7.68 Obtain an isomorphic factorization of the Petersen graph such that each factor is a path consisting of three edges.

Solution. In Fig. 7-20, each edge joining an outer vertex to the corresponding inner vertex belongs to a 1-factor, and the five blue edges belonging to this 1-factor are labeled 1, 2, 3, 4, and 5. The 2-factor consists of two components: one is the cycle $A \to B \to C \to E \to A$ and the other is $F \to H \to J \to G \to I \to F$. As we move from vertex A in the counterclockwise direction, edge AB gets label 1. As we move from F in the counterclockwise direction, edge FH gets label 1. Thus the path $H \to F \to A \to B$ is one copy of the isofactor. The remaining four copies correspond to the four paths labeled 2, 3, 4, and 5 respectively.

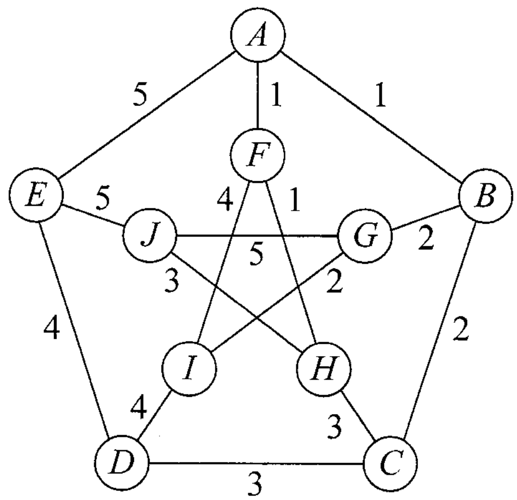


Fig. 7-20

7.69 Give examples of bipartite and nonbipartite cubic bridgeless graphs with a Hamiltonian cycle as a 2-factor.

Solution.

(*i*) The cube graph Q_3 with eight vertices [see Fig. 7-21(*a*)] is a bipartite bridgeless graph that can be factored into a Hamiltonian cycle and a 1-factor consisting of the pairs (1, 4), (2, 7), (3, 6), and (5, 8).

(*ii*) The cubic graph with eight vertices [see Fig. 7-21(*b*)] is a nonbipartite bridgeless graph that can be factored into a Hamiltonian cycle and a 1-factor consistng of the pairs (1, 3), (2, 4), (5, 7), and (6, 8).

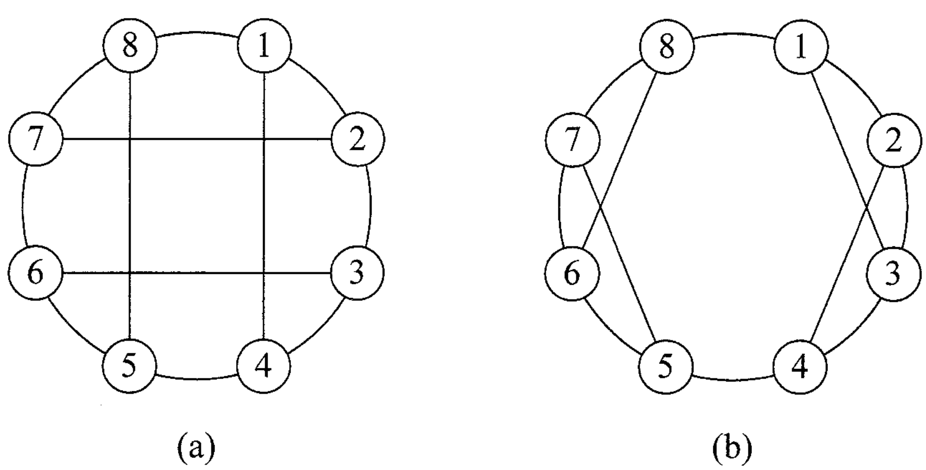


Fig. 7-21

7.70 Construct a nonbipartite cubic bridgeless graph of order 2^k with a Hamiltonian cycle as a 2-factor.

Solution. Label the vertices 1, 2, . . . , 2^k, and place them clockwise around a cycle. Join vertex i and vertex $i + 1$ for each i. Thus a 2-factor is constructed. To obtain the 1-factor, join vertex i and $i + 2$. Each $(i, i + 2)$ is a matched pair. See Figure 7-22, when $k = 4$.

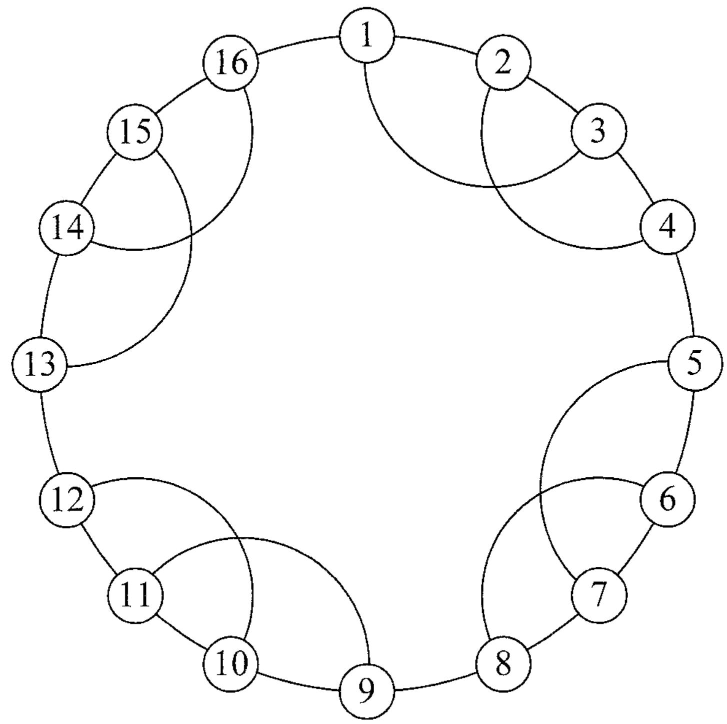


Fig. 7-22

7.71 Show that if a cubic graph G does not have a 1-factor, it will have at least three bridges, not all of them belonging to the same path. (In other words, if the bridges of a cubic graph lie on a single path, it has a 1-factor.)

Solution. By Tutte's theorem, G will have at least one bridge. Furthermore, there exists a set S of vertices such that $o(G - S) = k > |S| = s$. It has been already shown (see Problem 7.5) that k is odd if and only if s is odd. Hence, $k \geq (s + 2)$. Let the odd components of $(G - S)$ be $H_1, H_2, \ldots, H_k$, and let F_i be the set of edges of G between vertices in H_i and vertices in S for each i. Since the sum of the degrees (in G) of all the vertices in H_i is odd and the sum of the degrees (in H_i) of all the vertices in H_i is even, the number $f_i = |F_i|$ is necessarily odd. Let p be the number of odd components for which $f_i = 1$. Then $3s \geq p + 3(k - p) = 3k - 2p \geq 3(s + 2) - 2p = 3s + -2p$. So $p \geq 3$. Hence, the number of bridges in the graph is at least 3. Obviously no three bridges can belong to the same path.

7.72 If G is any $(r - 1)$ edge connected, r-regular graph, where $r \geq 3$ and odd, G has a 1-factor. (If $r = 3$, we recover one part of Theorem 7.6.)

Solution. We have to prove that $o(G - S) = k \leq |S| = s$, where S is any set of vertices of the graph. Let the odd components of $(G - S)$ be $H_1, H_2, \ldots, H_k$, and let F_i be the set of edges of G between vertices in H_i and vertices in S for each i, as in Problem 7.71. Since the sum of the degrees (in G) of all the vertices in H_i is an odd multiple of r and the sum of the degrees (in H_i) of all the vertices in H_i is even, $(f_i - r)$, where $f_i = |F_i|$, is necessarily even. But $(r - 1)$ edge connectivity implies that $f_i \geq (r - 1)$. Hence, $f_i \geq r$ for each odd component H_i. So $(f_1 + f_2 + \cdots + f_k) \geq rk$. The $(r - 1)$ edge connectivity also implies that $(r)(|S|) \geq (f_1 + f_2 + \cdots + f_k)$. Thus $|S| \geq k$. So by Tutte's theorem, G has a 1-factor.

7.73 If G is any $(r - 1)$ connected, r-regular graph, where $r \geq 3$ and odd, G has a 1-factor.

Solution. Any $(r - 1)$ connected graph is $(r - 1)$ edge connected. So the result follows from Problem 7.72.

7.74 Show that the Petersen graph is the complement of the line graph of the complete graph with five vertices.

Solution. The graph K_5 shown in Fig. 7-23 has 10 edges labeled as 1, 2, . . . , 10. Suppose e is the edge joining vertex u and vertex v in this graph. Then u is incident to three edges other than e. Likewise, v is adjacent to three other edges. Edge e is the only edge common to both u and v. In other words, the linegraph is a 6-regular graph with 10 vertices. Its complement is a 3-regular graph with 10 vertices. In the line graph, vertex 1 is adjacent to vertices 3, 4, 10, 7, 8, and 9. So in the complement of the line graph vertex 1 is adjacent to 2, 5, and 6. Likewise, vertex 2 is adjacent to vertices 1, 3 and 7. Thus the complement of the line graph is the graph shown in Fig. 7-3.

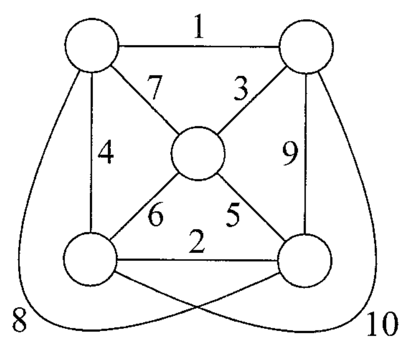


Fig. 7-23

7.75 Show that the Petersen graph has cycles of length 5, 6, 8, and 9.

Solution. In Fig. 7-3, we have cycles $C_5: 1 \to 2 \to 7 \to 10 \to 5 \to 1$, $C_6: 1 \to 2 \to 7 \to 10 \to 8 \to 6 \to 1$, $C_8: 1 \to 6 \to 8 \to 3 \to 4 \to 9 \to 7 \to 2 \to 1$, and $C_9: 1 \to 2 \to 3 \to 4 \to 9 \to 7 \to 10 \to 8 \to 6 \to 1$.

7.76 Show that the Petersen graph is not Hamiltonian. (This proof is by D. West.)

Solution. In Fig. 7-3, we see a 2-factor of the graph consisting of an "outer" cycle with vertices 1, 2, 3, 4, and 5 and an "inner" cycle with vertices 6, 7, 8, 9, and 10. In addition, there is a 1-factor consisting of five "links" connecting an outer vertex and its matching inner vertex. Suppose the graph is Hamiltonian. Then there is a Hamiltonian cycle that should contain either two or four of these connecting links. If the number of such links is 2, their end vertices are not adjacent in at least one of these two cycles. For instance, if the links are (1, 6) and (2, 7), end vertices 6 and 7 are not adjacent in the inner cycle. If the links are (1, 6) and (3, 8), end vertices 1 and 3 are not adjacent in the outer cycle. So if the number of links is 2, there are two end vertices u and v that are adjacent in one cycle and not adjacent in the other cycle. In the cycle in which u and v are not adjacent, there is no path between them that contains the remaining three vertices of that cycle as intermediate vertices. So the number of connecting links in the Hamiltonian cycle is not 2. If the number of links is 4, the number of vertices in a cycle consisting of these four links is 9; hence, there is no Hamiltonian cycle. So PG is not Hamiltonian.

7.77 A ***g*-cage** is a cubic graph with as few vertices as possible such that the number of edges in its smallest cycle is exactly g. Show that the Petersen graph is a 5-cage and that any 5-cage is isomorphic to it.

Solution. The number of edges in the smallest cycle in the Petersen graph PG is 5. To show that PG is a 5-cage, it is enough to prove that the number of vertices in any 5-cage G is at least 10. Let v_1 be a vertex in G that is adjacent to vertices v_2, v_3, and v_4, as in Fig. 7-24. Now v_2 will be adjacent to vertices v_5 and v_6, v_3 will be adjacent to v_7 and v_8, and v_4 will be adjacent to vertices v_9 and v_{10}. These 10 vertices are necessarily distinct since there is no 3-cycle or 4-cycle in the graph. So a 5-cage should have at least 10 vertices. Hence, *PG* (with 10 vertices) is indeed a 5-cage. Since *PG* is a 5-cage and has 10 vertices, any other 5-cage *G* should also be of order 10. Suppose *G* is any 5-cage with vertices as in Fig. 7-24. Since it is a cubic graph, vertex v_5 has to be adjacent to two more vertices. Without any loss of generality whatsoever, we join vertex v_5 to v_7 and v_9. Then we join vertex v_6 to v_8 and v_{10}. We cannot join v_7 to v_9. So v_7 and v_{10} are joined. Finally, we join v_8 and v_9. Thus the cubic graph *G* is constructed. If we relabel the vertices $v_i (i = 1, 2, \ldots, 10)$ as vertices 1, 5, 6, 2, 4, 10, 9, 8, 3, and 7, we have the Petersen graph displayed in Fig. 7-3. Thus the Petersen graph is the unique 5-cage.

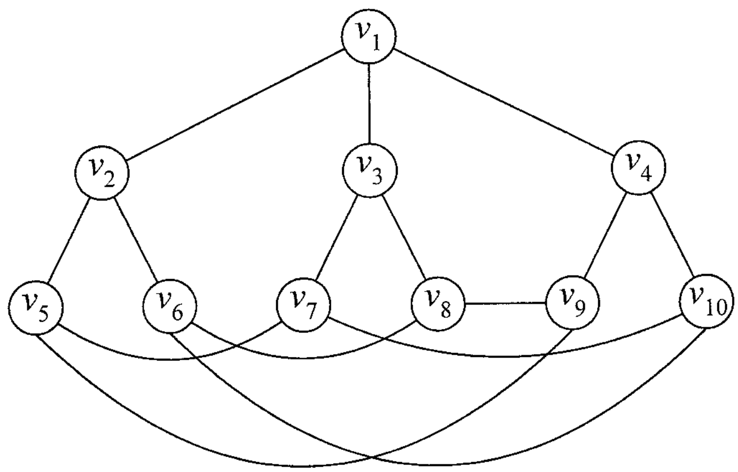


Fig. 7-24

7.78 The **girth** of a graph *G* that is not acyclic is the number of edges in a shortest cycle in *G*. An **(*r*, *g*)-cage** is a *r*-regular graph of girth *g* with as few vertices as possible. (Thus a *g*-cage is a (3, *g*)-cage.) Show that $K_{r,r}$ is the unique $(r, 4)$-cage $(r > 1)$.

Solution. The girth of $K_{r,r}$ is 4, and its order is $2r$. Consider any *r*-regular graph *G* with girth 4. Let *x* be any vertex of *G*, and let *y* be one of the *r* vertices adjacent to *x*. Since the girth of *G* is 4, *y* cannot be adjacent to any of the remaining $(r - 1)$ vertices adjacent to *x*. So there should be another set of $(r - 1)$ vertices adjacent to *y*. Thus *G* should have at least $1 + r + (r - 1) = 2r$ vertices. So any $(r, 4)$-cage should have at least $2r$ vertices. But $K_{r,r}$ is a *r*-regular graph with girth 4 and order $2r$.

7.79 Obtain the unique $(2, g)$-cage, the unique $(r - 1, 3)$-cage, the unique 3-cage, and the unique 4-cage.

Solution. Obviously, the unique $(2, g)$-cage is the cyclic graph of order *g*, the unique $(r - 1, 3)$-cage is the complete graph of order *r*, the unique 3-cage is the complete graph of order 4, and the unique 4-cage is $K_{3,3}$.

7.80 Show that the **Heawood graph *HG*** is the unique 6-cage.

Solution. The 3-regular graph *HG* of order 14 is constructed [see Fig. 7-25(*a*)] as follows. The 14 vertices labeled 1 to 14 are placed clockwise on a circle. Each vertex *i* is adjacent to $(i + 1)$ and $(i + 13)$, where addition is modulo 14. If *i* is odd, it is adjacent to $(i + 9)$. If *i* is even, it is adjacent to $(i + 5)$. Obviously, its girth is 6 and order is 14. It is easy to see that the girth of *HG* is 6. Let *G* be any 6-cage, and let v_1 and v_2 be two adjacent vertices in *G* (see Fig. 7-25(*b*)]. The other vertices adjacent to these two are v_3, v_4, v_5, and v_6, as shown in the figure. These four vertices, in turn, are adjacent to eight more vertices. No two of these 14 vertices can coincide since the girth is 6. So a 6-cage should have at least 14 vertices. Since the order of *HG* is 14, we conclude that *HG* is a 6-cage. To show that it is the unique 6-cage, we first construct a 3-regular graph using the 14 vertices, as shown in Fig.

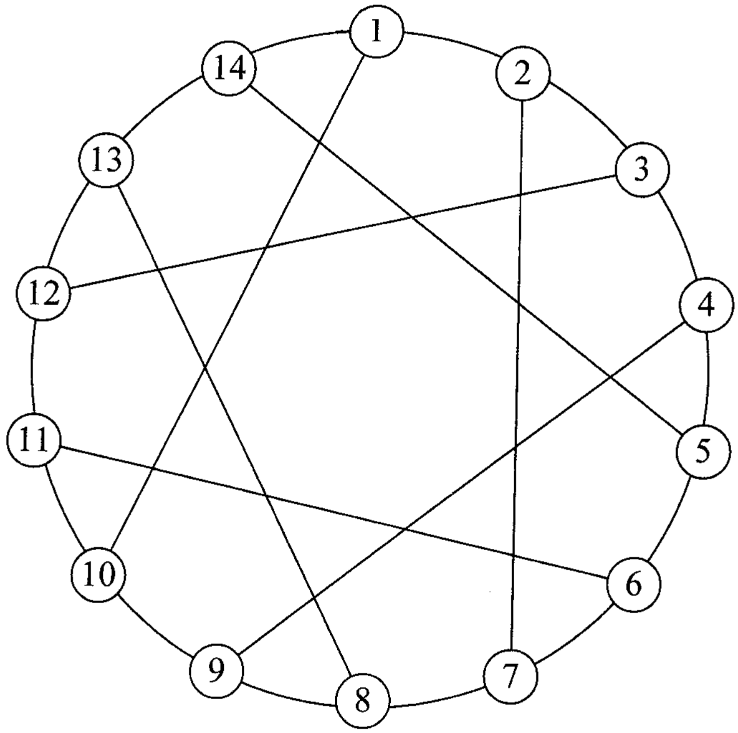


Fig. 7-25a

7-25(*b*), and then prove that the two graphs in Fig. 7-25(*a*) and (*b*) are isomorphic. By relabeling the 14 vertices v_i (i = 1, 2, . . . , 14) as 1, 14, 10, 2, 5, 13, 9, 11, 3, 7, 4, 6, 8, and 12, one can see that the graphs in Fig. 7-25(*a*) and (*b*) are isomorphic.

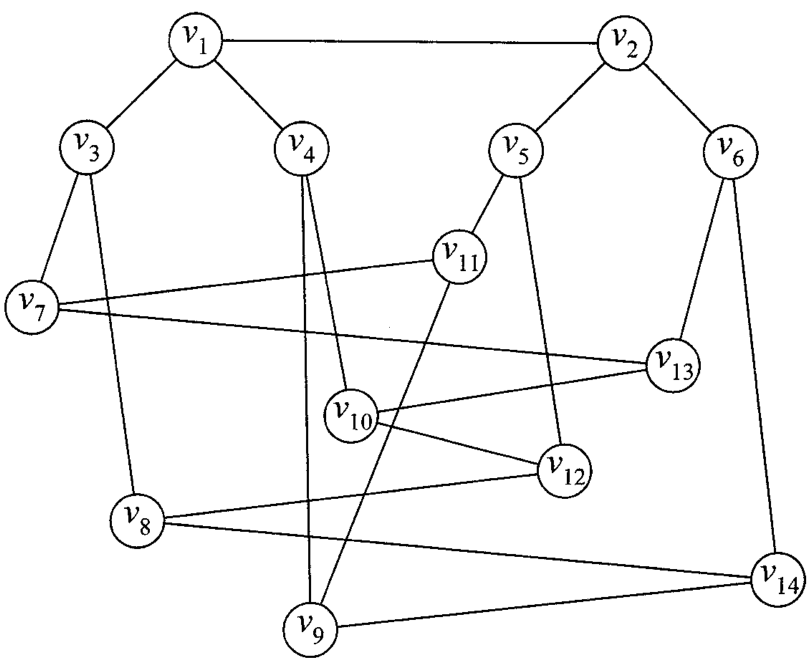


Fig. 7-25b

7.81 Show that the Heawood graph is 1-factorable.

Solution. Since HG is cubic and bridgeless, it can be factored into a 2-factor and a 1-factor. In this case, the 2-factor is a Hamiltonian cycle [see Fig. 7-25(a)], and the seven cross edges that do not belong to the outer circle (the Hamiltonian cycle) is the 1-factor. Since the Hamiltonian cycle has an even number of vertices, it can be factored into two 1-factors. Thus HG is 1-factorable.

Supplementary Problems

7.82 Solve the following optimal (minimization) assignment problem:

$$\begin{bmatrix} 8 & 3 & 2 & 10 & 5 \\ 9 & 6 & 9 & 5 & 5 \\ 3 & 8 & 3 & 1 & 8 \\ 6 & 8 & 3 & 1 & 1 \\ 8 & 4 & 7 & 4 & 8 \end{bmatrix}$$

Ans. The elements are (1, 3), (2, 4), (3, 1), (4, 5), and (5, 2) (not unique).

7.83 Solve the following optimal (minimization) assignment problem:

$$\begin{bmatrix} 21 & 18 & 20 & 14 & 19 \\ 15 & 15 & 12 & 13 & 20 \\ 15 & 17 & 20 & 22 & 20 \\ 15 & 12 & 14 & 12 & 12 \\ 20 & 10 & 20 & 20 & 23 \end{bmatrix}$$

Ans. The elements are (1, 1), (2, 2), (3, 4), (4, 3), and (5, 5).

7.84 Solve the following optimal (maximization) assignment problem:

$$\begin{bmatrix} 13 & 15 & 15 & 14 & 11 & 12 & 13 \\ 9 & 11 & 11 & 10 & 9 & 8 & 10 \\ 12 & 10 & 12 & 12 & 12 & 12 & 11 \\ 10 & 12 & 12 & 12 & 11 & 10 & 12 \\ 11 & 12 & 11 & 13 & 11 & 13 & 11 \\ 16 & 18 & 18 & 15 & 18 & 17 & 18 \\ 12 & 14 & 14 & 11 & 10 & 13 & 13 \end{bmatrix}$$

Ans. The elements are (1, 4), (2, 3), (3, 1), (4, 7), (5, 6), (6, 5), and (7, 2).

7.85 The weight matrix of an undirected network is

$$\begin{bmatrix} - & 1 & - & - & - & 1 & 4 \\ 1 & - & 2 & - & - & - & 1 \\ - & 2 & - & 2 & - & - & 4 \\ - & - & 2 & - & 3 & - & - \\ - & - & - & 3 & - & 9 & 3 \\ 1 & - & - & - & 9 & - & - \\ 4 & 1 & 4 & - & 3 & - & - \end{bmatrix}$$

Construct an extra edge joining vertices 1 and 6 with a weight of 7 units and another extra edge joining vertices 3 and 7 with a weight of 5 units. Obtain an optimal postman route in the enlarged network.
Ans. $1 \to 6 \to 1 \to 7 \to 2 \to 1 \to 6 \to 5 \to 7 \to 5 \to 4 \to 3 \to 7 \to 3 \to 2 \to 1$

7.86 Express the following doubly stochastic matrix as a convex combination of permutation matrices:

$$\begin{bmatrix} 0.21 & 0.13 & 0.38 & 0 & 0.28 \\ 0.31 & 0 & 0.15 & 0.38 & 0.14 \\ 0 & 0.73 & 0.14 & 0.05 & 0.08 \\ 0.38 & 0.14 & 0.05 & 0.36 & 0.07 \\ 0.08 & 0 & 0.28 & 0.21 & 0.43 \end{bmatrix}$$

[*Hint:* This matrix can be written as $0.05P_1 + 0.07P_2 + 0.08P_3 + 0.14P_4 + 0.28P_5 + 0.38P_6$, where the matrices in the convex linear combination are all permutation matrices.]

7.87 Obtain an optimal Hamiltonian cycle in the network with the weight matrix

$$\begin{bmatrix} — & 2 & — & 5 & — & — \\ — & — & 1 & — & 2 & 1 \\ 2 & — & — & — & — & 5 \\ 4 & — & — & — & — & 2 \\ — & 9 & — & 2 & — & — \\ — & — & 2 & — & 2 & — \end{bmatrix}$$

Ans. $1 \to 2 \to 3 \to 6 \to 5 \to 4 \to 1$

7.88 Obtain an optimal Hamiltonian cycle in the network with the weight matrix

$$\begin{bmatrix} — & 12 & 10 & 9 & 10 & 13 & 9 \\ 10 & — & 17 & 10 & 10 & 11 & 10 \\ 9 & 14 & — & 11 & 9 & 12 & 11 \\ 11 & 10 & 11 & — & 12 & 11 & 10 \\ 9 & 11 & 11 & 9 & — & 14 & 12 \\ 10 & 10 & 10 & 10 & 10 & — & 10 \\ 9 & 10 & 9 & 10 & 9 & 10 & — \end{bmatrix}$$

Ans. $1 \to 7 \to 6 \to 3 \to 5 \to 4 \to 2 \to 1$

Chapter 8

Graph Embeddings

8.1 PLANAR GRAPHS AND DUALITY

A graph is a **planar graph** if it is possible to represent it in the plane (that is, to draw it as a diagram on a piece of paper) such that no two edges of the graph intersect except possibly at a vertex to which they are both incident. Any such drawing of a planar graph G in a plane is a **planar embedding** of G. A **plane graph** is a particular representation of a planar graph in the plane drawn in such a way that any pair of edges meet only at their end vertices (if at all they meet). The graph in Fig. 8-1(a) is a planar graph since it is isomorphic to the plane graph in Fig. 8-1(b).

Theorem 8.1 **(Fary–Stein–Wagner Theorem).** Any simple planar graph has a straight-line representation: it has an embedding in a plane such that each edge in the embedding is a straight line. (See Solved Problem 8.32.)

If x is any point in the plane of a plane graph that is neither a vertex nor a point on an edge, the set of all points in the plane that can be reached from x by traversing along a curve that does not have a vertex of the graph or a point of an edge as an intermediate point is the **region of the graph** that contains x. Thus a plane graph G partitions the plane into the regions of G, and among these regions is exactly one region (the **exterior** or **infinite region**), whose area is not finite. Every other region is an **interior region.** The **boundary** of a region is the subgraph formed by the vertices and edges encompassing that region. If the boundary of the exterior region of a plane graph is a cycle, that cycle is known as the **maximal cycle** of the graph.

If a planar graph G has an embedding on the plane such that the *boundary* of each region (including the unbounded region) is a convex polygon, G is said to have a **convex embedding.** Notice that K_4 has a convex embedding, whereas $K_{2,n}$ does not have a convex embedding whenever $n \geq 4$.

Theorem 8.2. A planar graph G (i) can be embedded on the plane such that each region including the exterior region is a polygon if and only if G is 2-connected and (ii) has a convex embedding if G is 3-connected. (See Solved Problems 8.5, 8.29, and 8.31.)

The **degree of a region** is the number of edges in a (closed) walk that encloses it. Since a bridge belongs to the boundary of only one region, it contributes to the size of the boundary twice. Thus the sum of the degrees of all the regions in a plane graph is twice the size of the graph. In the plane graph in Fig. 8-1(b) are four interior regions of degree 3 and two interior regions of degree 4. The degree of the exterior region is 10. The sum of the degrees (of the regions) is 30, and the graph has 15 edges.

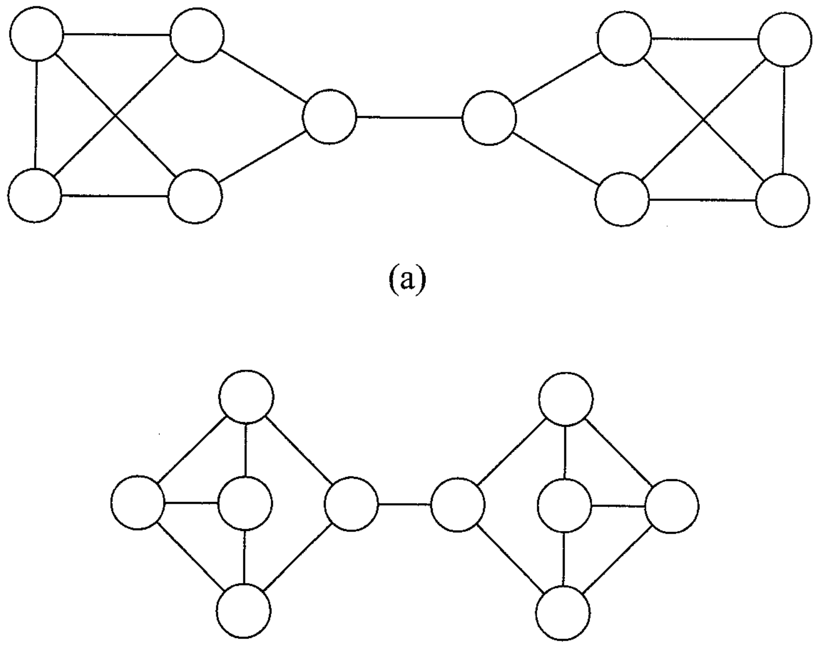
If G_1 and G_2 are two planar graphs, we can take vertex u from the boundary of the exterior face of G_1 and vertex v from the exterior face of G_2 and merge these two vertices to form a new vertex x, creating a new planar graph G with x as a cut vertex. In this case, G is obtained by a **vertex merging** of G_1 and G_2. Likewise, edge e from the boundary of the exterior face of G_1 joining vertices a and b can be merged with edge f from the boundary of the exterior face of G_2 joining the vertices p and q (by identifying a with p and b with q), creating a planar graph G. In this case, G is obtained by an **edge merging** of G_1 and G_2.

If H_1 and H_2 are any two plane graphs isomorphic to a planar graph G, both the plane graphs have the same number of regions. This result is an immediate consequence of the following theorem, due to Euler.

Theorem 8.3 **(Euler's Formula for Plane Graphs).** If a connected plane graph of order n and size m has f regions, $n - m + f = 2$. (See Solved Problem 8.1.)

For example, in Fig. 8-1(b), $n = 10$, $m = 15$, and $f = 7$, verifying this formula.

A simple planar graph is called a **maximal planar graph** if the graph becomes nonplanar when any two nonadjacent vertices in it are joined by an edge. A maximal planar graph is necessarily a connected graph. Any planar graph is a spanning subgraph of a maximal planar graph. A planar graph G is a maximal planar graph if and only if the degree of every region (interior as well as exterior) of G is 3. Moreover, if the order of a maximal planar graph is at least 4, the degree of every vertex is at least 3. Since every planar graph has a

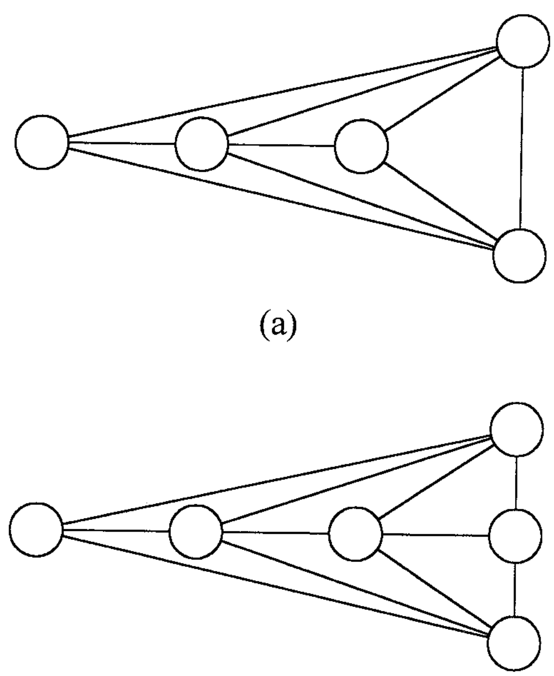


(a)

(b)

Fig. 8-1

straight-line representation, a maximal planar graph is also known as a **triangulation** or a **triangulated plane graph.** The graph in Fig. 8-2(a) is a triangulation, but the graph in Fig. 8-2(b) is not.



(a)

(b)

Fig. 8-2

Theorem 8.4. The number of edges in a triangulation of order n (where $n \geq 3$) is $3n - 6$. (See Solved Problem 8.13.)

For example, in the triangulation shown in Fig. 8-2(a), $n = 5$ and $m = 9$.

A **nonplanar graph** is a graph that is not planar. Clearly, a subgraph of a planar graph is planar, and any graph that has a nonplanar graph as a subgraph is nonplanar.

Theorem 8.5. Both K_5 and $K_{3,3}$ are nonplanar graphs. (See Solved Problem 8.15.)

As a consequence of Theorem 8.5, any graph that contains K_5 or $K_{3,3}$ as a subgraph is nonplanar. See Fig. 8-3, which represents the former as a pentagon and the latter as a hexagon. Any nonplanar graph *contains* one of these two graphs as a subgraph. Given edge e joining two vertices u and v in a graph G, a new graph H can be obtained from G by deleting e and introducing a new vertex x and two new edges, one joining u and x and the other joining v and x. This operation of replacing an edge by two edges and a new vertex of degree 2 is called **edge subdivision.** A graph H, obtained from a graph G by a sequence of edge subdivisions, is called a **subdivision** (or a **homeomorph**) of G. Notice that graph G is planar if and only if every homeomorph of G is planar. Two graphs are said to be **homeomorphic** if each is a homeomorph of some graph G; they are subdivisions of the same graph. If two graphs are homeomorphic, it is not necessary that each be a homeomorph of the other; see Solved Problem 8.39.

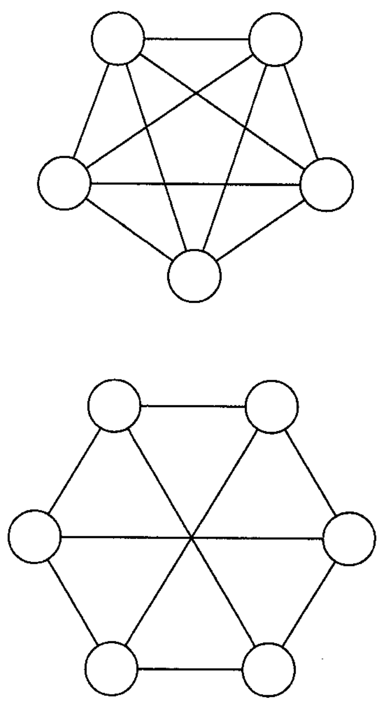


Fig. 8-3

Any homeomorph of K_5 or $K_{3,3}$ is known as a **Kuratowski graph.** A graph is said to have a ***K*-subgraph** if it has a Kuratowski graph as a subgraph. The following characterization of planar graphs is one of the basic theorems in graph theory.

Theorem 8.6 **(Kuratowski's Theorem).** A graph is planar if and only if it does not have a K-subgraph. (This theorem is also known as the **Kuratowski–Pontryagin** theorem.) (See Solved Problem 8.31.)

If e is an edge joining two vertices u and v of a graph G, an **elementary contraction of *G* by *e*** is the process of obtaining a simple graph ***G.e*** from G by deleting e and by introducing a new vertex (by merging u and v) such that this new vertex is adjacent to those vertices that were adjacent to u or v in G after deleting any multiple edges that may appear in this process. A graph G is **contractible** to graph H if H can be obtained from G by a sequence of elementary contractions; in this case, we say that H is a **contraction** of G. If H is a homeomorph of G, G is a contraction of H. But if H is a contraction of G, it is not necessary that G be a homeomorph of H.

Graph H in Fig. 8-4 is a contraction of G. It can be obtained from G by contracting the edge joining v_3 and v_4 and then contracting the edge joining the merged vertex and v_5. The set of vertices of G can be partitioned into $\{v_1\}$, $\{v_2\}$, $\{v_3, v_4\}$, and $\{v_5, v_6\}$, and the one-to-one correspondence is between these sets and w_1, w_2, w_3, and w_4, respectively. Here H is a contraction of G, but G is not a homeomorph of H.

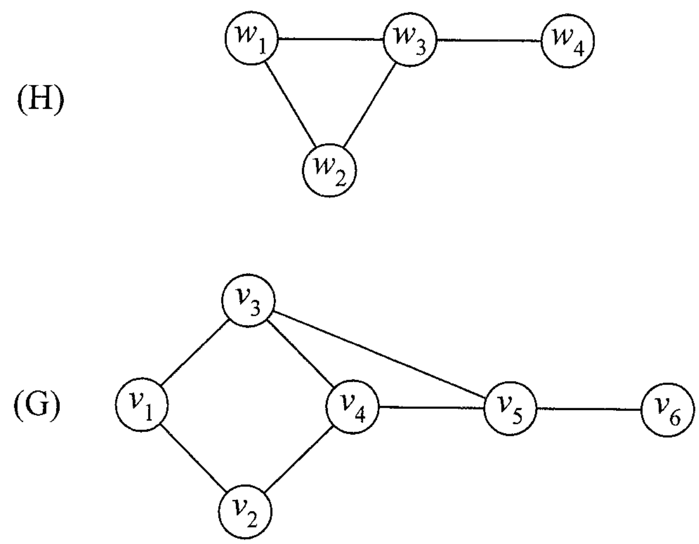


Fig. 8-4

Notice that $H = (W, F)$ is a contraction of $G = (V, E)$ if and only if there exists a partition of V and a one-to-one correspondence between the sets in this partition and the vertices in W such that two elements in W are adjacent if and only if the subgraph of G induced by the union of their corresponding images in the partition is a connected graph.

A **subcontraction** of a graph G is a contraction of a subgraph of G. If G has a subgraph that is a homeomorph of H, H is a subcontraction of G. We now have another important characterization of planar graphs involving K_5 and $K_{3,3}$.

Theorem 8.7 **(Harary–Tutte–Wagner Theorem).** A graph is planar if and only if neither K_5 nor $K_{3,3}$ is a subcontraction of G. (See Solved Problem 8.42.)

Planarity and Duality

Let G be an embedding of a planar graph on the plane. Using this embedding, we can construct a plane graph G', called the **geometric dual of the plane graph G,** as follows. Each region of G corresponds to a vertex of G'. If e is an edge of G that has region X on one side and region Y on the other (the two regions could be the same), the corresponding **dual edge e'** is an edge joining vertices x and y, which correspond to X and Y, respectively. By the way the dual graph is defined, there is a path between every pair of vertices in the dual graph. In other words, G' is a connected graph. Once the plane graph is embedded, its dual is uniquely defined. More generally, if is G any planar graph, the geometric dual of any plane embedding of G is called **a geometric dual of the planar graph.** It is not at all necessary that two different geometric duals of a planar graph G (corresponding to two different embeddings) are isomorphic. See Fig. 8-5. Both G_1 and G_2 are embeddings on the plane of a planar graph, but their geometric duals G_1' and G_2' are not isomorphic.

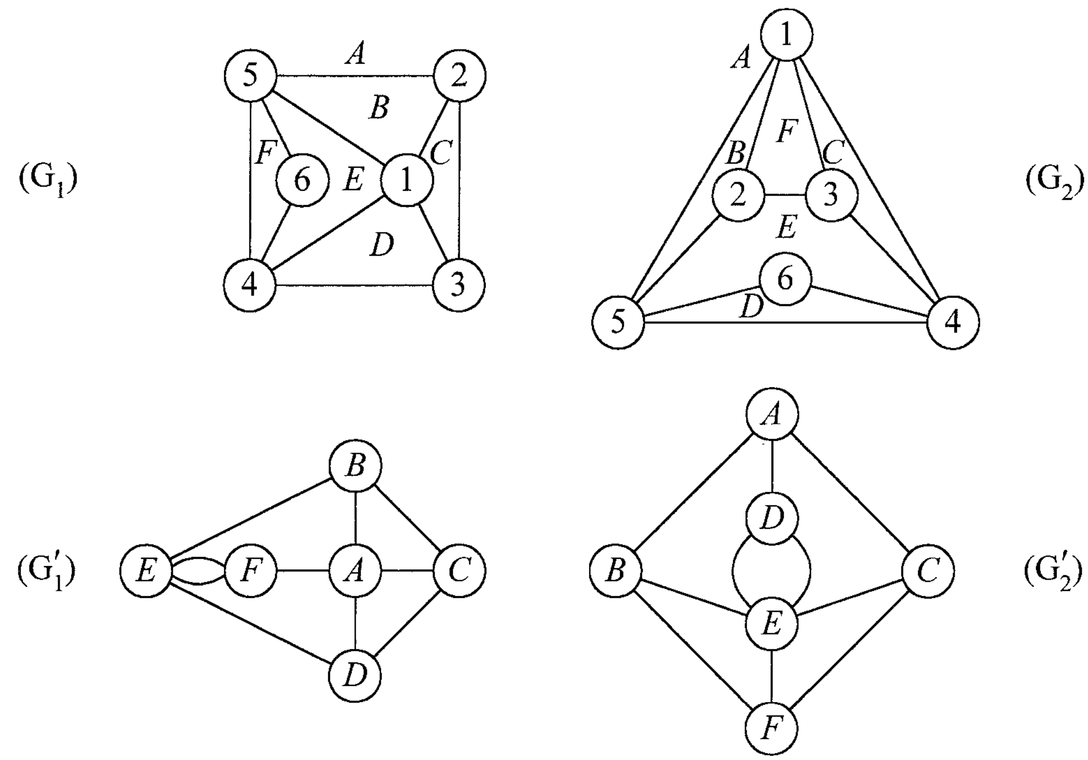


Fig. 8-5

This definition of a geometric dual makes sense only in the context of plane graphs and planar graphs. For a characterization of planar graphs using the concept of duality, it is desirable to have an abstract definition of duality for an arbitrary graph such that an abstract dual thus defined is the same as the geometric dual defined earlier. A graph $G^* = (V^*, E^*)$ is called an **abstract dual** of a graph $G = (V, E)$ if there is a bijection ϕ between E and E^* such that the set $C^* = \phi(C)$ of edges forms a cut set in G^* whenever the set C of edges forms a cycle in G. As in the case of geometric duals, it is possible for a graph to have two nonisomorphic abstract duals. In Fig. 8-6, both G_1^* and G_2^* are nonisomorphic abstract duals of G_1. It can be shown (see Solved Problem 8.67) that in the case of planar graphs, these two notions of duality are equivalent. It can also be shown

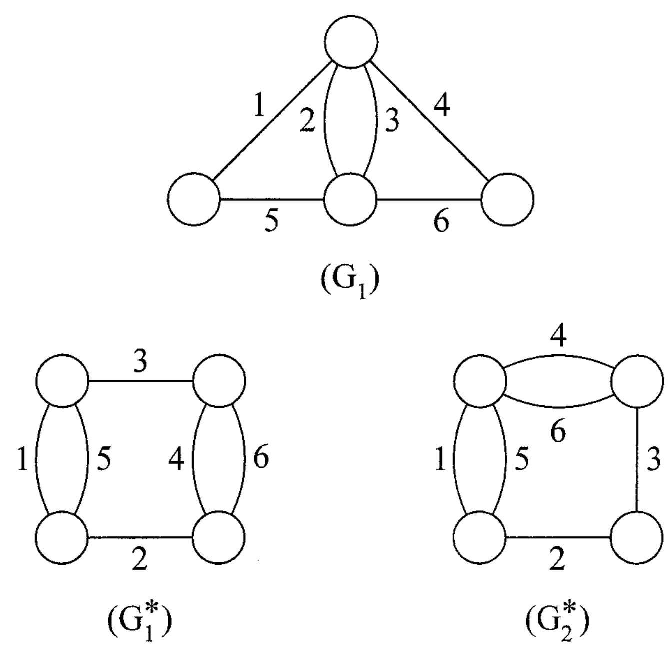


Fig. 8-6

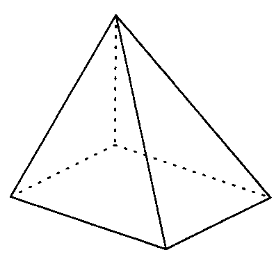
(see Solved Problem 8.70) that if G^* is the abstract dual of G, G is the abstract dual of G^*. Thus two graphs are **abstract duals to each other** if there is a bijection between their sets of edges such that cycles in one graph correspond to cut sets in the other and vice versa.

