of every other intermediate vertex in the trail is even. Conversely, according to Problem 3.10, if there are exactly two odd vertices, there is a trail between these two vertices consisting of all the edges of the graph.

3.12 Prove that a weakly connected digraph is Eulerian if and only if the indegree of each vertex is equal to its outdegree.

Solution. Whenever a circuit passes through a vertex, two distinct arcs are used: one to the vertex and one from the vertex. So each such passing results in a contribution of one to the outdegree and one to the indegree. Thus the existence of a directed circuit containing all the arcs implies that the outdegree of each vertex equals its indegree. Conversely, let G be a weakly connected digraph with m arcs in which the indegree of each vertex equals its outdegree. We prove by induction on m that the digraph is Eulerian. The result is obviously true when $m = 2$. Assume that the theorem is true for all weakly connected digraphs with $(m - 1)$ or fewer arcs in which outdegree and indegree are the same for every vertex, and let D be one such digraph. Observe that the outdegree (and therefore the indegree) of each vertex in D is positive. Let T be any trail from an arbitrary vertex u to another vertex v in this digraph. There will be at least one arc incident from v that is not an arc in T. So it is possibe to extend the trail and end up with a trail that terminates at u. If the closed trail T' thus obtained contains all the arcs of D, we are done. Otherwise, delete from D all the arcs belonging to T' as well as the vertices that become isolated as a result of this deletion. Each component of the resulting digraph D' is weakly connected with fewer arcs in which the outdegree and the indegree are the same. So by the induction hypothesis, each component has a directed Eulerian circuit. Since D is weakly connected, each weak component of D has a vertex in common with the closed circuit T'. An Eulerian circuit for D can now be constructed by inserting an Eulerian circuit of each weak component F of D' at a vertex common to both F and T'.

3.13 A weakly connected digraph is semi-Eulerian if and only if there are two vertices v and w in it such that (1) (outdegree of v) = (indegree of v) + 1, (2) (outdegree of w) = (indegree of w) − 1, and (3) the outdegree of every other vertex is equal to its indegree. Show that in a weakly connected digraph satisfying these three properties, any directed Eulerian trail is from v to w.

Solution. If the digraph G has an Eulerian trail T from vertex v to vertex w, the indegree and outdegree of each intermediate vertex in T are necessarily the same. Since v is the initial vertex, it contributes 1 to the outdegree in the beginning. Thereafter, whenever T passes through v, it contributes 1 to the outdegree and to the indegree. Thus the outdegree of v is its indegree plus 1. Likewise, the outdegree of w is its indegree minus 1. To prove the converse, let G be a weakly connected digraph satisfying the given conditions. Construct a new vertex x and two new arcs: one from the terminal vertex w to the new vertex x and another from x to the initial vertex v. The resulting digraph G' is weakly connected in which the outdegree of each vertex is equal to its indegree, so it is Eulerian. Let C be an Eulerian circuit in G' that starts from v and ends in v. The last two arcs in this circuit are (w, x) and (x, v). If we delete these two arcs and the new vertex from C, we have an Eulerian trail from v to w.

3.14 If every vertex in a graph G is even, no edge in that graph is a bridge.

Solution. Every vertex in any component $H = (W, F)$ of G is also even. So H is an Eulerian graph. The set F is the union of edge-disjoint cycles. So no edge is F; therefore, no edge in G is a bridge.

3.15 Find all positive integers n such that K_n is (*a*) Eulerian and (*b*) semi-Eulerian.

Solution.

(*a*) The degree of any vertex in a complete graph with n vertices is $(n - 1)$, so the graph is Eulerian if and only if n is odd.

(*b*) Since a complete graph is regular, it cannot be semi-Eulerian.

3.16 If $L(G)$ is the line graph of a simple graph G, show that $L(G)$ is Eulerian whenever G is Eulerian.

Solution. Let e be an edge in G joining two vertices x and y. By definition, the degree of the vertex in $L(G)$ that corresponds to e is equal to (degree of x in G) + (degree of y in G) $-$ 2. Since G is Eulerian, every vertex in it is even. So the degree of every vertex in the line graph is also even.

3.17 Show that if the line graph of a simple graph G is Eulerian, it is not necessary that G is Eulerian.

Solution. The complete graph with four vertices is not Eulerian since it has odd vertices. But the degree of each vertex of its line graph is even, so the line graph is Eulerian.

3.18 Show that a digraph that has an Eulerian circuit is a strongly connected digraph. Is the converse true?

Solution. Since the digraph is Eulerian, there is a closed directed trail emanating from every vertex that returns to it after traversing through each arc exactly once. Such a closed trail no doubt passes through each vertex of the graph at least once. So there is a directed trail (and therefore a directed path) from every vertex to every other vertex, establishing the strong connectivity of the digraph. The converse is not true; a strongly connected digraph need not be an Eulerian graph. As a counterexample, consider the digraph G' obtained by introducing a new arc from a vertex to a nonadjacent vertex in the cyclic digraph $1 \to 2 \to 3 \to 4 \to 1$.

3.19 Show that a digraph that has an Eulerian trail is a unilaterally connected digraph. Is the converse true?

Solution. If a digraph has an Eulerian trail, there is a directed path from a unique vertex in the digraph to another unique vertex in the digraph such that every other vertex is an intermediate vertex in this path. So if u and v are any two arbitrary vertices in the digraph, there is either a path from u to v or from v to u. Thus the digraph is unilaterally connected. The converse is not true. The digraph G', obtained by introducing a new vertex labeled 4 and the new arc (3, 4) to the cyclic graph $1 \to 2 \to 3 \to 1$, is unilaterally connected but has no Eulerian trail.

3.20 (*a*) Is there an Eulerian graph of even order and odd size? (*b*) Is there an Eulerian graph of odd order and even size?

Solution.

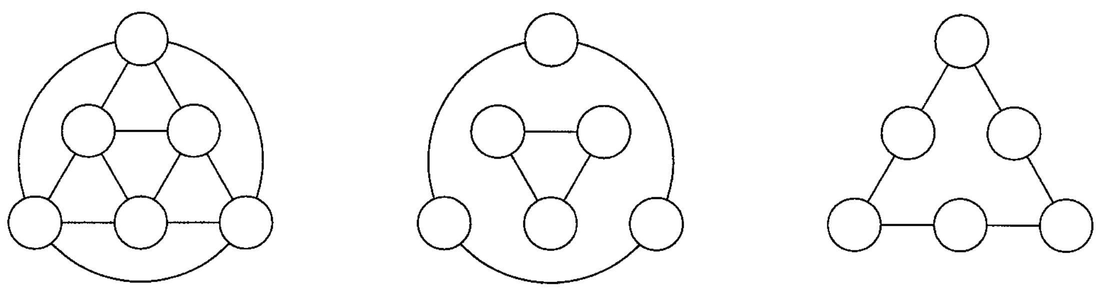
(*a*) Yes. Suppose C is a cycle of even order in which v is a vertex. Now consider a cycle C' of odd order passing through v such that the two cycles have no edge in common. The circuit G that consists of the edges of these two cycles is a graph in which each vertex is even.

(*b*) Yes. In part (*a*), suppose both C and C' are odd cycles. Then every vertex in the circuit G that has an odd number of vertices and even number of edges is even.

3.21 Show that if the degree of every vertex of a connected multigraph $G = (V, E)$ is 4, the graph has two spanning subgraphs such that (1) the degree of each vertex in these two subgraphs is 2, (2) the two subgraphs have no edge in common, and (3) E is the union of the sets of edges of these two subgraphs.

Solution. Since each vertex is even, G is Eulerian. If the graph has n vertices and m edges, $4n = 2m$, implying that the number of edges is even. We construct an Eulerian circuit starting from an arbitrary vertex and alternately color the edges of this circuit red and green. Then the red edges constitute a spanning subgraph in which the degree of each vertex is 2, and the blue edges constitute another spanning subgraph in which the degree of each vertex is also 2. Construct an Eulerain circuit. (Any k-regular spanning a subgraph of a graph is k-factor. We have shown that **every 4-regular graph can be decomposed into two factors.**)

3.22 Obtain a 2-factor decomposition of the 4-regular graph shown in Fig. 3-20.



(a) (b) (c)

Fig. 3-20

Solution. A 2-factor decomposition of the connected graph in Fig. 3-20(a) is as shown in Fig. 3-20(b) and (c).

3.23 Obtain a 2-factor decomposition of the complete bipartite graph $K_{4,4}$.

Solution. A decomposition is shown in Fig. 3-21.

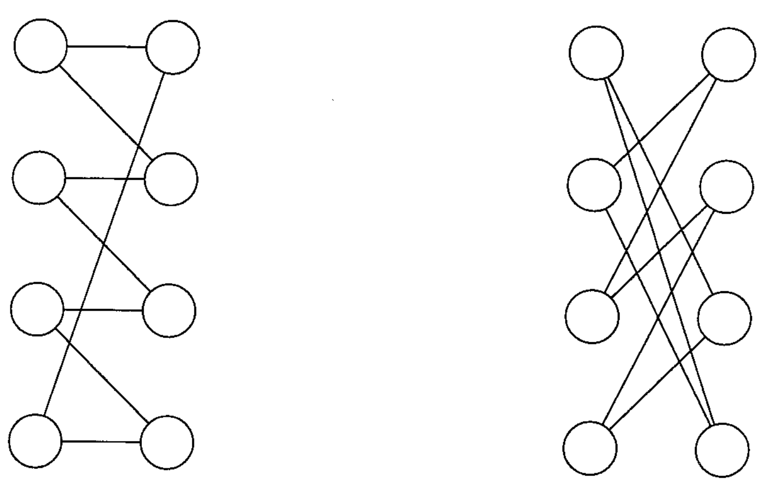


Fig. 3-21

3.24 A graph is said to be an **even graph** if the degree of every vertex in the graph is even. Find the number of nonequivalent labeled even graphs with $(n + 1)$ vertices with labels $1, 2, \ldots, (n + 1)$.