Theorem 8.8 **(Whitney's Theorem).** A graph is planar if and only if it has an abstract dual. (See Solved Problem 8.75.)

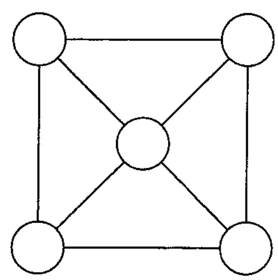
Convex Polyhedra in Three Dimensions

We conclude this section with a brief look at convex polyhedra and their associated graphs. For each positive integer n greater than 2, there is a convex polygon in the plane with n sides. This notion of a convex polygon can be extended to higher dimensions. A **convex polyhedron** (in three dimensions) is a solid bounded by a finite number of surfaces (known as **faces**), each of which is a plane such that no point in the line joining any two points of the solid lies outside the solid. Corresponding to a convex polyhedron P that has a finite number of corners, sides, and faces, we can construct a simple connected plane graph $G(P)$ as follows. Any face of the solid can be considered as its base. Assume that the faces of the solid are made of rubber. Then we can hold the sides of the base and stretch them out to transform the three-dimensional solid into a two-dimensional flat sheet. Thus with every convex polyhedron P, there is an associated plane graph $G(P)$ called a **1-skeleton** of P in which each vertex corresponds to a corner of the solid and each edge corresponds to a side of the solid. The graph $G(P)$ is necessarily a connected graph. Furthermore, the exterior region corresponds to the base of the solid, and every interior region corresponds to a face of the solid. Each interior region of the plane graph can be represented as a convex polygon. The degree of each vertex is at least 3, and the degree of each region is also at least 3.

Figure 8-7(a) shows a convex polyhedron, and Fig. 8-7(b) shows its associated plane graph.



(a)



(b)

Fig. 8-7

If the convex polyhedron P has n corners, m sides, and f faces, the connected graph $G(P)$ has n vertices, m edges, and f regions. So by Theorem 8.1, we get the same relation $n - m + f = 2$. Thus this equation is also known as **Euler's polyhedron formula.**

Theorem 8.9 **(Steinitz's Fundamental Theorem on Convex Types).** A graph G is isomorphic to the graph $G(P)$ of a convex polyhedron P if and only if G is planar and 3-connected. (Hence, any planar 3-connected graph is known as a **polyhedral graph.**) [The necessity part of the proof is quite straightforward. The proof of the (intuitively obvious) sufficiency part, on the other hand, is rather complicated and is outside the scope of most books on graph theory. For an elegant treatment of this topic in a more general setting, the reader is referred to *Convex Polytopes* by B. Grunbaum.]

A **regular polyhedron** is a convex polyhedron, all of whose faces are congruent polygons and at each of whose vertices the same number of polygons meet. Even though the number of regular polygons is infinite, there are only five regular polyhedra (see Solved Problem 8.23), known as the **Platonic solids:** the regular tetrahedron (in which the four faces are congruent equilateral triangles), the cube (in which the six faces are congruent squares), the octahedron (in which the eight faces are congruent equilateral triangles), the dodeca-

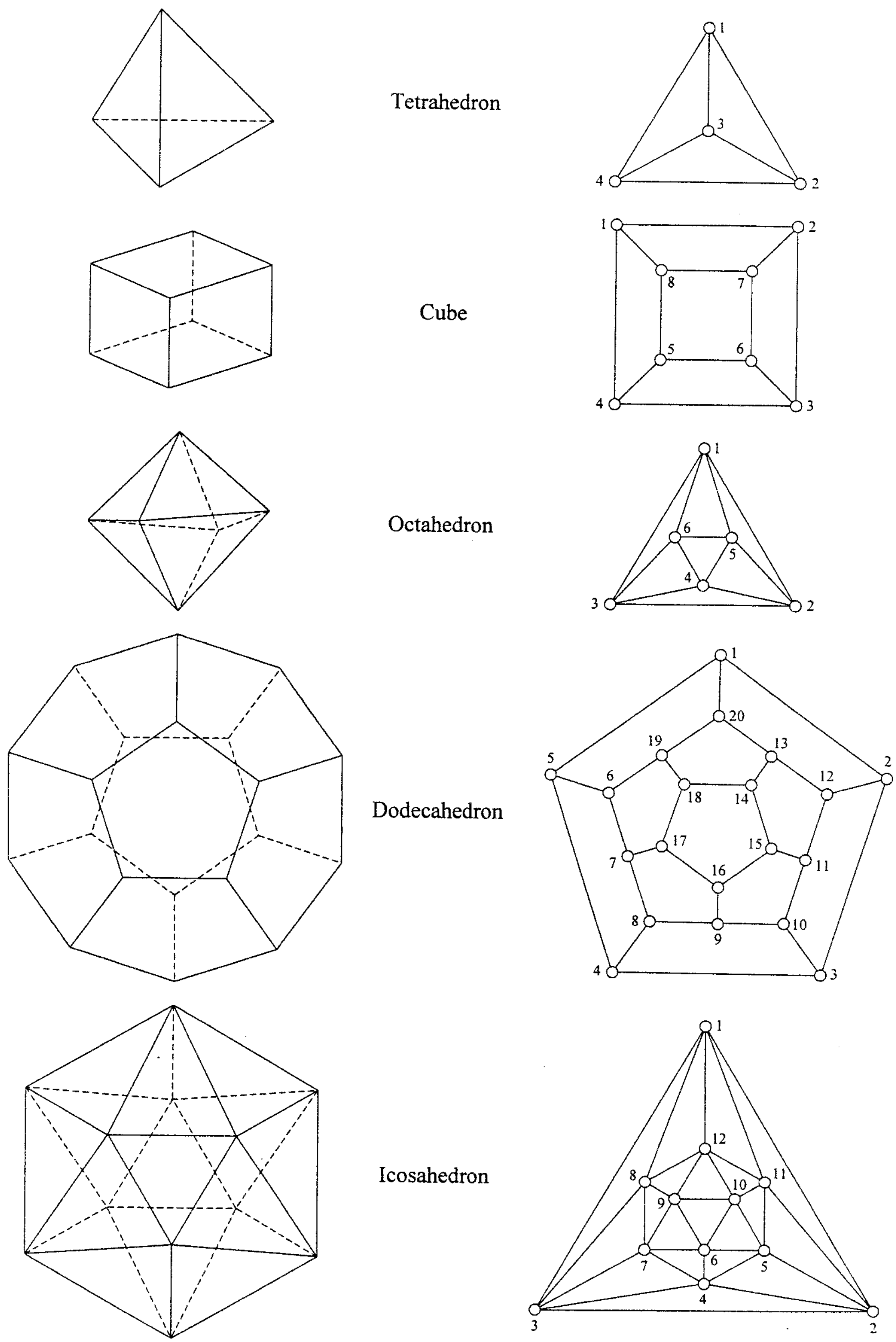


Fig. 8-8

hedron (in which the 12 faces are congruent regular pentagons), and the icosahedron (in which the 20 faces are congruent equilateral triangles). These five solids and their corresponding polyhedral graphs (the five **Platonic graphs**) are shown in Fig. 8-8.

The polyhedral graph $G(P)$ of each Platonic solid P has a unique embedding on the plane; therefore, it has a unique geometric dual. It can be easily verified that the (polyhedral) graph of a tetrahedron is its own dual, the graph of the cube and the graph of the octahedron are dual to each other, and the graph of the dodecahedron and the icosahedron are dual to each other. A regular graph is said to be **completely regular** if its dual also is regular. The polyhedral graph of any Platonic solid is completely regular.

8.2 HAMILTONIAN PLANAR GRAPHS

The 3-connected planar graphs corresponding to each of the five platonic solids are Hamiltonian: in each graph in Fig. 8-8, start from vertex 1 and return to vertex 1 after moving sequentially from one vertex to the next vertex. In general, however, an arbitrary 3-connected planar graph (known as a **3CP graph** in the lore, whereas a *cubic* 3-connected planar graph is a **C3CP graph**) need not be Hamiltonian; the **Herschel graph** shown in Fig. 8-9 is a counterexample. It is obviously a 3CP-graph, and since it has no odd cycles, it is a bipartite graph. But it cannot be a Hamiltonian graph since it has an odd number of vertices.

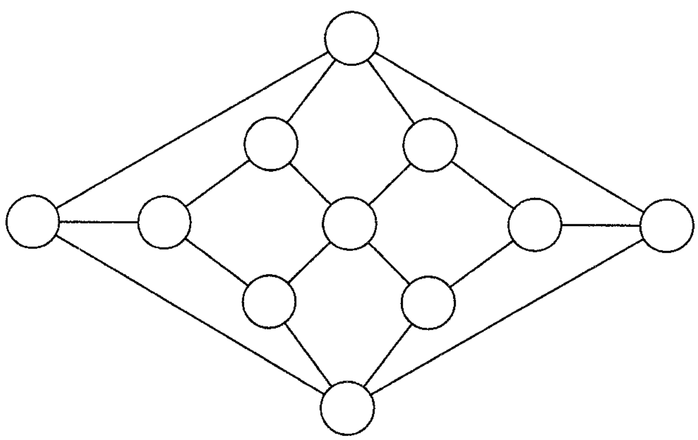


Fig. 8-9

In Chapter 3, several sufficient conditions for an arbitrary graph to be Hamiltonian were presented. If a graph is planar, we have a necessary condition (due to Grinberg and Kozyrev) to be satisfied by it if it is Hamiltonian. Needless to say, this theorem serves a very useful purpose when we would like to show that a given plane graph is not Hamiltonian. If r is the degree of a region in a plane graph, the **index** of that region is the integer $(r - 2)$. If the plane graph G has a Hamiltonian cycle C, C divides the plane into an interior part and an exterior part. The regions known as the **inner regions relative to C** are in the interior part, and the remaining regions, including the exterior region in the exterior part, are the **outer regions relative to C.**

Theorem 8.10 **(Grinberg–Kozyrev Theorem).** If C is any Hamiltonian cycle in a Hamiltonian plane graph of order n, the sum of the indices of the inner regions relative to C and the sum of the indices of the outer regions relative to C are both equal to $(n - 2)$. (See Solved Problem 8.76.)

Example 1. In the plane graph of order 11 shown in Fig. 8-10, each edge is either a thick line or a dashed line. The cycle sequentially passing through the 11 vertices starting from 1 and terminating in 1 is a Hamiltonian cycle; the 11 edges of this cycle are represented by thick lines. The inner regions are A, B, C, and D with degrees 4, 5, 4, and 4, respectively. So the sum of their indices is $2 + 3 + 2 + 2 = 9$. The outer regions are A', B', C', D', and the exterior region of the graph with degrees 3, 3, 3, 4, and 6, respectively. The sum of their indices is equal to $1 + 1 + 1 + 2 + 4 = 9$. Thus the theorem is easily verified.

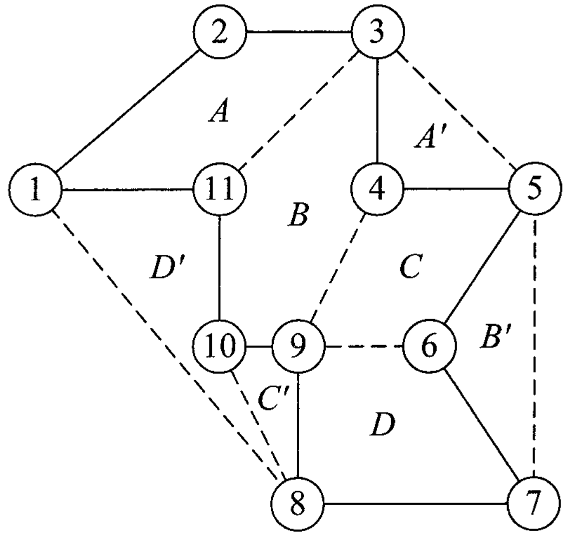


Fig. 8-10

The 3-connected graph shown in Fig. 8-9 is not a cubic graph. A C3CP graph (the celebrated **Tutte graph**), which is not Hamiltonian, was first exhibited by Tutte in 1946. Thus **Tait's conjecture** (which six decades earlier asserted that every C3CP graph is Hamiltonian) was proved false. See Solved Problem 8.80. It is not known whether every cubic 3-connected *bipartite* planar graph is Hamiltonian; this unresolved problem is known as **Barnette's conjecture.** The fact that every 4-connected planar (4CP) graph is Hamiltonian proved by Tutte in 1956 is now a particular case of Thomassen's theorem (1983), which asserts that every 4CP graph is actually Hamiltonian-connected.

8.3 MAXIMUM FLOW IN PLANE NETWORKS

Consider a connected undirected plane graph $G = (V, E)$ with a nonnegative weight defined on each edge as its capacity. On the boundary of the exterior face, two nonadjacent vertices are singled out: one vertex is designated source vertex s and another vertex is designated sink vertex t. In this context, the network G is known as an **s, t planar network.** If $V = \{1, 2, \ldots, n\}$, we may take vertex 1 as the source and vertex n as the sink in some cases. Then the optimization problem is the problem of finding a feasible s, t flow in the network such that the flow value is a maximum. Since vertices s and t are not adjacent, we construct an artificial edge e joining them that does not intersect any other edge so that the enlarged graph $G + e$ is also a plane graph. Then we construct the geometric (abstract) dual of the enlarged network $G + e$. The vertex in the dual that corresponds to the finite region in $G + e$ containing edge e is designated s^*, and the vertex that corresponds to the exterior face of $G + e$ is designated t^*. Once the dual is constructed, the edge joining s^* and t^* is deleted. The weight of dual edge f^* in the dual corresponding to edge f is, by definition, the weight of f. If there is more than one edge between two vertices, the one with minimum weight is chosen. The plane simple network thus defined is called the **dual network G^*.**

Recall that any s, t cut in G is a set of edges such that their deletion creates a subgraph in which there is no path between s and t. By our construction, the dual edges corresponding to the edges in an s, t cut in G constitute a path between s^* and t^* and vice versa. So the edges in a shortest path between s^* and t^* correspond to the edges in a minimum cut in G.

Example 2. In Fig. 8-11(a), the network G is an s, t network (where s is 1 and t is 6). An artificial edge joining s and t is constructed (as a dashed curve); Fig. 8-11(b) shows the dual network G^*. In each network, the capacities are indicated on their edges. For example, in G, the weight of the edge common to region A and region C is 8. In G^*, the weight of the dual edge joining vertices A and C is 8. Of the two edges between A and B with weights 3 and 6, the edge with weight 3 is chosen. A shortest path from s^* to t^* has edges with weights 5, 3, and 2. The corresponding minimum cut in G consists of edges $\{1, 2\}$, $\{1, 5\}$, and $\{1, 4\}$. Hence, the maximum flow value from the source to the sink (or the other way around) is $5 + 3 + 2 = 10$.

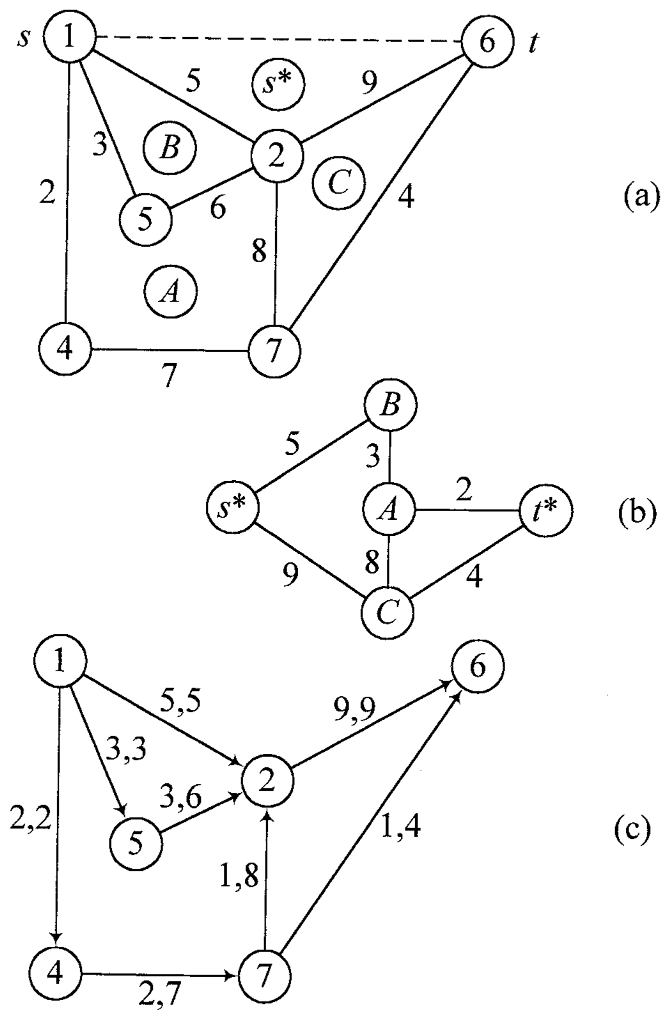


Fig. 8-11

The shortest distance between s^* and t^* in the dual network is the maximum flow value, as was seen in Example 2. Any shortest path between these two vertices in the dual network can be used to obtain a maximum flow in the given network; this is the content of the following assertion.

Theorem 8.11. Let G be an s, t network, and let G^* be its dual. Let $d(k^*)$ be the shortest distance between s^* and k^* in G^*. Define $x_{ij} = d(j^*) - d(i^*)$, where $\{i^*, j^*\}$ is the dual edge that corresponds to edge $\{i, j\}$ in G. Then the vector $[x_{ij}]$ is a maximum flow in G. (See Solved Problem 8.84.)

Example 3. In Fig. 8-11(b), the shortest distances are $d(s^*) = 0$, $d(A) = 7$, $d(B) = 5$, $d(C) = 9$, and $d(t^*) = 10$. The components of the maximum flow along the six edges are computed as follows:

$$
\begin{aligned}
x_{12} &= d(B) - d(s^*) = 5 - 0 = 5 \qquad & x_{14} &= d(t^*) - d(A) = 10 - 8 = 2\\
x_{15} &= d(A) - d(B) = 8 - 5 = 3 & x_{25} &= d(A) - d(B) = 8 - 5 = 3\\
x_{26} &= d(C) - d(s^*) = 9 - 0 = 9 & x_{27} &= d(C) - d(A) = 9 - 8 = 1\\
x_{47} &= d(t^*) - d(A) = 10 - 8 = 2 & x_{76} &= d(t^*) - d(C) = 10 - 9 = 1
\end{aligned}
$$

Using this information, each edge is converted into an arc. The maximum flow is displayed in Fig. 8-11(c). Along each arc, are two numbers; the first is the flow along the arc, and the second is its capacity, which is actually the weight of the edge that has now become an arc.

8.4 GRAPHS ON SURFACES (AN INFORMAL TREATMENT)

We can represent the different countries of the world visually in two ways: we can denote them on a sheet of paper as a flat map, or we can denote them (more realistically) on the surface of a sphere as a global map. Likewise, it makes no difference whether a graph is represented on the surface of a globe or on a flat sheet: these two representations are structurally equivalent. Hence, we can take it for granted that a graph has a plane embedding if and only if it can be embedded on the surface of a sphere such that no two edges intersect, except possibly at vertices.

Now consider the representation of a nonplanar graph on the surface of a sphere. If we attach a **handle** to the sphere at each crossing in the graph so that one of the crossing edges goes over the handle while the other goes under the handle, it is possible to draw the graph on this surface such that no two edges intersect. (For example, when two roads intersect at a traffic junction, instead of installing a traffic light to control the traffic, we sometimes construct an overpass so that the two roads do not intersect. This results in a smooth flow of traffic.) At a location where a handle is to be constructed, we cut two distinct circular holes in the sphere and join the edges of the holes by a circular tube, making no visible cuts on the surface. Equivalently, we can dig a hole (a tunnel) at the location. A sphere with one handle (or hole) thus constructed can be continuously transformed to look like a doughnut. This is called a **torus** (or a **toroid**); a **double torus** is a sphere with two handles.

A **graph G is embeddable on a surface S** if it is possible to represent it on S such that no two edges intersect except possibly at vertices. So if the crossing number of a graph is k, it can be embedded on a surface S, which, in this case, is a sphere with k handles. The possibility of embedding the same graph on a sphere with fewer handles cannot be ruled out, however. For example, the crossing number of the Petersen graph is 2, but it can be embedded on a torus. See Solved Problems 8.54 and 8.89. The **genus of a graph G** is the minimum number of handles to be attached to a sphere so that G can be embedded on the surface S (consisting of the sphere and these handles), and it cannot exceed the crossing number of the graph. Of course, the genus of a graph is 0 if and only if it is planar. A graph is **toroidal** if its genus is 1 and **double toroidal** if its genus is 2. Every planar graph is toroidal and every toroidal graph is double toroidal. Unlike planar graphs, there is no known characterization of toroidal graphs.

Theorem 8.12. The nonplanar graphs K_5 and $K_{3,3}$ are toroidal. (See Solved Problem 8.86 for an actual embedding of these two graphs on a torus.)

For an embedding of a graph on a surface of positive genus, regions and boundaries are defined in the same way they are defined for an embedding on the plane. There is a vital difference between an embedding on a plane (or on a sphere) and on an arbitrary surface insofar as the nature of a region is concerned, however. A region R is called a **2-cell** if any simple closed curve (like a circle or ellipse) lying completely in R can be continuously shrunk into a single point in that region where at any stage, every point on the resulting curve continues to lie on R. (In other words, a region is a 2-cell if and only if it is homeomorphic to an open disk.) If a connected graph is embedded on a sphere, every region is obviously a 2-cell. This need not be the case if a graph is embedded on surface of positive genus, however. Consider an embedding of the complete graph with four vertices on a torus. It is easy to visualize a situation in which three of the four regions are 2-cells and one region is not. An embedding of a graph on a surface is a **2-cell embedding** (also known as a **cellular embedding**) if every region defined by the embedding is a 2-cell. Any embedding of a connected planar graph on the sphere is no doubt a 2-cell embedding. An embedding of a graph on a surface is a **minimal embedding** if the genus of the graph is equal to the genus of the surface. It turns out (see Solved Problem 8.91) that every minimal embedding of a connected graph is indeed a 2-cell embedding.

Theorem 8.13 **(Generalized Euler Formula).** If an embedding of a connected graph of order n and size m on a surface of genus g defines r regions and if each region thus defined by the embedding is a 2-cell, $n - m + r = 2(1 - g)$. (See Solved Problem 8.92.)

Solved Problems

PLANAR GRAPHS AND DUALITY

8.1 Prove Theorem 8.3 (Euler's formula for plane graphs): If a connected plane graph of order n and size m has f regions, $n - m + f = 2$.

Solution. The proof is by induction on m. If $m = 0$, $n = 1$ and $f = 1$. So the result holds when $m = 0$. Suppose the result is true for any connected graph with fewer than m edges, where $m \geq 1$. Let G be any connected plane graph with n vertices and m edges. If G is a tree, $n = m + 1$ and $f = 1$. So the result is true for any tree with m edges. If G is not a tree, it has a cycle C. Let e be an edge of C. The connected graph $G' = G - e$ has n vertices and $(m - 1)$ edges. By the induction hypothesis G', has f' faces, where $n - (m - 1) + f' = 2$. Hence, $f' = 1 - n + m$. This implies that G has f faces, where $f = 2 - n + m$. In other words, the connected planar graph with n vertices and m edges has f faces, where $n - m + f = 2$. Thus the result is true for m.

8.2 Show that if a plane graph of order n and size m has f regions and k components, $n - m + f = k + 1$.

Solution. Suppose the components are G_i with n_i vertices, m_i edges, and f_i faces (where $i = 1, 2, \ldots, k$). Then $n_i - m_i + f_i = 2$ for each i. The exterior region is the same for all components. If the exterior region is not considered, $n_i - m_i + f_i = 1$ for each i, and on summation, we get $n - m + f = k$. So with the inclusion of the common exterior region, we obtain the relation $n - m + f = k + 1$.

8.3 Verify Euler's formula for the plane graphs shown in Fig. 8-12.

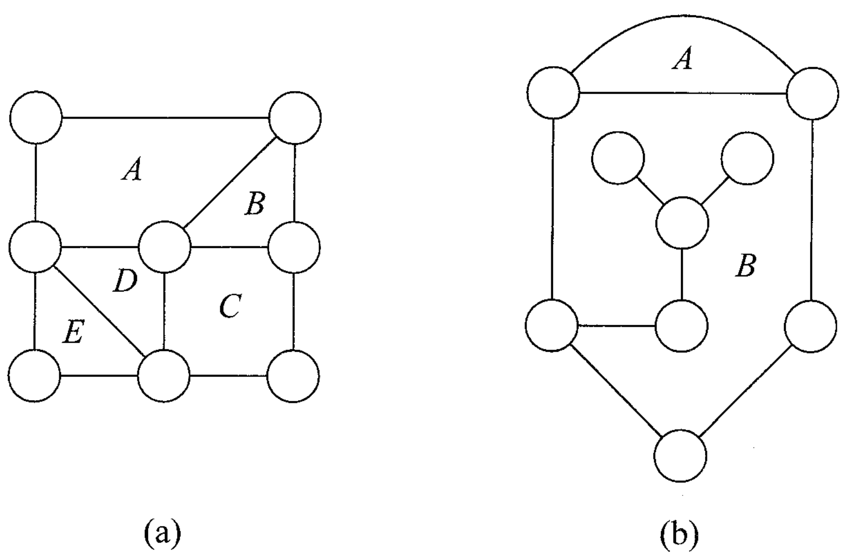


Fig. 8-12

Solution. (*a*) $n = 8$, $m = 12$, and $r = 6$; (*b*) $n = 9$, $m = 10$, and $r = 3$. In both cases, $n - m + r = 2$.

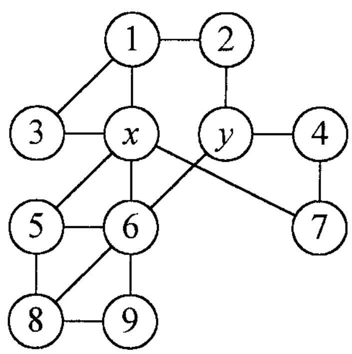
8.4 Find the degrees of the boundaries of the regions of the graphs shown in Fig. 8-12.

Solution. (*a*) The degrees of A, B, C, D, and E are 4, 3, 4, 3, and 3, respectively. The degree of the exterior region is 7. (*b*) The degree of A is 2, the degree of B is 13, and the degree of the exterior region is 5.

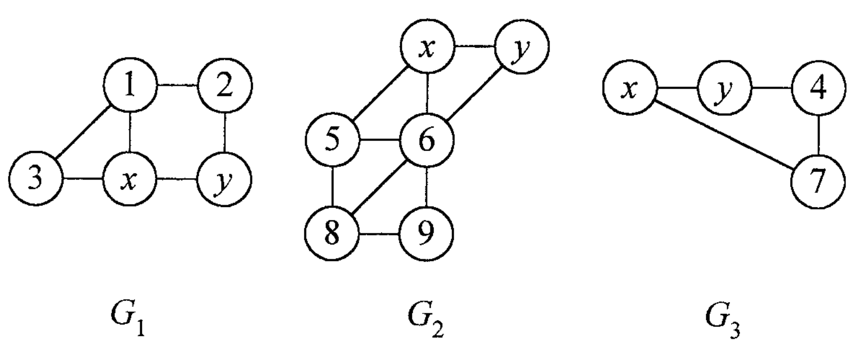
8.5 Show that a plane graph is 2-connected if and only if the boundary of each region is a cycle.

Solution. If the boundary of every region (including the exterior region) of a plane graph is a cycle, there will be at least two internally disjoint paths between every pair of vertices. So the graph is 2-connected. On the other hand, let G be a 2-connected plane graph. We have to establish that the boundary of every region of G is a cycle. Suppose there is a region of G whose boundary is not a cycle. Since G is 2-connected, it has at least one region bounded by a cycle. Let H be a maximal subgraph of G, the boundaries of whose regions are cycles. If there is a region in H whose boundary is a cycle and if two nonadjacent (in H) of vertices of this cycle are adjacent in G, H will have one more region whose boundary is a cycle, violating the maximality condition. We assume that this is not the case. $H = (W, F)$ is a proper subgraph of the 2-connected graph $G = (V, E)$. So whenever u and v are two vertices in W and w is a vertex in $V - W$, there exists a path between u and v that passes through w, as established in Solved Problem 6.32. This implies that there is a path joining two nonadjacent vertices in H such that every intermediate vertex in this path is in $V - W$. Any such path will partition some region of H into two regions, violating the maximality requirement. So the boundary of every region of a 2-connected plane graph is a cycle.

8.6 Let $G = (V, E)$ be a graph with vertex-connectivity number $\kappa(G) = 2$, and let x, y be two vertices such that $G - x - y$ is disconnected with components $H_i (i = 1, 2, \ldots, k)$, where $H_i = (V_i, E_i)$ and $W_i = V_i \cup \{x\} \cup \{y\}$. For each i, G_i is the graph induced by the set W_i if vertices x and y are adjacent in G. If these two vertices are not adjacent, G_i is the graph induced by W_i along with edge e joining x and y. The family $\{G_i : i = 1, 2, \ldots, k\}$ is called the **hammock decomposition** of G with respect to the separating set $\{x, y\}$. Obtain a hammock decomposition for the graph in Fig. 8-13(a) with respect to vertices x and y indicated in the diagram.



(a)



(b)

Fig. 8-13

Solution. The hammock decomposition consists of the three graphs shown in Fig. 8-13(b).

8.7 Show that a graph G with $\kappa(G) = 2$ is planar if and only if each graph in a hammock decomposition of G with respect to any separating set consisting of two vertices is planar.

Solution. Suppose G is planar and $\{G_i\}$ is a hammock decomposition with respect to a separating set consisting of x and y. If these two vertices are adjacent in G, each graph in the decomposition is a subgraph of G; therefore, it is planar. Let us examine the case when x and y are not adjacent. Observe that $\kappa(G_i) = 2$ for each i. So between vertices x and y are at least two paths in G_i for each i. Hence, graph G_i' has a path P_i between x and y, where G_i' is the subgraph of G induced by the set of vertices of G_i. Each G_i' is planar. Construct a new path Q_i in G_i' by introducing new vertices of degree 2 between x and y duplicating P_i such that $G_i' \cup Q_i$ is planar. Then G_i is homeomorphic to $G_i' \cup Q_i$. Thus each graph in the decomposition is planar. To prove the converse, assume that each graph in the decomposition $\{G_1, G_2, G_3, \ldots, G_k\}$ is planar. These graphs can be sequentially merged by edge merging to form a planar graph H; first merge G_1 and G_2 to form a planar graph. Then merge this new graph and G_3 to form another planar graph. Continue this process until all the k graphs are merged to obtain H, which is planar. Graphs G and H differ at most by an edge. So G is planar.

8.8 Show that any triangulation with at least four vertices is 3-connected.

Solution. Suppose the triangulation G is not 3-connected. This implies that $\kappa(G) = 2$, and there exist vertices u and v defining a Hammock decomposition $\{G_i : i = 1, 2, \ldots, k\}$ such that each G_i is planar and $\kappa(G_i) = 2$ for each i. Let $(k - 1)$ of these graphs be merged (by sequentially edge merging two at a time) to obtain a planar graph H. The remaining graph in the decomposition is G_k, which is also planar. Suppose the edge used in merging these two graphs joins vertices p and q. Let $C: p \to q \to v_1 \to v_2 \to \cdots \to v_m \to p$ be a cycle in H, and let $C': p \to q \to w_1 \to w_2 \to \cdots \to w_n \to p$ be a cycle in G_k. No vertex v_i is adjacent to vertex w_j. So we can have a planar embedding consisting of G and an edge joining nonadjacent vertices v_i and w_j. This violates that G is a maximal triangulated graph. So G is 3-connected.

8.9 If X is the set of blocks and Y is the set of cutvertices in a graph $G = (V, E)$, the **block–cut vertex graph (BC graph)** of G is the bipartite graph $H = (X, Y, F)$ in which there is an edge joining block B and cutvertex v if and only if v is a vertex in B. Construct the BC graph of graph G shown in Fig. 8-14(a).

Solution. Graph G shown in Figure 8-14(a) has six blocks as subgraphs induced by the following sets of vertices: $\{1, 2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{6, 7, 8, 9\}$, and $\{8, 10, 11, 12, 13\}$. The corresponding BC graph is shown in Fig. 8-1(b).

8.10 Show that if a graph is connected, its BC graph is a tree.

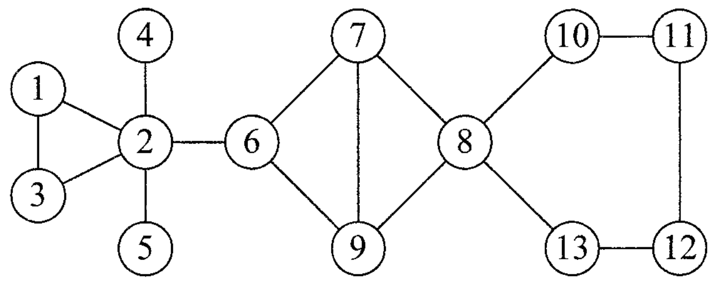
Solution. Let H be the BC graph of graph G. If G is connected, H is also connected. Suppose C: $v_1, B_1, v_2, B_2, \ldots, v_i, B_i, v_{i+1}, \ldots, B_k, v_1$ is a cycle in H. In each block B_i is a path between v_i and v_{i+1}. The union of these paths form a cycle C' in G. Among the k blocks in C, there should be at least two blocks that have edges in common with C'. Now if a block contains two vertices x and y, it contains all the paths joining those two vertices. So these two blocks should contain C', implying that they have at least three vertices in common. This is a contradiction since two blocks can have at most one vertex in common. So H is acyclic.

8.11 A block B in a graph is called a **pendant block** if B contains exactly one cut vertex. Show that if a graph has a cut vertex, it has at least two pendant blocks.

Solution. If G has a cut vertex, its BC-graph is nontrivial and acyclic with at least two vertices of degree 1. Any vertex of degree 1 in the BC-graph is a pendant block. The graph shown in Fig. 8-14(a) has four pendant blocks, which are displayed as leaves of its BC-tree in Fig. 8-15.

8.12 Show that a graph G is planar if and only if each of its blocks is planar.

(a)



(b)

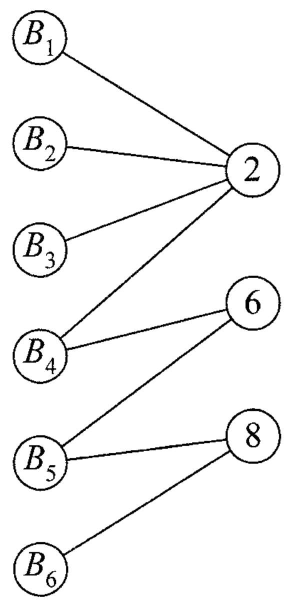


Fig. 8-14

Solution. We may assume without loss of generality that G is connected. Clearly, if a graph is planar, every block in it is planar. The converse is proved by an inductive argument on the number k of blocks in G. The result is true when $k = 1$. Assume that every graph G with fewer than k blocks (each being planar) is planar, where $k \geq 2$. Let G be a graph with k planar blocks, and let B be any pendant block (it exists, as proved in Problem 8.11) in G. Since it is a pendant block, there is a unique cutvertex v of G common to B. Let G' be the graph obtained from G after deleting all the vertices of B other than v. By the induction hypothesis, G' is planar. Since B is a pendant block, it can be adjoined to G', which is planar, by embedding both B and G' on the plane (where B is in the exterior region of G') with v as a connecting vertex. Thus G is planar. So the result is true for k as well.

8.13 Prove Theorem 8.4: The number of edges in a triangulation of order n (where $n \geq 3$) is $3n - 6$.

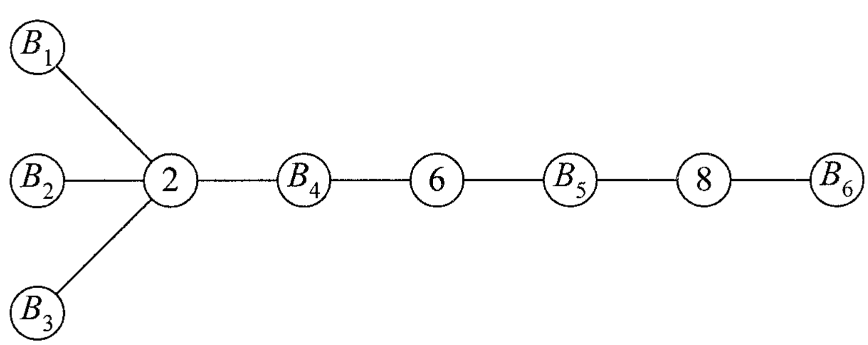


Fig. 8-15

Solution. Suppose there are m edges and f faces in a triangulation. The boundary of each face consists of three edges. If we add the edges of all the f faces, we get $3f$ edges. In this adding process, each edge is counted twice. So $3f = 2m$. On substituting $f = \frac{2}{3}m$ in Euler's formula $n - m + f = 2$, we get the relation $m = 3n - 6$.

8.14 Show that the number of edges in a simple planar graph of order n is at most $3n - 6$.

Solution. Let G be a planar embedding of the given planar graph, and suppose it has k components each with n_i vertices and m_i edges, where $i = 1, 2, \ldots, k$. Since the graph is simple, it follows from Problem 8.15 that $m_i \leq 3\,n_i - 6$ for each i. By adding over all their components, $m \leq 3n - 6k \leq 3n - 6$.

8.15 Prove Theorem 8.5: Both K_5 and $K_{3,3}$ are nonplanar.

Solution.

(*i*) If K_5 is a planar graph, it cannot have more than $(3)(5) - 5 = 9$ edges, according to the result established in Problem 8.14. But it has 10 edges, so it is not planar.

(*ii*) Since $K_{3,3}$ is bipartite, it cannot have an odd cycle. So the boundary of each face has at least four edges. If we add the edges of the boundaries of all the f faces, the sum is at least $4f$. But the number of edges is 9. So $4f \leq 18$, since each edge is counted twice in the adding process. But, by Euler's formula, f should be 5.

8.16 Show that in a simple planar graph G, there are at least four vertices with degrees at most 5. In particular, every convex polyhedron has at least four corners that are adjacent to three, four, or five corners.

Solution. We may assume that G is of order n and size m and is a plane graph. By joining nonadjacent vertices as and when needed, a maximal plane graph G' is constructed. Suppose the number of vertices of degree i in G' is n_i. Since the degree of each vertex in G' is at least 3, $3n_3 + 4n_4 + 5n_5 + \cdots = 2m = 2(3n - 6)$. Hence, $3(n_3 + n_4 + n_5) + 6(n_6 + n_7 + n_8 + \cdots) \leq 6n - 12$. But $(n_6 + n_7 + n_8 + \cdots) = n - (n_3 + n_4 + n_5)$. So $(n_3 + n_4 + n_5) \geq 4$, which implies that G' (and hence G) has at least four vertices of degree at most 5. In the case of the graph $G(P)$ of a convex polyhedron, the degree of each vertex is at least 3. So there are at least four vertices in $G(P)$ with degree 3, 4, or 5.

8.17 Exhibit a simple planar connected graph in which the degree of each vertex is at least 5.

Solution. In the plane graph corresponding to the icosahedron (see Fig. 8-8), the degree of each vertex is 5. Furthermore, it has 12 vertices.

8.18 Exhibit a simple planar connected graph in which the degree of each vertex is exactly 4.

Solution. In the plane graph of the octahedron (with eight vertices), the degree of each vertex is 4.

8.19 Show that if a simple graph G has at least 11 vertices, both G and its complement cannot be planar graphs.

Solution. Suppose G has n vertices and m edges and its complement has m' edges. Then $m + m' = n(n - 1)/2$. If both are planar, $n(n - 1)/2 \leq (6n - 12)$. This inequality is true if and only if $n < 11$.

8.20 Suppose n_i be the number of vertices of degree i in a triangulation G of order n. Establish the relation $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + 3n_9 + \cdots + (n - 6)n_k + 12$, where k is the maximum degree in G.

Solution. The degree of each vertex is at least 3. The sum of the degrees is twice the number of edges, which is equal to $6n - 12$. So $4n_3 + 4n_4 + \cdots + kn_k = 6n - 12$. But $n_3 + n_4 + \cdots + n_k = n$. Hence, $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + 3n_9 + \cdots + (n - 6)n_k + 12$.

8.21 Show that every convex polyhedron has at least one face whose boundary consists of three, four, or five edges. (This is similar to the fact established in Problem 8.16 that every convex polyhedron has a corner that is adjacent to three, four, or five corners.)

Solution. Assume that the convex polyhedron has n corners (vertices), m sides (edges), and f faces (regions). Let f_k be the number of faces of degree k. Suppose $f_k = 0$ when $k = 3, 4$, or 5. Then $6f_6 + 7f_7 + \cdots = 2m$. Hence, $6(f_6 + f_7 + \cdots) \le 2m$, which implies that $6f \le 2m$.

Suppose there are n_i corners of degree i. Since each corner is adjacent to at least three corners, $n_i \ge 3$ for each i. The degree sum $3n_3 + 4n_4 + \cdots = 2m$ implies that $3n \le 2m$. Now $n - m + f = 2$ implies that $m = n + f - 2 \le \frac{2}{3}m + \frac{1}{3}m - 2 = m - 2$. This contradiction establishes that there is at least one face surrounded by at most five faces.

8.22 Show that the total number of corners of degree 3 and faces of degree 3 in convex polyhedron is at least eight.

Solution. Assume that the convex polyhedron has n corners (vertices), m sides (edges), and f faces (regions). Let f_k be the number of faces of degree k and n_k be the number of vertices of degree k. Then $(3n_3 + 4n_4 + \cdots) + (3f_3 + 4f_4 + \cdots) = 4m = 4n + 4f - 8$, implying that $n_3 + f_3 \ge 8$.

8.23 Show that there are only five regular polyhedra.

Solution. Suppose the regular convex polyhedron P under consideration has n corners, m sides and f faces. Then its associated simple planar graph G has n vertices, m edges and f regions. Since P is regular, the degree each vertex of G is equal to a fixed integer k (≥ 3) and the degree of each region of G is a fixed integer $r (r \ge 3)$. Then $kn = rf = 2m$. But $n - m + f = 2$. So $8 = 4n - 2m - 2m + 4f = 4n - kn - rf + 4f = n(4 - k) + f(4 - r)$. Hence either $(4 - k) \ge 0$ or $(4 - r) \ge 0$. In other words $3 \le k \le 4$ or $3 \le r \le 4$.

Case (i): Let $k = 3$. Then $rf = 3n = 2m$ and $n + f(4 - r) = 8$. Hence, $24 = 3n + 12f - 3rf = rf + 12f - 3rf = f(12 - 2r)$, implying that $f(6 - r) = 12$. Again, $8 = n + f(4 - r) = n + 4f - rf = n + 4f - 3n$, implying that $n = 2f - 4$. Three choices arise: (*a*) $r = 3, f = 4, n = 4$, and $m = 6$ (tetrahedron); (*b*) $r = 4, f = 6, n = 8$, and $m = 12$ (cube); and (*c*) $r = 5, f = 12, n = 20$, and $m = 30$ (dodecahedron).

Case (ii): Let $k = 4$. In this case, $rf = 4n = 2m$ and $(4 - r)f = 8$. This implies that $r = 3, f = 8, n = 6$, and $m = 12$ (octahedron).

Case (iii): Let $r = 3$. Then $3f = kn = 2m$ and $n(4 - k) + f = 8$. Hence, $24 = 12n - 3nk + 3f = 12n - 2kn$, implying that $n(6 - k) = 12$. Also, $8 = 4n - nk + f = 4n - 3f + f$, implying that $f = 2n - 4$. Three choices arise: (*a*) $k = 3$, $n = 4$, $f = 4$, and $m = 6$ (tetrahedron again); (*b*) $k = 4$, $n = 6$, $f = 8$, and $m = 12$ (octahedron again); and (*c*) $k = 5$, $n = 12, f = 20$, and $m = 30$ (icosahedron).

Case (iv): Let $r = 4$. In this case, $k = 3, n = 8, f = 6$, and $m = 12$ (cube again).

8.24 (*Kotzig's Theorem*) Show that every polyhedral graph has two adjacent vertices such that their degree sum is at most 13.

Solution. It is enough if this result is proved for a triangulation. Let p be the number of edges in the graph joining two vertices such that their degree sum is at most 12, and let q be the number of edges joining two vertices such that their degree sum is 13. We have to prove that $p + q$ is positive. The result is true if p is positive. So what remains to be proved is that q is positive when p is zero. As usual, let n_i be the number of vertices of degree i. Suppose the degree of vertex v is 3 and that e is the edge joining v and vertex w. Since $p = 0$, the degree of w is 10 or more than 10. Thus $3n_3 \le q + \frac{1}{2}\Sigma_{k \ge 11} kn_k$. Similarly, $3n_3 + 4n_4 \le \frac{1}{2}\Sigma_{k \ge 9} kn_k$ and $3n_3 + 4n_4 + 5n_5 \le \frac{1}{2}\Sigma_{k \ge 8} kn_k$. Multiplying these three inequalities by 5, 3, and 2, respectively, and then adding gives $30n_3 + 20n_4 + 10n_5 \le 5q + 8n_8 + 24n_9 + 25n_{10} + 5\Sigma_{k \ge 11} kn_k$. Using Euler's formula, we then obtain the inequality

$$120 + 10[\Sigma_{k \ge 7} (k - 6)n_k] \le 5q + 8n_8 + 24n_9 + 25n_{10} + 5[\Sigma_{k \ge 11} kn_k]$$

It follows that $120 \le 5q$, which implies that $q > 0$, as we wished to prove.

8.25 A graph G is said to be **minimally nonplanar** if every proper subgraph of G is planar. Show that a minimally nonplanar graph is a block.

Solution. If G is not connected, each component is planar, which implies that G itself is planar. So G is connected. Suppose G has cut vertex v. Then the deletion of v will give subgraphs that are planar, implying that G is planar. Hence, G is 2-connected. Since G is nonplanar, at least one of its blocks is nonplanar. If G has two blocks, both will be planar with at most one vertex in common, implying that G is planar. Hence, G is a block.

8.26 Suppose G is a nonplanar graph that does not have a K-subgraph such that the size of any other nonplanar graph that does not have a K-subgraph is more than the size of G. Show that G is 3-connected.

Solution. By our assumption, if there is a graph H such that the size of H is less than the size of G and if it has no K-subgraph, it has to be planar. So no edge can be deleted from G without violating its nonplanarity property. Hence, G is minimally nonplanar, therefore, it is a block, as established in Problem 8.25. Suppose G becomes disconnected by deleting two vertices, x and y. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of G such that $G_1 \cup G_2 = G$ and $V_1 \cap V_2 = S$, where S is the separating set $\{x, y\}$. Let H_i be the graph obtained by joining vertices x and y to G_i for $i = 1, 2$ by edge e. Suppose both H_1 and H_2 are planar. Then each can be embedded on the plane such that e is a boundary of the exterior face. This implies that $H_1 \cup H_2 - e$ is a plane embedding of the nonplanar graph G. This contradiction implies that either H_1 or H_2 is nonplanar. The minimality assumption furthermore requires that one of these graphs, say H_1, is a nonplanar graph containing a K-subgraph. In that case, edge e is an edge of K; otherwise, K will be a subgraph of G, which is a contradiction. If we replace edge e by a simple path (in H_2) joining x and y, we get a homeomorph of K as a subgraph of G, which is a contradiction. So G does not have a 2-vertex separating set.

8.27 (*Thomassen's Theorem*) Show that if G is a 3-connected graph with at least five vertices, it has edge e such that $G.e$ is 3-connected.

Solution. Suppose $G.e$ is not 3-connected for every edge e of the graph. Let t be the vertex obtained by contracting edge e joining u and v. Since graph G is 3-connected, there are at least two internally disjoint paths between every pair of vertices in $G.e$. Hence, $G.e$ is 2-connected. Let x and y be two vertices in so that $G.e - x - y$ is disconnected. Suppose x, y, and t are distinct. Since G is 3-connected, $G - x - y$ is a connected graph. Notice that $G.e - x - y$ can be obtained from $G - x - y$ by contracting edge e, so it is a connected graph. This is a contradiction since $G.e - x - y$ is a disconnected graph. So x, y, and t cannot be distinct. In other words, whenever $G.e$ is constructed by contracting an edge into a single vertex, this contracted vertex is necessarily one of the two vertices in any separating set of $G.e$. Thus, corresponding to edge e joining u and v in G is vertex w (call it a mate of e) in G such that $G - \{u, v, w\}$ is disconnected. Let H_1 be a component of $G - \{u, v, w\}$ with the smallest number of vertices. By our hypothesis, these three vertices are adjacent to three distinct vertices in H_1. Suppose e_1 is the edge joining w and vertex p in H_1. As before, we use edge e_1 to contract G to obtain $G.e_1$. Let the mate of e_1 be q. Then $\{w, p, q\}$ is a minimum disconnecting set of G. The component H_2 of $G - \{w, p, q\}$ with the minimum number of vertices is a proper subgraph of H_1. By continuing this process, we ultimately get a component with a single vertex and edge f joining it to one of three vertices in the previous separating set. Then $G.f$ is 3-connected, which is a contradiction.

8.28 Show that if the contraction $G.e$ of a graph G has a K-subgraph, G also has one.

Solution. Let a homeomorph H of K_5 or $K_{3,3}$ be a subgraph of $G.e$, where e is the edge joining u and v in G that are merged together to form the new vertex w in $G.e$. If w is not a vertex of H, H is a subgraph of G. If w is a vertex in H of degree 2, by expanding vertex w into the edge (in G) joining u and v, we get a Kuratowski graph H' (a homeomorph from H) as a subgraph of G. If w is a vertex in H of degree more than 2, either its degree in H is 3 (in which case H is a homeomorph of $K_{3,3}$) or its degree in H is 4 (in which case H is a homeomorph of K_5). Suppose H is a homeomorph of the complete bipartite graph with partite sets $X = \{1, 2, 3\}$ and $Y = \{4, 5, w\}$. If vertex w is expanded into edge e joining u and v in G, we get a homeomorph H' from H as a subgraph of G. This H' is a homeomorph from the complete bipartite graph (X, Y', E'), where $Y' = \{4, 5, u\}$ or $\{4, 5, v\}$. Finally, suppose H is a homeomorph of K_5. Consider the case when w is expanded into the edge joining u and v: two of the edges incident to w become edges incident to u, and the other two become edges incident to v. So there are two vertices in H (say u_1 and u_2) of degree 4 and two paths P_1 and P_2 (in G) between u and each of these two vertices such that these paths have no internal vertices of degree 4. Likewise, there are two vertices v_1 and v_2 in H with paths P_3 and P_4 joining v and each of these two vertices. Delete those edges that belong to H from these four paths. We then find that G has a subgraph H' that is a homeomorph of the complete bipartite graph with partite sets $\{u, v_1, v_2\}$ and $\{v, u_1, u_2\}$. In the only remaining case, when the vertex u has at most one edge incident to it out of the four edges incident to w, we see that G has a subgraph homeomorphic to K_5.

8.29 *(Tutte's Theorem and Its Proof by Thomassen)* Show that if a 3-connected graph G has no K-subgraph, it has a convex straight-line embedding on the plane.

Solution. The proof is by induction on the order n of G. The induction hypothesis is that any 3-connected graph of order less than n with no K-subgraph has a convex embedding. The order of any 3-connected graph is at least 4. The complete graph of order 4 is the only 3-connected graph with four vertices, and it has a convex straight-line embedding. So the theorem is true when $n = 4$. Let $n \geq 5$. Since G is 3-connected, it has edge e joining u and v such that $G.e$ is 3-connected, as shown in Problem 8.27. If $G.e$ has a K-subgraph, G should also have a K-subgraph, as shown in Problem 8.28, violating our hypothesis. So $G.e$ does not have a K-subgraph; hence, by the induction hypothesis, it has a convex embedding. The proof is complete if we show that G has a convex straight-line embedding. If w is the vertex obtained by merging u and v, graph $G' = G.e - w$ is 2-connected. So the boundary of every region (including the exterior region) is a cycle in G', as proved in Problem 8.5. Any cycle in G' is a cycle in G. If we delete the edges incident to w from $G.e$, vertex w will be in one of the regions of G'. Let C be the cycle in which lies the boundary of the region that contains w.

Case (i): There are three vertices p, q, and r in C that are adjacent to both u and v in G. See Fig. 8-16. Then these five vertices are the vertices of a subgraph of G, which is a homeomorph of K_5, which is a contradiction.

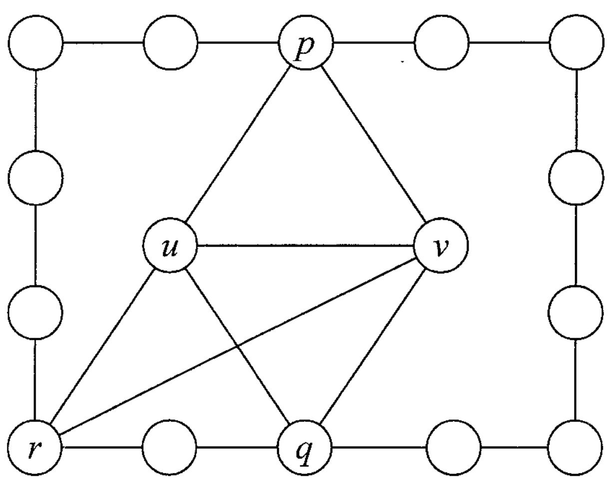


Fig. 8-16

Case (ii): There are four vertices p, q, r, and s that appear in that order on cycle C such that u is adjacent to p and r and v is adjacent to q and s. See Fig. 8-17. These six vertices are the vertices of a subgraph of G, which is a homeomorph of $K_{3,3}$, which is a contradiction.

Case (iii): This case is neither case (*i*) nor case (*ii*). So the number of vertices in C that are adjacent to both u and v is at most 2. We also have the situation in which the vertices adjacent to u and the vertices adjacent to v do not appear on the cycle as stipulated in case (*ii*). So if the vertices adjacent to u are $u_1, u_2, \ldots, u_k$ and the vertices adjacent to v are $v_1, v_2, \ldots, v_m$, these vertices are cyclically ordered on C as $u_1, u_2, \ldots, u_k$ first and then $v_1, v_2, \ldots, v_m$. (It is possible that $u_k = v_1$ or $v_m = u_1$, showing that the number of vertices adjacent to both u and v is at most 2.) See Fig. 8-18(a), with $k = 4$ and $m = 5$. This scenario as it is may give a convex straight-line

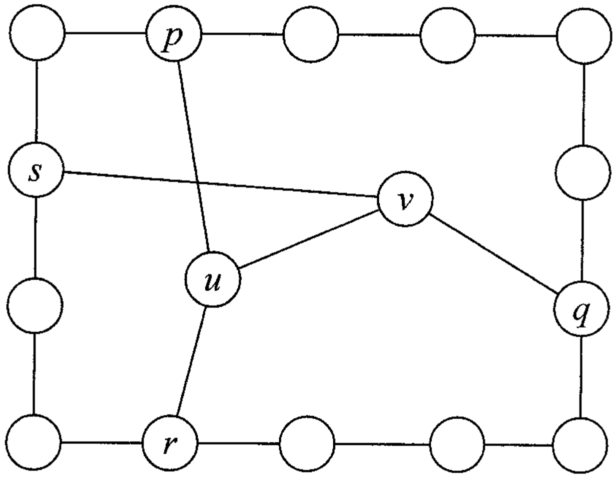


Fig. 8-17

embedding. Otherwise, we move vertex v to another region enclosed by a part of C that contains all the vertices adjacent to v, resulting a convex straight-line embedding for G. See Fig. 8-18(b). Thus, in this case, we have a convex embedding of G by placing v inside the appropriate region.

8.30 Show that 3-connectivity is necessary in Tutte's theorem.

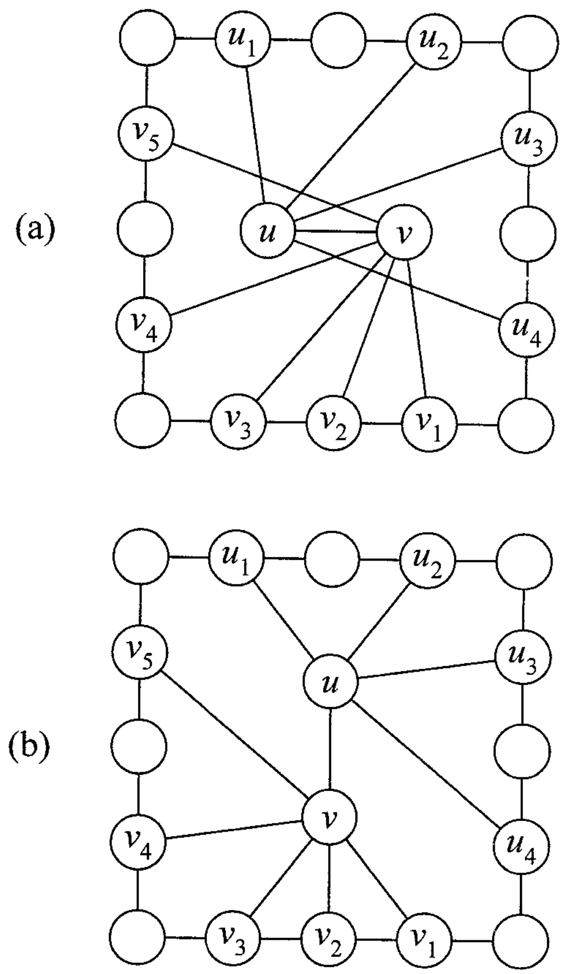


Fig. 8-18

Solution. The 2-connected complete bipartite graph $K_{2,n}$ (where $n \geq 4$) does not have a Kuratowski subgraph. It has an embedding in the plane in which each region is a polygon, but it does not have a convex embedding in the plane.

8.31 Prove Theorem 8.6 (the Kuratowski–Pontryagin theorem): A graph is planar if and only if it does not have a K-subgraph.

Solution. If G has a K-subgraph, G is obviously nonplanar. To establish the converse, suppose there is a nonplanar graph with no K-subgraph. Among such nonplanar graphs, let G be a graph with as few edges as possible. Then G is 3-connected, as established in Problem 8.26. But any 3-connected graph with no K-subgraph has a convex embedding, as established in Problem 8.29; hence, it is planar. This contradicts the assumption that G is nonplanar.

8.32 Prove Theorem 8.1 (the Fary–Stein–Wagner theorem): Any simple planar graph has a straight-line representation: it has an embedding in a plane such that each edge in the embedding is a straight line.

Solution. Every triangulation is a 3-connected graph, as shown in Problem 8.8. In other words, every triangulation is a 3-connected graph with no K-subgraph. So, by Tutte's theorem, it has a convex straight-line embedding, as established in Problem 8.27. But every simple planar graph is a spanning subgraph of a triangulation. So every simple planar graph has a straight-line embedding. The bipartite graph $K_{2,n}$ (where $n \geq 4$) is a simple planar graph that has a straight-line embedding but no convex embedding; it is a 2-connected graph that is not 3-connected.

8.33 *(The Conflict Graph of a Graph with Respect to One of Its Cycles)* Let C be a cycle in a graph $G = (V, E)$. A **piece** of G relative to C is either the subgraph consisting of an edge (in G) joining two nonadjacent vertices in C or a subgraph formed by a component H of $G - C$ and all the edges of G adjacent to the vertices in H. Vertex v in piece P is a **contact vertex** of P if v is a vertex of the cycle. Any piece containing more than one contact point is called a **segment** of G relative to C. Two segments S and S' are **in conflict** if an edge in S and an edge in S' necessarily intersect (not at a vertex) when these two segments are embedded on the same side (interior or exterior) of C. Let X be the set of all segments relative to cycle C. The **conflict graph** with X as the set of vertices is constructed as follows. Join two segments by an edge if and only if they are in conflict. Construct the conflict graph shown in Fig. 8-19(a) with respect to the cycle C: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 1$.

Solution. There are five pieces relative to C, as shown in Fig. 8-19(b). In P_1, the only contact vertex is 3, so this piece is not a segment. The remaining four pieces are segments. In P_2, both the vertices are contact vertices. The contact vertices in P_3 are 1 and 6; the contact vertices in P_4 are 1, 2, and 4; and the contact vertices in P_5 are 3, 5, and 8. The conflict graph (not shown) relative to C consists of four vertices defined by these four segments. The segments P_4 and P_5 are in conflict; in the interior of the cycle, it is impossible to embed them without an edge from one crossing an edge from the other. Thus the only edge in the conflict graph is the edge joining the vertices corresponding to segments P_4 and P_5.

8.34 Show that if a graph G contains K_5 or $K_{3,3}$ as a subgraph, there is a cycle in G such that the conflict graph relative to that cycle is not bipartite.

Solution. Suppose G contains the subgraph K_5 as a subgraph. Let the vertices of this subgraph be 1, 2, 3, 4, and 5. Then C: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ is a cycle in G. Among the pieces of the graph we have three segments: S_1 consists of the single edge joining 1 and 3, S_2 consists of the single edge joining 2 and 4, and S_3 is a tree consisting of edges $\{5, 1\}$, $\{5, 2\}$, $\{5, 3\}$, and $\{5, 4\}$. These three segments are pairwise in conflict, so they form an odd cycle in the conflict graph relative to cycle C. Thus the conflict graph is not bipartite. If G contains $K_{3,3} = (X, Y, E)$ as a subgraph, let the vertices of the subgraph be defined by $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$. Then C: $1 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 6 \rightarrow 3 \rightarrow 1$ is a cycle in G. Among the pieces relative to this cycle are three segments, in conflict pairwise, as edges $\{1, 5\}$, $\{2, 6\}$, and $\{3, 4\}$. So once again the conflict graph is not bipartite.

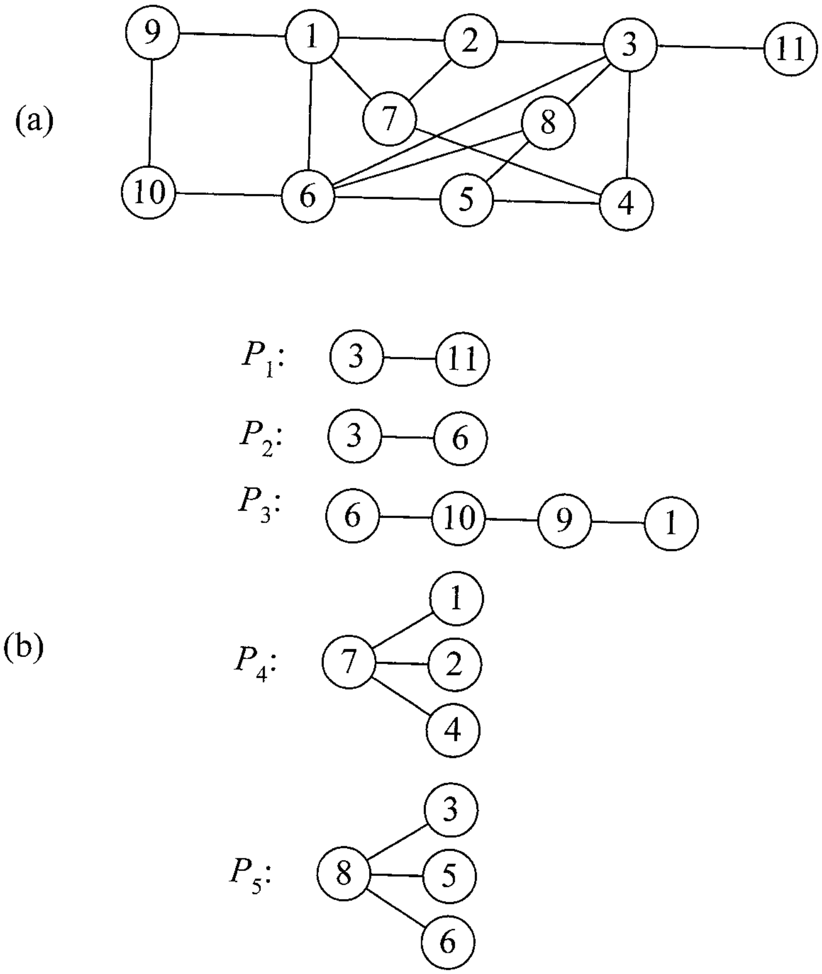


Fig. 8-19

8.35 (*Tutte's Conflict Graph Theorem*) Show that a graph is planar if and only if its conflict graph relative to every cycle in it is bipartite.

Solution. Suppose the graph is planar. Take any cycle C in it. First, C is drawn with no intersecting edges. The pieces that are not segments can be easily embedded so that the edges do not intersect. Since the graph is planar, the segments can be embedded in such a way that segments in conflict lie on either side of the cycle. So no two segments in the interior of C are adjacent vertices in the conflict graph. Similarly, no two segments in the exterior of C are adjacent. In other words, the conflict graph relative to any cycle in a planar graph is bipartite. To prove the converse, suppose G is a nonplanar graph. Then, by Kuratowski's theorem, it contains a subgraph homeomorphic to either K_5 or $K_{3,3}$. It follows from Problem 8.35 that there is a cycle in the graph such that the conflict graph relative to that cycle is not bipartite. In other words, if the conflict graph with respect to every cycle is bipartite, the graph is planar.

8.36 Prove that the conflict graph relative to any cycle in the graph shown in Fig. 8-19(a) is bipartite.

Solution. An embedding of the graph as a plane graph is shown in Fig. 8-20. So the given graph is planar. Hence, its conflict graph relative to any cycle is necessarily bipartite.

8.37 Show that the Heawood graph (see Fig. 7-25) is nonplanar.

Solution. This cubic graph has 14 vertices, as shown in Fig. 7-25(a). Consider the cycle C as shown passing consecutively through these 14 vertices. Each edge of the graph not belonging to the cycle is a segment defining

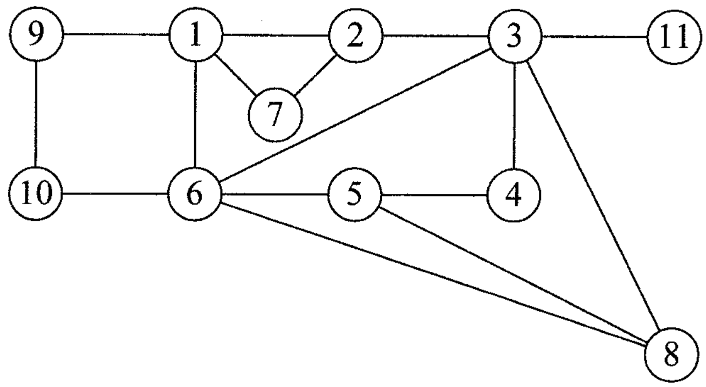


Fig. 8-20

a vertex in the conflict graph relative to this cycle. Let S_1 be the segment defined by the edge joining 1 and 10, let S_2 be the segment joining 3 and 12, and let S_3 be the segment defined by the edge joining 5 and 14. These three segments are in conflict pairwise, so they define an odd cycle in the conflict graph.

8.38 Show that the complement of the 3-cube is not planar.

Solution. The 3-cube is a planar graph G, as shown in Fig. 8-21. Two vertices in its complement are adjacent if and only if they are not adjacent in G. Using this information, we can locate a cycle C: $1 \to 3 \to 5 \to 4 \to 8 \to 2 \to 7 \to 6 \to 1$ in its complement. We now construct the conflict graph relative to this cycle in the complement. Three segments that are in conflict pairwise are the edges (in the complement) joining 1 and 8, 2 and 5, and 4 and 6. These three segments create an odd cycle in the conflict graph.

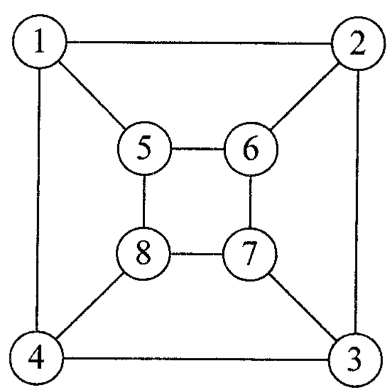


Fig. 8-21

8.39 Give an example of two homeomorphs of a graph such that each is not a homeomorph of the other.

Solution. In Fig. 8-22, both G' and G'' are homeomorphs of G, but G' is not a homeomorph of G'', and G'' is not a homeomorph of G'.

8.40 Show that if a graph G has $K_{3,3}$ as a subcontraction, G is nonplanar.

Solution. Suppose $G = (V, E)$ has a subgraph $H = (W, F)$ that is contractible to the complete bipartite graph in which the two partite sets are $X = \{u_i : i = 1, 2, 3\}$ and $Y = \{u_i' : i = 1, 2, 3\}$. So set W has a partition $\{W_1, W_2, W_3, W_1', W_2', W_3'\}$ such that u_i is matched with W_i and u_i' with W_i' for each i. Furthermore, since u_i and u_j' are adjacent, the subgraph G_i induced by W_i has vertex v_{ij}, and the subgraph G_i' induced by W_i' has vertex v_{ij}'

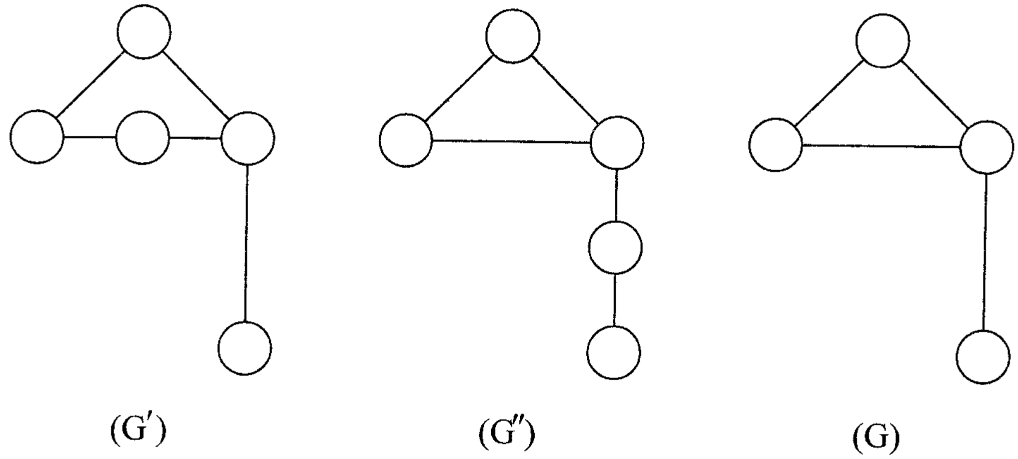


Fig. 8-22

such that these two vertices are adjacent. If the three vertices v_{i1}, v_{i2}, and v_{i3} are the same vertex, let each be labeled v_i. Otherwise, let v_i be a vertex in G_i that has internally disjoint paths connecting it to the distinct vertices among v_{i1}, v_{i2}, and v_{i3}. Thus each W_i has a vertex v_i with three internally disjoint paths from it to vertices in W'_1, W'_2, and W'_3. Similarly, each W'_i has vertex v'_i with three internally disjoint paths from it to vertices in W_1, W_2, and W_3. Graphs G_i and G_j are all connected graphs. Hence, G has a subgraph that is a homeomorph of $K_{3,3}$, so it is nonplanar.

8.41 Show that if a graph G has K_5 as a subcontraction, G is nonplanar.

Solution. Suppose $G = (V, E)$ has a subgraph $H = (W, F)$ that is contractible to a complete graph with a vertex set $\{u_i: 1 \le i \le 5\}$. Then $\{W_i: 1 \le i \le 5\}$ is a partition of W such that u_i is matched with W_i for each i. Let G_i be the subgraph induced by the set W_i. If there is a vertex v_i for each i such that there are four internally disjoint paths between v_i and v_j $(i \ne j)$, H is a homeomorph of K_5; hence, G is nonplanar. As in Problem 8.40, there exists vertex v_{ij} in G_i and vertex v_{ji} in G_i such that these two vertices are adjacent, where $i \ne j$, $1 \le i, j \le 5$. Let us fix $i = 1$. Then there are several mutually exclusive possibilities concerning the vertices v_{12}, v_{13}, v_{14} and v_{15}.

Case (i): These four vertices are the same, in which case let v_1 represent this common vertex; or three of the four vertices are the same, in which case also label this common vertex v_1. In both situations, there are four internally disjoint paths from v_1 to vertices in the other four sets.

Case (ii): Two of the four are the same (labeled v_1), and the other two are distinct. Also suppose that there are internally disjoint paths from v_1 to the other two vertices. In this case, there are also four internally disjoint paths from v_1 to vertices in the other four sets.

Case (iii): The four vertices are distinct. Also suppose there is vertex v_1 in W_1 such that there are four internally disjoint paths from v_1 to these four vertices so that there will be internally disjoint paths from v_1 to vertices in the other sets.

Case (iv): None of the above.

So if the scenario presented in case (*i*), (*ii*), or (*iii*) exists for each of five sets of the partition, we are done.

We now scrutinize case (*iv*). There is at least one graph, say G_1, among the five graphs in which case (*iv*) holds. This implies that there are two vertices w_1 and w'_1 in G_1 joined by path P_1 in which no intermediate vertex is v_{1j} such that each of them is connected to two of these four vertices by internally disjoint (probably trivial) paths. [Specifically, if $v_{12} = v_{13}$ and $v_{14} = v_{15}$ are two distinct vertices, we take $v_{12} = v_{13} = w_1$ and $v_{14} = v_{15} = w'_1$. If $v_{12} = v_{13} = w_1$ and if v_{14} and v_{15} are distinct, and since there are no internally disjoint paths from w_1 to these two vertices, w'_1 is the last common vertex in a path between w_1 and v_{14} and in a path between w_1 and v_{15}. If the four vertices are distinct and if there is no vertex with internally disjoint paths from that vertex to these four vertices as stipulated in case (*iii*), we can locate two vertices w_1 and w'_1 such that there is a path P_1 between these two vertices that does not pass through any of these four vertices. At the same time we can also locate two internally disjoint paths from w_1 to v_{12} and to v_{13} and two internally disjoint paths from w'_1 to v_{14} and to v_{15}.] Thus we have path P_1 joining w_1 and w'_1 and four internally disjoint paths (two from w_1 and two from w'_1) to the other four sets. Let E_1 be the set of edges of these paths. If case (*iv*) does not hold for any of the remaining four sets W_i, vertices

v_i (i = 2, 3, 4, 5) are well defined. Then the edges in E_1 define a subgraph that is a homeomorph of $K_{3,3}$ since there are internally disjoint paths between every vertex in $\{w_1, v_4, v_5\}$ to every vertex in $\{w_1', v_2, v_3\}$. So the conclusion is that if case (*iv*) does not hold for any of the five sets, H is a homeomorph of K_5. Otherwise, it is a homeomorph of $K_{3,3}$. In either case, G is not planar.

8.42 Prove Theorem 8.7 (the Harary–Tutte–Wagner theorem): A graph is planar if and only if neither K_5 nor $K_{3,3}$ is a subcontraction of G. (In other words, a graph is planar if and only if it does not have a subgraph contractible to K_5 or $K_{3,3}$.)

Solution. If a graph is nonplanar, it has a K-subgraph that can be contracted to either K_5 or $K_{3,3}$. So if neither K_5 nor $K_{3,3}$ is a subcontraction of G, G is planar. The converse follows from Problems 8.40 and 8.41.

8.43 Using Euler's formula, show that the Petersen graph is not planar.

Solution. The graph has 10 vertices and 15 edges. Every cycle in the graph has at least five edges. So if it is planar and has f regions, $5f \leq 30$, implying that the graph has at most six regions. Since it is connected and planar, it should have exactly seven regions.

8.44 Show that the Petersen graph is nonplanar by establishing that it has a K-subgraph.

Solution. Consider the subgraph of the Petersen graph obtained by deleting one edge. Fig. 8-23 shows a subgraph H of the Petersen graph after deleting the edge joining vertex 3 and vertex 4 as well as the edge joining vertex 7 and vertex 10. Vertices 3, 4, 7, and 10 become vertices of degree 2. It is easy to see that the subgraph H is a homeomorph of $K_{3,3} = (X, Y, E)$, where $X = \{1, 8, 9\}$ and $Y = \{2, 6, 5\}$. So the Petersen graph has a K-subgraph; hence, it is nonplanar.

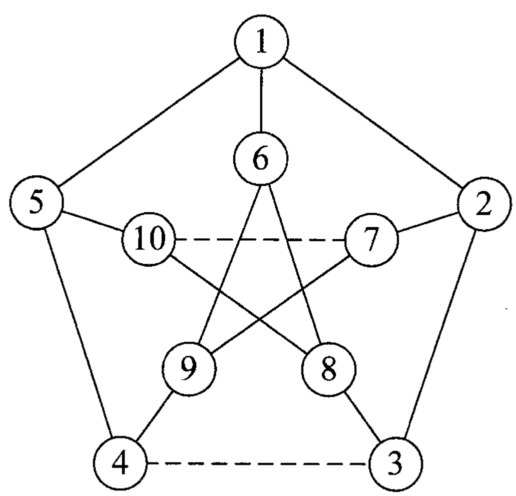


Fig. 8-23

8.45 Show that the Petersen graph is not planar by showing that the conflict graph relative to one of its cycles is nonbipartite.

Solution. In the graph shown in Fig. 8-23, consider the cycle (a longest possible cycle) $1 \to 2 \to 3 \to 4 \to 5 \to 10 \to 7 \to 9 \to 6 \to 1$. Edge e_1 joining 1 and 5 will intersect edge e_2 joining 2 and 7 if we embed both edges inside the cycle. So segment S_1 consisting of edge e_1 and segment S_2 consisting of edge e_2 are in conflict. These two segments are both in conflict with segment S_3, which consists of the edge joining 4 and 9. So these three segments form an odd cycle in the conflict graph. Thus the conflict graph relative to this cycle is not bipartite.

8.46 Show that the Petersen graph is nonplanar by showing that it is contractible to K_5.

Solution. Consider the representation of the Petersen graph with an outer pentagon and inner pentagon, as shown in Fig. 8-23. Contract this graph by merging a vertex in the outer pentagon with its mate in the inner pentagon. The resulting graph is K_5.

8.47 (*Theorem of H. Peyton Young*) Show that a 4-connected graph is nonplanar if and only if it has K_5 as a subcontraction.

Solution. Any graph is nonplanar if and only if it has either K_5 or $K_{3,3}$ as a subcontraction. It remains to be shown that if a 4-connected graph is nonplanar, it has K_5 as a subcontraction. Let G be a 4-connected graph with $K_{3,3} = (X, Y, E)$ as a subcontraction, where $X = \{A, B, C\}$ and $Y = \{D, E, F\}$. Let H be the subgraph consisting of nine internally disjoint paths joining the vertices in X and Y. These paths are called H-paths. By contracting the intermediate vertices, contract each of these H-paths into paths with two edges each. The contracted graph will have these six vertices (of degree 3) as before and nine intermediate vertices (of degree 2), and we also denote this graph by H. Since G is 4-connected, $G - Y$ is connected. Let P be a path between A and B in $G - Y$. Traversing P from A, let w be the last vertex on an H-path from A, and let x be the first vertex after w on an H-path from B or C, say from B. Let P_{wx} be the section of P between w and x, including terminal vertices w and x. No intermediate vertex in this subpath is a vertex of H. Among the three vertices in Y, there is a vertex (say F) such that neither w nor x is an intermediate vertex of any H-path emanating from F. Let Q be any path in $G - X$ from F to either D or E, say E. As we traverse Q from F, let y be the last vertex in this path belonging to any H-path emanating from F. Let z be the next vertex in this path that belongs to $P_{wx} \cup H$. (Here H is the contracted graph.)

If $z \in P_{wx}$, z is a vertex on an H-path from E. The four edges $\{A, w\}$, $\{B, x\}$, $\{E, z\}$, and $\{F, y\}$ are contracted into single vertices A, B, E, and F. At this stage, we have the contracted graph shown in Fig. 8-24(a). If we contract the edge joining C and D in this graph, we get the complete graph K_5.

If $z \in P_{wx}$, three cases need to be examined.

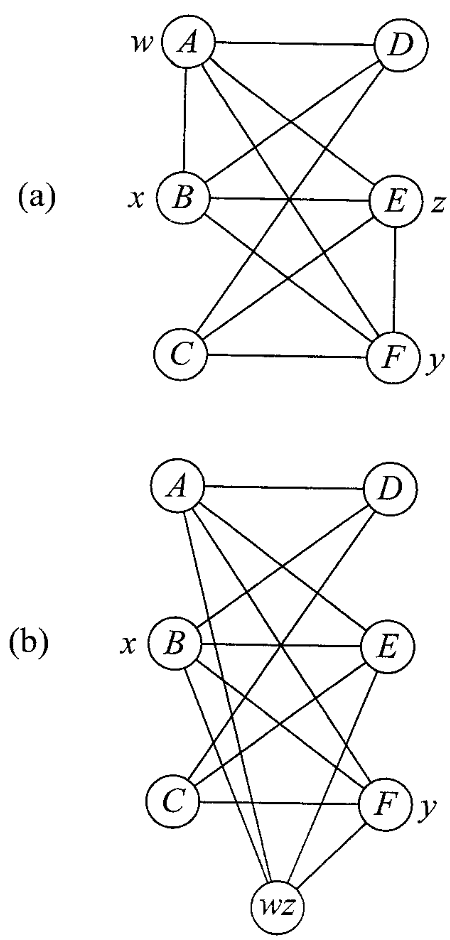


Fig. 8-24

***Case (i):* w $\neq$ A *and* z $\neq$ x.** If we contract edges $\{w, z\}$, $\{B, x\}$, and $\{F, y\}$, we get the graph shown in Fig. 8-24(*b*). If edges $\{A, D\}$ and $\{C, E\}$ are contracted in this graph, we get K_5.

***Case (ii)* w $\neq$ A *and* z = x.** Since z is a vertex in path Q that belongs to the graph $G - \{A, B, C\}$, vertices z and B are distinct. The common vertex $z = x$ is denoted as z. Contract edges $\{w, z\}$ and $\{F, y\}$. We get the graph shown in Fig. 8-24(*b*).

***Case (iii)* w = A *and* z = x.** While traversing the path Q from F to E, let v be the next vertex after z that is in H. This vertex v by definition is not in an F-path. Observe that z is a vertex in P between A (which coincides with w) and B (which coincides with x), and observe that it is also a vertex in Q between v and y. Let t be the vertex obtained by contracting all the intermediate vertices between A and B in P and all the intermediate vertices between v and y in Q so that t is adjacent to A, B, v, and y in the contracted graph. Then contract edges $\{E, v\}$ and $\{F, y\}$. We again get the graph shown in Fig. 8-24(*b*), which can be contracted to K_5.

Outerplanar Graphs

8.48 (*Chartrand–Harary Theorem*) A planar graph is an **outerplanar graph** if it has an embedding on the plane such that every vertex of the graph is a vertex belonging to the boundary of the same (usually exterior) region. Show that a graph is outerplanar if and only if it has no subgraph that is a homeomorph of K_4 or $K_{2,3}$. (These two complete graphs are the "forbidden graphs" for outerplanarity and play more or less the same role as K_5 and $K_{3,3}$ play in topics related to planarity.)

Solution. If a graph G has a subgraph that is a homeomorph of one of these two complete graphs, G is definitely not outerplanar. To establish the converse, assume that G has no subgraph that is a homeomorph of K_4 or $K_{2,3}$. Hence, it does not have a K-subgraph, so it is planar. Since a graph is planar if and only if every block in it is planar, we can assume that G is a block. Suppose G is embedded on the plane, with the maximum number of its vertices belonging to the boundary C of the exterior face. If G is not outerplanar, there is a vertex u in the interior of C. Let v_1 be a vertex in C that is adjacent to u. Since G is a block (with more than two vertices), there is a path P between u and vertex v_2 of cycle C. There are two possibilities to be considered.