Solution. Consider any labeled graph with n vertices with labels $1, 2, \ldots, n$. Construct a new vertex with label $(n + 1)$, and join this vertex to every odd vertex in G. The resulting graph G' is even and is labeled with $(n + 1)$ vertices. This correspondence, which is clearly 1 to 1, implies that the number of labeled even graphs of order $(n + 1)$ is equal to the number of labeled graphs of order n.

Randomly Eulerian Graphs

3.25 A graph G is **randomly Eulerian from a vertex** v if every trail in the graph starting from v can be extended to a circuit terminating at v that consists of all the edges of the graph. Show that the graph in Fig. 3-22 is randomly Eulerian only from vertex 1.

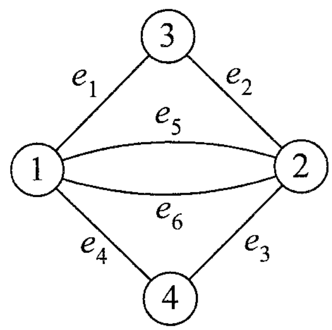


Fig. 3-22

Solution. It is easy to see that if we start from vertex 1 (or from vertex 2) and traverse along arbitrarily selected unused edges, we will return to the starting vertex after exhausting all the edges. But if we leave vertex 3, we will not be this lucky. For example, if we start from vertex 3 and choose edges e_2, e_5, and e_1 arbitrarily, we are back at vertex 3 without obtaining an Eulerian circuit and with no way of selecting an edge. The same situation exists if we start from vertex 4.

3.26 (*a*) Give an example of a graph that is randomly Eulerian from every vertex in it. (*b*) Give an example of an Eulerian graph that is not randomly Eulerian from any of its vertices.

Solution.

(*a*) Any cyclic graph is randomly Eulerian from any of its vertices.

(*b*) If we construct two parallel edges between vertices 3 and 4 shown in Fig. 3-22, we have an Eulerian multigraph that is not randomly Eulerian from any one of its four vertices.

3.27 Prove that (*a*) if G is randomly Eulerian from vertex v of G, every cycle in G passes through v; and (*b*) if every cycle in a Eulerian graph G passes through one of its vertices, G is randomly Eulerian from that vertex.

Solution.

(*a*) Suppose G is randomly Eulerian from v, and suppose there is a cycle C in G that does not contain v. If $G' = G - C$, every vertex in G' also is even, so every component of G' is Eulerian. In particular, the component H that contains v is Eulerian. Any Eulerian circuit in H contains all the edges of G incident at v. So a trail that starts from v and picks unused edges in G at random could return to v after exhausting all edges incident to it without using edges from the cycle; this contradicts the assumption that G is Eulerian from v.

(*b*) Suppose every cycle in an Eulerian graph G passes through vertex v. Assume that G is not randomly Eulerian from v. So there exists a non-Eulerian circuit T that contains all the edges incident at v, implying that there is edge e incident to vertex u not in T. Since every vertex in $G' = G - T$ is even, every component of G' (and, in particular, the component H that contains u) is Eulerian. This component has a circuit that does not pass through v; therefore, it has a cycle that does not pass through v, which contradicts our assumption.

3.28 Show that if G is randomly Eulerian from v, $G - v$ is acyclic. Also show that if v is a vertex in an Eulerian graph G such that $G - v$ is acyclic, G is randomly Eulerian from v.

Solution. If G is randomly Eulerian from v, every cycle in G passes through v, as established in Problem 3.27. So $G - v$ is acyclic. On the other hand, if G is Eulerian and if $G - v$ is acyclic, every cycle in G passes through v. Consequently, G is Eulerian, as shown in Problem 3.27.

3.29 If graph G is randomly Eulerian from a vertex, show that the degree of that vertex is equal to $\Delta(G)$, which is the maximum among the degrees of all its vertices. Show that an arbitrary Eulerian graph need not be randomly Eulerian from a vertex of maximum degree.

Solution. Suppose a graph is Eulerian from v, and let w be any other vertex. Since the graph is randomly Eulerian from v, every cycle in the graph passes through v. Thus between any two consecutive occurrences of the vertex w in an Eulerian circuit in the graph there is a cycle that contains the vertex v. So the degree of w cannot exceed the degree of v. The converse is not true. Consider the cyclic graph 1 — 2 — 3 — 4 — 1 and the Eulerian multigraph G obtained by constructing two new edges between vertices 1 and 3 and four new edges between vertices 2 and 4. Vertex 2 is a vertex with maximum degree, but the graph is not randomly Eulerian from that vertex.

3.30 Show that if G is randomly Eulerian from v and if w is another vertex such that both v and w have the same degree, G is randomly Eulerian from w as well.

Solution. Between any two consecutive occurrences of w in an Eulerian circuit randomly obtained by starting from v, there is an occurrence of v. In the same circuit, there is an occurrence of w between any two consecutive occurrences of v since the degrees of the two vertices are the same. So every cycle in the graph passes through the vertex w, implying that the graph is randomly Eulerian from w also.

3.31 A graph is **randomly Eulerian** if it is randomly Eulerian from every vertex in the graph. Obtain a necessary and sufficient condition for a graph to be randomly Eulerian.

Solution. Notice that a cyclic graph is randomly Eulerian. If an arbitrary graph is randomly Eulerian from a vertex, every cycle in that graph passes through that vertex. Thus if a graph is randomly Eulerian, it has to be a cyclic graph.

3.32 Show that if a graph is not randomly Eulerian, it is randomly Eulerian from at most two of its vertices.

Solution. If a graph is randomly Eulerian from two distinct vertices v and w, in any two consecutive occurrences of v in an Eulerian circuit there is an occurrence of w and vice versa. If there is a third vertex u from which the graph is also randomly Eulerian, we obviously have a cyclic graph, which is a contradiction.

3.33 A digraph is a **randomly Eulerian digraph from a vertex v** if every directed trail of the digraph starting from v can be extended to an Eulerian circuit. Prove that (*a*) an Eulerian digraph is randomly Eulerian from a vertex v if and only if every directed cycle in the digraph passes through v; (*b*) if a digraph is randomly Eulerian from vertex v, the maximum outdegree among its vertices is equal to the outdegree of v; (*c*) if a digraph is randomly Eulerian from v and if w is another vertex such that both v and w have the same outdegree, the digraph is randomly Eulerian from w also; and (*d*) if an Eulerian digraph of order n is not randomly Eulerian from every vertex, it is randomly Eulerian from at most $n/2$ vertices.

Solution. These are the direct generalizations of Problems 3.27 through 3.32.

De Bruijn Digraphs and Sequences

3.34 If W is the set $\{0, 1, 2, \ldots, p-1\}$, any linear arrangement (repetitions are allowed) using some of or all these numbers is called a **word** in the **alphabet** W. Any word with n numbers from W is called an ***n*-letter** word in W. The set of all n-letter words in the alphabet W with p numbers is denoted by $W(p, n)$. The **de Bruijn digraph $G(p, n)$** is constructed by the following inductive procedure. Suppose all the words in $W(p, k)$ are known for $k = 1, 2, \ldots, (n-1)$. The vertex set of $G(p, n)$ is $W(p, n-1)$. If t is any $(n-2)$-letter word, and if i and j are numbers (not necessarily distinct) in W, draw arcs (for each i) from vertex it to vertex tj as j varies from 0 to $p-1$. Then the arc from xt to ty represents the n-letter word xty. Find the order and size of the de Bruijn digraph $G(p, n)$, and show that it is an Eulerian digraph.

Solution. In a word with $(n - 1)$ numbers, each number can be chosen in p ways. So the order of $G(p, n)$ is $p^{(n-1)}$. The outdegree of each vertex by our construction is p. The indegree is also p. Thus the graph is Eulerian, and its size is p^n.

3.35 Construct the de Bruijn digraph $G(3, 2)$.

Solution. Here $W = \{0, 1, 2\}$. Each vertex is a one-letter word, and each arc is a two-letter word. There are three vertices (representing the three one-letter words) and eight arcs (representing the eight two-letter words) in the digraph, as shown in Fig. 3-23.

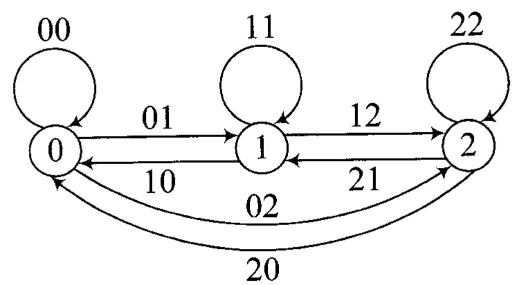


Fig. 3-23

3.36 If $x = 21134$ is a word in $W(4, 5)$, construct the arcs (a) incident from the vertex x and (b) incident to x in $G(4, 5)$.

Solution.

(a) Take $t = 1134$. We draw arcs from x to $t0$, $t1$, $t2$, $t3$, and $t4$ to denote the five six-letter words 211340, 211341, 211342, 211343, and 211344, respectively.

(b) Take $t = 2113$. We draw arcs from $0t$, $1t$, $2t$, $3t$, and $4t$ to x to denote the five six-letter words 021134, 121134, 221134, 321134, and 421134, respectively.