Case (i): v_2 is not adjacent to v_1. This implies there are two vertices x and y in the cycle: x between v_1 and v_2 on one part of the cycle and y between v_1 and v_2 on the other part. There are three internally disjoint paths from v_1 to u, x, and y, and there are three internally disjoint paths from v_2 to u, x, and y. So the graph G has a subgraph that is a homeomorph of $K_{2,3}$. This is a contradiction.

Case (ii): Suppose v_1 and v_2 are adjacent vertices in the cycle C. As before, let path P be between u and v_2. Let P' be the path between v_1 and v_2 obtained by adding the edge joining u and v_1 to P. If the degree of every intermediate vertex in P' is 2, we can obtain another planar embedding of the graph by deleting P' and drawing it outside cycle C, in which case the boundary of the new exterior region will have more vertices than the number of vertices in C, violating the maximality requirement. So there is at least one vertex w in this path of degree 3. So from w, in addition to the two internally disjoint paths to v_1 and v_2, there will be another path (with no vertex in common with the other two paths) to vertex v_3 of the cycle. Thus we have a subgraph that is a homeomorph of K_4. This also is a contradiction. Thus the proof is completed.

8.49 Show that a graph G is outerplanar if and only if neither K_4 nor $K_{2,3}$ is a subcontraction of G.

Solution. If G is not outerplanar, it has a subgraph that can be contracted to one of these complete graphs, as established in Problem 8.57. In other words, if neither K_4 nor $K_{2,3}$ is a subcontraction, G is outerplanar. To prove the converse, it is enough if we show that the graph is not outerplanar whenever one of these two graphs is a subcontraction. If $K_{2,3}$ is a subcontraction, as in Problem 8.40, it follows that G has a subgraph that is a homeomorph of $K_{2,3}$. If K_4 is a subcontraction, there is a subgraph that is a homeomorph of K_4. In either case, G is not outerplanar.

8.50 An outerplanar graph is **maximal outerplanar** if it loses its outerplanarity if any two nonadjacent vertices are joined by an edge. If G is an outerplanar graph of order n and size m with f regions, show that the following properties hold: (*a*) $m = 2n - 3$ and $f = n - 1$, (*b*) there are at least three vertices of degree not exceeding 3, (*c*) there are at least two vertices of degree 2, and (*d*) the vertex-connectivity number $\kappa(G) = 2$.

Solution.

(*a*) The degree of the exterior region is n, and the degree of each interior region is 3. So $3(f - 1) + n = 2m$. Then, by applying Euler's formula, we find that $f = (n - 1)$ and $m = (2n - 3)$.

(*b*) Let n_i be the number of vertices of degree i. Obviously, $n_1 = 0$. Then $2n_2 + 3n_3 + 4n_4 + \cdots = 2n_2 + 2n_3 + n_3 + 4n_4 + \cdots = 2m = 4n - 6$. Hence, $2n_2 + 2n_3 + 4(n_4 + n_5 + \cdots) \leq 4n - 6$. But $n_4 + n_5 + \cdots = n - (n_2 + n_3)$. Thus $2n_2 + 2n_3 + 4(n - n_2 - n_3) \leq 4n - 6$, implying that $n_2 + n_3 \geq 3$.

(*c*) $n_2 = 0$ implies that $3n_3 + 4n_4 + \cdots = 4n - 6$. So $3n_3 + 4(n - n_3) \leq 4n - 6$, which implies that $n_3 \geq 6$, which is a contradiction. So there is at least one vertex of degree 2 that is obvious. Suppose $n_2 = 1$. Then $3n_3 + 4(n - n_3) \leq 4n - 4$, which implies that $n_3 \geq 4$ which is also a contradiction. Hence, $n_2 \geq 2$.

(*d*) Suppose all the vertices are situated along the corners of a polygon. Each vertex is adjacent to its two neighboring vertices. The deletion of a single vertex will not disconnect the graph. So $\kappa(G) > 1$. But $\kappa(G) < 3$ since there is a vertex of degree 2. Hence, $\kappa(G) = 2$.

8.51 Show that the three conditions listed in Problem 8.50 are necessary but not sufficient for a graph to be maximal outerplanar.

Solution. The graph in Fig. 8-25 satisfies parts (*a*) through (*d*). It is not outerplanar.

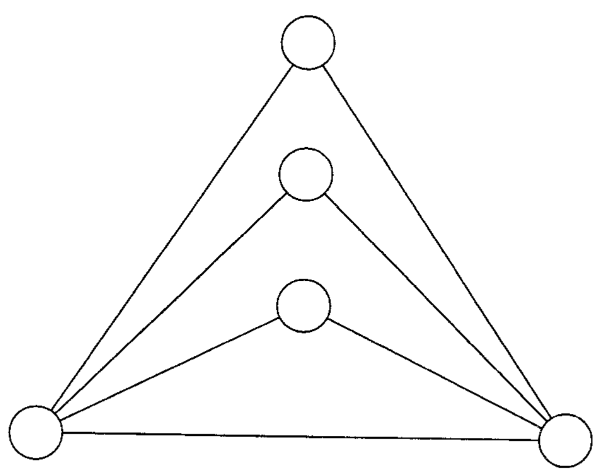


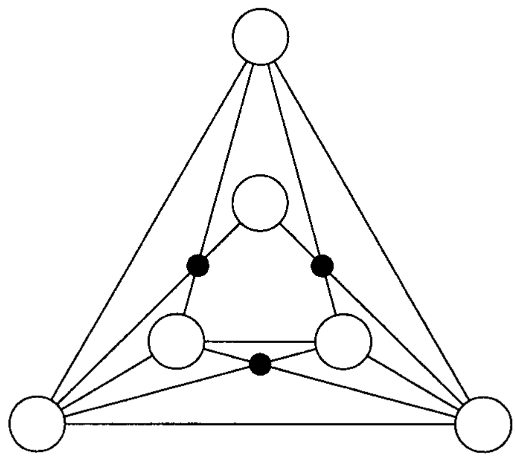
Fig. 8-25

Crossing Numbers and Thickness

8.52 Two edges in a graph create a **crossing** if there is an embedding of the graph in which the two edges intersect at a point that is not a vertex. If two edges meet at a crossing, a third edge should not pass through that crossing or it should not intersect either of these edges at a different point. The minimum number of crossings of the edges among all the embeddings of a graph G with this requirement is its **crossing number $\nu(G)$,** which is zero when G is planar. Find the crossing numbers of K_5, $K_{3,3}$, K_6, and $K_{3,4}$.

Solution. Both K_5 and $K_{3,3}$ are nonplanar, and each can be embedded with one crossing. So $\nu(K_5) = \nu(K_{3,3}) = 1$. Consider a drawing of K_6 on the plane with c crossings. We know that $c \geq 1$. At each crossing (the point of intersection), we introduce a new vertex, creating a planar graph. Each crossing will create two new edges. Thus we have $6 + c$ vertices and $15 + 2c$ edges. Since the graph is planar, $15 + 2c \leq 3(6 + c) - 6$, which implies that $c \geq 3$. But there is an embedding, as shown in Fig. 8-26(*a*) with three crossings. So its crossing number is 3. Any crossing in a bipartite graph (X, Y, E) involves two edges e and f, where e joins vertex p from X and q from Y. Similarly, f joins x from X and y from Y. If the crossing number is 1, the deletion of one of these vertices would have produced a planar graph, but that is not the case here. So the crossing number is more than 1. See Fig. 8-26(*b*), which shows an embedding of $K_{3,4}$ with two crossings. So the crossing number of $K_{3,4}$ is 2.

(a)



(b)

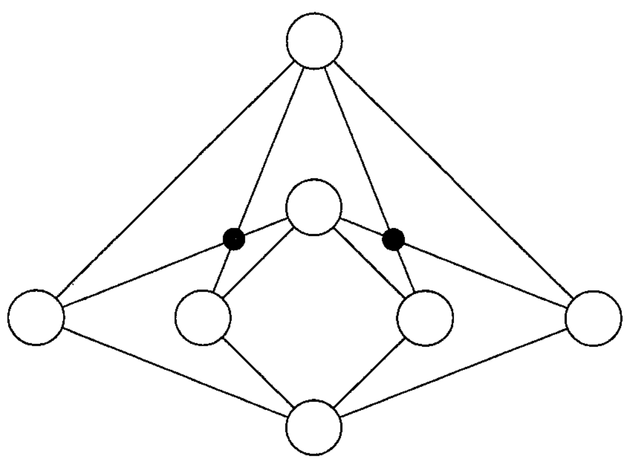


Fig. 8-26

8.53 A graph $G = (V, E)$ is ***k*-partite** if there is a partition of V into k subsets V_i such that every edge in E joins some vertex in V_i to some vertex in V_j, where $i \neq j$. If each V_i has r_i vertices and if there is an edge from each vertex in V_i (for every i) to every vertex in V_j for every j (where $i \neq j$), we have the complete k-partite graph $K(r_1, r_2, \ldots, r_k)$. Find the crossing number of $K(3, 2, 2)$.

Solution. It is easy to see that $K_{3,3}$, which can now be described as $K(3,3)$, is a subgraph of $G = K(3, 2, 2)$. So the crossing number c of G is at least 1. Suppose there is an embedding with just one crossing at the intersection of edges $\{p, u\}$ and $\{q, v\}$. At least one of these four vertices should belong to a partite set with two vertices. Suppose p is one such vertex. So $G - p$ is planar since there is only one crossing number. But $G - p$ is isomorphic to $K(1, 2, 3)$; hence, it is nonplanar. So the crossing number is at least 2. Figure 8-27 shows an embedding of the graph with two crossings. Thus the crossing number is 2.

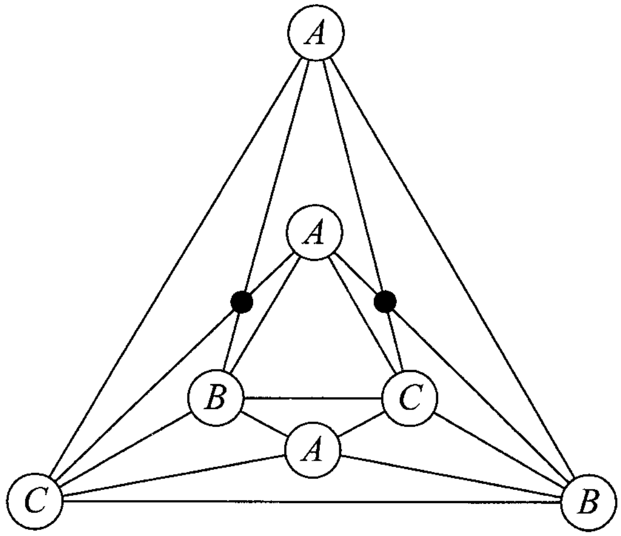


Fig. 8-27

8.54 Find the crossing number of the Petersen graph.

Solution. Since the Petersen graph is nonplanar, its crossing number is at least 1. It is not possible to embed it on the plane with one crossing since the deletion of any vertex leaves a homeomorph from $K_{3,3}$. An embedding with two crossings is shown in Fig. 8-28. So the crossing number is 2.

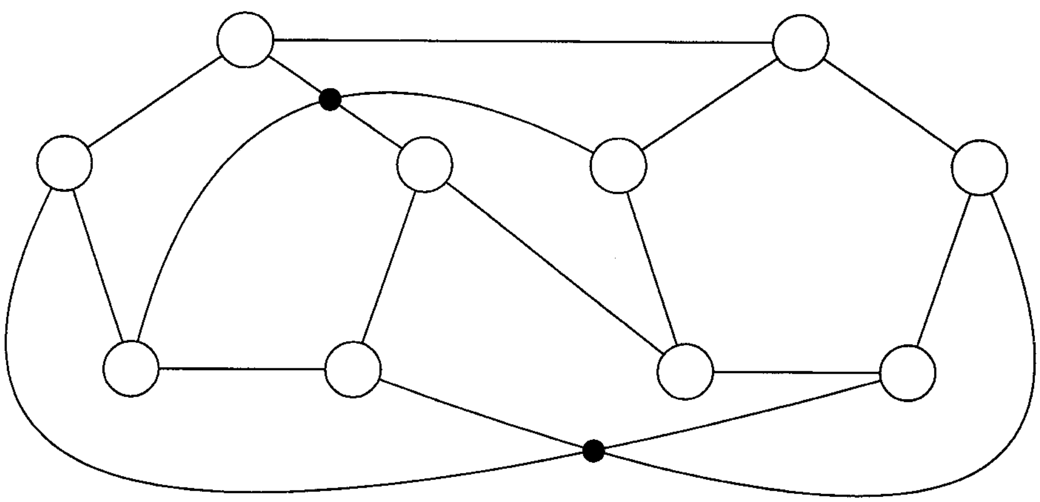


Fig. 8-28

8.55 The **thickness** $\theta(G)$ of a graph G is the minimum number of pairwise edge-disjoint spanning subgraphs in a decomposition of the graph. Find the thickness of the complete graph with n vertices, where $n \leq 8$.

Solution. The thickness of a graph is 1 if and only if the graph is planar. So the thickness of K_n is 1 if $n \leq 4$. Graph K_8 has a planar decomposition consisting of two spanning subgraphs, as shown in Fig. 8-29. So the thickness of K_n is 2 if $5 \leq n \leq 8$.

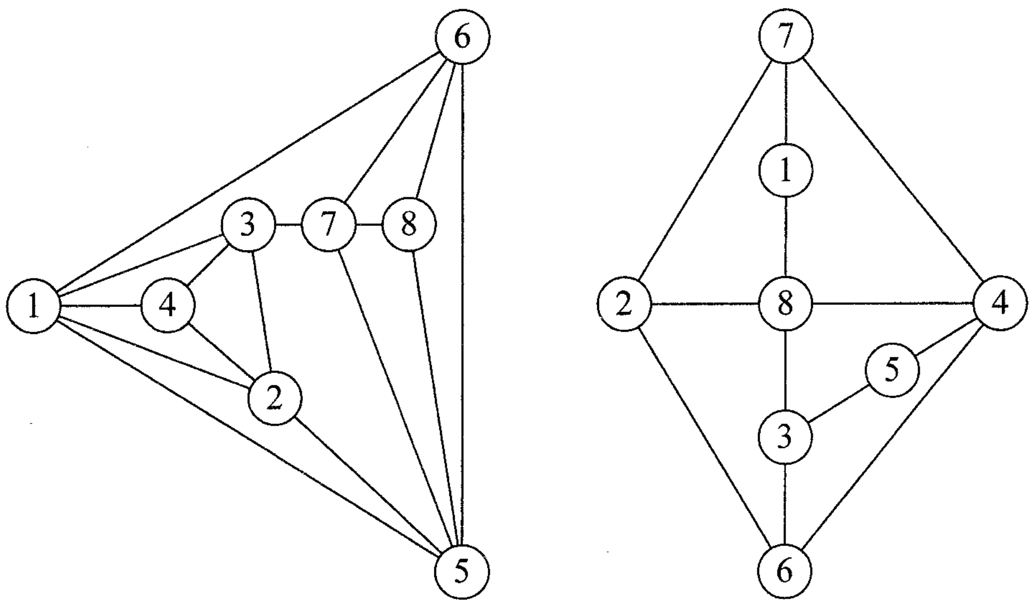


Fig. 8-29

8.56 Find the thickness of the Petersen graph.

Solution. The Petersen graph can be decomposed into two planar edge-disjoint spanning subgraphs: one planar spanning subgraph consisting of two disjoint cycles with five vertices each (the outer pentagon and the inner pentagon) and another planar spanning subgraph consisting of five copies of K_2 (the five "spokes"). So the thickness is 2.

8.57 A graph is **biplanar** if its thickness is 2. Show that if G is any nonplanar graph, there exists a biplanar graph H that is a homeomorph of G.

Solution. Construct a homeomorph H of G by inserting two vertices of degree 2 on each edge of G. So each edge of G is now split into three edges with one middle edge for each subdivision. Let H_1 be the spanning subgraph of H in which the only edges are these middle edges. Let H_2 be the spanning subgraph of H induced by all the edges of H other than the middle edges. Both these edge-disjoint spanning subgraphs are planar, and H is their union.

8.58 If a graph G with n vertices has m edges, show that its thickness $\theta(G) \geq m/(3n - 6)$. If the graph is bipartite, show that $\theta(G) \geq m/(2n - 4)$.

Solution. Suppose the thickness is r. So there are spanning subgraphs G_i $(1 \leq i \leq r)$ such that each subgraph is planar. If G_i has m_i edges, $m_i \leq (3n - 6)$ for each i. Hence $m = m_1 + m_2 + \cdots + m_r \leq r(3n - 6)$. So $r \geq m/(3n - 6)$. If the graph is bipartite, $m_i \leq (2n - 4)$ since the boundary of every region is at least 4.

8.59 Find a lower bound for the thickness of the complete graph of order n, where $n \geq 5$, using the result of Problem 8.58.

Solution. Here the size m is $n(n - 1)/2$. So from the result obtained in Problem 8.58, $\theta(K_n) \geq m/(3n - 6) = [n(n - 1)]/[6(n - 2)] = t$. Now $n(n - 1) = (n - 2)(n + 7) - 6(n - 2) + 2$. Hence, $t = (n + 7)/(6 - 1) + 1/[3(n - 2)]$. Let us write $(n + 7)/6 = q + r/6$, where $0 \leq r \leq 5$. Then $t = q + r/(6 - 1) + 1/[3(n - 2)]$. Now $n \geq 5$ implies that $1/[3(n - 2)] \leq \frac{1}{9} \leq \frac{1}{6}$ and $r \leq 5$ implies that $r/6 \leq \frac{5}{6}$. Hence, $r/6 + 1/[3(n - 2)] \leq 1$. Thus a lower bound for the thickness of the complete graph of order n is the smallest integer greater than or equal to t that is equal to the greatest integer less than or equal to $(n + 7)/6$. In particular, when $n \geq 5$, the thickness is at least 2, and when $n \geq 11$, the thickness is at least 3.

8.60 Find a lower bound for the thickness of the complete bipartite graph $K_{m,n}$ using the result of Problem 8.58.

Solution. The order is $(m + n)$, and the size is mn. Hence, a lower bound is $mn/2(m + n - 2)$.

Geometric Duals

8.61 Show that there is no plane graph with five regions such that there is an edge between every pair of regions.

Solution. If there is a plane graph G with five regions such that each pair shares a common edge, its geometric dual G' is nonplanar since it has K_5 as a subgraph. This is a contradiction.

8.62 Show that a planar graph is bipartite if and only if its dual is Eulerian.

Solution. Let G be an embedding of a planar bipartite graph on the plane. Since every cycle in G is an even cycle, the degree of each vertex in the dual graph (which is connected) is even. Hence the dual graph is Eulerian. Conversely, suppose the dual of a plane bipartite graph G is Eulerian. So every face in G is bounded by an even cycle. Suppose G is not bipartite. Then there is an odd cycle C in G that cannot be the boundary of a face. Suppose there are k faces in the interior of C. The edge sum of these k faces is even. In computing this edge sum, each edge in the interior of C is counted exactly twice and each edge in C is counted exactly once. So the number of edges belonging to C is even, which contradicts the assumption that C is an odd cycle. So G is bipartite.

8.63 Show that if G' is the geometric dual of a connected planar graph G, G is the geometric dual of G'.

Solution. Suppose G has m edges, r regions, and k components. Then it has $2 + k + m - r$ vertices. The dual graph G' is connected and has r vertices, m edges, and $2 + m - r$ regions. So connected graph $(G')'$ has $2 + m - r$ vertices. Thus both G and $(G')'$ are of the order if and only if G is connected. Each edge in the boundary of a region in G' intersects an edge of G joining two vertices of G, one of which has to be in the interior of the region. In other words, every region of G' "contains" at least one vertex of G. If there is a region in G' that contains more than one vertex of G, it implies that G is not connected. Thus the assumption that G is connected implies that each region X in G' contains exactly one vertex x from G. The number of edges in the boundary of X in G' is equal to the number of edges incident to vertex x since each boundary edge intersects an edge that is incident to x and vice versa. So if we construct the dual of the plane graph G' using the same procedure of constructing the dual of G from G, we recover the connected graph G intact as long as G is connected.

8.64 Show that if a planar graph is 3-edge-connected, its geometric dual is a simple graph.

Solution. If the geometric dual of a 3-edge-connected graph G is not simple, G will have a vertex of degree less than 3, and parallel edges in the dual will make a cut set of edges in G.

8.65 Show that a set of edges in a connected plane graph forms a spanning tree if and only if the set of duals of the remaining edges forms a spanning tree in a geometric dual of the graph.

Solution. Suppose the connected plane graph G has n vertices and m edges. Since any spanning tree in G will have $(n - 1)$ edges, there are $m' = (m - n + 1)$ remaining edges. The dual graph G' is a connected graph with $2 + m - n$ vertices. Hence, the set of m' dual edges forms a spanning tree in G'. The reverse implication holds since $G = (G')'$.

8.66 Show that a set of edges of a plane graph forms a cycle if and only if the set of dual edges forms a cut set in a geometric dual of the graph.

Solution. Suppose $G = (V, E)$ is a plane graph. If C is a cycle in the graph enclosing one or more regions of the graph, it contains in its interior a nonempty set S of vertices of the dual graph $G' = (V', E')$. The dual edges corresponding to the edges of cycle C are precisely those edges joining vertices in S and its complement $T = V' - S$, forming a cut set in the dual graph. The proof of the converse implication is similar.

Abstract Duals

8.67 Show that a geometric dual of any planar graph of G is the same as an abstract dual of G.

Solution. Let G be a planar embedding of a planar graph and let G' be a geometric dual of G. Then, as shown in Problem 8.66, there is a bijection from the set of edges in G to the set of edges in the dual graph G' such that a set of edges forms a cycle in C if and only if the corresponding set of dual edges forms a cut set in G'.

8.68 Show that the number of edges common to a cycle and a cut set in a graph is always even.

Solution. Let C be any cycle, and let $D = (X, Y)$ any cut set in G. If all the vertices of C are in X (or in Y), C and D have no edges in common. Suppose C has two vertices x and y, where x is in X and y is in Y. Then cycle C (which starts from x and ends in x) will necessarily have an even number of edges in common with the set of edges in D.

8.69 Let X be a set of edges in a graph $G = (V, E)$. Show that (*a*) if X has an even number of edges in common with every cut set of the graph, the edges in X constitute an edge-disjoint union of cycles, and (*b*) if X has an even number of edges in common with every cycle of the graph, the edges in X constitute an edge-disjoint union of cut sets.

Solution.

(*a*) Assume without loss of generality that G is connected. Let D be any cut set. By hypothesis, $|X \cap D)|$ is even. Let $T = (V, F)$ be any spanning tree in G, and let $X \cap (E - F) = \{e_1, e_2, \ldots, e_r\}$. Each e_i defines a unique cycle C_i (called a fundamental cycle) consisting of e_i and some edges of T. So, by Problem 8.68, $|C_i \cap D|$ is even for each i. Let C be the ring sum $C_1 \oplus C_2 \oplus \cdots \oplus C_r$ of these r cycles, and let $X' = X \oplus C$. [The **ring sum** of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_3 = G_1 \oplus G_2 = (V_3, E_3)$, where $V_3 = V_1 \cup V_2$ and $E_3 = (E_1 \cup E_2) - (E_1 \cap E_2)$. It can be easily verified that the ring sum operation is both commutative and associative.] Then $|X' \cap D|$ is even. If e is any edge in X', it has to be an edge belonging to the spanning tree. This edge e defines a unique cut set (called a fundamental cut set) in G, and the only edge common to this cut set and X' is edge e, contradicting that $|X' \cap D|$ is even. So set X' is empty, implying that set X and set C are the same. Hence, X is an edge-disjoint union of cycles.

(*b*) The proof is similar.

8.70 Show that if G^* is an abstract dual of G, G is an abstract dual of G^*. (Notice that G need not be connected here, in contrast to Problem 8.63.)

Solution. Let $D = (X, Y)$ be a cut set in G. Then $|D \cap C|$ is even for any cycle C in G, as shown in Problem 8.68. So C^* and D^* will also have an even number of edges in common. Since G^* is an abstract dual of G, C^* is a cut set in G^*. In other words, D^* has an even number of edges in common with every cut set in G^*, which implies that D^* is either a cycle or a disjoint union of cycles in G^*, as proved in Problem 8.69. Thus, using part (*a*) of Problem 8.69, we have shown that cut sets in G correspond to edge-disjoint unions of cycles in G^*. Similarly, using part (*b*) of Problem 8.69, it can be shown that cycles in G^* correspond to edge-disjoint unions of cut sets in G. So if D^* is not a single cycle, each cycle in the disjoint union will give a disjoint union of cut sets in G, which implies that D itself is an edge-disjoint union of cut sets instead of a single cut set. So D^* is a cycle. Thus cut sets in a graph correspond to cycles in the dual graph. So we have an isomorphism between G and $(G^*)^*$.

8.71 Give an example of a plane graph for which the geometric dual of the geometric dual and the abstract dual of the abstract dual are not the same.

Solution. The geometric dual G' and the abstract dual G^* are the same for any plane graph G. Also, $(G^*)^*$ is the same as G^*. Furthermore, if G is connected, both G and $(G')'$ are the same. Thus we are looking for a disconnected plane graph. If G is the plane graph that is the disjoint union of two cycles with three edges each, graph $(G')'$ has five vertices, whereas graph $(G^*)^*$ has six vertices.

8.72 Show that neither (*a*) $K_{3,3}$ nor (*b*) K_5 has an abstract dual.

Solution.

(*a*) Graph $K_{3,3}$ has no cut set consisting of two edges. So if it has an abstract dual G, it has no multiple edges. Graph $K_{3,3}$ has cycles of lengths 4 and 6. So G has no cut set consisting of fewer than 4 edges. This implies that the degree of each vertex in G is at least 4. Hence, it should have at least five vertices since G is simple. In that case, it should have at least 10 edges since the degree of each vertex is 4. But G can have only nine edges since $K_{3,3}$ has only nine edges. So $K_{3,3}$ has no abstract dual.

(*b*) The graph K_5 has no cut set consisting of two edges. So if it has an abstract dual G, it is a simple graph. It has cycles of length 3, 4, or 5. So the degree of each vertex in G is at least 3. Since the only cut sets of K_5 are sets of four edges, the cycles in G are all of size 4. So G is bipartite. If the order of G is less than 7, its size is at most 9. But it should have 10 edges since K_5 has 10 edges. So the order of G is at least 7. This implies that it should have at least (7)(3)/(2) edges. This is a contradiction again.

8.73 Show that if a graph G has an abstract dual, every subgraph of G has an abstract dual.

Solution. Let e be an edge of $G = (V, E)$ joining u and v, and let e^* be the dual edge in the abstract dual $G^* = (V^*, E^*)$ joining p and q. Let H be the subgraph of G obtained by deleting e, and let H' be the graph obtained from G^* by contracting e^*. Now every cycle C in H is a cycle in G; therefore, it corresponds to a cut set C^* in G^*. This cut set partitions V^* into two sets V_1^* and V_2^*. Since e^* is not an edge in C^*, both p and q are in one of these two sets. So C^* is a cut set in H'. Thus every cycle in H corresponds to a cut set in H'. Let D be a cut set in H'. Since e^* is not in D, set D is a cut set in G^* that should correspond to a cycle in H'. So every cut set in H' corresponds to a cycle in H. Since a subgraph H of G can be obtained from G by deletion of edges that do not belong to H, it follows that H has an abstract dual whenever G does.

8.74 Show that if G is a homeograph of H and if G has an abstract dual, H has an abstract dual.

Solution. Any two edges of G incident to a vertex of degree 2 correspond to two parallel edges in the abstract dual. If these two edges are replaced by a single edge, one of the two parallel edges will disappear. So it follows that if G has an abstract dual, H does also.

8.75 Prove Theorem 8.8 (Whitney's theorem): A graph is planar if and only if it has an abstract dual.

Solution. If a graph is planar, it has a geometric dual that is an abstract dual, as proved in Problem 8.67. To establish the reverse implication, we prove that a nonplanar graph G has no abstract dual. Suppose G has an abstract dual. Then any subgraph H of G has an abstract dual. If this H is a homeomorph of another graph H', H' also should have an abstract dual. Since G is nonplanar, there is a subgraph that is a homeomorph of either K_5 or $K_{3,3}$. Thus the existence of an abstract dual for a nonplanar graph would imply the existence of an abstract dual for K_5 or $K_{3,3}$. But there is no abstract dual for either of these two graphs, as shown in Problem 8.72. This contradiction proves that a nonplanar graph has no abstract dual.

HAMILTONIAN PLANAR GRAPHS

8.76 Prove Theorem 8.10 (the Grinberg–Kozyrev theorem): If C is any Hamiltonian cycle in a Hamiltonian plane graph of order n, the sum of the indices of the inner regions relative to C and the sum of the indices of the outer regions relative to C are both equal to $(n - 2)$.

Solution. Let the number of edges in the interior of the Hamiltonian cycle be q. Then the number of inner regions is $q + 1$. If the number of inner regions of degree k is r_k, the total number of inner regions is also equal to $(r_3 + r_4 + \cdots + r_n)$. Thus $q = (r_3 + r_4 + \cdots + r_n) - 1$. Now $(3r_3 + 4r_4 + \cdots + nr_n) = 2q + n$ by counting the edges in the boundary of each inner region. Hence, $(3r_3 + 4r_4 + \cdots + nr_n) = 2(r_3 + r_4 + \cdots + r_n) - 2 + n$. Hence, $(3 - 2)(r_3) + (4 - 2)r_4 + \cdots + (n - 2)(r_n) = n - 2$. The left-hand side is the sum of the indices of the inner regions. The proof is similar when we consider the outer regions.

8.77 Use the Grinberg–Kozyrev theorem to establish that the plane graph shown in Fig. 8-30 is not Hamiltonian.

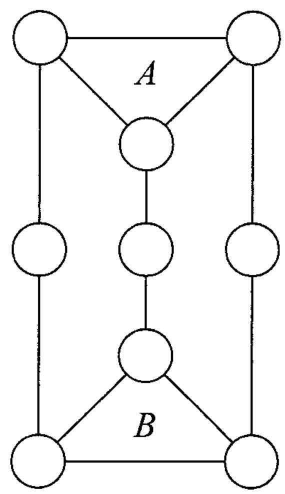


Fig. 8-30

Solution. There are five regions with indices 1, 1, 4, 4, and 4. Suppose there is a Hamiltonian cycle. There are two cases.

(*i*) Of two regions A and B, one is an inner region and the other is an outer region. Their indices are equal and cancel out each other in the counting process. There is no way that the indices of the three remaining regions cancel out.

(*ii*) Both regions A and B are either inner or outer. Suppose both are inner. Their indices add up to 2. There is no way of classifying the three remaining regions such that the sum of the indices of the inner regions is the same as the sum of the outer regions. So the graph is not Hamiltonian.

8.78 Show that the 3-connected cubic plane graph known as the **Grinberg–Kozyrev graph** shown in Fig. 8-31 is not Hamiltonian.

Solution. The degrees of the regions are 5, 8, or 9. So their indices are 3, 6, or 7. Suppose the graph is Hamiltonian. Let C be a Hamiltonian cycle. There is only one region with index 8, and it is an outer region with respect to C. Suppose the number of outer regions of indices 3 and 6 are x and y, respectively. Suppose the number of inner regions of indices 3 and 6 are a and b, respectively. Then we have the equation $3x + 6y + 7 = 3a + 6b$, which has no solution in nonnegative integers.

8.79 Show that in the Hamiltonian graph G shown in Fig. 8-32, any Hamiltonian cycle that contains edge e does not contain edge f.

Solution. Suppose there is a Hamiltonian cycle C in G that contains both the edges. Let G' be the graph obtained by subdividing these two edges by introducing one vertex (of degree two) in each of these edges. Since any Hamiltonian cycle in G that contains e and f is necessarily a Hamiltonian cycle in G', the graph G' is Hamiltonian. The graph G' has six regions with index 3 (out of which one is exterior) and one region with degree 2. Suppose there are x inner regions with index 3 and y inner regions with index 2. The sum of the indices for the inner regions is $3x + 2y$. The sum of the indices for the outer regions is $3 + 3(5 - x) + 2(1 - y)$. Since G' is

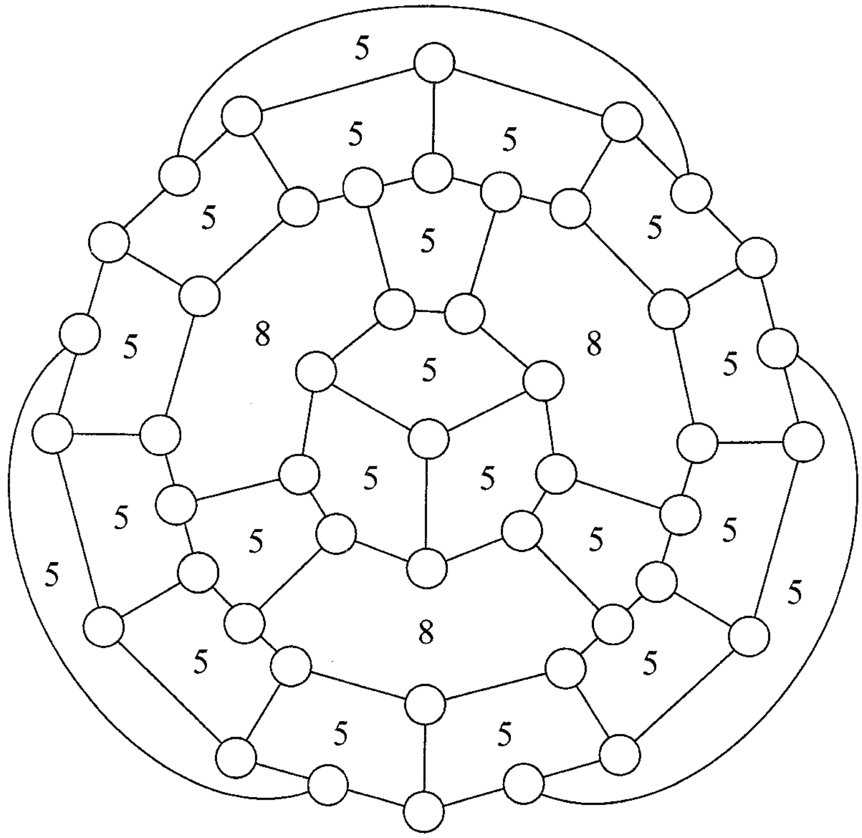


Fig. 8-31

Hamiltonian, these two sums are equal. So $3x + 2y = 10$, where $y = 0$ or 1. In either case, x is not an integer. So G' is not Hamiltonian, which is a contradiction.

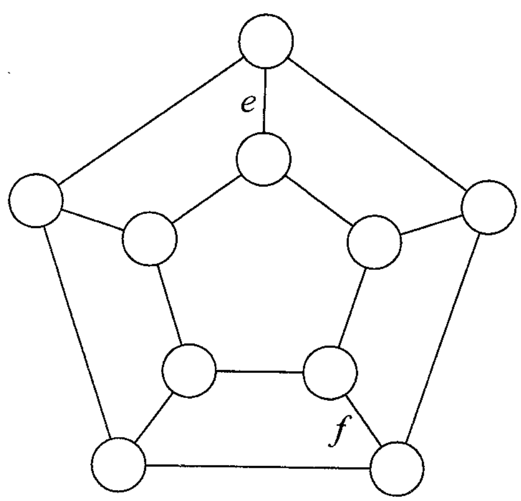


Fig. 8-32

8.80 Show that the Tutte graph shown in Fig. 8-33 is not Hamiltonian.

Solution. Suppose there is a Hamiltonian cycle D in the graph. Let two of the three regions marked A, B, and C (say regions A and B) be outer regions (relative to D) if possible. So edge a is not an edge of D. Since the exterior region is an outer region, both b and c are also not in D. But D should contain two of these three edges.

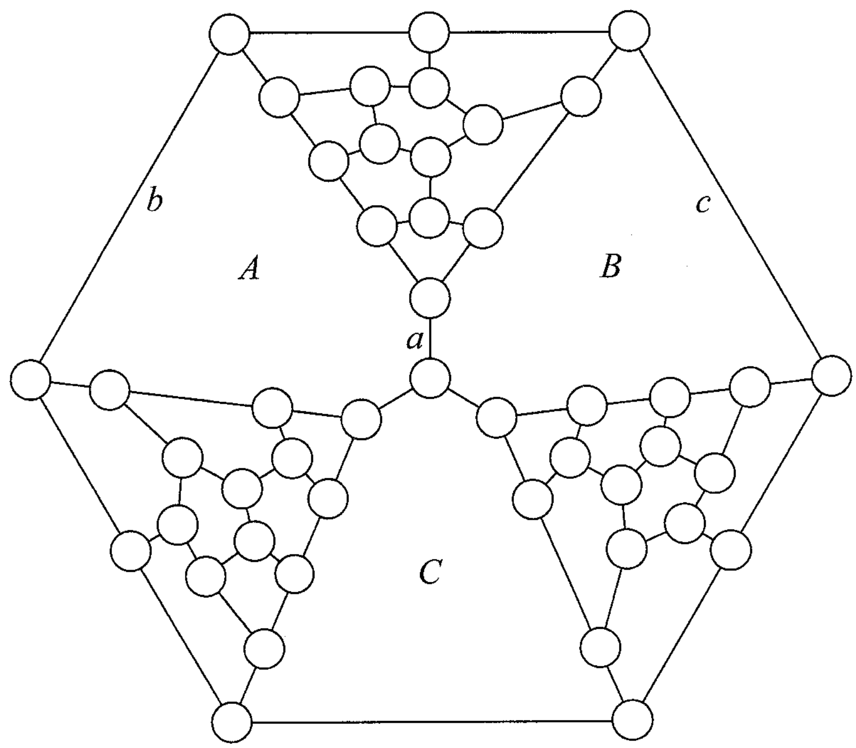


Fig. 8-33

This contradiction implies that at least two of the three regions are inner regions. Suppose A and B are inner regions. In this case also, a is not in the cycle D. Since the exterior region is an outer region, both b and c are edges of D. Let G' be the subgraph induced by the set of vertices in the "triangular section" of the graph whose end vertices are x, y, and z as marked in the figure. That edges b and c are edges of the Hamiltonian cycle D implies that there is a Hamiltonian path between x and y in G'. If we join x and y by a new edge e (not shown in the figure), $G' + e$ is a Hamiltonian graph with one region of degree 8, five regions of degree 5, two regions of degree 4, and one region of degree 3. Suppose C' is a Hamiltonian cycle in $G' + e$. Relative to this cycle, the region of degree 3 is an inner region, and the region of degree 8 is an outer region. Suppose there are x inner regions of degree 5 and y inner regions of degree 4. The index equation implies that $3x + 2y = 12$, giving the solution $x = 4$ and $y = 0$. Hence, the region that contains vertex z is an outer region, implying that the Hamiltonian cycle C' does not pass through z. This contradiction implies that $G' + e$ is not Hamiltonian. Hence, the Tutte graph G is not Hamiltonian.

8.81 Show that the plane graph shown in Fig. 8-34 is not Hamiltonian.

Solution. There are 42 vertices. If the graph is Hamiltonian, there is a Hamiltonian cycle such that the sum of the indices of the inner regions is exactly 40. Of the two regions of degree 11 in the graph, one has to be an outer region with index 9. The other could be inner or outer. The only region of degree 8 has to be an inner region, and its index is 6. Suppose the number of inner regions with degrees 11, 5, and 4 are x, y, and z, respectively. Then we have the equation $9x + 3y + 2z = 34$ for the sum of the inner regions, where x and z are binary variables. This equation has no solution in integers.

8.82 Show that is there is no Hamiltonian planar graph having regions of degrees 5 and 8 and one region of degree 7.

Solution. Suppose the graph is Hamiltonian. Then the order n is 8. Suppose there are m edges and r regions. Since the graph is Hamiltonian, it is connected. Therefore, by Euler's formula, $8 - m + r = 2$. If there are x

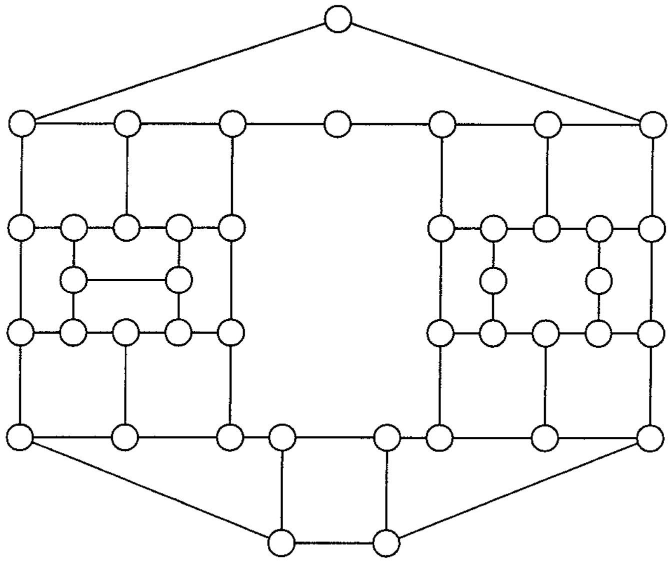


Fig. 8-34

regions of degrees 5 and y regions of degree 8, we have $5x + 8y + 7 = 2m$ and $x + y + 1 = r$. These three equations together will imply that $3x + 6y = 7$, which has no solution in nonnegative integers.

8.83 Give an example of a maximal planar graph that is not a Hamiltonian graph.

Solution. A maximal planar graph is a planar graph that becomes nonplanar whenever a pair of nonadjacent vertices is joined by a new edge. Graph G shown in Fig. 8-35 is a maximal planar graph. Suppose G is a Hamiltonian graph. If we delete the set W of five vertices labeled 1, 2, 3, 4, and 5, we get a disconnected graph $G - W$ with six components that each consist of a single vertex. Since G is Hamiltonian, the number of components of $G - W$ cannot exceed the cardinality of W. This contradiction implies that G is not Hamiltonian.

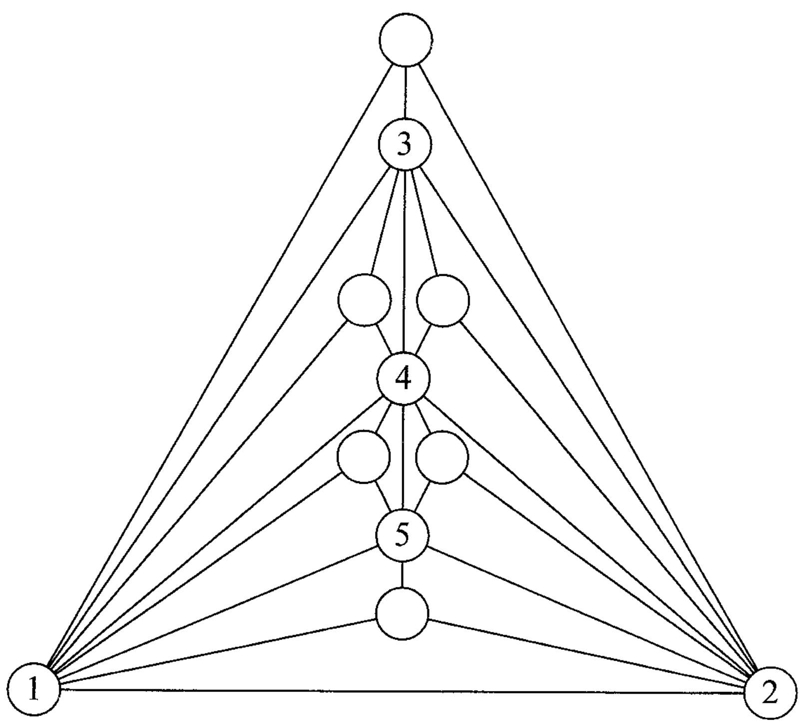


Fig. 8-35

MAXIMUM FLOW IN PLANE NETWORKS

8.84 Prove Theorem 8.11: Let G be an s, t network, and let G^* be its dual. Let $d(k^*)$ be the shortest distance between s^* and k^* in G^*. Define $x_{ij} = d(j^*) - d(i^*)$, where $\{i^*, j^*\}$ is the dual edge that corresponds to edge $\{i, j\}$ in G. Then the vector $[x_{ij}]$ is a maximum flow in G.

Solution. The component x_{ij} is negative if and only if x_{ji} is positive. So whenever the flow from i to j is negative, we consider it as a flow in the opposite direction. Thus, without loss of generality, we assume that each component is nonnegative. Since $d(j^*) \le d(i^*)$ + [capacity of the arc from i^* to j^*], we can conclude that the component x_{ij} cannot exceed the capacity of the arc from i^* to j^*. So the flow $[x_{ij}]$ is a capacitated flow. Let k be

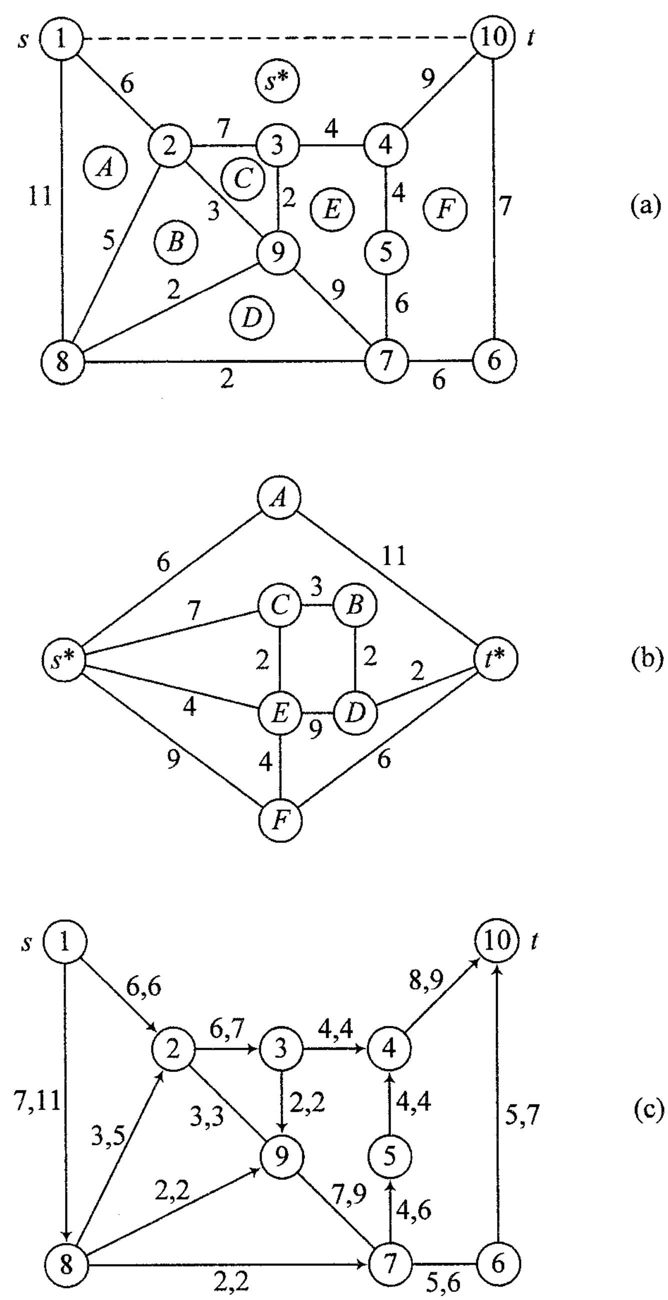


Fig. 8-36

any vertex in G other than the source or the sink. Consider the cut $\{\{k\}, V - k\}$ in G. The edges in this cut correspond to a cycle in G^*. The sum of the flow components along the edges of the cycle is zero. So the net flow at vertex k is zero. Suppose P^* is a shortest path between s^* and t^* in the dual network. Then $d(j^*) = d(i^*)$ + [capacity of the (i^*, j^*)]. So each edge in P^* is saturated. This implies that the flow is a maximum flow.

8.85 Obtain a maximum flow in the plane network shown in Fig. 8-36(a) with vertex 1 as the source and vertex 10 as the sink.

Solution. An artificial edge joining the source and the sink that encloses the finite region s^* is constructed. The exterior region is t^*. There are six other finite regions. The dual network G^* is shown in Fig. 8-36(b). The shortest distances from s^* to vertices A, B, C, D, E, F, and t^* are 6, 9, 6, 11, 4, 8, and 13, respectively. The maximum flow value is 13. The components of a maximum flow along the edges are computed using the formula established in Theorem 8.11. Using these components, each edge is converted into an arc. The undirected network now becomes a directed network, as shown in Fig. 8-36(c), displaying a maximum flow along the arcs with a flow value of 13.

GRAPHS ON SURFACES

8.86 Prove Theorem 8.12: The nonplanar graphs K_5 and $K_{3,3}$ are toroidal.

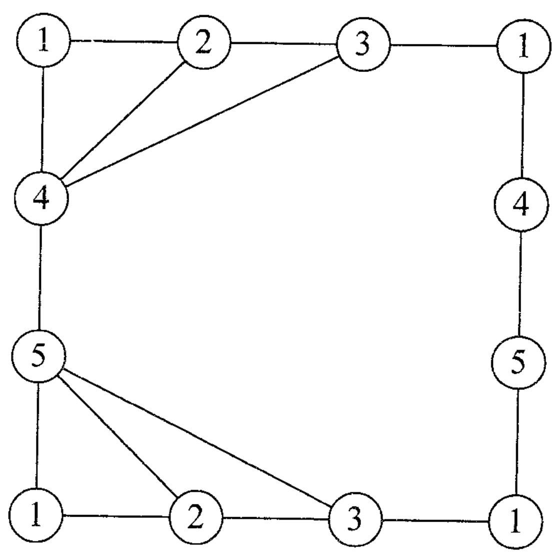


Fig. 8-37a

Solution. Any rectangle of length x and width y can be rolled into a circular tube of length x with two circular holes on either side. When these holes are glued together, we get a torus. This process is reversible. So embedding a graph of genus 1 on a torus can be easily visualized with the help of a rectangle. If vertex v is on a side of the rectangle, it is also represented symmetrically (as a duplicate) on the opposite side. If a vertex is inside the rectangle, it is not duplicated. Once the vertices are marked, the edges are identified. If the graph is toroidal, it can be embedded such that no two edges intersect. Figure 8-37(a) shows an embedding of K_5 on a torus, and Fig. 8-37(b) shows an embedding of $K_{3,3}$ on a torus.

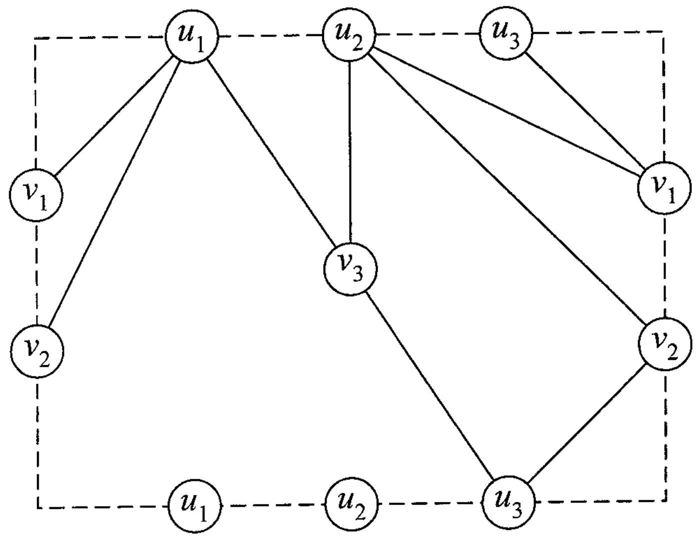


Fig. 8-37b

8.87 Find the genus of K_7.

Solution. Since K_7 is nonplanar, its genus is at least 1. Figure 8-38 shows an embedding of this complete graph on a torus. So its genus is 1.

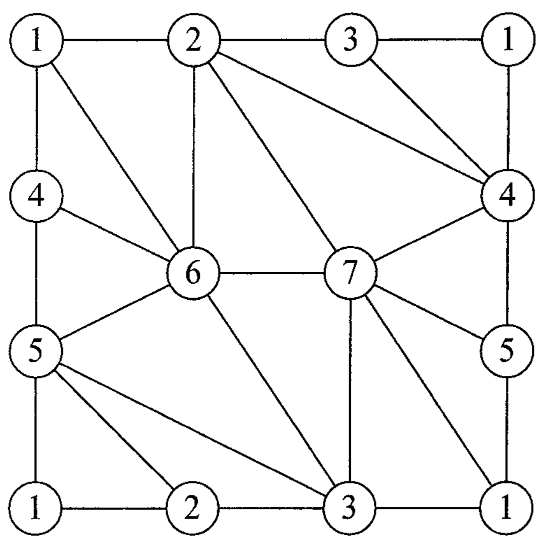


Fig. 8-38

8.88 Show that $K_{4,4}$ is a toroidal graph.

Solution. Graph $K_{4,4}$ can be embedded on a torus, as shown in Fig. 8-39. So it is a toroidal graph. Its genus is 1 since it is not planar.

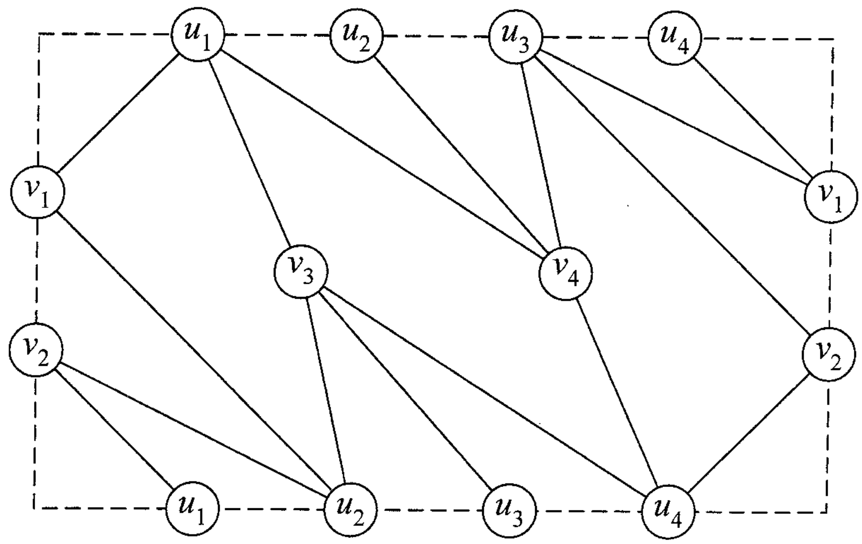


Fig. 8-39

8.89 Give an example of a graph for which the genus is less than the crossing number.

Solution. The Petersen graph comes to our rescue as usual. In Problem 8.54, it was shown that its crossing number is 2. But it is toroidal, as can be seen from Fig. 8-40. So its genus is only one. Since the crossing number of K_6 is 3 (see Problem 8.52), the crossing number of K_7 is at least 3. But its genus is only 1 since it can be embedded on a torus, as shown in Problem 8.88.

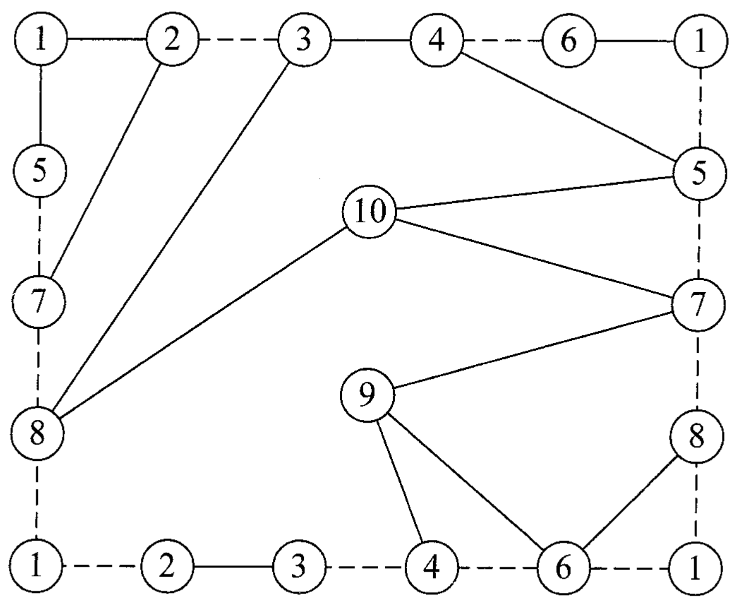


Fig. 8-40

8.90 Show that any graph can be embedded in a space of three dimensions (not on a surface) such that no two edges intersect except possibly at a vertex.

Solution. Let $V = \{1, 2, \ldots, n\}$ be the set of vertices in a graph. Each vertex i can be identified with a point (i, i^2, i^3) in 3-space. If i and j are adjacent in the graph, join the points (i, i^2, i^3) and (j, j^2, j^3) by a line segment. The resulting configuration is an embedding in 3-space.

8.91 Show that if a connected graph G of genus g is embedded on a surface of genus g, every region defined by the embedding is a 2-cell.

Solution. [What follows is a very informal explanation. For a rigorous proof, see J. W. T. Youngs, "Minimal Imbeddings and the Genus of a Graph," *Journal of Mathematics and Mechanics,* 12(1963): 303–315.] Let the surface on which graph G is embedded be a sphere with g handles. Assume that all the vertices lie on the surface of the sphere, and assume that only edges are embedded on each handle. Suppose there is a region R that is not simply connected. So there is a closed curve C in this region R that cannot be continuously deformed into a point in R. If the genus g is 0, the surface is a sphere, which implies that curve C divides the surface of the sphere into two parts, each of which contain parts (components) of G. This violates that G is connected. So the genus g is positive. There are two cases to be considered.

Case (i): Curve C lies entirely on the surface of the sphere, dividing the sphere into two parts. Since C cannot be squeezed into a single point in R, there is a handle T such that its two bases are on the two different parts defined by C. Then we can cut the surface along C and ignore the existence of T. In other words, we are left with a sphere with $(g - 1)$ handles and a graph of genus g, making an embedding impossible. This is a contradiction.

Case (ii): Curve C lies on handle T. Since the closed curve C cannot lie completely on a handle, it can be divided into two arcs out of which one arc, say arc C', does not lie on T. Then the edges of the graph lying on T can be redrawn along arc C' in region R. In this case, we also have an embedding of G on the surface without making use of handle T. This also is a contradiction, as in case (i).

8.92 Prove Theorem 8.13 (generalized Euler formula): If an embedding of a connected graph of order n and size m on a surface of genus g defines r regions and if each region thus defined by the embedding is a 2-cell, $n - m + r = 2(1 - g)$.

Solution. The proof is by induction on g. The result is true if $g = 0$, as established in Theorem 8.3. Assume that the generalized Euler formula is true for all connected graphs embedded on surfaces with genus $(g - 1)$. Let G be any connected graph of order n and size m with a cellular embedding on a sphere with g handles. We can assume without loss of generality that all vertices lie on the surface of the sphere. Each handle should contain at least one edge since every region is a 2-cell. Let T be the handle that contains edges labeled $e_i (i = 1, 2, \ldots, k)$. Draw two disjoint simple curves C and C' (going around the handle), intersecting each e_i exactly once. At each point of intersection, a new vertex is located. The points of intersection of C with k edges are labeled u_i, and the points of intersection of C' with these edges are labeled v_i, where $i = 1, 2, \ldots, k$. Join vertices u_i and u_{i+1}, where $i = 1, 2, \ldots, k$, with $k + 1 = 1$. Also do the same for vertices v_i. Thus if $k > 2$, there will be $2k$ vertices, $4k$ new edges, and $2k$ new regions. If $k = 1$, there is only one edge on handle T such that C intersects it at u_1 and C' intersects it at v_1. To make the graph simple, we define two new vertices u_2 and u_3 on C and two new vertices v_2 and v_3 on C'. Thus there are six new vertices, 10 new edges, and four new regions. If $k = 2$, we define a new vertex u_3 on C in addition to u_1 and u_2 and do the same on C' to avoid parallel edges. In this case, there will be six new vertices, 11 new edges, and five new regions. Let G' be the enlarged graph thus constructed.

Obviously G', is also embedded on the sphere with g handles such that each region in G' also is a 2-cell. Now delete all edges joining u_i and v_i, and let G'' be the graph thus defined. As far as G'' is concerned, handle T is superfluous. In other words, G'' can be embedded on a sphere with $(g - 1)$ handles. So we can use the induction hypothesis.

Case (i): $k > 2$. The order of G'' is $n + 2k$, and its size is $m + 4k - k = m + 3k$. The number of regions in G'' will be $r + 2k - (k - 2) = r + k + 2$. Thus $(n + 2k) - (m + 3k) + (r + k + 2) = 2 - 2(g - 1)$. So $n - m + r = 2(1 - g)$.

Case (ii): $k = 1$. The order of G'' is $n + 6$, its size is $m + 10 - 3 = m + 7$, and the number of regions is $r + 4 - 1 = r + 3$. Thus $(n + 6) - (m + 7) + (r + 3) = 2 - 2(g - 1)$, which implies that $n - m + r = 2(1 - g)$.

Case (iii): $k = 2$. The order of G'' is $n + 6$, its size is $m + 11 - 3 = m + 8$, and the number of regions is $r + 5 - 1 = r + 4$. Thus $(n + 6) - (m + 8) + (r + 4) = 2 - 2(g - 1)$, which also implies that $n - m + r = 2(1 - g)$.

8.93 Show that if a connected graph of order n, size m, and genus g is embedded on a surface of genus g and if the number of regions in the embedding is r, $n - m + r = 2(1 - g)$.

Solution. Since this is a minimal embedding, it is a cellular embedding, as proved in Problem 8.92. The result follows from what was established in Problem 8.92.

8.94 Find the number of 2-cells created if the Petersen graph is embedded on a torus.

Solution. The genus of the torous is 1, and the genus of the Petersen graph is also 1. So the embedding is minimal, and each region is a 2-cell. The graph is of order 10 and size 15. Using the generalized Euler's formula, there should be five regions on the surface.

8.95 If a simple connected graph G is embedded on a surface, find a lower bound for the genus of the graph. Examine the case when G is bipartite.

Solution. Let the genus of G (of order n and size m) be g. For embedding G on the surface, the genus of the surface should be at least g. If both have the same genus, it is a cellular embedding. Suppose there are r regions. Since the graph is simple, $3r \le 2m$. Then, from the generalized Euler formula, we have the inequality $g \ge \frac{1}{6}(m - 3n + 6)$. If G is bipartite, there are no odd cycles; hence, $4r \le 2m$. So the generalized Euler formula implies that $g \ge \frac{1}{4}(m - 2n + 4)$ in this case.

8.96 If a simple connected graph G is embedded on a surface, find an upper bound for the number of edges. Examine the case when the graph is bipartite.

Solution. The inequality obtained in Problem 8.95 also implies that $m \le (3n - 2 + 2g)$. In the case of bipartite graphs, $m \le (2n - 4 + 4g)$.

8.97 A simple connected graph of order n, size m, and genus g is a **maximal g-graph** if $m = (3n - 2 + 2g)$ if it is not bipartite and $m = (2n - 4 + 4g)$ if it is bipartite. Show that K_7 and $K_{4,4}$ are **maximal toroidal graphs** but K_6 is not.

Solution. In both cases, $g = 1$. For K_7, $n = 7$ and $m = 21$, and it is not bipartite. For $K_{4,4}$ $n = 8$ and $m = 16$, and it is bipartite. Both are maximal toroidal. In the case of K_6, which is toroidal, $m = 15$, which is less than 18. So it is not maximal toroidal.

8.98 Find a lower bound for the genus of the complete graphs (*a*) K_n and (*b*) $K_{m,n}$.

Solution.

(*a*) For K_n, $m = \frac{1}{2}(n)(n - 1)$. From Problem 8.95, the inequality is $g \ge \frac{1}{12}(n - 3)(n - 4)$.

(*b*) For $K_{m,n}$, the order is $(m + n)$, and the size is mn. This time, the inequality for the bipartite graph is $g \ge \frac{1}{4}(m - 2)(n - 2)$. [That the inequality is actually an equality (an achievement indeed) was established by Ringel and Youngs in 1968 for the nonbipartite case and by Ringel in 1965 for the other case.]

8.99 Find a lower bound for the genus of the k-cube.

Solution. The order is 2^k, and the size is $(k)(2^{k-1})$. The graph is bipartite. Using the same inequality as in Problem 8.95, $g \ge (k \cdot 2^{k-3} - 2^{k-1} + 1) = 1 + (2^{k-3})(k - 4)$. (In this case, that inequality is an equality was established by Beineke, Harary, and Ringel.)

Supplementary Problems

8.100 If the boundary of every face in a plane graph of order n has four edges, show that the size of the graph is $2n - 4$.

Ans. Suppose there are m edges and f faces in the graph under consideration. Then, as in Problem 8.99, $4f = 2m$. Using Euler's formula, $m = 2n - 4$.

8.101 If a plane graph of order n is 2-connected and if no face is a triangle, show that the size of the graph cannot exceed $2n - 4$.
Ans. By hypothesis, the boundary of each face has at least four edges. Thus if there are m edges and f faces, $4f \leq 2m$, which implies that $m \leq 2n - 4$.

8.102 If a 4-regular plane graph has eight faces, find the number of its vertices and edges.
Ans. If there are n vertices and m edges, $4n = 2m$. Using Euler's formula, $n = 6$ and $m = 12$.

8.103 If a 4-regular plane graph has 10 faces, find the number of its vertices and edges.
Ans. If there are n vertices and m edges, $4n = 2m$. Using Euler's formula, $n = 8$ and $m = 16$.

8.104 If the boundary of each region of a connected plane graph of order n and size m has r edges, show that $m(r - 2) = r(n - 2)$.
Ans. $rf = 2m$. Using Euler's formula, $r(2 + m - n) = 2m$. So $m(r - 2) = r(n - 2)$.

8.105 If a 3-regular connected plane graph has 12 regions, find the number of its vertices and edges.
Ans. If there are n vertices and m edges, $3n = 2m$. Using Euler's formula, $n = 20$ and $m = 30$.

8.106 If the girth (number of edges in a cycle with the minimum number of edges) of a connected of order n and size m is g, prove the inequality $m(g - 2) \leq g(n - 2)$. [*Hint:* Use the inequality $gf \leq 2m$ (f is the number of regions) and Euler's formula.]

8.107 Show that a graph with fewer than nine edges is planar. [*Hint:* Kuratowski says at least nine edges.]

8.108 Find the number of regions in a planar graph of order n if (*a*) it is a triangulation and (*b*) it is a maximal outerplanar graph.
Ans. Use Euler's formula. The number of edges is known.

8.109 Find the crossing number of $K_{3,4}$. *Ans.* 2

8.110 A plane graph is 2-connected if and only its geometric dual is 2-connected. [*Hint:* Use Solved Problem 8.5.]

8.111 Show that there is no Hamiltonian planar graph having regions of degrees 4 and 6 and one region of degree 9. [*Hint:* Proceed as in Problem 8.82.]

8.112 Show that the planar graph shown in Fig. 8-41 is not Hamiltonian. [*Hint:* Use the Grinberg–Kozyrev theorem.]

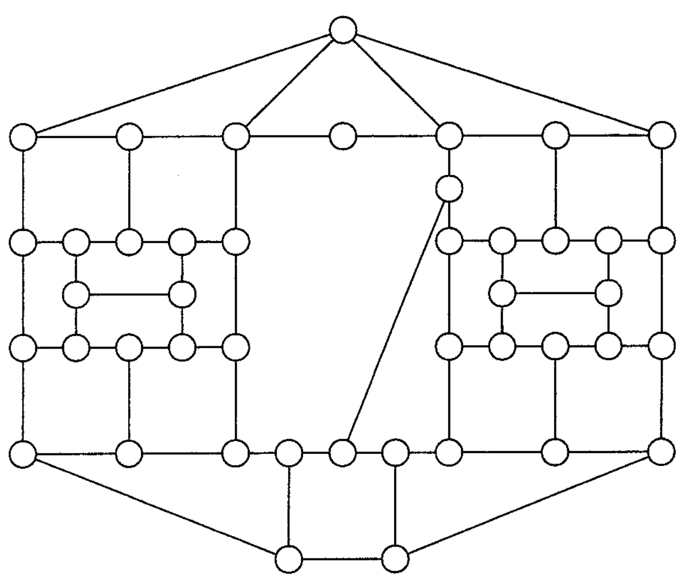


Fig. 8-41

8.113 Show that the planar graph shown in Fig. 8-42 is not Hamiltonian. [*Hint:* Use the Grinberg–Kozyrev theorem.]

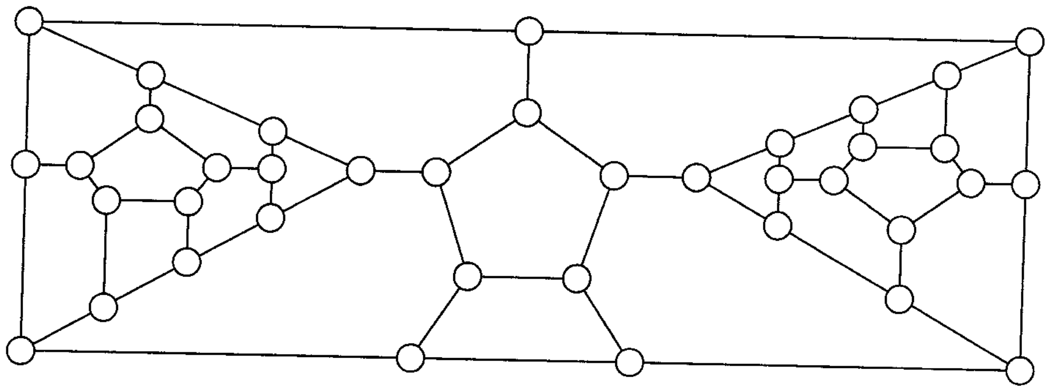


Fig. 8-42

8.114 Show that the planar graph in Fig. 8-43 is not Hamiltonian. [*Hint:* Use the Grinberg–Kozyrev theorem.]

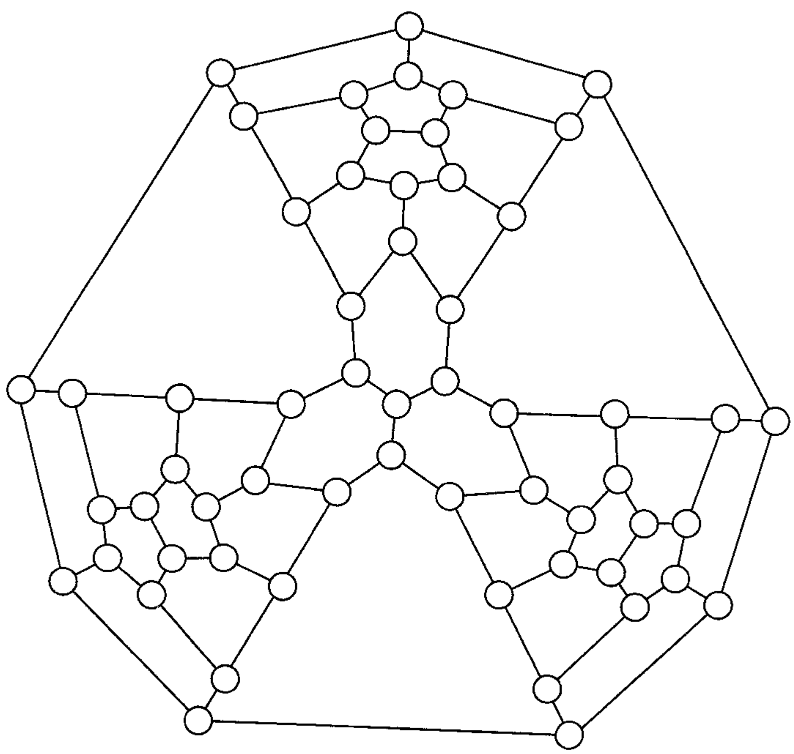


Fig. 8-43

Chapter 9

Colorings of Graphs

9.1 VERTEX COLORING OF GRAPHS

A graph is said to be ***k* vertex colorable** (or ***k*-colorable**) if it is possible to assign one color from a set of k colors to each vertex such that no two adjacent vertices have the same color. If the graph G is k-colorable but not $(k - 1)$-colorable, we say that G is a ***k*-chromatic** graph and that its **chromatic number $\chi(G)$** is k. So the chromatic number is the minimum number k such that G is k-colorable. Hence, graph G is k-colorable if and only if $\chi(G) \le k$. In other words, a k-chromatic graph is a graph that needs at least k colors, whereas a k-colorable graph is a graph that does not need more than k colors.

Obviously, the chromatic number of G is 1 if and only if G is trivial, and $\chi(G) = 2$ if and only if G is bipartite. Even though there are many known examples of such graphs, it is not known under what conditions the chromatic number of a graph is 3. For instance, any cycle with an odd number of vertices is 3-chromatic. Likewise, it is easy to see that the chromatic number of a wheel of even order is 3. The complete graph K_n is n-chromatic, and if G has K_n as a subgraph, $\chi(G) \ge n$.

No efficient and convenient procedure is known for finding the chromatic number of an arbitrary graph. But there are some well-known bounds for this number in the general case and some formulas in a few special cases involving the degrees of G. The maximum and minimum among the degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If the vertex set of G is $V = \{v_1, v_2, \ldots, v_n\}$, the degree of v_i is denoted by $\deg(v_i)$ or d_i for each i.

Theorem 9.1. $\chi(G) \le \Delta(G) + 1$ for any graph G. (Notice that the inequality becomes an equality when G is a complete graph or an odd cycle.) (See Solved Problem 9.4.)

An obvious coloring procedure to obtain an upper bound for $\chi(G)$ is by adopting a greedy method that is also known as a **sequential (incremental) coloring algorithm:** for any ordering $v_1, v_2, \ldots, v_n$ of the n vertices of a graph and any sequence $c_1, c_2, \ldots, c_n$ of n colors, the color to be assigned to v_i is the smallest-indexed color not already assigned to one of its lower-indexed neighboring vertices. Since each vertex has at most $\Delta(G)$ neighboring vertices, we do not need more than $\Delta(G) + 1$ colors to color the vertices of G. This once again confirms the content of Theorem 9.1.

Example 1. The maximum degree is 3, and so $\chi(G)$ cannot exceed 4. The graph is not bipartite. Hence, $\chi(G)$ is either 3 or 4. The vertices of the graph shown in Fig. 9-1 are assigned subscripts $1, 2, \ldots, 10$. The list of 10 colors is c_i ($i = 1, 2, \ldots, 10$). Vertex v_1 is assigned color c_1. For vertex v_2, the only lower-indexed neighboring vertex is v_1, which has been assigned color c_1. The smallest-indexed color not already assigned is c_2, which goes to v_2. For vertex v_3, the only forbidden color is c_2. The smallest-indexed available color for v_3 is c_1. We continue this process and color each vertex, as shown in Fig. 9-1. We see the graph is 3-colorable, and $\chi(G) = 3$.

Theorem 9.2 **(Brooks's Theorem).** If G is a connected graph that is neither a complete graph nor an odd cycle, $\chi(G) \le \Delta(G)$. (See Solved Problem 9.6.)

Example 2. The Petersen graph is neither complete nor an odd cycle. Since it is not bipartite, it is not 2-colorable. Its maximum degree is 3. So, by Brooks's theorem, its chromatic number cannot exceed 3. Hence, the chromatic number is 3.

If we modify the sequential coloring method by ordering the set of vertices based on their degrees in nonincreasing order, we have a variation of the algorithm known as the **largest first sequential algorithm.**

Example 3. In the graph shown in Fig. 9-2, the set of seven vertices is an ordered set with degrees 4, 4, 3, 3, 3, 3, and 2. Since the graph is not bipartite, $\chi(G)$ is at least 3. By Brooks's theorem, $\chi(G)$ cannot exceed the maximum degree 4 since G is neither an odd cycle nor a complete graph. Thus $\chi(G)$ is either 3 or 4. The largest first sequential algorithm will not need more than five colors. Using this algorithm, the vertices get the colors

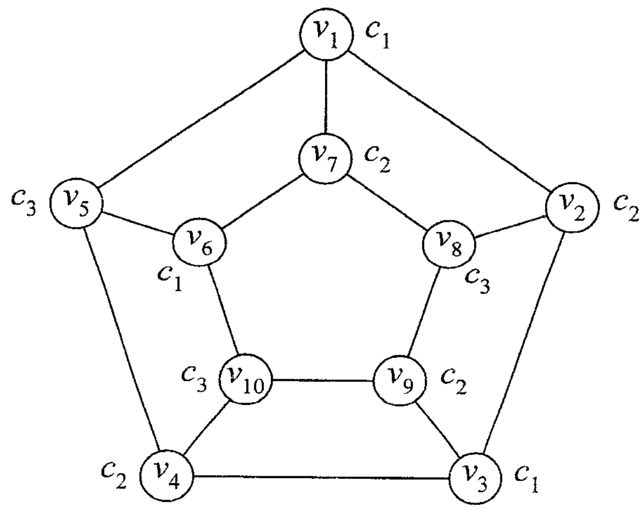


Fig. 9-1

c_1, c_2, c_1, c_2, c_3, c_4, and c_3, respectively. So the graph is 4-colorable. But it is easy to see that it is 3-colorable: color v_1 and v_6 red, color v_2 and v_4 blue, and the remaining three vertices green.

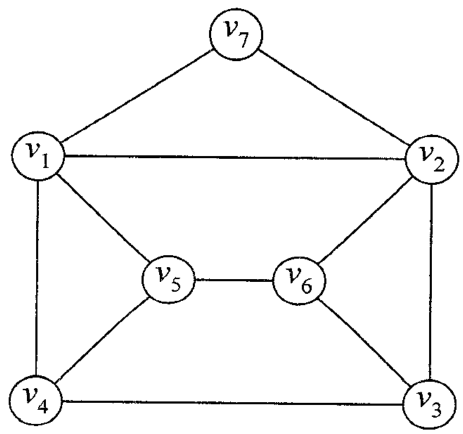


Fig. 9-2

Lower Bounds for the Chromatic Number

A set of A of vertices in a graph G of order n is an **independent set** in G if no two vertices in A are adjacent (in G), and the cardinality of a largest independent set in G is the **vertex-independence number** (or **independence** or **internal stability number),** denoted by $\alpha(G)$. See Problems 1.18 and 1.19. At the other extreme, a set Z of vertices in G is called a **clique** in G if there is an edge (in G) between every pair of vertices in Z, and the cardinality of a largest clique in G is the **clique number** of G, denoted by $\omega(G)$. Obviously, both $\omega(G)$ and $n/[\alpha(G)]$ are lower bounds for $\chi(G)$.

Perfect Graphs

A graph G is **perfect** if for every induced subgraph H of G, $\omega(H) = \chi(H)$. The **perfect graph theorem (PGT) of Lovasz** asserts that the complement of a perfect graph is perfect. See Solved Problem 9.100. An **odd hole** in a graph is a cycle that is an induced subgraph with k vertices, where k is odd. An **antihole** is the complement of an odd hole. The chromatic number of an odd hole with more than three vertices is 3, whereas its clique number is only 2. So any graph that has an odd hole with more than three vertices cannot be perfect. The **perfect graph conjecture** (also known as the **strong perfect graph** conjecture) asserts that a graph G is perfect if and only if neither G nor its complement has an odd hole with more than three vertices. An equivalent assertion of this conjecture is that a graph is perfect if and only if it does not have an odd hole or odd antihole with more than three vertices. See Solved Problems 9.90 through 9.101 for more on perfect graphs.

Mycielski Construction

If a connected graph G does not have a triangle as a subgraph, the lower bound $\omega(G)$ is only 2 and the chromatic number could be large, in which case this lower bound estimate is not very helpful. Specifically, starting from a triangle-free k-chromatic graph $G = (V, E)$, it is always possible to obtain (by the **Mycielski construction** method) a triangle-free $(k + 1)$-chromatic triangle-free graph for any choice of k. If $V = \{v_1, v_2, \ldots, v_n\}$, introduce two new sets $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w\}$ so that the union of the three sets V, U, and W becomes the set of vertices for the new graph G' that contains G as a proper subgraph. Set U is an independent set in G'. Join w to each u_i. Also join each u_i to every vertex adjacent to v_i. It can be shown (see Solved Problem 9.33) that the graph G' thus constructed with $2n + 1$ vertices is a $(k + 1)$-chromatic graph that is triangle-free.

Example 4. We start from the triangle-free subgraph K_2 consisting of vertices v_1 and v_2. The new nonadjacent vertices are u_1 and u_2 along with w, which is adjacent to both of them. We join u_1 to v_2 and u_2 to v_1. At the end of the first iteration, we get a cycle with five vertices that is a 3-chromatic triangle-free graph. For the next iteration, we begin with a cycle with vertices v_i $(i = 1, 2, 3, 4, 5)$ and introduce an independent set of five new vertices u_i $(i = 1, 2, 3, 4, 5)$ and a new vertex w that is adjacent to each of these new vertices. The resulting triangle-free 4-chromatic graph (known as the **Grotsch graph**) obtained at the end of the second iteration is shown in Fig. 9-3.

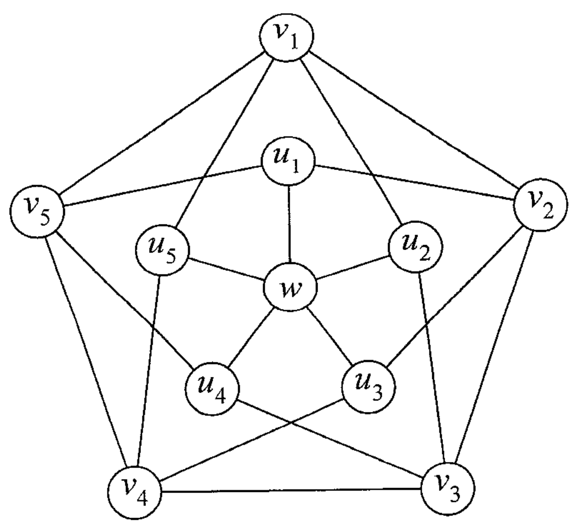


Fig. 9-3

Chromatic Polynomials (Chromials)

If the number of ways of coloring the vertices of a graph G (in such a way that no two adjacent vertices are assigned the same color) using at most x colors is denoted by $P(G, x)$, the smallest value of x such that $P(G, x)$ is not zero is the chromatic number of G since $P(G, x) = 0$ if and only if G is not x-colorable. For example, if the graph is K_2 and if there are x colors available, one of the vertices can be colored in x ways and the other can be colored in $(x - 1)$ ways. So $P(K_2, x) = x(x - 1)$. The smallest (positive integer) value of x such that the expression $x(x - 1)$ is not zero is 2, which is the chromatic number of K_2. More generally,

$P(K_n, x) = x(x - 1)(x - 2) \ldots (x - n + 1) = x_{(n)}$. Suppose there are $f(r)$ ways of partitioning the vertex set of a graph into r independent sets. Then for each such partition, the number of ways of coloring the vertices of G is $x_{(r)}$; consequently, $P(G, x) = \Sigma f(r)\, x_{(r)}$. If the order of G is n, $f(r) = 0$ when $r > n$. Thus $P(G, x)$ is a polynomial in x (known as the **chromatic polynomial or chromial of G**) of degree n with *integer coefficients* in which the coefficient of the leading term x^n is 1 since there is only one way of partitioning the vertex set of n elements into n nonempty (independent) subsets. So it is a **monic polynomial.** Moreover, $f(0) = 0$; therefore, the constant term in the chromatic polynomial is 0. It can be shown that the absolute value of the coefficient of x^{n-1} is the size of the graph (see Solved Problem 9.39) and that the coefficients in the polynomial alternate (see Solved Problem 9.44) in sign. It is not known under what conditions a given polynomial will be the chromatic polynomial of a graph. See the Solved Problems for more on chromatic polynomials.

9.2 EDGE COLORING OF GRAPHS

A graph G with no loops is said to be ***k* edge colorable** if it is possible to assign to each edge one color from a set of k colors such that no two edges with a vertex in common get the same color. A k edge colorable graph is a ***k* edge chromatic graph** if it is not $(k - 1)$ edge colorable and if its **chromatic index $\chi'(G)$** is k. In other words, the chromatic index of G is the minimum number k such that it is k edge colorable. Obviously, the maximum degree $\Delta(G)$ of any graph G is necessarily a lower bound for its chromatic index, whereas by Brooks's theorem, $\Delta(G)$ is an upper bound of the chromatic number of any graph that is neither a complete graph nor an odd cycle. The surprising fact (see Theorem 9.4) is that $\Delta(G) + 1$, which is the chromatic number of G when G is complete or an odd cycle, is an upper bound for the chromatic index of any simple graph. Furthermore, the edges in an edge coloring of the graph that get the same color constitute a matching in that graph; therefore, the chromatic index of the graph is also equal to the minimum number of matchings into which the edge set of the graph can be partitioned. In particular, the edges of a regular graph G can be colored using $\Delta(G)$ colors if and only if G is 1-factorable. Notice also that the chromatic index of a simple graph G is the same as the chromatic number of its line graph $L(G)$ since two edges in G have a vertex in common if and only if the vertices corresponding to these edges are adjacent in $L(G)$. In particular, both the chromatic number and the chromatic index are the same for any cyclic graph.