3.37 If p and n are two positive integers and if $p^n = r$, a sequence $\langle a(0), a(1), \ldots, a(r-1)\rangle$, where each $a(i)$ is in $W = \{0, 1, 2, \ldots, p-1\}$, is called a **de Bruijn sequence,** denoted by $B(p, n)$, if and only if any n-letter in W is of the form $a(i)a(i+1) \cdots a(i+n-1)$, where i is at most r and where addition of the subscripts is modulo r. (Equivalently, these r numbers in the sequence form a circular arrangement such that any choice of n consecutive (in the clockwise direction) numbers in this arrangement gives a unique word.) Show that a de Bruijn sequence exists for every choice of p and n.

Solution. Construct the de Bruijn digraph $G(p, n)$. If the sequence of arcs that appear in an Eulerian circuit of this digraph is $\langle e(0), e(1), \ldots, e(r-1)\rangle$, where $r = p^n$ defines the sequence $\langle a(0), a(1), \ldots, a(r-1)\rangle$, where $a(i)$ is the first number that appears in $e(i)$, any word of the form $a(i)a(i+1) \cdots a(i+n-1)$, where i is at most r and where addition of the subscripts is modulo r, is obviously an n-letter word. Conversely, suppose $w = d(1)d(2) \cdots d(n)$ is any word. Now the arcs in the digraph are defined such that a Eulerian circuit in this graph must go from vertex $d(1)d(2) \cdots d(n-1)$ to vertex $d(2)d(3) \cdots d(n)$ and then to $d(3)d(4) \cdots d(n+1)$, and so on. Thus the first letters of the arcs in this sequence will constitute the word w.

3.38 Obtain a de Bruijn sequence such that any three-letter word using 0, 1, and 2 can be obtained from this sequence.

Solution. There are nine two-letter words and 27 three-letter words using the numbers 0, 1, and 2. So the digraph $G(3, 2)$ has nine vertices (representing the two-letter words) and 27 arcs (representing the three-letter words). Both the indegree and outdegree of each vertex is three. The vertices are 00, 01, 02, 10, 11, 12, 20, 21, and 22. From 00, we draw arcs to 00, 01, and 02 representing 000, 001, and 002. Continue like this until all the arcs are drawn and labeled. An Eulerian circuit starting from 00 and ending at 00 is the following sequence of 27 words: ⟨000, 001, 010, 101, 011, 111, 112, 122, 222, 221, 212, 121, 210, 102, 021, 211, 110, 100, 002, 022, 220, 202, 020, 201, 012, 120, 200⟩. If we select the first number from each arc in this Eulerian sequence, we get the following sequence: ⟨0 0 0 1 0 1 1 1 2 2 2 1 2 1 0 2 1 1 0 0 2 2 0 2 0 1 2⟩, which is a $B(3, 3)$ sequence for this problem. Any three-letter words using the numbers 0, 1, and 2 can be obtained from this sequence by choosing any three "consecutive" numbers from this sequence. The three consecutive numbers starting from the last 1 in the sequence are 1, 2, and the first element in the sequence, which is 0. So this choice gives the word 120. If we start from the last element, we choose 2 and the first and second numbers from the sequence. This choice gives the word 200. The other 25 words can be easily obtained by starting from any number and selecting that number and the two numbers following it. For example, if we start from the 1 that appears for the first time in the sequence, the word 101 is obtained.

3.39 (*Rotating Drum Problem*) A rotating drum has 2^p sectors. The problem is to assign each sector the label 0 or 1 such that no two sequences of p consecutive labels are the same. Solve this problem when (*a*) $k = 3$ and (*b*) $k = 4$.

Solution. To solve this problem, we construct the de Bruijn graph $G(2, p - 1)$ and find an Eulerian circuit in this graph. Then the corresponding de Bruijn sequence will give an assignment of labels to the sectors.

(*a*) The graph $G(2, 2)$ is shown in Fig. 3-24(*a*) in which ⟨000 001 010 101 011 111 110 100⟩ is an Eulerian circuit. By choosing the first letter from each arc, we obtain the corresponding de Bruijn sequence, which assigns the following eight labels consecutively: 0, 0, 0, 1, 0, 1, 1, and 1.

(*b*) The graph $G(2, 3)$ is shown in Fig. 3-24(*b*). An Eulerian circuit in this digraph has the following 16 arcs consecutively: 0000, 0001, 0011, 0111, 1111, 1110, 1101, 1010, 0101, 1011, 0110, 1100, 1001, 0010, 0100, and 1000. A de Bruijn sequence corresponding to this Eulerian circuit gives the following assignment of labels to the 16 consecutive sectors: 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1.

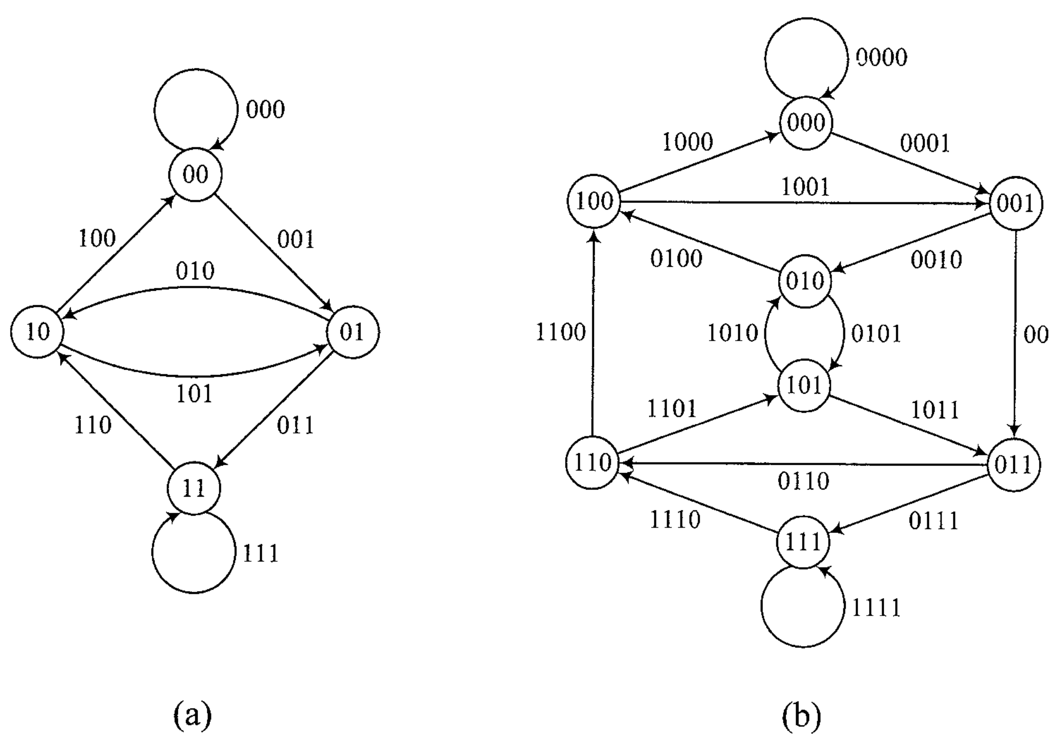


Fig. 3-24

3.40 Show that there are two unique four-letter binary words that can be deleted from the set of all such words so that any of the remaining words can be obtained by selecting four consecutive elements in a binary sequence placed in a circular arrangement. Obtain such a binary sequence.

Solution. The set of arcs in $G(2, 3)$ is in 1-to-1 correspondence with the set of all four-digit binary words. The only loops in the digraph are the arcs 0000 and 1111. If we delete these two arcs, we still have an Eulerian digraph. So the words that can be deleted are 0000 and 1111. Once they are deleted, we have the Eulerian sequence 0001, 0011, 0111, 1110, 1101, 1010, 0101, 1011, 0110, 1100, 1001, 0010, 0100, and 1000, consisting of 14 arcs. If we choose the first element of each of these 14 words and make a circular arrangement of these elements, any four-letter word (other than 0000 and 1111) can be obtained by selecting any four consecutive (clockwise) elements from this circular arrangement. Notice that any such arrangement will have seven zeros and seven ones.

HAMILTONIAN GRAPHS AND DIGRAPHS

3.41 *(Bondy and Chvátal)* Let u and v be two nonadjacent vertices in a simple graph $G = (V, E)$ of order n such that the sum of their degrees is at least n, and let $G + uv$ be the graph obtained from G by joining these two nonadjacent vertices. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Solution. If G is Hamiltonian, $G + uv$ is Hamiltonian. To prove the converse, assume that $G + uv$ is Hamiltonian but G is not. Then there is a Hamiltonian path in G between u and v. Let us consider this undirected path as a path from u to v. For each vertex adjacent to vertex u, the vertex in the path immediately preceding that vertex cannot be adjacent to v; otherwise, there will be a Hamiltonian cycle in G. So the degree sum of u and v cannot exceed $(n - 1)$ as established in the proof of Theorem 3.5. In other words, the degree sum of these two nonadjacent vertices is less than n, which is a contradiction.

3.42 The **closure $c(G)$** of a graph G of order n is obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists. Show that every graph has a unique closure.