Theorem 9.3. (*i*) The chromatic index of a bipartite multigraph G is $\Delta(G)$. In particular, the chromatic index of the complete bipartite graph $K_{m,n}$ is $\max\{m, n\}$. (*ii*) If G is the complete graph with n vertices, its chromatic index is $\Delta(G) = n - 1$ if n is even, and it is $\Delta(G) + 1 = n$ if n is odd. (See Solved Problems 9.52, 9.55, and 9.56.)

Theorem 9.4 **(Vizing's Theorem).** The chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G) + 1$. (See Solved Problem 9.59.)

(In view of Vizing's theorem, the collection of all simple graphs can be grouped into two classes. A graph is in class 1 if its chromatic index is equal to its maximum degree. Otherwise it belongs to class 2. Even though the **classification problem** of determining which graphs belong to which class remains unsolved, P. Erdös and R. J. Wilson have proved that the more the number of vertices in a graph, the more its probability to claim membership in class 1.)

Example 5. In the graph shown in Fig. 9-4(*a*), the letters marked on the edges represent their colors in an optimal edge coloring, and the letters marked on the vertices represent their colors in an optimal vertex coloring. Since graph G is neither complete nor an odd cycle, its chromatic number, $\chi(G)$ by Brooks's theorem, cannot exceed the maximum degree, which is 3. Since the graph has an odd cycle, $\chi(G)$ is at least 3. The chromatic index $\chi'(G)$ is at least 3, and by Vizing's theorem, it cannot exceed 4. Any partitioning of the edge set will have at least four sets. The line graph is shown in Fig. 9-4(*b*), and its chromatic number is 4.

Theorem 9.5. If a cubic graph has a bridge, its chromatic index is 4. (See Solved Problem 9.62.)

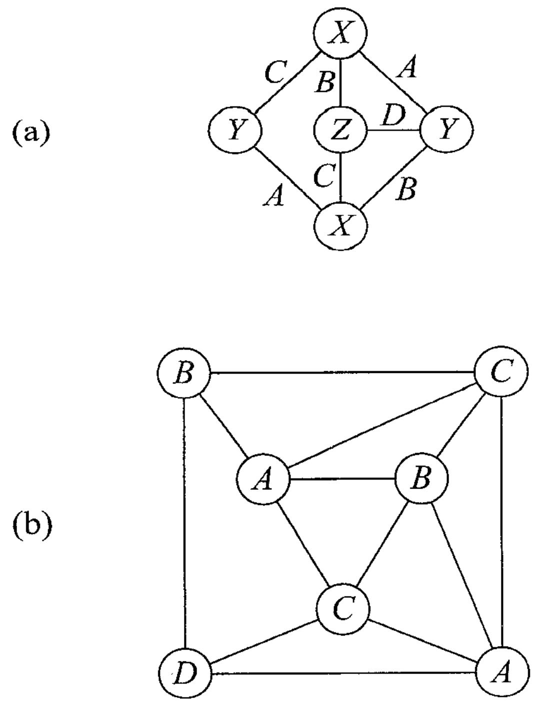


Fig. 9-4

9.3 COLORING OF PLANAR GRAPHS

In a political (or geographical) map showing various countries on a sheet of paper or on the surface of a globe, the boundary of a country is usually a closed simply connected curve, and two countries are neighbors if they share at least one boundary of nonzero length. The map is said to be colored if each country gets a color such that no two neighboring countries have the same color. Any map can be viewed as a connected plane graph in which each country corresponds to a region in the graph and where two countries are neighbors if and only if the corresponding regions in the graph have at least one edge in common. It is reasonable to assume that this connected graph has no loops and also no bridges; if there were a bridge after crossing it (which is actually a boundary), one would not have left the country. Thus in graph-theoretical terminology, a **map** is a connected bridgeless plane multigraph. The problem of determining the minimum number of colors needed to color a map is one of the most famous problems in graph theory. In the words of K. Appel, for almost 150 years, a "Holy Grail of graph theory has been a simple incisive proof" of the **four-color theorem (4CT),** which asserts that no map needs more than four colors. In an attempt to prove the 4CT in 1890 (see Solved Problem 9.3), Heawood was able to show that no map needs more than five colors. Finally, in 1976, K. Appel and W. Hakken proved the 4CT, but theirs was not a "simple incisive" proof. At this point, nobody knows whether the grail exists. A presentation of the awe-inspiring proof (or even a sketch of the proof) of this famous theorem is beyond the scope of this book. A few important results related to the saga of the 4CT will be presented here, however.

If G is a map, its geometric dual G' is a connected plane graph in which the degree of each vertex is at least 2. Since G' has no loops, one can color its vertices such that no two adjacent vertices are of the same color. Any coloring of the regions of G defines a coloring of the vertices of G' if we assign each vertex of G' the same color that the corresponding region in G gets. In this coloring of the vertices, no two adjacent vertices have the same color since two vertices are adjacent if and only if the two corresponding regions are neighbors. So if the regions of G are colored using k colors, the vertices of G' can also be colored using k colors. The reverse implication is obvious. Thus an alternative statement of the 4CT is that the **chromatic number of a planar graph cannot exceed 4** since any planar graph is isomorphic to some plane graph. It is in this (dual) form the 4CT is usually stated. Notice that by the coloring of a *map,* we mean the coloring of its regions (including the unbounded exterior region), whereas by the coloring of a *graph* (unless otherwise explicitly stated), we mean the coloring of its vertices.

A planar graph is said to be **irreducible** if it is 5-chromatic and if the chromatic number of any graph with fewer vertices is less than 5. (The 4CT implies that there are no irreducible graphs.) A graph is **reducible** if it is not a subgraph of an irreducible graph. A **configuration** in a planar graph G is a subgraph of G consisting of a cycle C in G and the vertices, edges, and regions of G interior to C. A set of graphs is an **unavoidable set** if every planar graph contains at least one graph from the set as a configuration.

Suppose there exists a finite unavoidable set $X = \{H_1, H_2, \ldots, H_k\}$ of reducible configurations. Assume that there is a planar graph that needs at least five colors. Then there should be an irreducible planar graph G. Since G is planar, it should contain at least one graph, say H_i, from X as a subgraph. Since H_i is reducible, it cannot be a subgraph of the irreducible graph G.

So the existence of an unavoidable set of reducible configurations will imply that no map (planar graph) needs more than four colors. The starkest description of Appel and Hakken's proof is that they were able to exhibit an unavoidable set of about 2000 reducible configurations with the aid of a computer. (The size of the set has been reduced to 700 or so in the recent past.) It is not true that they first obtained a finite set of configurations and then proved that every configuration in that set is reducible. In fact, both concepts had their independent roles toward the construction of the final set of reducible configurations. For a lucid explanation of Appel and Hakken's truly impressive and monumental work, see their October 1977 *Scientific American* article, "The Solution of the Four-Color Map Problem."

More generally, when we consider a surface S_g of positive genus g, its chromatic number $\chi(S_g)$ is the maximum chromatic number among all graphs that can be embedded on S_g. It was proved (see Solved Problem 9.86) by G. Ringel and J. W. Youngs in 1968 that $\chi(S_g)$ is equal to the floor of the number $\frac{1}{2}\{7 + \sqrt{1 + 48g}\}$ for all $g > 0$. Known as the **Heawood Map Coloring Theorem,** this fact is true for all nonnegative g and is also a consequence of the four-color theorem.

Example 6. Figure 9-5(a) shows a map G depicting seven finite regions and one exterior region that can be labeled region 8 that is not labeled. Notice that between region 4 and region 5 are two frontiers. To investigate the

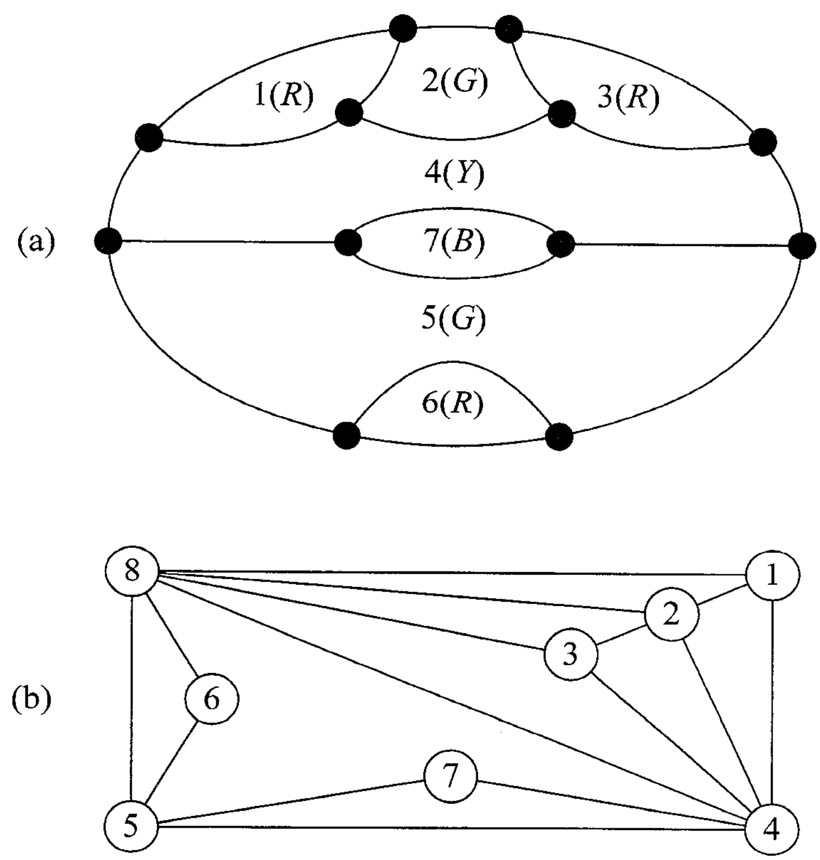


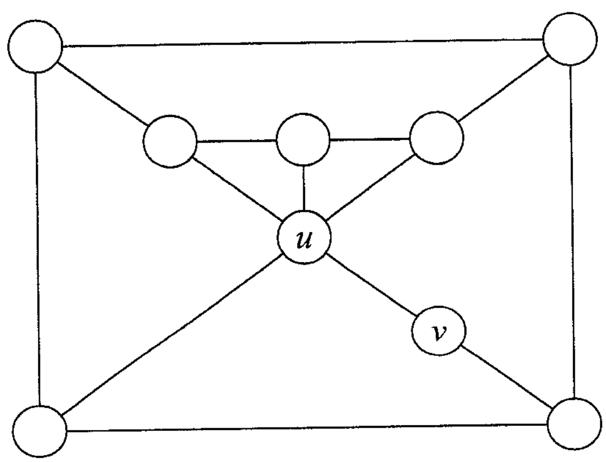
Fig. 9-5

vertex-colorability of the dual graph, we have to draw only one edge joining the vertices corresponding to these two regions. The connected graph G' shown in Fig. 9-5(b) is the dual graph thus modified. The subgraph induced by the set $\{2, 3, 4, 8\}$ of vertices in G' is K_4, and as such, at least four colors are needed to color the vertices of G'. So by the 4CT, the chromatic number of G' is 4. Consequently, four colors are needed to color all the regions, including the exterior region of the map. If the map represents an island, region 8 is the ocean, which can be colored blue, and the same color can be assigned to region 7, which could be a lake. Regions 1, 3, and 6 can be colored red. Regions 2 and 5 can be colored green. Finally, region 4 is colored yellow.

Cubic Maps and Edge-Colorings

A **cubic map** is a 3-regular bridgeless connected planar graph. Given any map G in which the minimum degree is at least 2, it is possible to construct a cubic map G' such that the k-colorability of one implies the k-colorability of the other. The procedure is as follows. If there is a vertex u in G whose degree r is more than 3, we introduce r new vertices "surrounding" u, construct a cycle passing through these r new vertices, and delete vertex u. Then join each vertex in the cycle to the vertex in G that was adjacent to u, making sure at the same time that the newly constructed graph continues to be planar. If v is a vertex of degree 2 joining two vertices, say p and q, insert a new vertex a on the edge joining p and u, insert a second new vertex c on the edge joining q and u, construct a third new vertex b and join it to a and c, construct a fourth vertex d and join it to a and c, delete u, and finally join b and d. Thus a vertex of degree 2 is effectively replaced by the planar graph K_4 minus one of its edges. When this process is completed (see Fig. 9-6) for each vertex of degree more

(a)



(b)

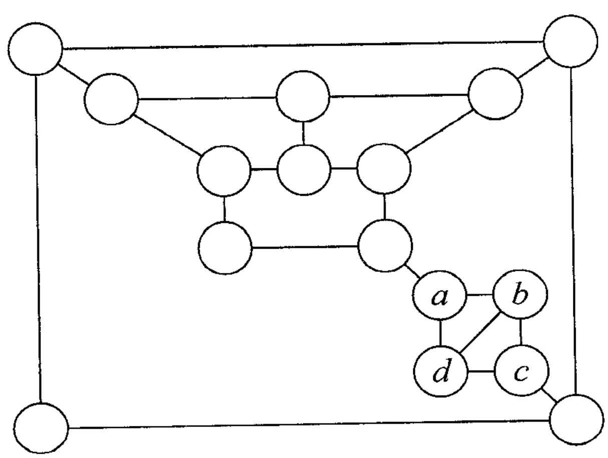


Fig. 9-6

than 3 and for each vertex of degree 2, we get a cubic bridgeless planar graph G'. This construction naturally leads to the following equivalence theorem.

Theorem 9.6. The four-color theorem is true if and only if every cubic map is 4-colorable.

Next we consider a cubic map G. Suppose its faces are 4-colored using colors A, B, C, and D. Then we can 3-color the boundaries of its regions using colors B, C, and D as follows. If there is a boundary between a region with color A and a region with color B, that boundary is assigned color B. This coloring assignment can be symbolically represented by the equation $A + B = B + A = B$. Similarly, we define $A + C = C$, $A + D = D$, $B + C = D$, $B + D = C$, and $C + D = B$. See the 4-coloring of the regions in the cubic graph shown in Fig. 9-7 and the 3-coloring of its edges. The color of the external region is B. On the other hand, suppose the edges of a cubic map are colored using the colors B, C, and D. To get a 4-coloring of its faces, we proceed as follows. Start from any region and label it A. When we move from that region to a neighboring region after crossing the edge colored, say B, we assign the color $A + B = B$ to that region. Thus, using the same addition, rule all the regions (including the external region) can be colored using these four colors. (This coloring scheme is taken from an article by M. Gardner's April 1976 *Scientific American* article under the column Mathematical Games). Thus we have the following characterization of the 4-colorability of planar graphs.

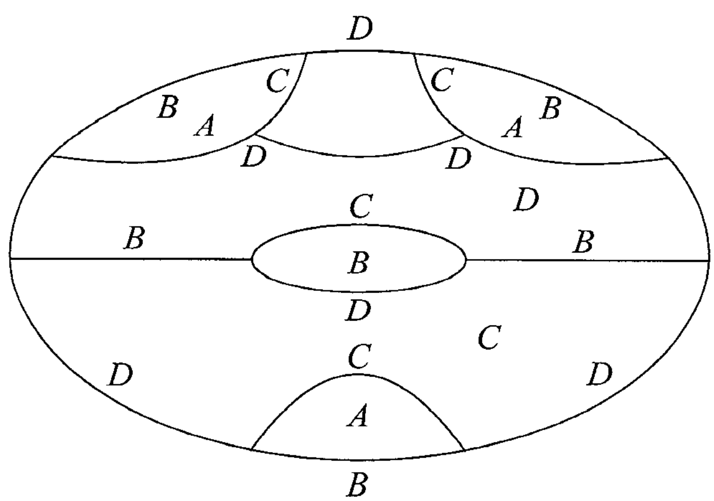


Fig. 9-7

Theorem 9.7 **(Tait's Theorem).** The following claims are equivalent:

(*i*) The chromatic number of a planar graph does not exceed 4.

(*ii*) The chromatic index of a cubic bridgeless planar graph (CBP graph) is 3.

(*iii*) Any CBP graph is 1-factorable.

(*iv*) Any CBP graph is a class 1 graph.

Any 3-coloring of the edges of a cubic graph is called a **Tait coloring,** and a cubic graph is said to be **uncolorable** if it does not have a Tait coloring. So by Vizing's theorem, the chromatic index of an uncolorable graph is 4. *Thus there are four kinds of cubic graphs:*

1. Cubic graphs with at least one bridge. Such graphs are uncolorable, as stated in Theorem 9.5. See Fig. 9-8(*a*).
2. Bridgeless planar cubic graphs. The four-color theorem implies that such graphs are colorable.

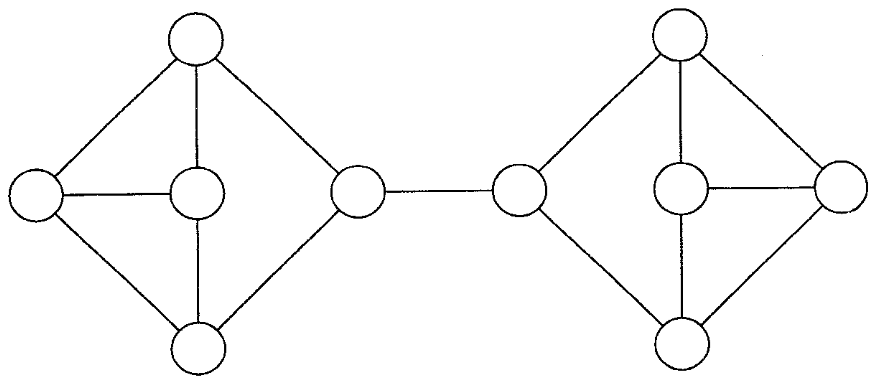


Fig. 9-8a

3. Bridgeless nonplanar cubic graphs that are colorable. See Fig. 9-8(*b*) in which the colors are marked 1, 2, and 3.
4. Bridgeless nonplanar cubic graphs that are uncolorable. The Petersen graph belongs to this category. It is no doubt a bridgeless nonplanar cubic graph. It is uncolorable because it is not 1-factorable. It also happens to be the smallest graph in this category.

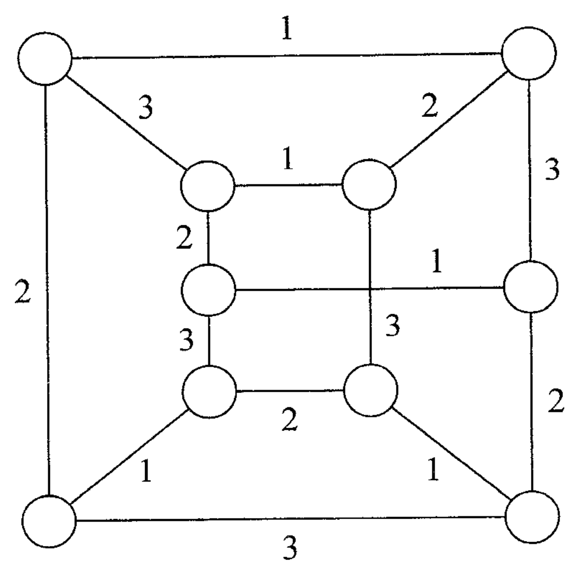


Fig. 9-8b

Finding such bridgeless cubic nonplanar uncolorable graphs is not easy. As R. Isaacs puts it, anyone who looks for them will be "vividly impressed with the maddening difficulty of finding" a single one. In the words of D. A. Holton and J. Sheehan, these graphs "do not appear to be thick on the ground." Such graphs were christened **snarks** by M. Gardner after Lewis Carroll's ballad entitled "The Hunting of the Snark." Given a cubic bridgeless graph G of chromatic index 4, one can obtain another such graph G' by a "trivial modification" (quoting B. Descartes) of G by replacing a vertex by a triangle or by inserting vertices on two of its edges having a vertex in common and joining them. Figure 9-9 shows two such trivial modifications of the Petersen graph. For a more exclusive definition of snarks (and the concepts leading to this definition) that precludes such

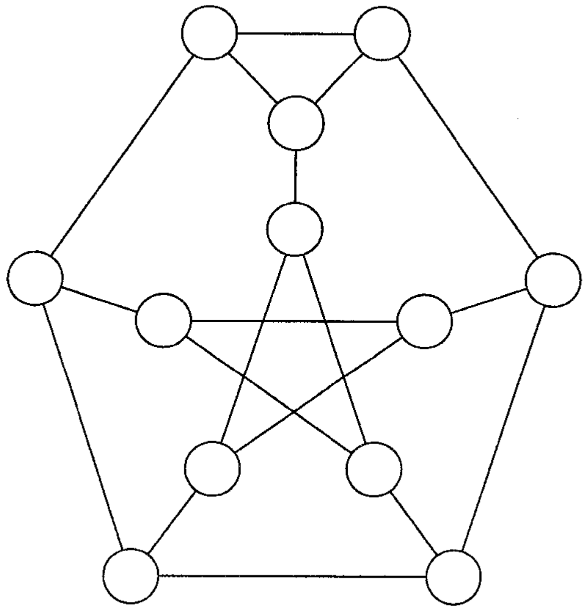


Fig. 9-9a

trivial generalizations, see solved Problems 9.61 through 9.70. We conclude this chapter with a statement of the celebrated **Tutte's conjecture:** Every snark has a subgraph that is homeomorphic (or contractible) to the Petersen graph. Observe that this conjecture implies the four-color theorem since the Petersen graph is non-planar; therefore, it is a genuine generalization of the theorem. Could an incisive proof of the verity of this conjecture be the Holy Grail of graph theory?

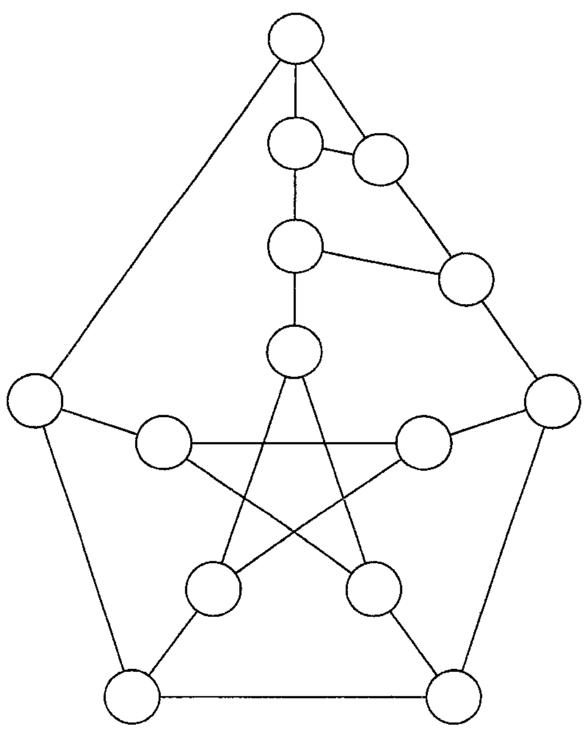


Fig. 9-9b

Solved Problems

VERTEX COLORING OF GRAPHS

9.1 Prove that a graph is k-colorable if and only if each block in it is k-colorable.

Solution. If a graph is k-colorable, each block in it is also k-colorable. The converse can be established by induction on the number of blocks. The clain is true if there is only one block. Assume that the theorem holds for any graph with r blocks. Consider any graph G with $(r + 1)$ blocks. Suppose B is one of its pendant blocks, and let H be the union of the remaining blocks. By the induction hypothesis, both B and H are k-colorable. Suppose the vertices of B and H are colored using at most k colors. Now there is exactly one vertex v common to both B and H. If v gets the same color in both B and H, G is k-colorable. Otherwise, we take another k-coloring of one of these components such that v gets the same color in both H and B.

9.2 Show that every planar graph is 6-colorable.

Solution. The proof is by induction on the order n of a planar graph. The result is true when $n \leq 6$. The induction hypothesis here is that any planar graph of order $(n - 1)$ is 6-colorable. Now any planar graph has a vertex of degree at most 5, as established in Solved Problem 8.16. Let G be any planar graph of order n, and let v be any vertex of degree at most 5. The graph $G' = G - v$ is 6-colorable. A 6-coloring of G is possible by coloring v with a color that is different from the colors of the (at most five) vertices adjacent to v. Hence, the chromatic number of a planar graph is at most 6.

9.3 Show that every planar graph is 5-colorable.

Solution. As in Problem 9.2, the proof is by induction on the order n of a planar graph. The result is true when $n \leq 5$. The induction hypothesis here is that any planar graph of order $(n - 1)$ or less is 5-colorable. Now any planar graph has a vertex of degree at most 5, as established in Solved Problem 8.16. Let G be any planar graph of order n, and let v be any vertex of degree at most 5. The graph $G' = G - v$ is 5-colorable. If the degree of v is less than 5, since $G - v$ is 5-colorable, G also is 5-colorable. If the degree of v is equal to 5, the five vertices adjacent to v cannot be pairwise adjacent since G is planar. So among these five, at least two are not adjacent. Suppose the vertices adjacent to v are a, b, c, d, and e, and let a and b be nonadjacent. Construct the planar graph G' of order $(n - 2)$ by contracting the edge joining v and a as well as the edge joining v and b. The graph G' is 5-colorable. The available colors for coloring the vertices of G' are p, q, r, s, and t. Suppose the merged vertex vab is assigned color p and vertices c, d, and e are assigned colors q, r, and s, respectively. Now unravel the merged vertex vab. Color the vertices of G such that nonadjacent vertices a and b both get color p and vertex v gets color t. The colors of the other vertices remain unchanged. So the planar graph G is also 5-colorable.

9.4 Prove Theorem 9.1: $\chi(G) \leq \Delta(G) + 1$ for any graph G.

Solution. Let n be number of vertices of G. The proof is by induction on n. Let G' be the graph obtained by deleting any vertex v from G. Then $\Delta(G') \leq \Delta(G)$. By the induction hypothesis, $\chi(G') \leq \Delta(G') + 1 \leq \Delta(G) + 1$. We can assign a color to v (in G) that is different from the colors (assigned in G') to the vertices adjacent to it. In that case, we do not need more than $\Delta(G) + 1$ colors to color the vertices of G since the number of vertices adjacent to v is at most $\Delta(G)$. Hence, $\chi(G) \leq \Delta(G) + 1$.

9.5 (*Szekeres–Wilf Theorem*) Show that for any graph G, $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs G' of G.

Solution. Let $\chi(G) = k$, and let H be the minimal induced subgraph such that $\chi(H) = k$. So for any vertex v in H, the graph $(H - v)$ is $(k - 1)$-colorable. Fix vertex v in H, and consider any $(k - 1)$ coloring of $(H - v)$. In that case, if the degree of v in H is less than $(k - 1)$, it is possible to color the vertices of H using at most $(k - 1)$ colors. Hence, the degree of any vertex v in H is at least $(k - 1)$. Thus $(k - 1) \leq \delta(H) \leq \max \delta(G')$.

[Observe that by replacing max $\delta(G')$ by $\Delta(G)$, we revert to the upper bound in Problem 9.4. Also, max $\delta(G')$ cannot be replaced by $\delta(G)$ for an arbitrary graph. See Problem 9.15.]

9.6 Prove Theorem 2 (Brooks's theorem): If G is a connected graph that is neither a complete graph nor an odd cycle, $\chi(G) \leq \Delta(G)$.

Solution. Let the order of G be n with $\chi(G) = k$ and $\Delta(G) = r$. If $r = 1$, G is complete. If $r = 2$, either the graph is bipartite, in which case $r = k$, or the graph is an odd cycle. So we assume that $r \geq 3$.

Case (i): The graph G is not r-regular. So there is a vertex v with degree less than r. Construct a spanning tree rooted at this vertex. Since there are n vertices, the root is labeled v_n, and the remaining vertices are labeled in descending order as and when a new vertex is added to the spanning tree, culminating at vertex v_1. Thus we have an ordered set of n vertices. Observe that each vertex v_i other than the root has a higher-indexed neighbor in the unique path (in the tree) from that vertex to the root. Therefore, the number of its lower-indexed neighbors is at most $(r - 1)$ since the maximum degree is only r. So if we use the greedy (sequential search) method to color the vertices of G, we need at most r colors; hence, $r \leq k$.

Case (ii): G is r-regular and w is a cut vertex of G. Suppose G' is one of the components of $G - w$. Let H be subgraph obtained by adjoining w and the edges joining w to the vertices in G'. Then the degree of w in H is less than r. As we saw in case (i), subgraph H is r-colorable. Then we can have a recoloring of the vertices in G so that G also is r-colorable. This argument holds for every cut vertex in G. Thus $r \leq k$.

Case (iii): G is r-regular and 2-connected. Since G is not complete, there are two nonadjacent vertices u, v, and a third vertex w that is adjacent to both u and v. If G is 3-connected, the graph $G - u - v$ is connected. If G is not 3-connected, there is a vertex w such that $(G - w)$ has a cut vertex. Each pendant block of $(G - w)$ has a vertex adjacent to w. We can then choose u and v from two different pendant blocks. Thus in any case, there are two nonadjacent vertices u and v and a vertex w adjacent to both of them such that $G - u - v$ is connected. Let us relabel the nonadjacent vertices u and v as v_1 and v_2, respectively, and their common neighbor as v_n. Now construct a spanning tree in the graph $G - v_1 - v_2$ with v_n as the root and terminating at vertex v_3 as in case (i). A coloring of the vertices using the greedy method can be now implemented. First, we have an ordering of the $(n - 3)$ vertices $\{v_3, v_4, \ldots, v_{n-1}\}$ defined by the tree. Each vertex in this set has at most $(r - 1)$ lower-indexed adjacent vertices. Since v_1 and v_2 are not adjacent, they can be assigned the same color. In addition to these two vertices, the root has $(r - 2)$ lower-indexed adjacent vertices in $\{v_3, v_4, \ldots, v_{n-1}\}$. So the greedy algorithm uses at most r colors to color G. Thus the proof is completed. (This proof is a modification by D. West of the proof given by Lovasz in 1975.)

9.7 (*Welsh–Powell Theorem*) Prove that if the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ of a graph G has the nonincreasing degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, $\chi(G) \leq \max_i \min\{d_i + 1, i\}$.

Solution. Apply the sequential search method to color the vertices with the given ordering of the vertices based on their degrees. When the ith vertex v_i is colored, the number of vertices already colored cannot exceed the minimum of $(i - 1)$ and its degree d_i for each i. So to color this vertex and the lower-indexed vertices, the number of colors needed does not exceed $1 + \min\{i - 1, d_i\}$. Hence, $\chi(G)$ cannot exceed the maximum among these n numbers. So $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\} \leq \max_i \min\{d_i + 1, i\}$.

9.8 Show that the number of colors needed to color the vertices of G using the largest first searching method does not exceed $1 + \Delta(G)$.

Solution. From Problem 9.7, $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\} \leq 1 + \Delta(G)$. So the number of colors needed if we use this algorithm cannot exceed $1 + \Delta(G)$.

9.9 Use the largest first algorithm to color the vertices of the graph shown in Fig. 9-10.

Solution. The set of vertices is ordered by degrees in nonincreasing order. Since there is an odd cycle, $\chi(G) \geq 3$. By Brooks's theorem, $\chi(G) \leq 4$. Using the algorithm, the vertices can be colored sequentially with colors 1, 2, 3, 1, 2, 3, and 4. So the graph is 4-colorable.

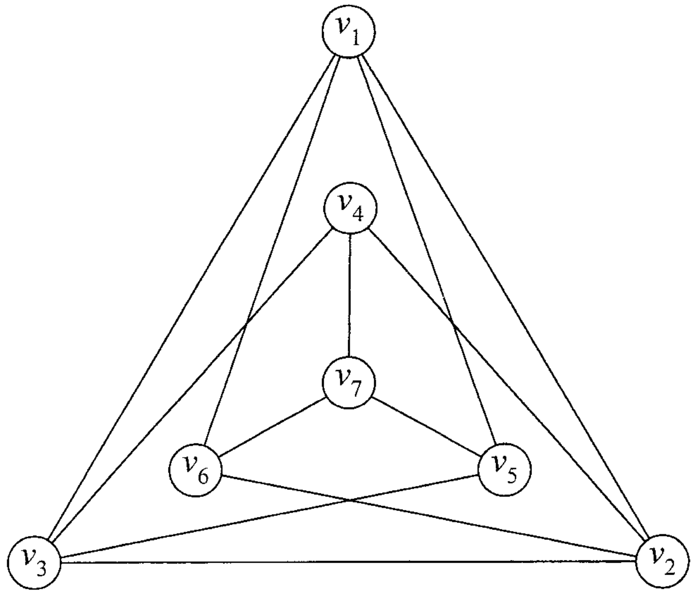


Fig. 9-10

Critical Graphs

9.10 A k-chromatic graph G is said to be a **critically k-chromatic** (or a **k-critical) graph** if $\chi(G - v) = k - 1$ for every vertex v of G. Show that a critically k-chromatic graph G is a block.

Solution. If G is not connected, the chromatic number of any component of G is less than k, which implies that the chromatic number of G also is less than k. So G is connected. If G has a cut vertex v giving a vertex partition $V_1, V_2, \ldots, V_r$ of the disconnected graph $G - v$, let G_i be the subgraph induced by $V_i \cup \{v\}$. Each G_i is $(k - 1)$-colorable. The only vertex common to any pair of these r subgraphs is v. G is also $(k - 1)$ colorable, which is a contradiction. So G has no cut vertex. Thus G is a block.

9.11 Give an example of a k-chromatic graph that is not critically k-chromatic.

Solution. Consider the graph consisting of a triangle and an edge joining one of its vertices to a fourth vertex. Its chromatic number is 3, and it is not critically 3-chromatic.

9.12 Characterize critically k-chromatic graphs when $k = 2$ and $k = 3$.

Solution. Obviously, a graph is critically 2-chromatic if and only if it is K_2, and it is critically 3-chromatic if and only if it is an odd cycle.

9.13 A k-chromatic graph G is said to be a **minimally k-chromatic graph** $\chi(G - e) = k - 1$ for every edge e of G. Give an example of (a) a minimally k-chromatic graph and (b) a critically k-chromatic graph that is not minimally k-chromatic.

Solution. Notice that every k-chromatic graph of minimum order is critically k-chromatic, whereas every k-chromatic graph of minimum size is minimally k-chromatic. Of course, every minimally k-chromatic graph without isolated vertices is a critically k-chromatic graph.

(a) Of course, K_2 and K_3 are minimally k-chromatic, where $k = 2$ and 3, respectively. For $k = 4$, the Grotzsch graph shown in Fig. 9-11(a) serves as an example.

(b) The Harary graph G shown in Fig. 9-11(b) is critically 4-chromatic but not minimally 4-chromatic since $\chi(G - e) = \chi(G) = 4$.

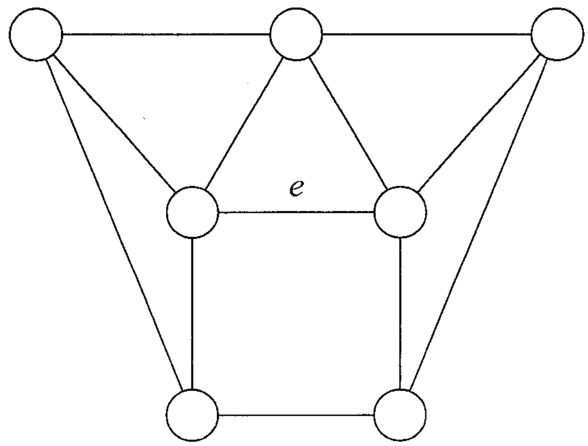


Fig. 9-11

9.14 Show that a k-chromatic graph contains a critically (minimally) k-chromatic graph.

Solution. Suppose the k-chromatic graph G is not critically k-chromatic. Then there exists a vertex v such that $G - v$ is k-chromatic. If $G - v$ is critically k-chromatic, we are done. Otherwise, there is a vertex w in $G - v$ such that $(G - v) - w$ is k-chromatic. Continue this process. We eventually arrive at a critically k-chromatic graph. The proof is similar for the case of minimally k-chromatic graphs.

9.15 Show that if G is critically k-chromatic, $\delta(G) \geq (k - 1)$. Give a counterexample to show that the converse does not hold.

Solution. Suppose $\delta(G) < (k - 1)$. Let v be a vertex in G such that the degree of v is equal to $d = \delta(G)$. The graph $G - v$ is $(k - 1)$-colorable, giving a vertex partition $\{V_1, V_2, \ldots, V_{k-1}\}$ of the graph G. By assumption, $d < (k - 1)$. So there is one set V_i of vertices in this partition such that no vertex in that set is adjacent to v. Thus it is possible to have another partition $\{V_1, V_2, \ldots, V_i \cup \{v\}, V_{k-1}\}$, which implies that G is $(k - 1)$-colorable. If e is an edge of K_4, $G = K_4 - e$ is 3-chromatic but not critically 3-chromatic. But the inequality holds in G.

9.16 Give an example of a k-chromatic graph G for which the inequality $\delta(G) \geq (k - 1)$ is not valid.

Solution. We are looking for k-chromatic graph (which is not critically k-chromatic) in which the inequality will not hold. Consider the example in Problem 9.11. The chromatic number is 3, but the inequality does not hold good because the minimum degree is 1.

9.17 Prove that a k-chromatic graph G has at least k vertices of degree at least $k - 1$.

Solution. Let H be a critically k-chromatic subgraph of G. The degree of every vertex in H is at least $k - 1$. Since the chromatic number of H is k, it should have at least k vertices.

9.18 Use the inequality obtained in Problem 9.15 to prove the Szekeres–Wilf theorem (Problem 9.5).

Solution. Let H be any induced critically k-chromatic subgraph of a k-chromatic graph G. Then $\chi(G) = \chi(H) \leq \delta(H) + 1 \leq 1 + \max \delta(G')$, where the maximum is taken over all the induced subgraphs G' of G.

9.19 Prove that a critically k-chromatic graph G with exactly one vertex whose degree exceeds $(k - 1)$ is minimally k-chromatic.

Solution. Since G is critically k-chromatic, the degree of each vertex is at least $(k - 1)$. In this case, the degree of each vertex is $(k - 1)$ except for one vertex. If e is any edge of G, $\delta(G - e) = k - 2$. Then

$\chi(G - e) \leq 1 + \max \delta(G')$, where G' is any induced subgraph of $(G - e)$. Thus $\chi(G - e) \leq 1 + (k - 2) = k - 1$. So $\chi(G - e) = k - 1$ since the minimum degree in $(G - e)$ is $k - 2$.

9.20 Show that if $G = (V, E)$ is a graph with $\chi(G) \geq k$ and with a vertex partition (X, Y) of V such that the subgraphs $G(X)$ and $G(Y)$ induced by X and Y, respectively, are $(k - 1)$-colorable, cut $[X, Y]$ consisting of those edges in E joining vertices in X and vertices in Y has at least $(k - 1)$ edges.

Solution. Since the two subgraphs are $(k - 1)$-colorable, sets X and Y can be partitioned into $\{X_i : i = 1, 2, \ldots, k - 1\}$ and $\{Y_i : i = 1, 2, \ldots, k - 1\}$ such that the vertices in any of these subsets get the same color. Consider the bipartite graph H with vertices $\{X_i : i = 1, 2, \ldots, k - 1\}$ on one side and $\{Y_i : i = 1, 2, \ldots, k - 1\}$ on the other in which vertex X_i and vertex Y_j are joined by an edge if and only if there is no edge between them in G. Suppose the cardinality of cut $[X, Y]$ is less than $(k - 1)$. Then H will have more than $(k - 1)(k - 2)$ edges. Now a set of $(k - 2)$ vertices in H can cover at most $(k - 2)(k - 1)$ edges. So any vertex cover of H should have at least $(k - 1)$ vertices. By Konig's theorem (Theorem 6.12), H has a matching of size $(k - 1)$, which implies that there is a complete matching in H. So we do not need more than $(k - 1)$ colors to color the vertices of G. This is a contradiction; therefore, cut $[X, Y]$ has at least $(k - 1)$ edges.

9.21 Show that a critically k-chromatic graph G is $(k - 1)$-edge connected. [Equivalently, any connected minimally k-chromatic graph is $(k - 1)$-edge connected.] Give an example to show that the converse is not true.

Solution. If $k = 2$, the graph is K_2, which is 1-edge connected. If $k = 3$, the graph is an odd cycle, which is 2-edge connected. Let $k \geq 4$. Suppose G is not $(k - 1)$ edge connected. Then there is a partition of the vertex set of G into two sets X and Y such that the cardinality of cut $[X, Y]$ is less than $(k - 1)$. So the subgraphs induced by X and Y are $(k - 1)$-colorable. Since the chromatic number of G is k, according to Problem 9.20, cut $[X, Y]$ should have at least $(k - 1)$ edges. This contradiction establishes that G is $(k - 1)$ edge connected. If e is an edge of K_4, $K_4 - e$ is a 2-edge connected graph that is not critically 3-chromatic.

9.22 Show that if G is a critically k-chromatic graph, there are no subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and, at the same time, such that $G_1 \cap G_2$ is complete.

Solution. Suppose there are subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and such that the intersection $H = G_1 \cap G_2$ is complete. The chromatic number of both G_1 and G_2 is at most $(k - 1)$. Consider a $(k - 1)$-coloring of G_1. In this coloring, no two vertices in H will have the same color. The same is also true in any $(k - 1)$-coloring of G_2. So the distinct colors that are assigned to the vertices in H (either in G_1 or in G_2) can be used for a coloring of these vertices in graph G. This implies that we do not need more than $(k - 1)$ colors for a coloring in G also, which is a contradiction.

9.23 Show that the graph induced by a separating set W of vertices of a critically k-chromatic graph is not a complete graph. In particular, if $|W| = 2$, the two vertices in W are not adjacent.

Solution. Let W be a separating set of vertices in a critical graph $G = (V, E)$ with $\chi(G) = k$, and let the vertex sets of $G - W$ be $V_1, V_2, \ldots, V_r$. Then the subgraph G_i induced by $(V_i \cup W)$, known as a **W-component** of G, being a proper subgraph of G for each i, is $(k - 1)$-colorable. Suppose each G_i has a $(k - 1)$-coloring in which no two vertices in W get the same color. If vertex w in W gets color c in a particular G_i, w can be assigned the same color c in each G_i and also in G. Then, when it comes to the coloring of the remaining vertices in set $(V - W)$, it is not necessary that two vertices in this set get different colors. This implies that G can be colored using at most $(k - 1)$ colors. This contradiction proves that there are two vertices in W that are not adjacent in G.

9.24 Give an example of a graph G with a separating set W consisting of two adjacent vertices.

Solution. Since the subgraph induced by W is complete, the graph G cannot be critically k-chromatic for any k. In Fig. 9-12(a), the separating set W consists of the two adjacent vertices u and v. There are three W-components for this graph, as shown in Fig. 9-12(b). Observe that G is 4-chromatic but not critically 4-chromatic.