Solution. Suppose G_1 and G_2 are two graphs obtained from G by this process. Let $S = \langle e_1, e_2, \ldots, e_k\rangle$ and $T = \langle f_1, f_2, \ldots, f_r\rangle$ be the sequences of edges added to G to obtain G_1 and G_2, respectively. If S and T are not equal, let edge e_t joining u and v be the first edge in S that is not in T. Let H be the graph obtained by adding edges $e_1, e_2, \ldots, e_{t-1}$ to G. Observe that H is a subgraph of both G_1 and G_2. By our construction, u and v are nonadjacent vertices in H, and since the edge joining these two vertices is selected as the next edge for inclusion in S, we conclude that their degree sum is at least n in graph H. This implies that their degree sum is also at least n in graph G_2 since H is a subgraph of G_2. But this is a contradiction since the degree sum of any pair of nonadjacent vertices in that graph is less than n. So S and T are the same, implying that every graph has a unique closure.

3.43 Show that a graph is Hamiltonian if and only if its closure is Hamiltonian.

Solution. The closure of G of order n is obtained by adding edges to G recursively by joining nonadjacent vertices u and v at each stage where their degree sum is at least n. It has been already established that G is Hamiltonian if and only if $G + uv$ is Hamiltonian. Hence G is Hamiltonian if and only if its closure is Hamiltonian.

3.44 Show that to establish a graph is Hamiltonian, it is sufficient if we show that its closure is complete.

Solution. If the closure of a graph G is complete, the closure is Hamiltonian since a complete graph is Hamiltonian. So G is Hamiltonian.

3.45 Give an example of a Hamiltonian graph whose closure is not complete.

Solution. The cyclic graph with more than four vertices is Hamiltonian and is its own closure.

3.46 *(Chvátal's Theorem)* If the n vertices $(n \geq 3)$ of a graph $G = (V, \mathrm{E})$ are labeled such that their degrees d_i $(i = 1, 2, \ldots, n)$ can be arranged as a sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, and if $d_{n-k} \geq (n - k)$ whenever $d_k \leq k < n/2$, G is Hamiltonian.

Solution. It is enough if we prove that the closure $c(G)$ is complete. We denote the degree of vertex v in the closure by $d'(v)$. Suppose the closure is not complete. We select two nonadjacent vertices u and v in the closure such that $d'(u) \leq d'(v)$ and the degree sum $d'(u) + d'(v)$ is as large as possible. Since these two vertices are nonadjacent, their degree sum is, of course, less than n.

Let $S = \{x \in V - v$: x and v are not adjacent in $c(G)\}$ and $T = \{x \in V - u$: x and u are not adjacent in $c(G)\}$. Now $x \in S \Rightarrow d'(x) + d'(v) \leq n \Rightarrow d'(x) + d'(v) \leq d'(u) + d'(v) \Rightarrow d'(x) \leq d'(u)$. Similarly, $x \in T \Rightarrow d'(x) \leq d'(v)$. Obviously, $|S| = (n-1) - d'(v)$ and $|T| = (n-1) - d'(u)$.

Now denote $d'(u)$ by the integer k. Then $k < n/2$ since $k \leq d'(v)$ and $k + d'(v) < n$. Now $d'(u) = k \Rightarrow d'(v) < (n-k) \Rightarrow |S| > (n-1) - (n-k) \Rightarrow |S| \geq k$. Again, $d'(u) = k \Rightarrow |T| = (n-1) - k \Rightarrow |T| = (n-1) - k \Rightarrow |T| < (n-k)$. Thus the closure $c(G)$ has at least k vertices of degree at most k and at least $(n-k)$ vertices of degree less than $(n-k)$. Since G is a spanning subgraph of the closure, the same degree conditions prevail in G. We have thus found an integer k satisfying the inequalities $d_k \leq k < n/2$ and $d_{n-k} < (n-k)$, contradicting the hypothesis. So the closure is complete.

3.47 (*Bondy's Theorem*) If the n vertices ($n \geq 3$) of a graph $G = (V, E)$ are labeled such that their degrees d_i ($i = 1, 2, \ldots, n$) can be arranged as a sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, and if $d_j + d_k \geq n$ whenever $d_j \leq j < k$ and $d_k \leq (k-1)$, G is Hamiltonian.

Solution. Suppose the hypothesis of this theorem does not imply the hypothesis of Chvátal's theorem. So there is an integer p such that $d_p \leq p < n/2$ and $d_{n-p} < (n-p)$. Then $d_p + d_{n-p} < n$. Also $p < n/2$ implies $p < (n-p)$. If $n - p = q$, we have the inequalities $d_p + d_q < n$, $d_p \leq p < q$, and $d_q \leq (q-1)$, which will contradict the hypothesis of Bondy's theorem. So Bondy's hypothesis implies Chvátal's hypothesis.

3.48 (*Posa's Theorem*) If the n vertices ($n \geq 3$) of a graph $G = (V, E)$ are labeled such that their degrees d_i ($i = 1, 2, \ldots, n$) can be arranged as a sequence S: $d_1 \leq d_2 \leq \cdots \leq d_n$, and if $d_k > k$ whenever $1 \leq k < n/2$, G is Hamiltonian.

Solution. Suppose the hypothesis of this theorem does not imply the hypothesis of Bondy's theorem. So there exist j and k such that $d_j \leq j < k$, $d_k \leq (k-1)$ and $d_j + d_k < n$. So $k + (k-1) < n$, which implies that $k < (n+i)/2$. Thus $d_j < n/2$.

Let d_j be denoted by p. Then $d_j \leq j \Rightarrow p \leq j \Rightarrow d_p \leq d_j \Rightarrow d_p \leq p$. But $p < n/2$. Thus there exists p such that $1 \leq p < n/2$ and $d_p \leq p$, which contradicts the hypothesis of Posa's theorem. So Bondy's hypothesis implies Posa's hypothesis.

3.49 Prove Ore's theorem (Theorem 3.5) using Posa's theorem.

Solution. The hypothesis in Ore's theorem is that the degree sum of any pair of nonadjacent vertices in a graph of order n is at least n. Suppose this hypothesis holds, and suppose there exists a positive integer p such that $p < n/2$ and $d_p \leq p$. If $q < p$, $d_q + d_q \leq 2p < n$. Thus $(q < p) \Rightarrow$ the vertices v_p and v_q are adjacent because of the hypothesis of Ore's theorem. So the subgraph induced by $W = \{v_i: i = 1, 2, \ldots, p\}$ is a clique. Since $d_p \leq p \Rightarrow d_i \leq p$ for every vertex in W, and since each v_i in W is adjacent to the remaining $(p-1)$ vertices in W, a vertex in W can be adjacent to at most one vertex in the complement of W. Now $n - p > p$ since $p < n/2$. So the complement has more vertices than W; consequently, there is vertex v_i in W and vertex v_j in the complement such that there is no edge joining them. Also, $d_j \leq (n-p) - 1$. So $d_i + d_j \leq p + (n-p) - 1 < n$. But vertices v_i and v_j are not adjacent. This contradicts Ore's hypothesis. So Ore's hypothesis implies Posa's hypothesis.

3.50 Point out the implications regarding sufficient conditions (established in this sequel) for the existence of a spanning cycle in a graph.

Solution. The theorems are due to Chvátal (1972), Bondy (1969), Posa (1962), Ore (1963), and Dirac (1952), in reverse chronological order. Chvátal's theorem is proved in Problem 3.46, and it implies Bondy's theorem as shown in Problem 3.47. This theorem in turn implies Posa's theorem as proved in Problem 3.48. It was shown in Problem 3.49 that Posa's theorem implies Ore's theorem. That Ore's theorem implies Dirac's theorem was established in Theorem 3.6.

3.51 Show that Chvátal's condition is not a necessary condition for the existence of a spanning cycle in a graph.

Solution. Consider the graph in Fig. 3-25. It is obviously Hamiltonian. Its degree sequence is [2, 2, 2, 3, 3, 4]. Here $n = 6$ and $d_2 = 2 < 3 = n/2$. But $d_{n-2} = d_4 = 3$, violating the requirement that $d_4 \geq (6 - 2)$.

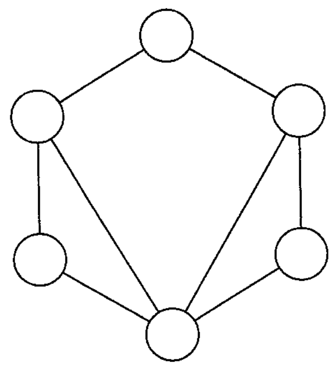


Fig. 3-25

3.52 Show that Chvátal's theorem is stronger than Bondy's theorem.

Solution. The degree sequence in the graph of Fig. 3-26 is [2, 2, 2, 4, 4, 4]. Here $d_2 = 2 \leq 2$ and $d_3 = 2 \leq 3$. But their sum is only 4, which is less than the order of the graph. So this Hamiltonian graph does not satisfy Bondy's hypothesis. It can be easily verified that it satisfies Chvátal's hypothesis, however.

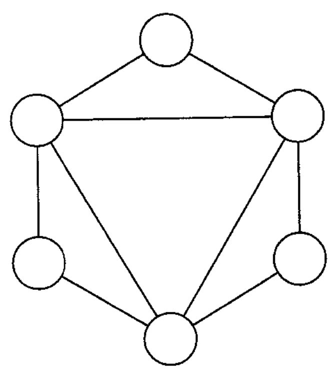


Fig. 3-26

3.53 Show that Bondy's theorem is stronger than Posa's theorem.

Solution. The degree sequence in the graph of Fig. 3-27 of order 7 is [2 2 4 5 5 5 5] in which d_2 is not more than 2 even through 2 is less than 7/2. Thus for this Hamiltonian graph, Posa's hypothesis is not satisfied. Bondy's hypothesis is satisfied, however.

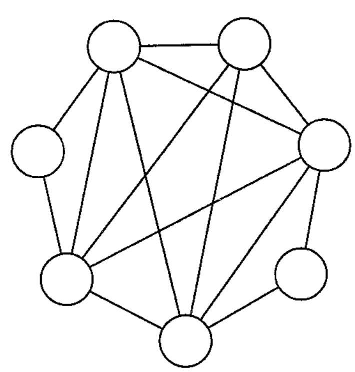


Fig. 3-27

3.54 Show that Posa's theorem is stronger than Ore's theorem.

Solution. In the graph of Fig. 3-28 of order 6, it is easy to locate two nonadjacent vertices whose degree sum is less than 6. This Hamiltonian graph satisfies Posa's hypothesis, however.

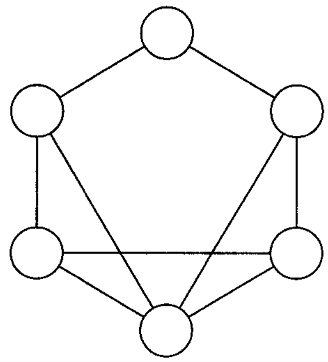


Fig. 3-28

3.55 Show that Ore's theorem is stronger than Dirac's theorem.

Solution. In the Hamiltonian graph of Fig. 3-29 of order 5, there is a vertex with degree less than 5/2. The degree sum of every pair of nonadjacent vertices is at least 5, however. Thus Ore's hypothesis is satisfied.

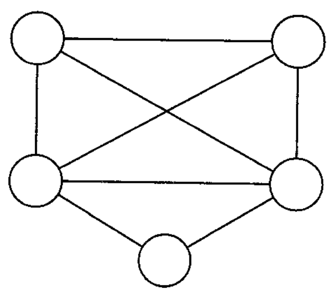


Fig. 3-29

3.56 (*a*) If $G = (V, E)$ is a graph with n vertices and m edges (where m is at least three), and if $m \geq [(n - 1)(n - 2)/2] + 2$, G is Hamiltonian. (*b*) Show that the converse is not true by exhibiting a counterexample. (*c*) Show that the inequality with this bound on the size of the graph is "sharp" in the sense that there is a non-Hamiltonian graph whose size is one less than this bound.

Solution.

(*a*) If G is complete, it is Hamiltonian. Otherwise, let u and v be two nonadjacent vertices in G with degrees x and y, respectively. If we delete these two vertices from G, we get a subgraph with $(n - 2)$ vertices and q edges, where $q \leq (n - 2)(n - 3)/2$. Now $m = q + x + y$ since u and v are nonadjacent. So $x + y = m - q \geq \{[(n - 1)(n - 2)/2] + 2\} - \{[(n - 2)(n - 3)/2]\} = n$. So by Ore's theorem, G is complete.

(*b*) The converse is not true. Any cyclic graph with four or more vertices is Hamiltonian. But the inequality does not hold.

(*c*) Consider the graph G obtained by creating a new vertex and joining it to exactly one vertex of the cyclic graph with three vertices.

3.57 If $G = (V, W, E)$ is a bipartite graph with $|V| = |W| = n$, and if the degree of each vertex is more than $n/2$, G is Hamiltonian.

Solution. Suppose G is not Hamiltonian. Add as many edges as possible joining vertices in V and W until we obtain a graph H that will become Hamiltonian if one more such edge is added. H cannot be the complete bipartite graph $K_{n,n}$. If the degree in G of each vertex is more than $n/2$, the degree of each vertex in H is also more than $n/2$. Let $u \in V$ and $v \in V$ be two nonadjacent vertices H. Obviously, there is a Hamiltonian path $u = v_1, v_2, \ldots, v_{2n} = v$ in H from u to v where v_i is in V if and only if i is odd. If there is an edge joining v_1 and v_i, there cannot be an edge joining v_{2n} and v_{i-i} since H is non-Hamiltonian. Since the degree of v_1 is more than $n/2$, we find that the degree of v_{2n} is less than $n - (n/2)$, which contradicts the hypothesis.

3.58 Show that if G is Eulerian, its line graph $L(G)$ is Hamiltonian. Give a counterexample to show that the converse is not true.

Solution. Each edge in G defines a vertex in the line graph $L(G)$ such that the vertices corresponding to two edges are adjacent in the line graph if and only if the two edges have a vertex in common. Suppose G is Eulerian with m edges. Let an Eulerian circuit in G be the sequence $\langle e(1), e(2), \ldots, e(m)\rangle$. Then this sequence of vertices defines a spanning cycle in the line graph. The converse is not true; $L(K_4)$ is Hamiltonian, but K_4 is not Eulerian. (So if a graph G under consideration is a line graph, we have an easily verifiable sufficient condition to test whether it is Hamiltonian. Unfortunately, most graphs are not line graphs.)

3.59 Show that if a graph G is Hamiltonian, its line graph $L(G)$ is Hamiltonian. Give a counterexample to show that the converse is not true.

Solution. Let C be a Hamiltonian cycle in G. If $L(C)$ is a Hamiltonian cycle in $L(G)$, we are done. If that is not the case, let e be an edge in G that is not in C, and let the edges in C adjacent to e be e_p and e_q. Then there is an edge f_p in $L(G)$ joining the vertices corresponding to E and e_p and an edge f_q in $L(G)$ joining the vertices corresponding to e and e_q. Now delete the edge joining the vertex corresponding to e_p and the vertex corresponding to e_q in the cycle $L(C)$, and enlarge the cycle by introducing two new consecutive edges f_p and f_q together with the intermediate vertex that corresponds to edge e. We continue this process until every edge not in C is taken care of. This completes the proof. The converse is not true; the bipartite graph $K_{1,3}$ is not Hamiltonian, but its line graph is Hamiltonian.

Hamiltonian-Connected Graphs

3.60 Prove Theorem 3.7: A graph G with n vertices (n is at least three) is Hamiltonian-connected if the sum of the degrees of any two nonadjacent vertices is more than n. In particular, it is Hamiltonian-connected if the degree of each vertex is more than $n/2$.

Solution. The hypothesis implies that G is Hamiltonian by Ore's theorem. Relabel the vertices (if necessary) so that C: $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle in the graph. If the graph is not Hamiltonian-connected, there will be two vertices v_i and v_j (with $i < j$) that are not joined by a Hamiltonian path. This implies that the vertices v_{i+i} and v_{j+1} (the addition of subscripts is modulo n) are not adjacent; otherwise, the path $v_i, v_{i-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{j+1}, v_{i+1}, v_{i+2}, v_j$ will be a Hamiltonian path between v_i and v_j. We now seek a lower bound on the number of vertices that are adjacent to v_{i+1}. Suppose there are p vertices $v_r (i + 1 < r < j + 1)$ and q vertices v_s ($s < i + 1$ or $s > j + 1$) that are adjacent to v_{i+1}. Thus the degree of v_{i+1} is $(p+q)$. That there is no Hamiltonian path between v_i and v_j implies that there is no edge between v_{r-1} and v_{j+1} in the first category and no edge between v_{s+1} and v_{j+1} in the second category. Observe that vertex v_{i+1} is common to both these sets. So there are at least $p + q - 1$ vertices to which v_{j+1} is not adjacent. Hence, (degree v_{j+1}) $\leq (n - 1) - (p + q - 1) = n -$ (degree v_{i+1}). This contradicts the hypothesis. So G is Hamiltonian-connected. In particular, if the degree of each vertex is more than $n/2$, the graph is Hamiltonian-connected. Thus the theorems of Ore and Dirac are generalized.

3.61 Give a counterexample to show that the sufficient conditions established in Problem 3.60 for a graph to be Hamiltonian-connected are not necessary conditions.

Solution. Figure 3-30 shows a Hamiltonian-connected graph with eight vertices. The degree of each vertex is only 3.

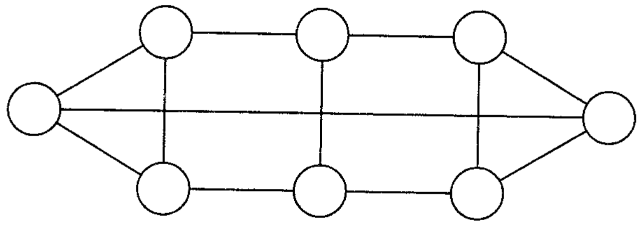


Fig. 3-30

3.62 Define the concept of closure in the context of graphs that are Hamiltonian-connected. Show that the closure of G thus defined is unique. State and prove the appropriate theorem in this context.

Solution. It is easy to show that if u and v are nonadjacent vertices of a graph of order n such that the degree sum of these two vertices is at least $(n + 1)$, G is Hamiltonian-connected if and only if $G + uv$ is Hamiltonian-connected, as in Problem 3.41. A closure of a graph G is obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $(n + 1)$. As in Problem 3.42, it can be shown that the closure that may be denoted by $c'(G)$ is unique. Thus the appropriate theorem is the assertion that G is Hamiltonian-connected if and only if $c'(G)$ is Hamiltonian-connected. This assertion can be verified as done in Problem 3.43. Consequently, a graph is Hamiltonian-connected if its closure $c'(G)$ is complete.

3.63 Give an example of a Hamiltonian-connected graph whose closure as defined in Problem 3.62 is not complete.

Solution. The graph in Fig. 3-30 is Hamiltonian-connected and is its own closure. But it is not complete.

3.64 (*a*) If $G = (V, E)$ is a graph with n vertices and m edges (where m is at least three), and if $m \geq [(n - 1)(n - 2)/2] + 3$, G is Hamiltonian-connected. (*b*) Show that the condition is sufficient but not necessary by exhibiting a counterexample.