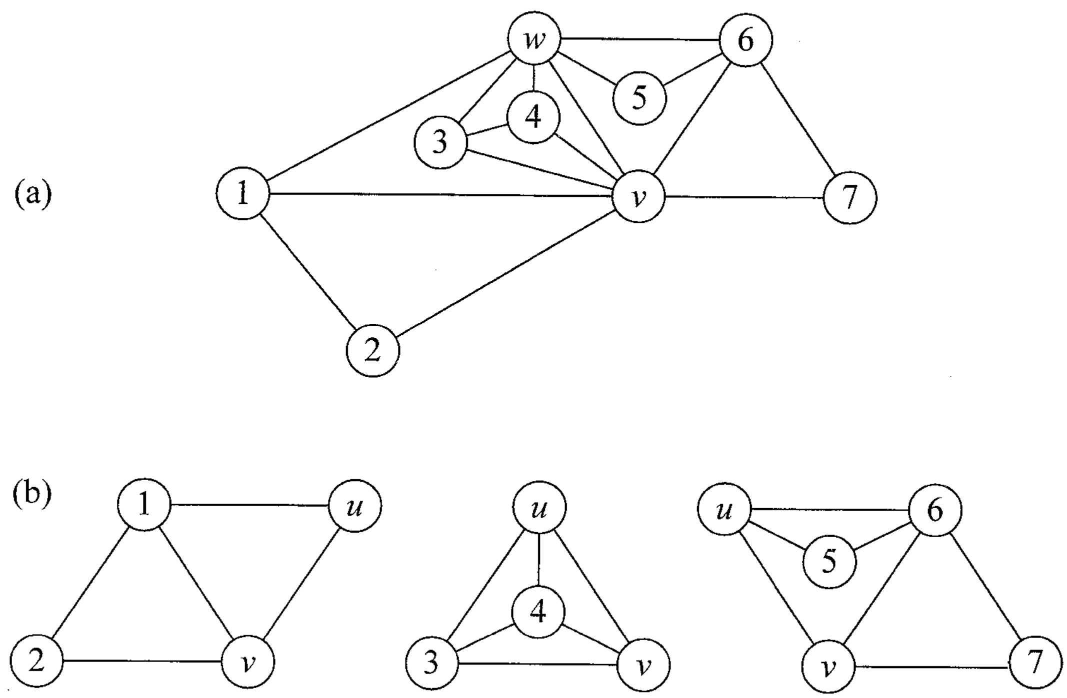


Fig. 9-12

9.25 (*Dirac's Theorem on Critical Graphs*) If a minimally k-chromatic graph G has a separating set W consisting of two vertices u and v, show that (a) there are exactly two W-components G denoted by G_1 and G_2 such that G is the union of these components, and (b) the graph $G_1 + e$ obtained by joining u and v by edge e and adding it to G_1 and the graph G_2' obtained from G_2 by joining u and v by an edge and then contracting it are both minimally k-chromatic.

Solution.

(a) u and v are not adjacent, as shown in Problem 9.24. Each W-component is $(k - 1)$-colorable. If there are $(k - 1)$-colorings of these components such that the two nonadjacent vertices u and v get the same color in each of these components, G also becomes $(k - 1)$-colorable. Thus there is a component, say G_1, such that both u and v get the same color under every $(k - 1)$-coloring, and there is a component, say G_2, such that these two vertices get different colors under any $(k - 1)$-coloring. In that case, the union of these components will need more than $(k - 1)$ colors. Since G is a critically k-chromatic graph, we then conclude that G is the union of these components and that there is no other component.

(b) In G_1, both vertices have the same color. If they are joined by edge e, they should have different colors. So $G_1 + e$ is k-chromatic. Let f be any edge of $G_1 + e$. We have to show that $G_1 + e - f$ is $(k - 1)$-colorable. This is obvious if $f = e$. In any $(k - 1)$-coloring of $G - f$, vertices u and v have different colors since component G_2 is a subgraph of G. The restriction of such a coloring to the vertices of G_1 is a $(k - 1)$-coloring of $G_1 + e - f$. So $G_1 + e$ is critically k-chromatic. For G_2', since any coloring assigns different colors to u and v, the graph obtained by merging them will need k colors. If f is any edge, $G - f$ is $(k - 1)$-colorable, and so G_2' is also.

9.26 Illustrate Dirac's theorem in the case of the graph shown in Fig. 9-13(a).

Solution. The graph shown in Fig. 9-13(a) is a minimally 4-chromatic graph. Set W of vertices consisting of u and v is a separating set defining the two components, as shown in Fig. 9-13(b). In the component where the degree of u is 2, any 3-coloring will assign the same color to both these vertices, whereas in the other component, any 3-coloring will have to assign different colors to these two vertices. Furthermore, if we join these two vertices in the first component, we get a minimally 4-chromatic graph. Likewise, if we merge these two vertices in the other component, we once again get a minimally 4-chromatic graph. Thus the theorem is verified.

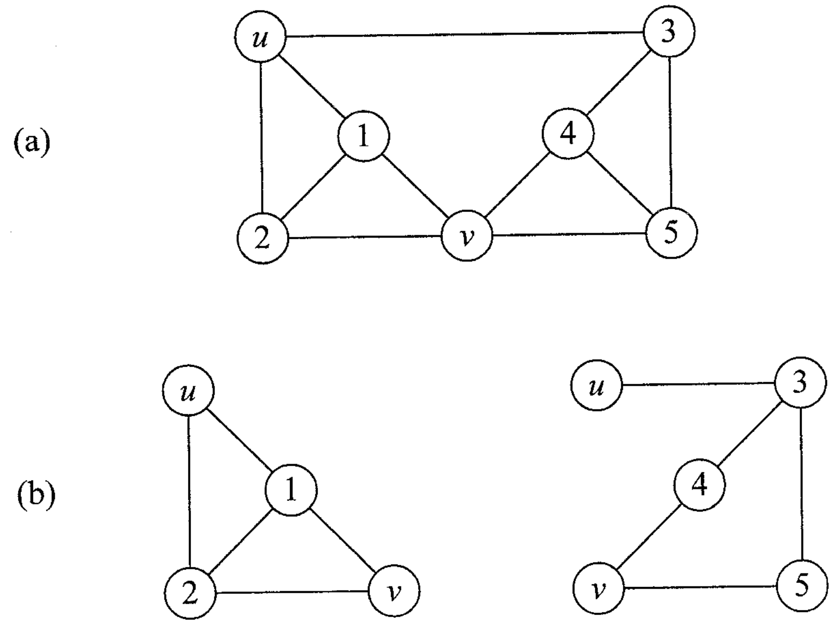


Fig. 9-13

9.27 Show that if a minimally k-chromatic graph has a separating set consisting of two vertices, the sum of their degrees is at least $(3k - 5)$.

Solution. Using the notation as in Problem 9.26, graph $G_1 + e$ is minimally k-chromatic. So (degree of u) + (degree of v) $\geq (k - 1) + (k - 1)$ in $(G_1 + e)$. This implies (degree of u) + (degree of v) $\geq (k - 2) + (k - 2) = (2k - 4)$ in G_1. Graph G_2' is minimally k-chromatic. So the degree of the vertex obtained by merging u and v is at least $(k - 1)$ in this graph. Now (degree of u) + (degree of v) $\geq (k - 1)$ in G_2. Thus (degree of u) + (degree of v) $\geq (2k - 4) + (k - 1) = (3k - 5)$ in G.

9.28 Use the inequality obtained in Problem 9.27 to prove that $\chi(G) \leq \Delta(G)$, where G is a connected graph that is neither a complete graph nor an odd cycle in the special case when G is not 3-connected. (This is a part of Brooks's theorem.)

Solution. We may assume without loss of generality that G is minimally k-chromatic. The hypothesis implies that $k \geq 4$ and G is 2-connected. Since it is not 3-connected, there is a separating set consisting of two vertices u and v. So $2\Delta(G) \geq$ degree of u + degree of v, which implies that $2\Delta(G) \geq (3k - 5) \geq (2k - 1)$ since $k \geq 4$. So $k \leq \Delta(G)$.

Uniquely Colorable Graphs

9.29 A k-chromatic graph G is **uniquely colorable** if any k-coloring of G induces the same partition of the vertex set of G. (*a*) List the k-chromatic graphs that are uniquely colorable when $k = 2$ and $k =$ the order of the graph. (*b*) Give an example of a graph that is not uniquely colorable. (*c*) Show that if the k-chromatic graph G is uniquely colorable, $\delta(G) \geq (k - 1)$.

Solution.

(*a*) The only uniquely colorable 2-chromatic graphs are the bipartite graphs. The complete graph of order n is the only uniquely n-colorable n-chromatic graph.

(*b*) The odd cycle of order 3 is a 3-chromatic graph that is not uniquely colorable.

(*c*) In any unique coloring, any vertex v should be adjacent to at least one vertex of every color different from that of v. Otherwise, it is possible to have a different vertex partition by changing the color of v.

9.30 Show that if a k-chromatic graph G is uniquely colorable, the subgraph induced by the union of any two sets in the vertex partition in a coloring is a connected graph.

Solution. Suppose V_1 and V_2 are two sets with colors c_1 and c_2, respectively, in a vertex partition of G, and let H be the subgraph induced by these two sets. If H is not connected, each component of H should have vertices of both colors c_1 and c_2. By interchanging these two colors, we have a different color assignment, contradicting the uniqueness assumption. The converse is not true, as seen from the 3-chromatic graph shown in Fig. 9-14.

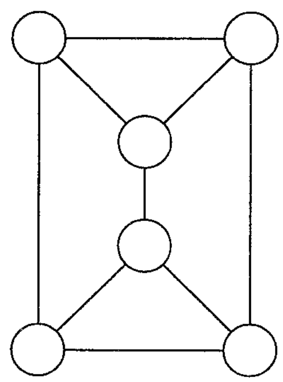


Fig. 9-14

9.31 Show that if a k-chromatic graph G is uniquely colorable, it is $(k - 1)$-connected.

Solution. This is true when G is the complete graph with k vertices. Consider the case when G is not complete. Suppose it is not $(k - 1)$-connected. So there is a set W of at most $(k - 2)$ vertices such that $G - W$ is disconnected. Since W has at most $(k - 2)$ vertices, there are at least two colors, say c_1 and c_2 (associated with sets V_1 and V_2 in a vertex partition of G), such that no vertex in W gets either of these two colors. At the same time, the subgraph H induced by the union of V_1 and V_2 is a connected subgraph contained in one of the components, say G_1, of the disconnected graph $G - W$. Now take any vertex v in a component of $G - W$ other than G_1 and assign it one of these two colors that no vertex in W has. So it is possible to have a new k-coloring of G, giving a different vertex partition. This is a contradiction.

9.32 Show that any 4-chromatic uniquely colorable planar graph G is a maximal planar graph.

Solution. Suppose the unique vertex partition corresponding to a 4-coloring defines the sets V_i $(i = 1, 2, 3, 4)$. Let G_{ij} be the subgraph (of order n_{ij} and size m_{ij}) induced by the union of V_i and V_j. There are six such connected subgraphs and six inequalities $m_{ij} \geq n_{ij} - 1$. By adding, we get $m \geq 3n - 6$, where m and n are the size and order of G. Since G is planar, $m \leq 3n - 6$. See Theorem 8.4. So equality that implies maximal planarity holds.

9.33 (*Zykov's Theorem*) Show that the graph obtained from a triangle-free k-chromatic graph by the Mycielski method is a triangle-free $(k + 1)$-chromatic graph.

Solution. Let $G = (V, E)$, where $V = \{v_i : i = 1, 2, \ldots, n\}$ be a triangle-free k-chromatic graph. Suppose $U = \{u_i : i = 1, 2, \ldots, n\}$ and $W = \{w\}$ are two new sets of vertices such that V, U, and W are pairwise disjoint. The new graph is G' with vertex set V', which is the union of V, U, and W. By definition, set U is an independent set in G', vertex w is adjacent to every vertex in U, and each u_i is adjacent to every vertex in G, which is adjacent to v_i. Obviously, the existence of a triangle in G' implies the existence of one in G. Thus G' is also triangle-free. Since the chromatic number of G is k and since both u_i and v_i can be assigned the same color, the same k colors that are used for a k-coloring of G can also be used for a k-coloring of $G' - w$. Once this is done, a new color can be assigned to w. Thus the chromatic number of G' cannot exceed $(k + 1)$. The crux of the problem then is to show that this is equal to $(k + 1)$. Suppose, to the contrary, that G' is k-colorable using the colors from the set $\{1, 2, \ldots, k\}$. Let the color of w be k. Then no vertex in U can have color k in any k-coloring of G'. Since the chromatic number of G is k, there should be at least one vertex in V with color k in any k-coloring of G using these

k colors. Once a k-coloring of G is accomplished, we can subsequently obtain a k-coloring of G' as follows. Let the color of w be k. If the color of v_i in G is k (in the k-coloring of G), we can recolor v_i (in G') using the color of u_i since these two vertices are not adjacent and since vertex v_j in V is adjacent to v_i in G if and only if v_j is adjacent to u_i. In that case, the restriction of this recoloring of G' to G uses at most $(k - 1)$ colors to color the vertices of G. This is a contradiction since the chromatic number of G is k.

Chromatic Polynomials (Chromials)

9.34 Find $P(G, x)$, where G is a cyclic graph with n vertices where $n = 3$ or $n = 4$.

Solution. If $n = 3$, no two vertices can have the same color. So the chromatic polynomial is $x(x - 1)(x - 2)$. If $n = 4$, suppose the four vertices are a, b, c, and d in a cyclic order. Let $f(r)$ be the number of ways of partitioning the vertex set into r independent subsets. Then $f(1) = 0$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 1$. So $P(G, x) = f(1)x_{(0)} + f(2)x_{(2)} + f(3)x_{(3)} + f(4)x_{(4)} = 0 + x(x - 1) + 2x(x - 1)(x - 2) + x(x - 1)(x - 2)(x - 3) = x^4 - 4x^3 + 6x^2 - 3x$.

9.35 If e is an edge of K_4, find the chromatic polynomial of $K_4 - e$.

Solution. Let the four vertices be a, b, c, and d. Suppose the only nonadjacent vertices are b and d. Let $f(r)$ be the number of ways of partitioning the vertex set into r independent subsets. Then $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, and $f(4) = 1$. So the chromatic polynomial is equal to $x(x - 1)(x - 2) + x(x - 1)(x - 2)(x - 3) = x(x - 1)(x - 2)^2$.

9.36 Show that if G is the union of two graphs G_1 and G_2 that share a single vertex, the chromatic polynomial of G is the product of the chromatic polynomials of G_1 and G_2 divided by x.

Solution. The number of ways of coloring the graph G_1 using x colors is $P(G_1, x)$. Once a coloring of this graph is done, the color that the common vertex gets remains the same for the coloring of the entire graph G. The number of ways of coloring the vertices of G that belong to G_2 (other than the common vertex) is $(1/x)P(G_2, x)$.

9.37 If G is the connected graph obtained by linking two triangles so that they share one vertex in common, find the chromatic polynomial of G.

Solution. The chromatic polynomial of each triangle is $x(x - 1)(x - 2)$. So, by Problem 9.36, $P(G, x) = (1/x)[(x)(x - 1)(x - 2)]^2 = x(x - 1)^2(x - 2)^2$.

9.38 Obtain the chromatic polynomial of the graph shown in Fig. 9-15.

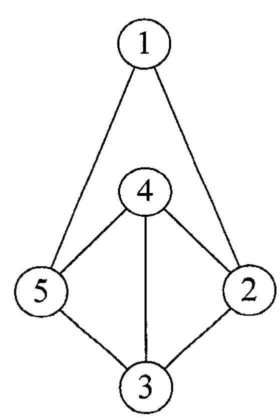


Fig. 9-15

Solution. Let $f(r)$ be the number of ways of partitioning vertex set $V = \{1, 2, 3, 4, 5\}$ into r independent subsets. There is at least one edge in the graph. So $f(1) = 0$. It is not possible to partition V into two independent subsets. So $f(2) = 0$. There are two ways of partitioning V into three independent subsets: $\{3\}, \{1, 4\}, \{2, 5\}$; and $\{4\}, \{1, 3\}, \{2, 5\}$. So $f(3) = 2$. There are three ways of partitioning V into four independent subsets: $\{1\}, \{3\}, \{4\}, \{2, 5\}$; $\{2\}, \{3\}, \{5\}, \{1, 4\}$; and $\{2\}, \{4\}, \{5\}, \{1, 3\}$. So $f(4) = 3$, and finally $f(5) = 1$. Thus $P(G, x)$ is $(2)(x)(x - 1)(x - 2) + (3)(x)(x - 1)(x - 2)(x - 3) + (x)(x - 1)(x - 2)(x - 3)(x - 4)$, which is equal to $x^5 - 7x^4 + 19x^3 - 23x^2 + 10x$.

9.39 Show that if G is a graph of order n and size m, the absolute value of the coefficient of x^{n-1} in the chromatic polynomial is equal to the size of the graph.

Solution. The coefficient of x^{n-1} in $f_{(n)}$ is $-\frac{1}{2}(n)(n - 1)$. So the coefficient of x^{n-1} in the chromatic polynomial is $f_{(n-1)} - \frac{1}{2}(n)(n - 1)$. Now $f_{(n-1)}$ is also equal to the number of nonadjacent pairs of vertices, so it is equal to $\frac{1}{2}(n)(n - 1) - m$. Thus the coefficient of x^{n-1} in the chromatic polynomial is $-m$.

9.40 Show that a graph of order n is a tree if and only if its chromatic polynomial is equal to $x(x - 1)^{n-1}$.

Solution. Let G be a tree with n vertices. We have to establish that the chromatic polynomial is $x(x - 1)^{n-1}$. The result is true if $n = 1$ and $n = 2$. Assume that this is true for all trees with $(n - 1)$ vertices. Let e be an edge joining vertex u to vertex v of degree 1 in a tree T of order $(n - 1)$. Then, by the induction hypothesis, the chromatic polynomial of $(T - v)$ is $x(x - 1)^{n-2}$. Vertex v can be assigned any color other than the color of u. So vertex v can be colored $(x - 1)$ ways. Thus the chromatic polynomial of T is $x(x - 1)^{n-2}(x - 1) = x(x - 1)^{n-1}$. On the other hand, let $x(x - 1)^{n-1}$ be the chromatic polynomial of a graph. Certainly the order of G is n. The coefficient of x^{n-1} is $-(n - 1)$, so the size is $(n - 1)$. If G is not connected, its chromatic polynomial is the product of the chromatic polynomials of its components; therefore, the coefficient of x in it has to be 0. But the coefficient of x in $x(x - 1)^{n-1}$ is not. So G is a connected graph with n vertices and $(n - 1)$ edges.

9.41 (*Reduction Theorem of Birkhoff and Lewis*) (*a*) If $G.e$ is the graph obtained from G by merging any two adjacent vertices u and v (joined by edge e) into a single vertex and by joining this new combined vertex to all those vertices to which either u or v were already adjacent, show that $P(G, x) = P(G - e, x) - P(G.e, x)$. (*b*) If $G.e$ is the graph obtained from G by merging any two nonadjacent vertices u and v into a single vertex and by joining this new combined vertex to all those vertices to which either u or v were already adjacent, show that $P(G, x) = P(G + e, x) + P(G.e, x)$, where $G + e$ is the graph obtained from G by joining u and v by new edge e.

Solution.

(*a*) The number of ways of coloring $G - e$ such that u and v do not get the same color is the same as the number of ways of coloring G. The number of ways of coloring $G - e$ such that u and v get the same color is the same as the number of ways of coloring $G.e$. Hence, $P(G - e, x) = P(G, x) + P(G.e, x)$.

(*b*) The proof is to (*a*).

9.42 Use the reduction theorem proved in Problem 9.41 to compute $P(G, x)$ of graph G in Problem 9.38.

Solution. The order of the graph is 5, and its size is 7. The complete graph with five vertices has 10 edges. We can delete the edges one at a time and apply the reduction rule (*a*) or add edges one at a time and apply the reduction rule (*b*). Let us use the second rule.

Iteration 1: Join vertices 2 and 5. Then contract the newly constructed edge. The resulting graphs are G_1 and G_2, as shown in Fig. 9-16.

Iteration 2: Join vertices 1 and 4 in G_1, and contract this edge. The resulting graphs are G_3 and G_4. Join vertices 1 and 4 in G_2, and contract the new edge. The resulting graphs are G_5 and G_6. Notice that G_4 is K_4 and G_6 is K_3.

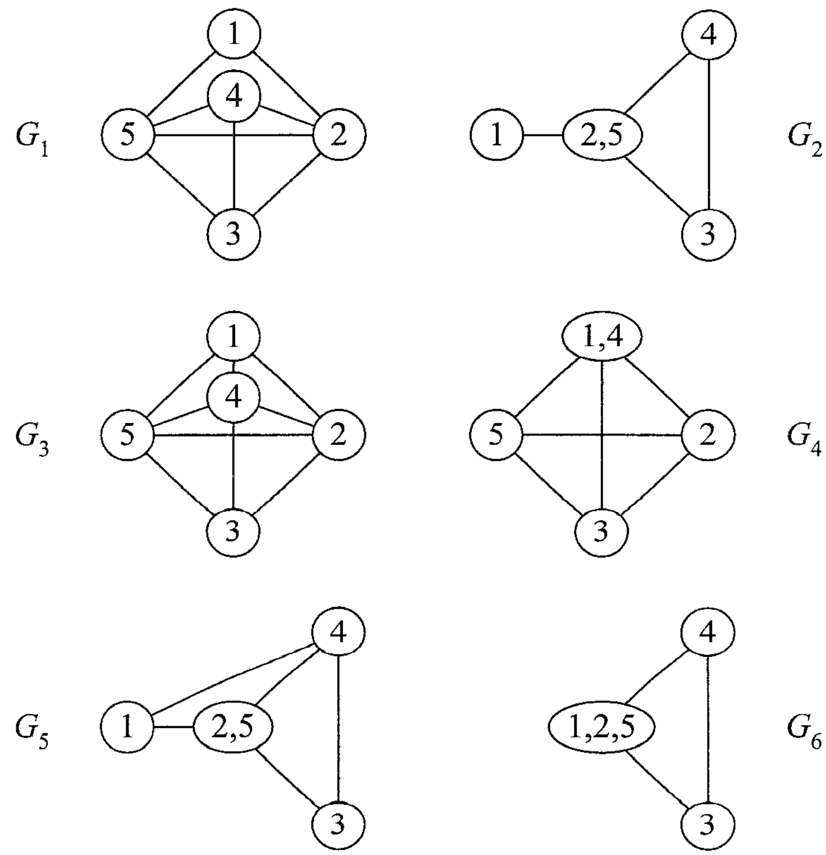


Fig. 9-16

Iteration 3: Join vertices 1 and 3 in G_3, and contract this edge. The resulting graphs are G_7 (which is actually K_5) and G_8 (which is K_4). Join vertices 1 and 3 in G_5, and contract this new edge. The resulting graphs are G_9 (which is K_4) and G_{10} (which is K_3). At this stage, all the graphs are complete graphs:

$$\begin{aligned} P(G, x) &= P(G_4, x) + P(G_6, x) + P(G_7, x) + P(G_8, x) + P(G_9, x) + P(G_{10}, x) \\ &= P(K_4, x) + P(K_3, x) + P(K_5, x) + P(K_4, x) + P(K_4, x) + P(K_3, x) \\ &= P(K_5, x) + 3\, P(K_4, x) + 2\, P(K_3, x) \\ &= x_{(5)} + 3x_{(4)} + 2x_{(3)} = x^5 - 7x^4 + 19x^3 - 23x^2 + 10x \end{aligned}$$

9.43 The chromatic polynomial of a cyclic graph of order n is $(x-1)^n + (-1)^n(x-1)$.

Solution. First, consider the case $n = 3$. The chromatic polynomial of the graph obtained by deleting an edge is $x(x-1)^2$. The chromatic polynomial of the graph obtained by contracting an edge is $x(x-1)$. So, by the reduction theorem, the chromatic polynomial of K_3 is $x(x-1)^2 - x(x-1) = (x-1)^3 + (-1)^3(x-1)$. Thus the formula is true when $n = 3$. Suppose this holds for all cyclic graphs of order $(n-1)$. Let G be any cyclic graph of order n, and let e be one of its edges. Graph $G - e$ is a tree (that is, a path) with n vertices, so its chromatic polynomial is $x(x-1)^{n-1}$. The graph obtained from G by contracting this edge is a cyclic graph of order $(n-1)$, and its chromatic polynomial is $(x-1)^{n-1} + (-1)^{n-1}(x-1)$ by the induction hypothesis. So, by applying the reduction theorem, the chromatic polynomial of G is $[x(x-1)^{n-1}] - [(x-1)^{n-1} + (-1)^{n-1}(x-1)] = (x-1)^n + (-1)^n(x-1)$. Thus the result is true for n as well.

9.44 Prove that the sum of the coefficients of the chromatic polynomial of a graph that has at least one edge is 0.

Solution. If a graph G has at least one edge, it is not possible to color its vertices with one color. In other words, $P(G, 1) = 0$. But $P(G, 1)$ is the sum of the coefficients of the chromatic polynomial of G.

9.45 Show that the coefficients of the chromatic polynomial alternate in sign.

Solution. The proof is by induction on the size m of graph G. If $m = 0$, the chromatic polynomial is x^n, and the result is true. Suppose the result is true for all graphs of order n and size less than m. Let G be any graph of order n and size m. Let e be any edge in G. Then, by the reduction theorem, $P(G, x) = P(G - e, x) - P(G.e, x)$. Now $(G - e)$ is a graph of order n with $(m - 1)$ edges, whereas $G.e$ is a graph of order $(n - 1)$ with $(m - 1)$ edges. So, by the induction hypothesis, the coefficients alternate in sign in their chromatic polynomials. Thus

$$P(G - e, x) = x^n - (m - 1)x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} + \cdots$$

and
$$P(G.e, x) = x^{n-1} - (m - 1)x^{n-2} + b_{n-3}x^{n-3} - b_{n-4}x^{n-4} + \cdots$$

where coefficients a_i and b_j are nonnegative.

On subtracting,

$$P(G, x) = x^n - mx^{n-1} + [a_{n-2} + (m - 1)]x^{n-2} - [a_{n-3} + b_{n-3}]x^{n-3} - \cdots$$

So the coefficients in $P(G, x)$ also alternate in sign.

9.46 Show that if $P(G, x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} + \cdots$ is the chromatic polynomial of a connected graph G, $1 < a_{n-1} < a_{n-2} < \cdots < a_r$, where r is the floor of $(n/2 + 1)$.

Solution. If G is a tree, $P(G, x) = x(x - 1)^{n-1}$, in which case the inequality is satisfied since the coefficients are the binomial numbers. So the theorem is true when $m = (n - 1)$. We now use induction on size m of the graph. Suppose the inequality condition is satisfied for any connected graph (with at least one cycle) with n vertices and m edges. Since G is not a tree, it has edge e belonging to a cycle; using this edge, we can construct two graphs $G - e$ and $G.e$, each with $(m - 1)$ edges, as in Problem 9.45:

$$P(G - e, x) = x^n - (m - 1)x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} + \cdots$$

and
$$P(G.e, x) = x^{n-1} - (m - 1)x^{n-2} + b_{n-3}x^{n-3} - b_{n-4}x^{n-4} + \cdots$$

where coefficients a_i and b_j are nonnegative and satisfy the inequality condition.

On subtracting,

$$P(G, x) = x^n - mx^{n-1} + [a_{n-2} + (m - 1)]x^{n-2} - [a_{n-3} + b_{n-3}]x^{n-3} - \cdots$$

It can be easily verified that the coefficients in $P(G, x)$ also satisfy the inequality condition, as stipulated in the problem. So the result holds for m as well.

9.47 Show that if $P(G, x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} + \cdots$ is the chromatic polynomial of a connected graph G, $a_i \geq \binom{n-1}{r-1}$ for each i.

Solution. The chromatic polynomial of any spanning tree T in the graph is $P(T, x)$, which is equal to $x(x - 1)^{n-1} = \sum_{r=1}^{r=n} (-1)^{n-r-1}\binom{n-1}{r-1}x^r$. If G' is the graph obtained by adding an edge of the graph to the spanning tree, the absolute value of the coefficient cannot decrease, as we saw in the reduction theorem. The entire graph G can be obtained from T by adding one edge at a time. This implies that $a_i \geq \binom{n-1}{r-1}$ for each i.

9.48 Show that the coefficient of x in the chromatic polynomial of a connected graph is not zero.

Solution. From Problem 9.47, this coefficient is at least 1.

9.49 Show that the smallest number k such that the coefficient of x^k is not zero in the chromatic polynomial of G is the number of components.

Solution. If G has k components, the chromatic polynomial of G is the product of k chromatic polynomials in each of which the coefficient of x is not zero.

9.50 Show that $x^5 - 7x^4 + 9x^3 - 3x^2$ cannot be the chromatic polynomial of a simple graph.

Solution. If it were the chromatic polynomial of a simple graph G, G should have two components, five vertices, and seven edges. No such graph exists.

9.51 The necessary conditions to be satisfied by the chromatic polynomial of a connected simple graph of order n and size m ($m > 0$) are (1) it should be a polynomial in x of degree n, (2) it is a monic polynomial, (3) the sum of the coefficients is zero, (4) the coefficients alternate in sign, (5) the constant term is zero, (6) the coefficient of x is not zero, (7) the coefficient of x^{n-1} is $-m$, and (8) the absolute values of the coefficients of $x^n, x^{n-1}, x^{n-2}, \ldots, x^r$ are strictly increasing, where r is the floor of $(n/2 + 1)$. Give an example of a polynomial that satisfies these eight conditions but is not the chromatic polynomial of a simple connected graph.

Solution. Consider the polynomial $x^5 - 11x^4 + 14x^3 - 6x^2 + 2x$. If this is the chromatic polynomial of a connected simple graph G, G should have five vertices and 11 edges. But the number of edges in a connected simple graph of order 5 is at least four and at most 10. So there is no graph for which this given polynomial is the chromatic polynomial.

EDGE COLORING OF GRAPHS

9.52 Show that if G is a bipartite multigraph, its chromatic index is $\Delta(G)$. In particular, show that the chromatic index of the complete bipartite graph $K_{m,n}$ is the maximum of $\{m, n\}$.

Solution. The proof is by induction on the number of edges in G. Let $\Delta(G) = r$. It is sufficient to prove that if every edge in $G - e$ (where e is an arbitrary edge in G) can be colored using colors from set S of r colors, the edges in G also can be colored using colors from this set. Suppose edge e joins vertices u and v. Let S_u be the set of colors in S that are not used to color the edges adjacent to u. Clearly, S_u and similarly S_v are nonempty subsets of S. If the intersection of these two subsets is nonempty, any color belonging to this intersection can be used to color edge e; therefore, the edges of G can be colored with r edges. Hence, $S_u \cap S_v$ is empty. Suppose $s \in S_u$ and $t \in S_v$. Let $H(s, t)$ be the connected subgraph of $G - e$ consisting of vertex u and all those vertices and edges in $G - e$ that can be reached from u by a path whose edges are either s-colored or t-colored. If v is a vertex in this component, edge e and any path joining u and v in H will form an odd cycle in G. So v is not a vertex in H. Now interchange the colors of the edges in H; every s-edge becomes a t-edge and vice versa. Then assign color s to edge e. This implies that the edges of G can be colored using r colors. Thus the chromatic index of a bipartite graph is its maximum degree. If the graph is $K_{m,n}$, the maximum degree is $\max\{m, n\}$.

9.53 If a graduate student in a department has taken k courses ($k \geq 0$) taught by a professor in that department, the professor has to give k oral examinations to the student at the end of the academic year. Each oral examination lasts exactly t hours. Find the minimum time required to complete all the departmental oral examinations if we know the number of courses each student has taken taught by each professor.

Solution. Construct a bipartite multigraph $G = (X, Y, E)$, where X is the set of graduate students and Y is the set of professors. Join vertex x in X and vertex y in Y by k edges if and only if x has taken k courses taught by y. The minimum number of time periods needed will be the chromatic index of G, which is the maximum degree $\Delta(G)$. So the minimum time taken is the product of t and $\Delta(G)$.

9.54 A **Latin square** of order n is an $n \times n$ matrix with entries from the set $\{1, 2, \ldots, n\}$ such that no entry appears twice in the same row and no entry appears twice in the same column. Show that a Latin square of order n can be constructed using an n edge coloring of the complete bipartite graph $K_{n,n}$.

Solution. Let $K_{n,n} = (X, Y, E)$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The set of colors is $S = \{c_1, c_2, \ldots, c_n\}$. Assume that the edges have been colored using these n colors. If the color of the edge joining x_i and y_k is c_j, $a_{ij} = k$ in the $n \times n$ matrix $A = [a_{ij}]$. The matrix A is obviously a Latin square of order n.

9.55 Show that if G is the complete graph with $2n$ vertices, its chromatic index is $2n - 1$.

Solution. The number of vertices is even. Let $V = \{1, 2, 3, \ldots, 2n\}$ be the vertex set. The maximum degree is $2n - 1$. Consider set $S = \{c_i: i = 1, 2, 3, \ldots, (2n - 1)\}$ of colors. An actual coloring of the edges of the complete graph using each color from S exactly n times can be obtained by the following procedure.

Consider the following arrangement of the $2n$ numbers as a $(2n - 1) \times 2n$ matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 2 & 3 & 4 & 5 & \ldots & 2n-2 & 2n-1 & 1 & 2n \\ 3 & 4 & 5 & 6 & \ldots & 2n-1 & 1 & 2 & 2n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2n-1 & 1 & 2 & 3 & \ldots & \ldots & 2n-3 & 2n-2 & 2n \end{bmatrix}.$$

Take the elements in row 1 of the matrix and arrange them as pairs $(1, 2n)$, $(2, 2n - 1)$, $(3, 2n - 2)$, and so forth. There will be n such pairs, with each pair corresponding to a unique edge in the complete graph. Assign color c_1 to each edge in this collection.

Take the elements in row 2 of the matrix and arrange them as pairs $(2, 2n)$, $(3, 1)$, $(4, 2n - 1)$, and so forth. Assign color c_2 to the edge corresponding to each pair in this row. Continue this process until the elements in all the rows are paired and the corresponding edges are colored.

(Another proof is as follows: The complete graph with $2n$ vertices is a regular 1-factorable graph. So its chromatic index is its maximum degree, which is $2n - 1$.)

9.56 Show that if G is the complete graph with $2n - 1$ vertices, its chromatic index is $2n - 1$.

Solution. The number of vertices is odd. Color the edges of K_{2n} with $(2n - 1)$ colors, as in Problem 9.55. Delete one vertex. Then we have graph K_{2n-1} with a coloring of its edges using $(2n - 1)$ colors. Suppose it is possible to color the edges of this graph with $(2n - 2)$ colors. There are $(n - 1)(2n - 1)$ edges in this graph. When $(n - 1)(2n - 1)$ is divided by $(n - 1)(2n - 1)$, we get $(n - 1) + (n - 1)/(2n - 2)$, which is greater than $(n - 1)$. This implies that some color from a set of $(2n - 2)$ colors has to be assigned to at least n edges that will need $2n$ vertices. But the number of vertices is only $2n - 1$. So we need at least $(2n - 1)$ colors to color the edges of K_{2n-1}.

9.57 (*Chromatic Index and the Lucas Schoolgirls Problem*) There are $2n$ schoolgirls in a boardinghouse. Each morning they walk to school in groups of two, as doubles. Find the maximum number of consecutive morning walks they can undertake such that each girl forms a pair with every other girl exactly once during these walks.

Solution. Since there are $(2n - 1)$ ways of pairing a girl with the remaining girls, the number of consecutive morning walks is at most $(2n - 1)$ such that no two girls form a pair more than once during the walk. Construct a complete graph with $2n$ vertices. Since the chromatic index is $(2n - 1)$, the edges can be colored using $(2n - 1)$ colors. Each color represents a walk. So it is possible to have $(2n - 1)$ walks.

9.58 (*Chromatic Number and the Kirkman Schoolgirls Problem*) There are 15 girls in a boardingschool who walk to school in groups of three (as triples) all seven days a week. Is it possible to form triples such that no two girls walk together more than once?

Solution. The answer is yes. Any three girls can be chosen out of 15 girls in 455 ways. Each day, the set of 15 is partitioned into five sets of triples. Since there are seven days, we are interested in first obtaining a set S (if it exists) of 35 triples such that no two triples in this collection have more than one girl in common. One could construct a graph G with 455 vertices in which each vertice represents a triple. Join two vertices by an edge if and only if they have two elements in common. Any independent set of 35 vertices in this graph will be a candidate to be chosen as set S. Once an independent set S of 35 is located, construct a graph G' with S as a vertex set. Join two vertices in G' by an edge if and only if the two vertices have one element in common. The chromatic number of G' cannot be less than 7. If it is equal to 7, the problem is solved since the vertices having the same color form a partition of the 15 girls. Otherwise, choose another independent set of 35 vertices and proceed. (A discussion and analysis of the existence theorem in this context, which is in the realm of design theory, is beyond the scope of this book.) If the first 15 letters of the alphabet represent the girls, a solution of the problem is as follows:

Sunday: ABE, CNO, DFL, GHK, IJM

Monday: ACI, BHO, DKM, ELN, FGJ

Tuesday: AFK, BGL, CHM, DIN, EJO

Wednesday: ALM, BCF, DEH, GIO, JKN

Thursday: AHJ, BMN, CDG, EFI, KLO

Friday: AGN, BDJ, CEK, FMO, HIL

Saturday: ADO, BIK, CJI, EGM, FHN

9.59 Prove Theorem 9.4 (Vizing's theorem): The chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G) + 1$.

Solution. Since the chromatic index $\chi'(G)$ of G cannot be less than $\Delta(G)$, it is enough if we show that $\chi'(G) \le \Delta(G) + 1$ whenever G is simple. If this is not true, there is a simple graph G such that $\chi'(G) > \Delta(G) + 1$ and $\chi'(G - e) \le \Delta(G - e) + 1 \le \Delta(G) + 1$, where e is any edge of G. Let the maximum degree $\Delta(G) = r$, and let $H = G - e$, where e is some fixed edge of G joining vertices u and v. Suppose the edges of H are colored using colors from set S of $(r + 1)$ colors. Color s from S is said to be *missing at a vertex* if s is not assigned to any edge incident to that vertex. Since the degree of each vertex in H is at most r and the number of colors in S is $(r + 1)$, there is at least color from S missing at every vertex of H. If there is a color in S missing at both u and v, that color can be used to color edge e, which implies that the edges of G can also be colored using the colors from set S.

So if there is no color missing at both these vertices (joined by deleted edge e) under the existing coloring scheme in H, the crux of the problem is as follows. Show that the edges of H can be *recolored* using these $(r + 1)$ colors from S such that once the new coloring scheme is implemented, there will be a color in S missing at u and at some adjacent vertex w. In this case, the edge f joining v and w can be deleted to redefine the graph $H = G - f$. This shows once again that the edges of G can also be colored using at most $(r + 1)$ colors, thereby contradicting the assumption that more than $(r + 1)$ colors are needed to color the edges of G.

Let e_1 be the edge joining u and v_1 in the graph, and let $H = G - e_1$. Suppose s is a color that is missing at u and t_1 is a color that is missing at v_1. If there is no edge incident to u with color t_1, t_1 will be missing at both u and v_1. In that case, we are done. So we assume that there is edge e_2 (with color t_1) joining u and vertex v_2 and that t_2 is a color that is missing at v_2. Thus we inductively define a sequence $v_1, v_2, \ldots, v_i$ of vertices adjacent to u and a sequence of colors $t_1, t_2, \ldots, t_i$ such that color t_i is missing at v_i and such that edge e_i joining u and v_i has color t_{i-1}. There is at most one edge with color t_i joining u and vertex v. If $v \notin \{v_1, v_2, \ldots, v_i\}$, we label v as v_{i+1}, and a color missing at this vertex is labeled t_{i+1}. These sequences can have at most r terms. Suppose the sequences terminate with vertex v_k and color v_k. One reason for the termination at this stage could be that there is no new vertex v (adjacent to u) such that the edge joining this vertex and u has color t_k. Then we can recolor the edges so that e_i gets color t_i for $i = 1, 2, \ldots, k$. This implies that the edges of G can be colored using at most $(r + 1)$ colors.

The only *other* reason for the termination of the sequence is that for some $j < k$, the missing color t_k at v_k is the same as color t_{j-1} of edge e_{j-1} joining u and v_j. In this case, we recolor edges e_i with colors t_i, where $1 \le i < j$. The edge joining u and v_j has lost its color. The missing color at v_j now becomes t_{j-1}, which is the same as t_k. The colors of the remaining edges in the sequence are unaffected. Thus every edge in G is colored using the colors from set S except edge e_{j-1} joining u and v_j in the sequence. Next we show that with some additional recoloring, we can locate a color that is missing at both u and v_j.

Consider the subgraph $G(s, t_k)$ consisting of all the edges (along with their incident vertices) of G that are colored either with s or with t_k. Each component of this subgraph is either a cycle or a path. Color s is missing at u, and color t_k is missing at both v_j and v_k. The degrees of three vertices in this subgraph are at most equal to 1. So these three vertices cannot belong to the same component of $G(s, t_k)$. There are two cases to be examined.

Case (i): Vertices u and v_j are in two different components. In the component that contains v_j, each s-edge is recolored as a t_k-edge and each t_k-edge is recolored as an s-edge so that s is a missing color at v_j. Since s is also missing at u, the uncolored edge e_{j-1} can be assigned color s.

Case (ii): The vertices u and v_k are in different components. Recolor the edge joining u and v_i with color t_i for each i, where $1 \le i < k$. This recoloring does not have any influence on the graph $G(s, t_k)$. At this stage, the only uncolored edge in G is the edge joining u and v_k. As before, interchange the colors of the edges in the component that contains v_k. As a result, color s becomes a missing color at v_k. Thus s becomes a missing color at the two vertices that join the only uncolored edge in G. This completes the proof in its entirety.

[This theorem has a generalization (also due to Vizing) as follows. If G is a multigraph, the chromatic index is at most equal to $\Delta(G) + m$, where m is the maximum of all edge multiplicities in G. Both Vizing's theorem and its generalization were independently proved by R. P. Gupta at about the same time.]

9.60 If G is an r-regular simple graph with an odd number of vertices, show that its chromatic index is $r + 1$. Is the converse true?

Solution. Since the maximum degree is r, the chromatic index is r or $r + 1$. The graph is not 1-factorable since it is of odd order. So the chromatic index is $r + 1$. The converse is not true. The chromatic index of the 3-regular Petersen graph is 4, but its order is not odd.

Uncolorable Cubic Graphs and Snarks

9.61 (*Blanusa's Theorem*) Let G be a 3-colorable cubic graph whose edges are colored using the colors $c_i (i = 1, 2, 3)$, and let F be a cut set in G. If the number of edges with color c_i in F are x_i, show that the three numbers x_1, x_2, and x_3 are all even or all odd.