Solution.

(*a*) The proof is similar to that of Problem 3.56 when we use the sufficient condition in Theorem 3.7.

(*b*) Consider the graph in Fig. 3-30, which is Hamiltonian-connected. Here $n = 8$ and $m = 12$. Thus $[(n - 1)(n - 2)/2] + 3 = 24$.

3.65 Show that if a graph with n vertices $(n > 3)$ and m edges is Hamilton-connected, it is necessary that $m \geq (3n)/2$. Show that this condition is by no means sufficient for a graph to be Hamiltonian-connected.

Solution. Observe that if a graph with more than three vertices is Hamiltonian-connected, the degree of each vertex has to be at least three. Thus the sum of all the degrees is at least $3n$. So $2m \geq 3n$. The converse is not true; the inequality is certainly satisfied for the complete bipartite graph $K_{3,4}$, but it is not Hamiltonian-connected.

3.66 A Hamiltonian graph is said to be **strongly Hamiltonian** if every edge in the graph belongs to a Hamiltonian cycle. Give an example of (*a*) a strongly Hamiltonian graph and (*b*) a Hamiltonian graph that is not strongly Hamiltonian.

Solution.

(*a*) Every cyclic graph is obviously strongly Hamiltonian.

(*b*) The graph obtained by joining two nonadjacent vertices of a cyclic graph with more than three vertices is not strongly Hamiltonian.

3.67 Show that a Hamiltonian-connected graph is a strongly Hamiltonian graph. Is the converse true?

Solution. Let e be an edge in a Hamiltonian-connected graph joining vertices u and v. There is a Hamiltonian path P between u and v. If we adjoin e to this path, we get a Hamiltonian cycle in the graph that contains edge e. The converse is not true since any cyclic graph with more than three vertices is not Hamiltonian-connected, even though it is strongly Hamiltonian.

3.68 Obtain a sufficient condition for a graph to be strongly Hamiltonian.

Solution. If G is a graph of order n, and if the degree sum of any pair of nonadjacent vertices is more than n (or if the degree of each vertex is more than $n/2$), G is Hamiltonian-connected and therefore strongly Hamiltonian.

3.69 Show that if the degree sum of every pair of nonadjacent vertices in a non–Hamiltonian-connected graph with three or more vertices is at least k (where k is some positive integer), the graph contains a path of length k.

Solution. Let P: $v_0, v_1, v_2, \ldots, v_p$ be a longest path in the graph. In other words, the number of edges in any path in the graph cannot exceed p. Since P is a longest path, neither of its terminal vertices can be adjacent to a vertex not in P. Let terminal vertex v_0 be adjacent to intermediate vertex v_i. Then we claim that the other terminal vertex v_p cannot be adjacent to the vertex v_{i-1}. Suppose this were the case. Then the p vertices in the path constitute the cycle C: $v_0, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1, v_0$, which cannot contain all the vertices of the non-Hamiltonian graph. This implies that there is a vertex w that is not a vertex in the cycle but that is adjacent to one of its vertices. So there is a path of length $(p + 1)$ in the graph violating the maximality of P. Since v_p is adjacent to v_{p-1}, we conclude that the two terminal vertices are nonadjacent. Hence, (degree of v_p) $\leq p -$ (degree of v_0). Thus $p \geq$ (degree of v_p) $+$ (degree of v_0). But (degree of v_p) $+$ (degree of v_0) $\geq k$ since the two terminal vertices of the longest path are nonadjacent. Since there is a path with p edges and since $p \geq k$, there exists a path with k edges.

3.70 If the degree sum of every pair of nonadjacent vertices in a graph of order n is at least $(n - 1)$, the graph has a Hamiltonian path. In particular, if the degree of each vertex is at least $(n - 1)/2$, the graph has a Hamiltonian path.

Solution. The hypothesis implies that the graph is connected. The result follows from Problem 3.69.

Randomly Hamiltonian Graphs (Chartrand–Kronk Theory)

3.71 A graph G is **randomly traceable** if a Hamiltonian path results upon starting from any vertex and successively proceeding to any adjacent vertex that is not yet in the path. Furthermore, if G has at least three vertices and the terminal vertex of each such path is adjacent to the initial vertex, the graph is called **randomly Hamiltonian.** Obviously, cyclic graphs and complete graphs with three or more vertices are randomly traceable and randomly Hamiltonian. Show that (*a*) G is randomly traceable if and only if it is randomly Hamiltonian, and (*b*) every randomly traceable graph is Hamiltonian.

Solution.

(*a*) Obviously, any randomly Hamiltonian graph is randomly traceable. To prove the converse, assume that G is randomly traceable, and let P be any Hamiltonian path $\langle v_1, v_2, \ldots, v_n\rangle$. It is enough if we show that there is an edge in the graph joining the first vertex and last vertex of this path. Since the graph is randomly traceable, once we have path $P' = \langle v_2, v_3, \ldots, v_n\rangle$, the next vertex has to be v_1 to complete this path into a Hamiltonian path. This implies that the vertices v_n and v_1 are adjacent.

(*b*) Every randomly traceable graph is randomly Hamiltonian; therefore, it is Hamiltonian.

3.72 Show that if $C = \langle v_1, v_2, \ldots, v_n, v_1\rangle$ is a Hamiltonian cycle in a randomly Hamiltonian graph and if there is an edge in the graph between v_j and v_k, there is an edge between v_{j+i} and v_{k+i} for $i = 1, 2, \ldots, n - 1$. (Here the addition of the subscripts is modulo n.)

Solution. The path $P = \langle v_{j+2}, v_{j+3}, \ldots, v_k, v_{j-1}, v_{j-2}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{k+1}\rangle$, which started from v_{j+2} and ended in v_{k+1}, passes through every vertex of the graph except v_{j+1}. Since the graph is randomly

Hamiltonian, there has to be an edge between v_{j+1} and v_{k+1}. Now consider the path $P' = \langle v_{j+3}, v_{j+4}, \ldots, v_k, v_{k+1}, v_{j+1}, v_j, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{k+2} \rangle$, which started from v_{j+3} and ended in v_{k+2} and which did not pass through v_{j+2}. So there is an edge between v_{k+2} and v_{j+2}. By repeating the process, we get the desired result.

3.73 Show that if $C = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a Hamiltonian cycle in a randomly Hamiltonian graph G and if there is an edge between v_p and v_{p+2} (for some p), G is a complete graph.

Solution. Select any two arbitrary vertices v_j and v_k in G. By Problem 3.72, there is an edge between v_{j-1} and v_{j+1}. Now consider the path $\langle v_{k+1}, v_{k+2}, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_k \rangle$, which starts from v_{k+1} and terminates in v_k, that contains all the vertices except v_j. So there has to be an edge between v_j and v_k. In other words, G is complete.

3.74 Let G be a randomly Hamiltonian graph of order n, and let C be any fixed Hamiltonian cycle in this graph. The edges in C are called **cycle edges,** and the edges not in C are called **diagonal edges.** Any cycle C' consisting of $(k - 1)$ cycle edges and exactly one diagonal edge is called an **outer k-cycle.** Show that the minimum value of k such that G has an outer k-cycle is either 3 or 4. Furthermore, if $k = 3$, the graph is a complete graph, and if $k = 4$, the number of vertices in the graph is even.

Solution. Let $C = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$ be a Hamiltonian cycle in G. Suppose there is no outer cycle of length 4, and suppose $C' = \langle v_1, v_2, \ldots, v_k, v_1 \rangle$ is an outer k-cycle, where $k > 4$. Since the edge joining v_1 and v_k is a diagonal edge, the edge joining v_2 and v_{k+1} is also a diagonal by Problem 3.72. Now consider path P, which starts from v_4 and ends in v_n, defined by the following sequence $\langle v_4, \ldots, v_k, v_1, v_2, v_{k+1}, \ldots, v_n \rangle$. This path goes through every vertex other than v_3. So there is an edge joining v_n and v_3. Hence, we have an outer cycle $\langle v_1, v_2, v_3, v_n \rangle$ consisting of four edges, which contradicts the hypothesis. Thus $k = 3$ or $k = 4$. In the former case, there is an edge between every pair of vertices in cycle C (see Problem 3.72); therefore, the graph is complete. Suppose n is odd in the latter case when the smallest outer cycle has four edges. So G contains all the diagonal edges joining v_i and v_{i+3} ($i = 1, 2, \ldots, n$). Consider the path starting from v_5 defined by the sequence $\langle v_5, v_4, v_7, v_6, v_9, v_8, \ldots, v_{n-1}, v_{n-2}, v_2, v_1 \rangle$. This path does not pass through v_3. Since the graph is randomly Hamiltonian, there is an edge between v_1 and v_3, creating an outer cycle with less than four edges. This contradiction establishes that the number of vertices is even.

3.75 Show that a graph with three or more vertices is randomly Hamiltonian if and only if it is a cyclic graph or a complete graph or a complete bipartite graph with equal number of vertices in each part.

Solution. If a graph is a cyclic, a complete, or a complete bipartite graph with equal number of vertices in each part, the graph is obviously randomly Hamiltonian. Conversely, let G be a randomly Hamiltonian graph with n vertices, and let C be a Hamiltonian cycle in G. If every edge in G is an edge of C, G is a cyclic graph. So for the remaining part of the theorem we assume that G is not a cyclic graph. The smallest value of k such that G has an outer k-cycle is either 3 or 4. If $k = 3$, by Problem 3.72, the graph is complete. If $k = 4$, the order of the graph is even, as established in Problem 3.74. Observe that there is a diagonal edge joining vertex v_i and vertex v_{i+3} for $i = 1, 2, \ldots, n$. Let v_r be any vertex where r is even. Consider the path from v_3 to v_1 defined by the following sequence: $\langle v_3, v_2, v_5, v_4, v_7, v_6, \ldots, v_{r-1}, v_{r-2}, v_{r+1}, v_{r+2}, \ldots, v_n, v_1 \rangle$. This path contains all the vertices except v_r. So there is an edge joining v_1 and v_r where r is even. By Problem 3.72, there is an edge joining v_2 and v_r where r is odd. More generally, there is an edge between v_i and v_j whenever i and j are of opposite parity. Next, we show that there is no edge in the graph joining v_i and v_j when i and j are of the same parity. It is enough if we show that there is no edge between v_1 and v_s when s is odd. Suppose there is an edge joining v_1 and v_s when s is odd. Now we construct the path $\langle v_{s+2}, v_{s+1}, v_{s+4}, v_{s+3}, v_{s+6}, v_{s+5}, \ldots, v_{n-1}, v_{n-2}, v_1, v_s, v_{s-1}, v_{s-2}, \ldots, v_3, v_2 \rangle$, which passes through every vertex other than v_n. So there is an edge joining v_2 and v_n, implying that we have an outer cycle consisting of three vertices, which contradicts the assumption.

3.76 Show that the cyclic graph is the only graph that is both randomly Eulerian and randomly Hamiltonian.

Solution. A graph is randomly Eulerian if and only if it is a cyclic graph. This assertion due to O. Ore was established in Problem 3.31. In conjunction with the Chartrand–Kronk theory, this assertion leads us to the conclusion that the only graph that is both randomly Eulerian and randomly Hamiltonian is the cyclic graph.

3.77 A graph G is **strongly randomly traceable** if for every two distinct vertices u and v, a Hamiltonian path between u and v exists whenever we start from u and successively proceed to another vertex not yet encountered, with the restriction that v will be chosen only when there is no other alternative. Show that a graph G of order n is a strongly randomly traceable graph if and only if it is the complete graph K_n.

Solution. Any complete graph is strongly randomly traceable. On the other hand, suppose G is strongly randomly traceable. If we delete a vertex from G, the resulting graph should be randomly Hamiltonian. So G can be neither a cyclic graph with more than three vertices nor a complete bipartite graph with equal number of vertices on its two parts. Thus the only alternative is that G is the complete graph with n vertices.

TOURNAMENTS

3.78 *(Landau's Theorem)* The outdegree of a vertex in a tournament can be considered as the score of the player represented by that vertex. A player with maximum score is a winner. If u is a player who has defeated a winner w in the tournament, show that w has defeated some player who has defeated u.

Solution. Let G be a tournament of order n. Suppose the outdegree of winner w is p. Thus w has defeated p players who constitute the set X, and at the same time, w has been defeated by $(n - 1 - p)$ players who constitute the set Y. Let u be any player in Y. If u has defeated every player in X, the outdegree of u will at least $(p + 1)$, which is a contradiction since the outdegree of no vertex can exceed p. So there is a vertex v in X such that there is an arc from v to u. This completes the proof.

[A set D of vertices in a graph (digraph) is called a ***k*-dominating set** if there is a path (directed path) from every vertex in D to every other vertex in the graph (digraph) such that the number of edges (arcs) in the path does not exceed k. If $k = 1$, D is a dominating set as defined in Solved Problem 1.21. Any player with the maximum outdegree is called a winner in the tournament. Landau's theorem asserts that the winners in a tournament constitute a 2-dominating set.]

3.79 *(Moon–Moser Theorem)* A graph (digraph) of order n $(n \geq 3)$ is **vertex-pancyclic** if every vertex is contained in a cycle (directed cycle) of length p for every p $(3 \leq p \leq n)$. Show that a strongly connected tournament is vertex-pancyclic.

Solution. Let v be any vertex in the graph $G = (V, E)$. The proof is by induction on p. Since the graph is strongly connected, both the outdegree and indegree of each vertex is necessarily positive.

Let $X = \{u \in V$: there is an arc from v to u in the tournament$\}$ and $Y = \{u \in V$: there is an arc from u to v in the tournament$\}$. Since the graph is strongly connected, there should be an arc from a vertex in X to a vertex in Y. So there is a cycle of length 3 passing through v. Hence, the theorem is true when $p = 3$.

Suppose there is a directed cycle $C = \langle v_1, v_2, \ldots, v_k, v_1 \rangle$, where $k < n$ passing through vertex v_1. Let v be any vertex not in this cycle. We consider two cases.

Case (i): There is an arc from v to one of the vertices in C, and there is an arc from one of the vertices in C to v. Then there must be two adjacent vertices in the cycle such that there is an arc from one of them to v_1 and an arc from v_1 to the other. Then using these two vertices and v, we can construct a cycle of length $(k + 1)$ passing through v_1.

Case (ii): No vertex exists, as in case (i). Let $X = \{x \in V$: there is an arc from each vertex in C to $x\}$ and $Y = \{y \in V$: there is an arc to y to each vertex in $C\}$. Since the graph is strongly connected, both X and Y are nonempty. The strong connectivity also implies that there has to be an arc from vertex u in X to vertex v in Y. Then the length of the cycle $C' = \langle v_1, v_2, v_3, \ldots, v_{k-2}, u, v, v_1 \rangle$ that passes through v_1 is $(k + 1)$, which completes the induction argument.

3.80 *(Camion's Theorem)* Prove Theorem 3.13: A tournament is Hamiltonian if and only if it is strongly connected.

Solution. If the tournament is strongly connected, it is vertex-pancyclic (as proved in Problem 3.79); hence, it is Hamiltonian. On the other hand, if it is Hamiltonian, there is a path from every vertex to every other vertex. So the graph is strongly connected.

3.81 Obtain a necessary and sufficient condition to be satisfied by a vertex in a tournament so that there is a Hamiltonian path starting from that vertex.

Solution. In a tournament $G = (V, E)$, any subset W of vertices is an **outclassed group** (as defined by O. Ore) if there is no arc in the tournament from a vertex in W to a vertex in $(V - W)$. So the initial vertex of any Hamiltonian path in a tournament cannot be a vertex in any outclassed group. We now claim that there is a Hamiltonian path from any vertex v_1 that does not belong to any outclassed group. Suppose we start from v_1 and get the path $P = \langle v_1, v_2, \ldots, v_k \rangle$. If this path has all the vertices, we are done. Let v_{k+1} be a vertex not in this path. If there is an arc from any vertex in P to v_{k+1}, we can enlarge the path into path P' (with $k + 1$ vertices) that includes this vertex either as an intermediate vertex or as the terminal vertex. Thus the path starting from v_1 is extended as much as possible. If w is a vertex not in the path thus extended, there is an arc from w to every vertex in the path that implies that the vertices in the path (including v_1) form an outclassed set, which is a contradiction. Thus the extended path contains all the vertices.

3.82 A tournament $G = (V, E)$ is **irreducible** whenever W is a subset of V, there should be an arc from a vertex in W to a vertex in $V - W$. Show that in a tournament $G = (V, E)$, the following concepts are equivalent: (*a*) G is Hamiltonian, (*b*) G is strongly connected, (*c*) G is irreducible.

Solution. The equivalence of (*a*) and (*b*) was already established in Problem 3.80. If a tournament is Hamiltonian, and if W is any set of vertices, there should be an arc from a vertex in W to a vertex in $(V - W)$. So any Hamiltonian tournament is irreducible. Now consider an irreducible tournament G. In any Hamiltonian path in the tournament, there should be an arc from the terminal vertex to some vertex v in the path. If v is the starting vertex of the path, we are done. Otherwise, we have cycle $C = \langle v_1, v_2, \ldots, v_k, v_1 \rangle$ through k vertices, where k is less than the order n of the tournament. Let X be the set of those vertices x not in C such that there is an arc from x to a vertex in C, and let Y be the set of those vertices y not in C such that there is an arc from y to a vertex in C. If X is empty, there is no arc from any vertex in Y to a vertex in $V - Y$, which is against the hypothesis. If Y is empty, there is no arc from C to $(V - C)$, which contradicts the hypothesis. Thus both X and Y are not empty. By the same reasoning, there is a vertex y in Y and a vertex x in X such that there is an arc from y to x. Then we can locate two vertices v_i and v_{i+1} in the cycle such that there is an arc from the former to y and an arc from the latter to x. Thus cycle C becomes enlarged with two intermediate vertices between v_i and v_{i+1}. We continue this process until we get a cycle that includes all the vertices.

3.83 Show that every tournament is either a strongly connected digraph or a digraph that can be converted into a strongly connected digraph by reversing the orientation of exactly one arc.

Solution. Suppose G is a tournament. Let u and v be any two vertices in G. So there is a directed Hamiltonian path from u to v. Either there is an arc from v to u or there is an arc from u to v. In the former case, the graph is strongly connected. In the latter case, change the orientation of the arc; the resulting digraph then becomes strongly connected.

3.84 The sequence $\langle s_1, s_2, \ldots, s_n \rangle$ of nonnegative integers is called a **score sequence** (of a tournament) if there exists a tournament of order n whose vertices can be labeled v_i so that the outdegree of each v_i is s_i for each $i = 1, 2, \ldots, n$. Show that a nondecreasing sequence of n nonnegative integers is the score sequence of a transitive tournament if and only if the sequence is $\langle 0, 1, 2, \ldots, n - 1 \rangle$.

Solution. If the score sequence of a tournament G is $\langle 0, 1, 2, \ldots, n \rangle$, the n vertices of G can be labeled $v_1, v_2, \ldots, v_n$ such that every arc is from vertex v_i to vertex v_j, where $i < j$. Obviously, this is a transitive tournament. On the other hand, suppose G is a transitive tournament. Thus it has a unique Hamiltonian path. Label the vertices v_i $(i = 1, 2, \ldots, n)$ such that this unique path is $v_1 \to v_2 \to \cdots \to v_n$. Since G is transitive, every arc is from vertex v_i to vertex v_j, where $i < j$. So the outdegree of v_k is $(k - i)$ for each k. [It follows from this problem that there is exactly one (up to isomorphism) transitive tournament of order n for every positive integer n.]

3.85 Show that the sequence $\langle s_1, s_2, \ldots, s_n \rangle$ of nondecreasing nonnegative integers is a score sequence of a tournament if and only if $s_1 + s_2 + \cdots + s_k \geq k(k-1)/2$ for $1 \leq k \leq n$ with equality holding when $k = n$.

Solution. Consider the subgraph H induced by the set $W = \{v_1, v_2, \ldots, v_k\}$ of k vertices. H is a subtournament. The sum of the outdegrees (in H) of these k vertices is $k(k-1)/2$. The sum $\Sigma_{i=1}^{k} s_i$ has to be at least equal to $k(k-1)/2$ since there could be arcs in the tournament from vertices in W to vertices in $V - W$. Thus the inequality holds for all $k \leq n$. Obviously, the inequality becomes an equality when $k = n$.

The converse is proved by contradiction. If there is a sequence satisfying the given conditions that is not the score sequence of a tournament, there should be a sequence with n elements, where n is as small as possible. Among such sequences, choose $S = \langle s_1, s_2, \ldots, s_n \rangle$, where the first component s_1 is as small as possible. We consider two cases.

Case (i): Suppose there exists $k\ (0 < k < n)$ such that $s_1 + s_2 + \cdots + s_k = k(k-1)/2$. Then $S_1 = \langle s_1, s_2, \ldots, s_k \rangle$ is the score sequence of some tournament $H_1 = (V_1, E_1)$ of order k because of the minimality assumption on n.

Consider the sequence $S_2 = \langle t_1, t_2, \ldots, t_{n-k} \rangle$, where $t_i = s_{k+1} - k$, $i = 1, 2, \ldots, (n-k)$. This is a nondecreasing sequence. Suppose m is any positive integer where $m \leq (n-k)$. Then $t_1 + t_2 + \cdots + t_m = (s_{k+1} + s_{k+2} + \cdots + s_{k+m}) - mk = (s_1 + s_2 + \cdots + s_{m+k}) - (s_1 + s_2 + \cdots + s_k) - mk \geq [(m+k)(m+k-1)/2] - [k(k-1)/2] - mk = m(m-1)/2$ and $t_1 + t_2 + \cdots + t_{n-k} = (n-k)(n-k-1)/2$. So, by hypothesis, S_2 is the score sequence of some tournament $H_2 = (V_2, E_2)$ of order $(n-k)$. Let $H = (V, E)$ be a digraph where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup F$, where F is the set of arcs obtained by drawing an arc from each vertex in V_2 to each vertex in V_1. Then it is easy to see that H is a tournament with the score sequence S, which contradicts the assumption that there is no tournament with S as the score sequence.

Case (ii): $s_1 + s_2 + \cdots + s_k > k(k-1)/2$ for every $k < (n-1)$. In particular, s_1 is positive. Then $S' = \langle s_1 - 1, s_2, \ldots, s_{n-1}, s_n + 1 \rangle$ is a another nondecreasing sequence satisfying the requirements of the hypothesis. Due to the minimality condition on s_1, we conclude that there is a tournament H' for which S' is the score sequence.

Let x, y be the vertices in H' with respective outdegrees $s_n + 1$ and $s_1 - 1$. Then $(s_1 - 1) + 2 \leq (s_n + 1)$, which implies that there is a vertex v in H' such that there is an arc from x to v and an arc from v to x. If we reverse the directions of these two arcs, we get a new tournament with S as a score sequence, which is again a contradiction. (The proof presented here is due to C. Thomassen.)

3.86 Show that the sequence $\langle s_1, s_2, \ldots, s_n \rangle$ of nondecreasing nonnegative integers is a score of a strongly connected tournament if and only if $s_1 + s_2 + \cdots + s_k > k(k-1)/2$ for $1 \leq k \leq (n-1)$ and $s_1 + s_2 + \cdots + s_n = n(n-1)/2$.

Solution. Let $S = \langle s_1, s_2, \ldots, s_n \rangle$ be the score sequence of a strongly connected tournament G, where s_i is the score (outdegree) of vertex v_i. Then, obviously, $s_1 + s_2 + \cdots + s_n = n(n-1)/2$. Now consider the subgraph H induced by $W = \{v_1, v_2, \ldots, v_k\}$, where $k < n$. The subgraph H is a strongly connected tournament, and the sum of the outdegrees (in H) of these k vertices is $k(k-1)/2$. As we saw in Problem 3.85, this sum is at most equal to $s_1 + s_2 + \cdots + s_k$. The strong connectivity implies that there is at least one vertex v_i in W such that the outdegree of v_i in G is more than the outdegree of the same vertex in H. Thus $s_1 + s_2 + \cdots + s_k > k(k-1)/2$ for $k = 1, 2, \ldots, (n-1)$.

To prove the converse, let the sequence $S = \langle s_1, s_2, \ldots, s_n \rangle$ satisfy the hypothesis of the theorem. By Problem 3.85, there is a tournament $G = (V, E)$ with S as the score sequence. We have to show that G is strongly connected. Suppose it is not. Then there is a set W of k vertices $(k < n)$ such that there is a directed path between every pair of vertices in W. The subgraph H induced by W is called a strong component of G. If u is any vertex in $(V - W)$ and w is any vertex in W, the arc between these two vertices has to be an arc from u to w. Consequently, the outdegree of every vertex in W is the same for both G and H. So the sum of the outdegrees (in G) of these k vertices in W is $k(k-1)/2$.

Now, $s_1 + s_2 + \cdots + s_k) \le$ (sum of outdegrees in G of the k vertices in W) since the sequence S is nondecreasing. So $(s_1 + s_2 + \cdots + s_k) \le k(k - 1)/2$, which contradicts the hypothesis. Thus the tournament is strongly connected.

Supplementary Problems

3.87 If G is a connected graph with k odd vertices, find the minimum number of trails in G such that every edge in the graph is an edge in exactly one of these trails. *Ans.* $(k/2)$

3.88 Show that if a multidigraph has an Eulerian walk but not an Eulerian circuit, exactly one vertex has an excess of one indegree and exactly one vertex has an excess of one outdegree. [*Hint:* Generalize Theorem 3.3.]

3.89 Show that if a graph has a circuit of odd length, it has a cycle of odd length. [*Hint:* Every circuit is a disjoint union of cycles.]

3.90 Suppose in a group of n people ($n > 3$), any two of them together know all the other people in the group. Show that these n people can be seated around a circular table so that each person is seated between two acquaintances. [*Hint:* Consider the acquaintance graph and use Problem 3.69.]

3.91 Show that a k-regular graph with $(2k - 1)$ vertices is Hamiltonian. [*Hint:* Use Dirac's theorem.]

3.92 Show that any k-regular simple graph with $(2k - 1)$ vertices is Hamiltonian. [*Hint:* Use Dirac's theorem.]

3.93 A nontrivial connected graph is Eulerian if and only if every block in the graph is Eulerian. [*Hint:* This follows from the definition of a block in a graph.]

3.94 Find the number of Hamiltonian graphs in K_n. *Ans.* $\frac{1}{2}[(n - 1)!]$

3.95 Find the number of Hamiltonian cycles in $K_{n,n}$. *Ans.* $\frac{1}{2}[(n - 1)!](n!)$

3.96 Show that if n is odd, the set of edges of K_n can be partitioned into $\frac{1}{2}(n - 1)$ disjoint Hamiltonian cycles. [*Hint:* If the vertices are $1, 2, 3, \ldots, n$, arrange these vertices in a cycle in that order. The different cycles are $1 — (1 + i) — (1 + i + i) — \cdots — 1$, where addition is modulo n and $1 \le i \le \frac{1}{2}(n - 1)$.]

3.97 Thirteen mathematicians are attending a six-day conference. Each night they sit around a circular table for dinner so that no two persons sit next to each other more than once and so that each person gets a chance to sit to next to each other person exactly once during those six nights.
Ans. List the six distinct Hamiltonian cycles in the complete graph with 13 vertices (as indicated in Problem 3.96), where the vertices are labeled $1, 2, 3, \ldots, 13$, representing the 13 individuals. The six cycles (giving the seating arrangements around the circular table) are

(*a*) 1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11 — 12 — 13 — 1
(*b*) 1 — 3 — 5 — 7 — 9 — 11 — 13 — 2 — 4 — 6 — 8 — 10 — 12 — 1
(*c*) 1 — 4 — 7 — 10 — 13 — 3 — 6 — 9 — 12 — 2 — 5 — 8 — 11 — 1
(*d*) 1 — 5 — 9 — 13 — 4 — 8 — 12 — 3 — 7 — 11 — 2 — 6 — 10 — 1
(*e*) 1 — 6 — 11 — 3 — 8 — 13 — 5 — 10 — 2 — 7 — 12 — 4 — 9 — 1
(*f*) 1 — 7 — 13 — 6 — 12 — 5 — 11 — 4 — 10 — 3 — 9 — 2 — 8 — 1