Solution. Suppose set F disconnects G, partitioning the set of vertices of G into two sets X and Y of cardinalities x and y, respectively, such that each edge in F joins a vertex in X and a vertex in Y. Fix i. The x_i edges in F with color c_i join x_i vertices in X and x_i vertices in Y. Each of the remaining vertices in X is adjacent to another remaining vertex in X such that the color of the edge joining these two vertices is also c_i. In other words, $x - x_i$ is even for each i. So the three numbers x_1, x_2, and x_3 are all even or all odd.

9.62 Prove Theorem 9.5: If a cubic graph has a bridge, its chromatic index is 4.

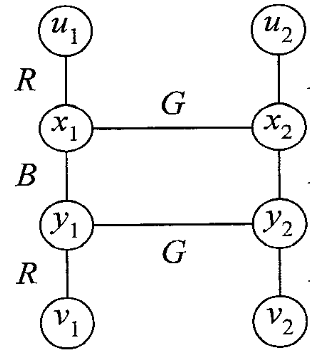
Solution. Let G be a cubic graph with a bridge. So it has a cut set consisting of one edge. Suppose G is 3-colorable. The cut set has exactly one edge having one color, say color c_1, and no edges having the other two colors. So the number of edges in the cut set with color c_1 is odd, and the number of edges in the cut set with any one of the remaining two colors is even. This is a contradiction to the fact established in Problem 9.61. So its chromatic index is 4.

9.63 Let G' be the graph obtained from a cubic graph G by contracting a triangle (a cycle of three vertices) in it into a single vertex. Show that $\chi'(G) = 3$ if and only if $\chi'(G') = 3$.

Solution. Consider a cycle in G consisting of three vertices u_i that is condensed into a single vertex u in G'. Let each u_i be adjacent in G to vertex v_i, which is not in the cycle. If the edge joining u_i and v_i is denoted by e_i, the set $F = \{e_1, e_2, e_3\}$ of three edges forms a cut set in G. If the chromatic index of G is 3, the three edges in F are necessarily of different colors, as proved in Problem 9.61. Assign these three colors to the three edges joining u and vertices v_i. Thus G' is 3-edge colorable. The reverse implication is obvious.

9.64 Let e_1 and e_2 be two edges with no vertex in common in a cubic graph G. Insert two vertices x_1 and y_1 on e_1 and two vertices x_2 and y_2 on e_2. Join x_1 and x_2 by an edge. Join y_1 and y_2 by an edge. If the chromatic index of the new graph G' thus constructed is 4, show that the chromatic index of G also is 4.

Solution. Suppose it is possible to color the edges of G using three colors, red (R), blue (B), and green (G). Let e_1 be the edge joining u_1 and v_1, and let e_2 be the edge joining u_2 and v_2 such that the new edges are as shown in Fig. 9-17. There are two cases to be considered.



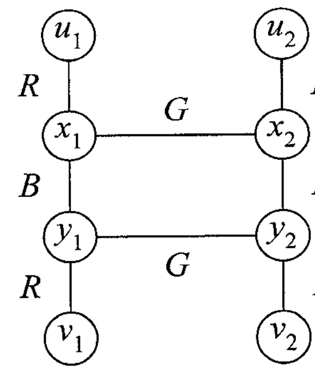


Fig. 9-17

(*i*) Both e_1 and e_2 have the same color, say R, in G. In this case, change the color scheme in G', as shown in Fig. 9-17(a).

(*ii*) Edges e_1 and e_2 have different colors; the first one is red and the other is blue. Now change the color scheme in G', as shown in Fig. 9-17(b).

In either case, the edges of G' can be colored using three colors, contradicting the hypothesis that the chromatic index of G' is 4.

9.65 Give a counterexample to show that the chromatic index of G' (in Problem 9.64) need not be 4 when the chromatic index of G is 4.

Solution. The converse is not true. In Fig. 9-18, we construct G' from the Petersen graph by inserting vertices on two nonadjacent edges. The chromatic index of G' is 3 (the three colors 1, 2, and 3, are indicated on the edges), but the chromatic index of the Petersen graph is 4.

9.66 Let G be a cubic graph with a cut set F consisting of three edges $e_i (i = 1, 2, 3)$ joining vertices u_i and v_i, where the six vertices are distinct such that $G - F$ has two subgraphs H_1 and H_2. Construct vertex u in H_1, and join it to the three end-vertices in H_1 of the cut set, creating a new cubic graph G_1. Likewise, create another cubic graph G_2 by constructing vertex v and joining it to the end-vertices in H_2 of the cut set. Show that the graph G is 3 edge colorable if and only if both G_1 and G_2 are 3 edge colorable.

Solution. Let G be 3-edge-colorable. By Problem 9.61, the three edges in the cut set are of three different colors. Assign the color of the edge joining u_i and v_i to the edge joining u and u_i as well as the edge joining v and v_i for each i. The coloring of the other edges remains unchanged. Thus we have a 3 edge coloring of the two new graphs. To prove the converse, assume both G_1 and G_2 are 3 edge colorable. Rearrange the coloring if necessary such that both the edge joining u and u_i and the edge joining v and v_i have the same color for each i. Then we have a 3-edge-coloring for G also.

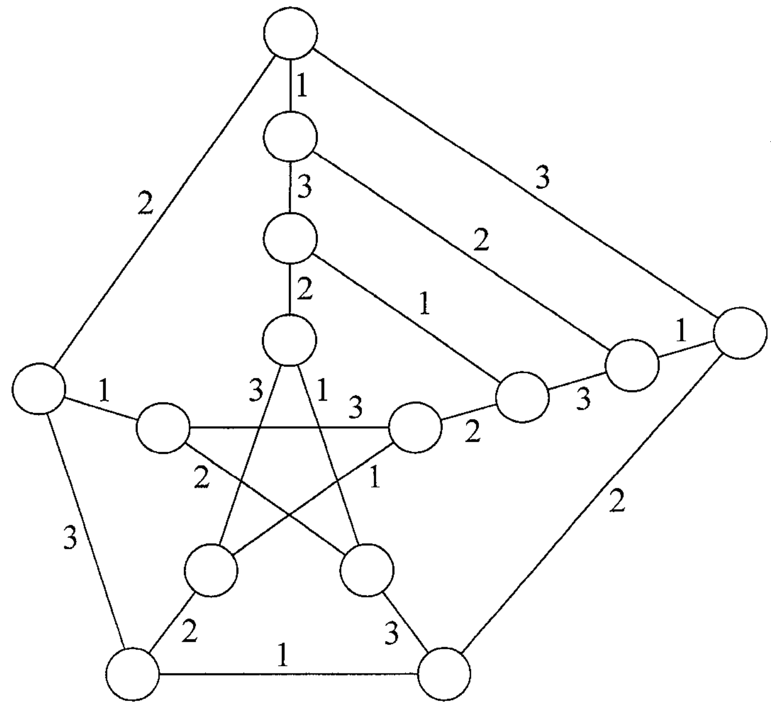


Fig. 9-18

9.67 Let G be a cubic bridgeless graph with a cut set F consisting of two edges, edge e_1 joining u_1 and v_1 and edge e_2 joining u_2 and v_2, so that $G - F$ has two components H_1 and H_2. Assume that u_1 and u_2 are in one component. Join u_1 and u_2 by an edge in H_1, creating a cubic (multi) graph G_1. Likewise, create G_2 from H_2 by joining v_1 and v_2. Show that the graph G is 3 edge colorable if and only if both G_1 and G_2 are 3 edge colorable.

Solution. Since the graph is bridgeless, the four vertices u_1, u_2, v_1, and v_2 are distinct. In any 3-coloring of G, two edges e_1 and e_2 are necessarily of the same color (say color c_1) as a consequence of the property established in Problem 9.61. Give the same color to the new edges. Thus both G_1 and G_2 are 3 edge colorable. On the other hand, if both these graphs are 3 edge colorable, we can always rearrange the colorings of their edges so that both the new edges have the same color, say c_2. Then we delete these two new edges and rebuild the deleted edges, which are assigned the same color c_1. Then we have a 3-coloring of the edges of G.

9.68 A cut set F in a graph G is called a **cyclic cutset** if $G - F$ has two components, each of which has a cycle. The **cyclic edge connectivity $\lambda_c(G)$** of G is the cardinality of the smallest cyclic cut set in G, and G is said to be **cyclically k edge connected** if $\lambda_c(G) \geq k$. Show that the Petersen graph is cyclically 4 edge connected.

Solution. Suppose the Petersen graph is not cyclically 4 edge connected. Since it is 3 edge connected, it is at least cyclically 3 edge connected. Let F be cyclic cutset consisting of three edges. Then $G - F$ has two components, each containing a cycle. Since the girth of G is 5, each component must have five vertices. Each component has three vertices of degree 2 and two vertices of degree 3, which implies that there is a cycle with fewer than five vertices in each component. This is a contradiction.

9.69 A **snark,** by definition, is an uncolorable, cyclically 4 edge connected cubic graph of girth at least 5. Show that this definition is more exclusive in the sense that a cubic bridgeless graph cannot be called a snark just because it is uncolorable. (The **girth conjecture** is the statement that every snark has a cycle consisting of five or six edges. This conjecture is now known to be false.)

Solution. First, the assumption that a snark is cyclically 4 edge connected implies that it has no bridges and by definition is not a multigraph. Following M. Gardner, let us use the notation NUT graph for a bridgeless (nontrivial), uncolorable, cubic (trivalent) graph. Any NUT graph that has a cycle of three or four edges can be obtained from a smaller NUT graph, as shown in Problems 9.63 and 9.64. Since the girth of a snark is more than 4, it cannot have a cycle with three or four edges. The cyclical 4 edge connectivity also implies that a snark is an NUT graph that does not have a cut set consisting of three edges, and as such, it cannot be obtained from a smaller NUT graph, as shown in Problem 9.66. The cyclical 4 edge connectivity coupled with a snark not having a cycle consisting of four edges implies that it does not have a cut set consisting of two edges. This gives rise to the possibility that it can be obtained from a smaller NUT graph, as shown in Problem 9.64. Thus the definition takes into account all the trivial modifications.

9.70 Let x and y be any two adjacent vertices in a cubic graph G of order n, where x is also adjacent to a and b. Likewise, y is adjacent to c and d. Four vertices a, b, c, and d are assumed to be distinct. Let e and f be two independent edges in a cubic graph G' of order n', where e joins vertices p and q and f joins vertices r and s. Delete x and y in G, and delete e and f from G'. Then connect the two graphs by constructing four new edges joining a and p, b and q, c and r, and d and s. The graph thus constructed is called a **dot product $G.G'$** of the two graphs, and it is obviously a cubic graph of order $n + n' - 2$. Show that a dot product of two snarks is a snark.

Solution. Obviously, a dot product of two snarks is cyclically 4 edge connected, and its girth is at least 5. The only thing to be shown is that the chromatic index of a dot product of two snarks is 4. Suppose the chromatic index is 3. Consider a 3-coloring of the edges of the dot product. By Blanusa's theorem (Problem 9.61), either all the four new edges are all the same color or two of them are one color and the other two are another color. This implies that G' is 3 edge colorable, which is a contradiction.

9.71 A **Blanusa snark** is a dot product of the Petersen graph with itself. Construct a Blanusa snark.

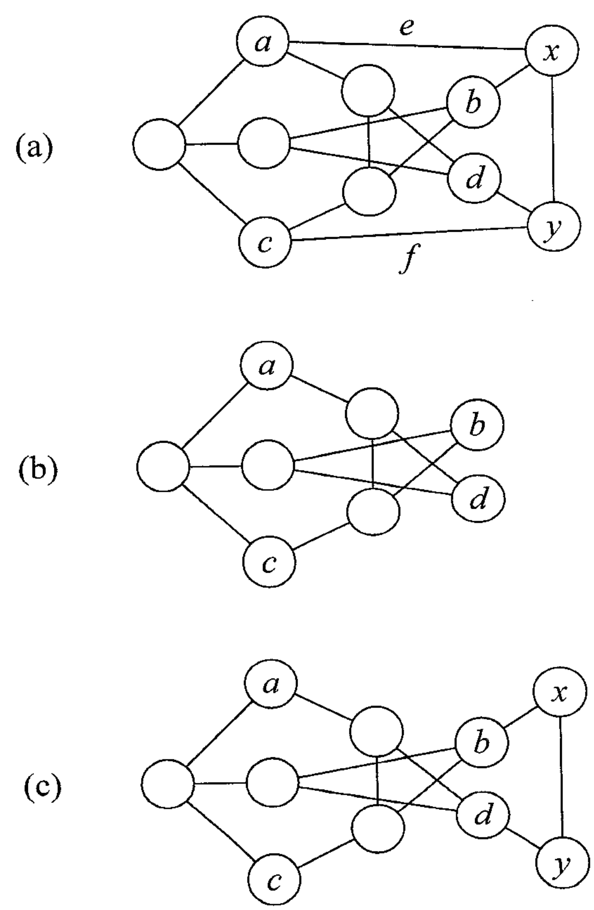


Fig. 9-19

Solution. The Petersen graph is shown in Fig. 9-19(a), with two adjacent vertices x and y and two independent edges e and f. Fig. 9-19(b) shows the graph obtained from G after deleting vertices x and y. Fig. 9-19(c) shows the graph obtained from G after deleting edges e and f.

A dot product of G with itself after making use of these deletions is shown in Fig. 9-20, in which the new edges are dashed lines. (It is possible to obtain another dot product by reconnecting the edges differently. Thus there are two Blanusa snarks.)

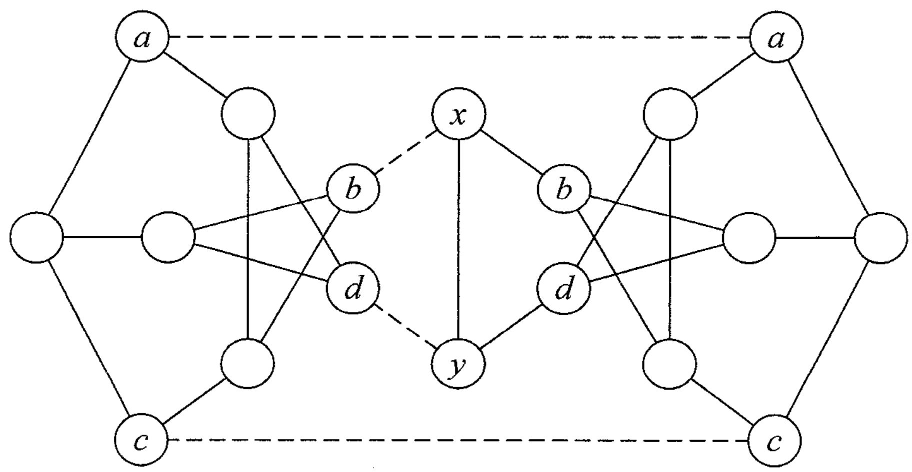


Fig. 9-20

9.72 Show that a snark is not a Hamiltonian graph.

Solution. Every snark has an even number of vertices. So if snark G is Hamiltonian, it has a Hamiltonian cycle with an even number of edges. The edges of this cycle can be colored using two colors. Once the edges of the cycle are colored, all the remaining edges can be colored using a third color. This implies that edges of snark G can be colored using three colors, which is a contradiction.

9.73 Give an example of a non-Hamiltonian cubic graph that is not a snark.

Solution. We are looking for a planar bridgeless cubic graph that is not Hamiltonian. Fig. 9-21 shows a graph of this kind.

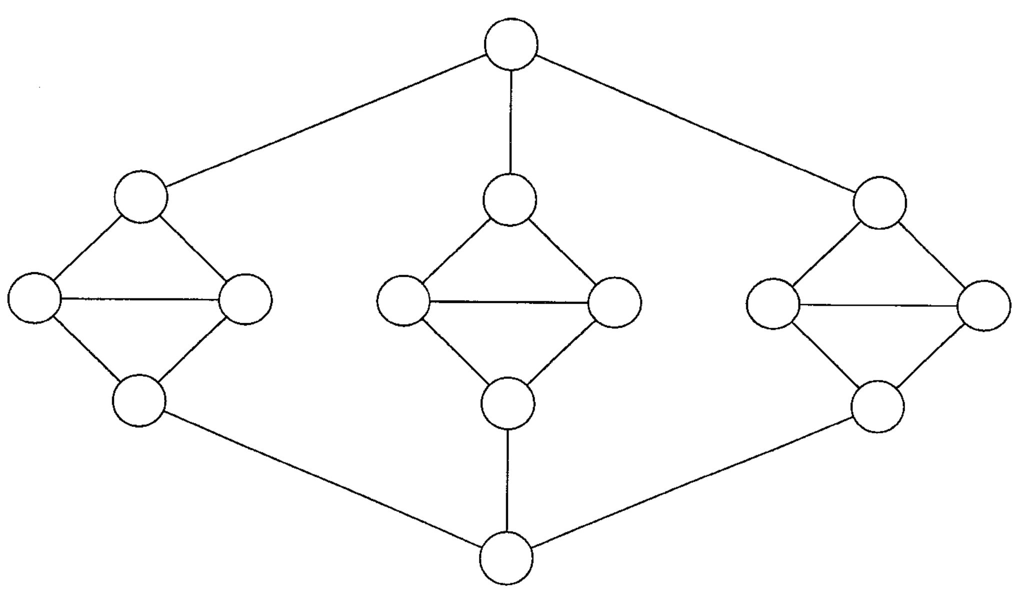


Fig. 9-21

9.74 Give an example of a "nonsnark," a cyclically 4-connected nonplanar cubic graph, the edges of which can be colored using three colors.

Solution. The graph shown in Figure 9-8(*b*) is a nonsnark.

COLORING OF PLANAR GRAPHS

Unavoidable Sets and Reducibility

9.75 (*Kempe's Theorem*) Show that in any cubic map, there is at least one region with fewer than six boundary edges. (See also Solved Problem 8.16.)

Solution. Let G be a cubic map of order n and size m with r regions. Let f_i be the number of regions with i vertices and i boundary edges. When we count the total number of vertices on all the regions, we obtain three times the total number of vertices. Hence,

$$3n = 2f_2 + 3f_3 + 4f_4 + 5f_5 + 6f_6 + \cdots \quad \text{and} \quad r = f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + \cdots$$

We also have the relation $3n = 2m$ and $n - m + r = 2$. So $6n - 6m + 6r = 12$.

Now $\quad 6n = 2f_2 + 6f_3 + 8f_4 + 10f_5 + 12f_6 + \cdots, \qquad 6r = 6f_1 + 6f_2 + 6f_3 + 6f_4 + 6f_5 + 6f_6 + \cdots$

Again, $$6m = 9n = 6f_2 + 9f_3 + 12f_4 + 15f_5 + 18f_6 + \cdots$$

Thus $$4f_2 + 3f_3 + 2f_4 + f_5 = 12 + f_7 + f_8 + \cdots$$

Since the right-hand side of this equation is positive, the left-hand side also is necessarily positive. So there exists a region with at most five borders.

9.76 Show that there exists an unavoidable set of four configurations, and list them.

Solution. That there is an unavoidable set of four configurations is a consequence of Kempe's theorem or of the equivalent (dual) property that there is a vertex of degree at most 5 in any planar graph. Each configuration can be considered a graph (as in Fig. 9-22) with one vertex v in the interior surrounded by a cycle of two, three, four, or five vertices such that v is adjacent to each vertex in the cycle. Every triangulation should contain at least one of these graphs.

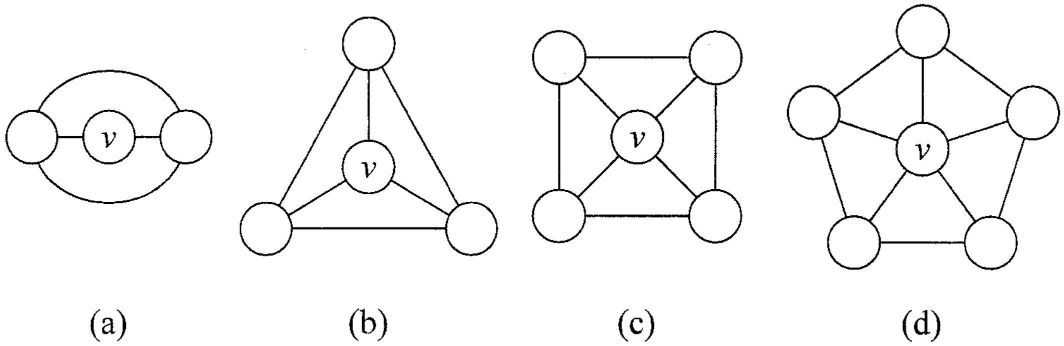


Fig. 9-22

9.77 Show that the first three graphs shown in Fig. 9-22 are reducible.

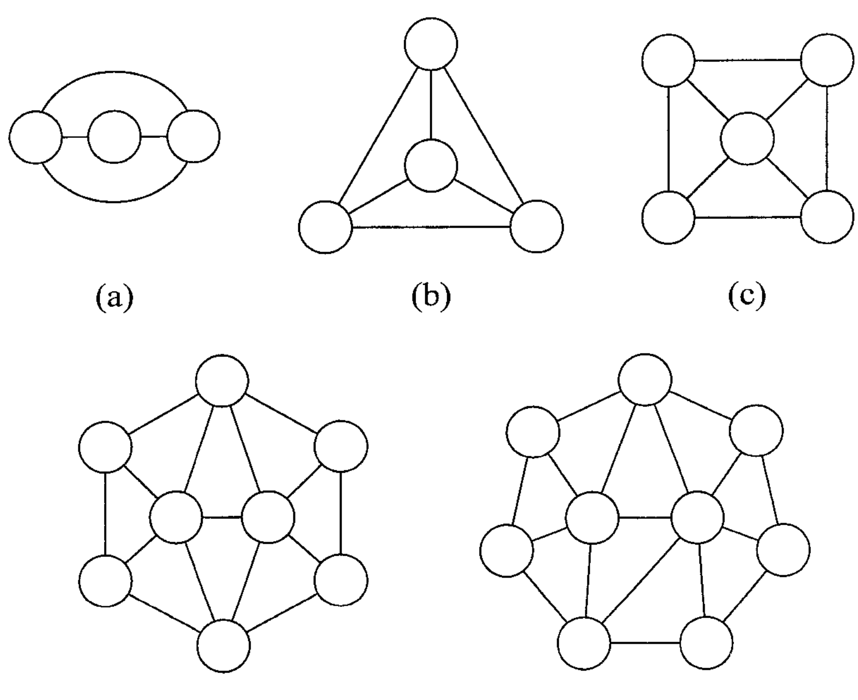
Solution. Let T be any planar graph with the minimum number of vertices that needs at least five colors. We can assume without loss of generality that T is a triangulation by joining any two nonadjacent vertices by an edge. So the assumption is that every planar graph with fewer vertices than T is 4-colorable.

(*i*) If T contains the configuration shown in Fig. 9-22(a) or (b), we could color $(T - v)$ with at most four colors and then assign a color to v that is not the same as any vertex adjacent to v in T. This implies that T also is 4-colorable, which is a contradiction. So both Fig. 9-22(a) and Fig. 9-22(b) are reducible.

(*ii*) If the four vertices a, b, c, and d together can be colored in fewer than four colors in any 4-coloring of $T - v$, T can also be 4-colored. If they have four different colors, we need a fifth color for vertex v if those four colors are not changed by a recoloring. So we have to recolor the vertices of $T - v$ such that a and c (or for that matter b and d) get the same color in the revised coloring. Suppose the colors of the four vertices, a, b, c, and d are A, B, C, and D, respectively. Let $H(A, C)$ be the subgraph of $T - v$ induced by the set of all vertices colored either A or B, and let $H(B, D)$ be the subgraph of $T - v$ induced by all vertices colored B or D. If both a and c belong to the same component of $H(A, C)$ and, at the same time, both b and d belong to the same component of $H(B, D)$, both these vertices have a vertex in common, which is not possible. So we may assume without loss of generality that a and c are not in the same component of the subgraph $H(A, C)$. In that case, we can interchange the colors in the component that contains a so that a get color c. Once this is done, we can assign color a to vertex v in T, which implies that T is 4-colorable.

[Any component of the subgraph induced by the vertices colored by two colors, as described in this problem, is called a **Kempe chain.** In a Kempe chain, one can always interchange the two colors of its vertices without changing the colors of the other vertices. This method of interchanging colors is called the **Kempe chain argument.** In 1879, Kempe erroneously claimed that the graph shown in Fig. 9-22(d) is also reducible, and he thereby presented an ostensible proof of the four-color theorem. Eleven years later, Heawood discovered a flaw in Kempe's proof of the reducibility of this last graph. The pentagon cannot be reduced. The four-color problem once again reverted to the status of a conjecture, and the quest began for a collection of reducible polygons in the place of a single pentagon.]

9.78 (*Wernicke's Unavoidable Set*) Show that the set consisting of the five graphs shown in Fig. 9-23 is an unavoidable set.



(a) (b) (c) (d) (e)

Fig. 9-23

Solution. Suppose T is a triangulation of order n and size m that does not contain any one of the five connected graphs shown in Fig. 9-23. So T has no vertices of degree 2, 3, or 4. If vertex v of degree 5 adjacent to another vertex w, the degree of w should be at least 7. Let n_i be the number of vertices of degree i. Then $n_5 + n_6 + \cdots = n$. Also

$$5n_5 + 6n_6 + 7n_7 + 8n_8 + \cdots = 2m$$

and since T is a triangulation, $2m = 6n - 12$. Thus

$$5n_5 + 6n_6 + 7n_7 + 8n_8 = 6(n_5 + n_6 + n_7 + n_8 + \cdots) - 12$$

Hence,

$$(6 - 5)n_5 + (6 - 6)n_6 + (6 - 7)n_7 + (6 - 8)n_8 + \cdots = 12 \qquad \textbf{(Kempe's equation)}$$

Each vertex of degree i initially has a "charge" of $(6 - i)$. Vertices of degree 7 or more are called major vertices; others are minor vertices. Major vertices initially have negative charges. Only vertices of degree 5 have positive charge. Vertices of degree 6 have 0 charge. We redistribute the charges by taking the positive charge of one unit from every vertex of degree 5 and equally distributing it via the five edges incident to other vertices, making sure that the total charge remains the same positive number 12. This method of redistributing charges is known as **discharging.** Since the five vertices adjacent to a vertex of degree 5 are all major, at the end, the charge at each vertex of degree 5 is 0. Since a vertex of degree 6 is not adjacent to a vertex of degree 6, the charge at a vertex of degree 6 remains 0 during discharging. If v is any vertex of degree i more than 6, the updated charge at v is at most $[6 - i + (i/5)]$, which is negative if i is more than 7. So after the redistribution, the charges at vertices of degree 8 and more are still negative. Vertex w of degree 7 will end up with positive charge only if it is adjacent to at least six vertices of degree 5. This will imply that vertices of degree 5 become adjacent. So once the charges are redistributed, the total charge becomes negative, contradicting the requirement that the total charge is a positive number, namely 12. Thus there is no triangulation that does not contain at least one of the connected graphs from the set in Fig. 9-23 a subgraph.

9.79 Show that the **Birkhoff diamond** (see Fig. 9-24) is reducible.

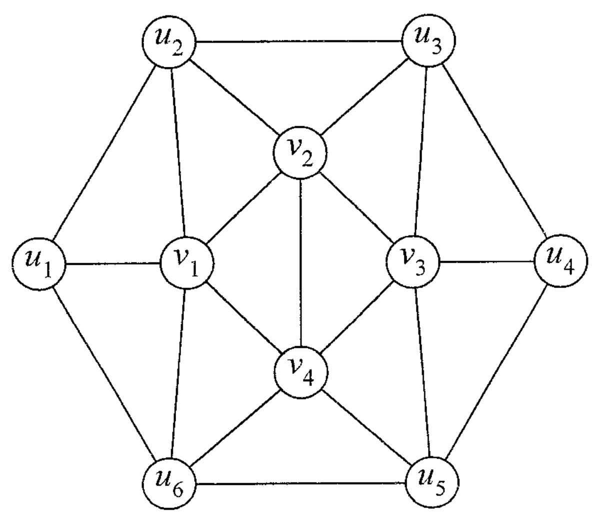


Fig. 9-24

Solution. Let T be any 5-chromatic triangulation with a minimum number of vertices that contains the Birkhoff diamond as a subgraph, and let T' be the 4-colorable triangulation obtained from T by deleting the four vertices inside the cycle (the **ring**) passing through the six vertices $u_i (i = 1, 2, \ldots, 6)$. The aim is to show that any 4-coloring of T' can be used to define a 4-coloring of T, thereby contradicting that it is 5-chromatic. The degree of every vertex in T is at least 5 since any vertex of degree less than 5 is reducible, as shown in Problem 9.77. Suppose there is a cycle C in T of length at most 4 such that the deletion of C disconnects T. This implies that there is a vertex in T of degree 3 or 4. So there cannot be a separating cycle of length at most 4 in T. If vertices u_3 and u_5 were adjacent, these two vertices and vertex v_3 together will form a separating set of length 3 whose deletion will cause a trivial component consisting of vertex u_4. Hence, u_3 and u_5 are not adjacent in T (and in T'); as such, there is no harm whatsoever in assuming (if necessary) that they both have the same color in any 4-coloring of T'. We can thus identify these two vertices in T' so that they both merge into a single vertex, say u. Then we join u and vertex u_1 by an edge. See Fig. 9-25. The resulting triangulation T'' is also 4-colorable.

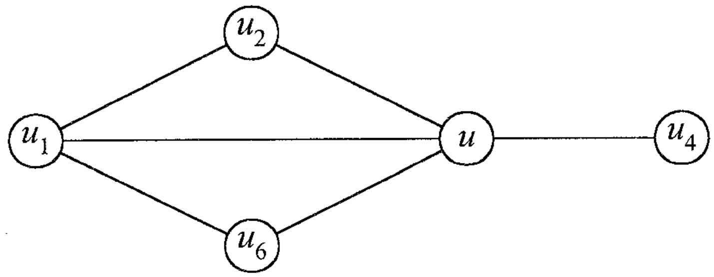


Fig. 9-25

Suppose the four available colors for coloring T'' are 1, 2, 3, and 4. There are exactly six (up to a permutation of these four colors) ways of coloring these five vertices u_1, u_2, $u_3 = u_5 = u$, u_4, u_6. They are 1, 2, 3, 1, 2; 1, 2, 3, 4, 2; 1, 2, 3, 4, 4; 1, 2, 3, 1, 4; 1, 2, 3, 2, 4; and 1, 2, 3, 2, 2. All these colorings except the last can be easily extended to a coloring of T. For example, the first coloring can be extended to the six vertices u_1, u_2, u_3, u_4, u_5, and u_6 by assigning colors 1, 2, 3, 1, 3, and 2, respectively. Then the four internal vertices v_1, v_2, v_3, and v_4 can be assigned colors 3, 1, 2, and 4, respectively. Thus we have a 4-coloring for T. But it is not possible to extend the last coloring—1, 2, 3, 2, 3, 2—of these six vertices to a 4-coloring of T without recoloring a vertex of the ring. At this stage, we invoke a Kempe chain argument. Suppose there is no path joining u_4 and u_2 (or joining u_4 and u_6) passing through vertices that are colored either color 2 or color 4. In that case, we change the color of u_4 from 2 to 4, defining the coloring scheme 1, 2, 3, 4, 3, 2 of the six vertices u_1, u_2, u_3, u_4, u_5, and u_6, which can be used to assign colors 3, 1, 2, and 4 to the four internal vertices v_1, v_2, v_3, and v_4. On the other hand, suppose there is a path P joining u_4 and u_6 in which the color of each vertex is either 2 or 4. If there is a path P' between u_1 and u_5 in which the color of every vertex is either 1 or 3, both P and P' will have a vertex in common. So if P exists, there is no P'; therefore, the color of u_5 can be changed from 3 to 1. Thus once again we have a coloring scheme 1, 2, 3, 2, 1, 2 for the six vertices u_1, u_2, u_3, u_4, u_5, and u_6 and a coloring scheme 4, 1, 4, 3 for the four internal vertices v_1, v_2, v_3, and v_4. The argument is analogous if there is a path Q joining u_4 and u_2 in which every vertex has either color 2 or color 4. This completes the proof.

Theorems of Gallai and Minty

9.80 (*Gallai's Theorem*) (*a*) Show that the chromatic number of a graph G cannot exceed $1 + t(D)$, where $t(D)$ is the number of arcs in a longest directed path in any acyclic orientation of G. (*b*) Show that for every graph G, there is an acyclic orientation D such that $\chi(G) = 1 + t(D)$. Hence, $\chi(G) = \min\{1 + t(D) : D$ varies through all the acyclic orientations of $G.\}$

Solution.

(*a*) Observe that there is at least one acyclic orientation. If the order of G is n, assign labels $1, 2, \ldots, n$ to the vertices, and then draw an arc from i to j if and only if $i < j$ and there is an edge joining i and j in G. Let D be any acyclic orientation of a graph G. For every vertex v of G, let $t(v)$ be the number of arcs in a directed path that starts from v and with as many arcs as possible. Assign color $1 + t(v)$ to vertex v. Obviously, we do not need more than $1 + t(D)$ colors to color the vertices of G.

(*b*) Suppose the chromatic number of G is k. Color the vertices using colors $1, 2, \ldots, k$. If there is an edge joining vertices u and v, convert this edge into an arc from u to v if and only if the color of u is more than the color of v. The orientation D thus obtained is acyclic for which $t(D)$ is at most $(k - 1)$. So $1 + t(D) \leq k$. But $k \leq 1 + t(D)$.

9.81 (*Minty's Theorem*) The **flow ratio** of a cycle C in an orientation D of a graph G is the ratio p/q (where $p \geq q \geq 0$), p being the number of arcs in one direction and q being the number of arcs in the opposite direction in C. Show that the vertices of a G can be k-colored if and only if there is orientation D of the graph in which the flow ratio of no cycle exceeds $(k - 1)$.

Solution. Suppose the vertices of G are colored using colors $0, 1, 2, \ldots, k - 1$. Define an orientation D for G as follows. If there is an edge in G joining u and v, change it into an arc from u to v if the color of v is more than the color of u. Otherwise, it is an arc in the opposite direction. Then no cycle in D can have a flow ratio of more than $(k - 1)$. To prove the converse, we start with an orientation D (of a connected graph G) in which no cycle has a flow ratio of more than $(k - 1)$. Fix vertex s (the source). Let v be any vertex in D, and consider a path (not necessarily a directed path) between these two vertices s and v in which an arc is *forward* if it is directed to vertex v. Otherwise, it is a *backward* arc. The value of any forward arc is one unit, and the value of any backward arc is $1 - k$. Choose a path P such that its value (the sum of the values of its arcs) is a maximum. Because of the upper bound constraint on the flow ratio, it is enough if we restrict our attention to simple paths if the intent is to maximize the value of the path. Hence, there is a simple path P between s and any vertex v such that the value of P is maximum. This maximum value, denoted by $g(P)$, is thus associated with vertex v, assigned color $g(v)$, which is $g(P) (\text{mod } k)$. If u and v are two adjacent vertices, $g(u)$ and $g(v)$ are unequal, and the absolute value of their difference is less than k. Consequently, the coloring thus defined is indeed a k-coloring of the vertices of G.

Conjectures of Hajos and Hadwiger and Heawood's Theorem

9.82 **Hajos's conjecture** claims that every k-chromatic graph has a subgraph homeomorphic to the complete graph of order k. Show that the conjecture is true if $k \leq 4$.

Solution. The conjecture is obviously true if $k = 1$ or 2. If $k = 3$, the graph has an odd cycle that is a subdivision of K_3. For $k = 4$, the proof due to Dirac is along the following lines. Let G be a minimally 4-chromatic graph of order n, where n is at least 5. The proof is by induction on n. If there is a separating set consisting of two vertices u and v, the two vertices are not adjacent (see Problem 9.25), and, by the induction hypothesis, the graph $G_1 + e$, where e is the (new) edge joining these two nonadjacent vertices, contains a subdivision of K_4. If we now replace edge e by a path P in G_2 joining these two vertices, $G_1 \cup P$ contains a subdivision of K_4. Thus G also contains a subdivision of K_4. If G is 3-connected, it has a cycle C of length at least 4 since the degree of each vertex is at least 3. Let u and v be two nonadjacent vertices in this cycle. Then there are vertices p and q on C and a p, q path P in $G - \{u, v\}$. Likewise, there is a u, v path P' in $G - \{p, q\}$. If these two paths have no vertex in common, the union of these two paths and C constitutes a subgraph (of G) that is a subdivision of K_4. If the two paths have a vertex in common, let w be the first vertex of P that belongs to P'. Then we have a path P'' joining p and w as a subpath of P. In this case, the union of P', P'', and C forms a subgraph that is homeomorphic to K_4.

(The largest integer n such that a given graph contains a subgraph homeomorphic to K_n is called the **Hajos number** of G. Another way of stating Hajos conjecture is that the chromatic number of a graph cannot exceed its Hajos number.)

9.83 Show that if Hajos's conjecture is true for $k = 5$, the four-color theorem is true.

Solution. The hypothesis is that every 5-chromatic graph contains a subgraph that is homeomorphic to K_5. Suppose there is a planar graph G that is not 4-colorable. So the hypothesis implies that G has a subgraph that is homeomorphic to K_5, which is a contradiction since G is planar. Thus every planar graph is 4-colorable if Hajos's conjecture holds when $k = 5$.

9.84 Show that Hajos's conjecture is false if $k \geq 7$.

Solution. Let H be the multigraph obtained by replacing each edge of cycle C_5 by three edges. The degree of each vertex in H is 6. Let G be the graph obtained by deleting two nonadjacent vertices from the line graph of H. Since a maximum independent set in G (which is of order 13) has cardinality 2, its chromatic number is at least 7. Now it is easy to see that K_6 is a subgraph of G. If w is any vertex that is not a vertex of this K_6, the number of internally disjoint paths from w to any vertex in K_6 is less than 6. In other words, the Hajos number of G is 6. So G is a counterexample of Hajos's conjecture when $k = 6$. Let $h(G)$ be the Hajos number of any graph G. Then $h(G + v) = h(G) + 1$ and $\chi(G + v) = \chi(G) + 1$. So if G is a counterexample for k for Hajos's conjecture, $G + v$ is a counterexample for $k + 1$. Thus Hajos conjecture is false for all $k \geq 7$.

(The status of this conjecture is unsettled when $k = 5$ or 6. P. Erdös and S. Fajtlowicz, in a joint paper, have proved that this conjecture is false for almost all graphs.)

9.85 **Hadwiger's conjecture** claims that every k-chromatic graph has a subgraph contractible to the complete graph of order k. Show that (*a*) the conjecture is true if $k \leq 4$ and (*b*) (**Wagner's theorem**) the conjecture is true when $k = 5$ if and only if the four-color theorem is true.

Solution.

(*a*) First notice that Hajos's conjecture implies Hadwiger's conjecture. Thus Hadwiger's conjecture holds if $k \leq 4$.

(*b*) As in Problem 9.84, if Hadwiger's conjecture holds for $k = 5$, the four-color theorem is true. To prove the converse, assume that the four-color theorem is true. Let G be any 5-chromatic graph. The four-color theorem implies that G is nonplanar. So there is a subgraph that is contractible to K_5 or $K_{3,3}$. If it is contractible to K_5, we are done. Suppose G has no subgraph contractible to K_5. So by P. Young's theorem (see Solved Problem 8.47), G is not 4-connected. We may assume that G is minimal in the sense that every contraction of G is 4-colorable. There exists a set W of at most three vertices such that $G - W$ has components $H_1, H_2, \ldots, H_k (k \geq 2)$ with the property that each vertex in W is adjacent to some vertex in H_i for every i. Let G_i be the subgraph induced by W and the set of vertices of H_i. Suppose T is a maximal independent subset of W. Then each G_i can be 4-colored such that the vertices in T all get the same color (say color 1) and the vertices in $W - T$ get colors other than 1. Then these 4-colorings can all be combined to obtain a 4-coloring of G, contradicting the assumption that the chromatic number of G is 5.

(It has been proved by B. Bollobas, P. A. Catlin, and P. Erdös that Hadwiger's conjecture is true for almost all graphs.)

9.86 (*Heawood's Map-Coloring Theorem*) Prove that $\chi(S_g) = H(g)$, where $H(g)$ is the floor of $\frac{1}{2}\{7 + \sqrt{1 + 48g}\}$ for any surface of positive genus g.

Solution. Let G be any triangulation with n vertices and m edges on a surface of genus g. Let d' be the average degree. Then $(n)(d') = (2)(m) = (3)(r)$, where r is the number of regions. Applying Euler's formula, we find $d' = (12/n)(g - 1) + 6$, which implies that $(n - 1) \geq (12/n)(g - 1) + 6$. Thus $n^2 - 7n - 12(g - 1) \geq 0$. On solving this quadratic, we have the lower bound $n \geq H(g)$. If $n \leq H(g)$, obviously $\chi(G) \leq H(g)$.

Suppose $n > H(g)$. Then $d' < [12/H(g)](g - 1) + 6 = H(g) - 1$. Now $g = 0$ will imply that $d' < 3$. So the genus g is necessarily positive. Thus there is a vertex v in G of degree at most $H(g) - 2$. Identify v and any vertex w adjacent to it by an elementary contraction to obtain a graph G' of order $n - 1$. If $n - 1$ is equal to $H(g)$, G' is $H(g)$-colorable, which will induce an $H(g)$-coloring on G. Otherwise, $n - 1 > H(g)$. We continue this process. We ultimately end up with an $H(g)$-colorable graph, which implies that G itself is $H(g)$-colorable. Thus the chromatic number of any graph that can be embedded on a surface of positive genus cannot exceed $H(g)$. Hence, $\chi(S_g) \leq H(g)$, where $g > 0$.

Let us write $p = H(g)$. Then $p \leq \frac{1}{2}\{7 + \sqrt{1 + 48g}\}$. On simplifying this inequality we find that $g' = \frac{1}{12}(p - 3)(p - 4) \leq g$, where g' is the genus of K_p (see Solved Problem 8.98). Hence, $\chi(S_{g'}) \leq \chi(S_g)$. Since K_p is embeddable on $S_{g'}$, we also have the inequality $\chi(K_p) \leq \chi(S_{g'})$. But $\chi(K_p) = p = H(g)$. So $H(g) \leq \chi(S_{g'}) \leq \chi(S_g)$. Thus the reverse inequality is established because the genus of a complete graph of order n is $\frac{1}{12}(n - 3)(n - 4)$. This completes the proof.

Two Colorability and Three Colorability

9.87 Show that a plane graph G is 2-colorable if and only if the degree of each region in G is even.

Solution. If the plane graph G is 2-colorable, it is bipartite, hence, the degree of each region is even. On the other hand, suppose G is a plane graph in which the degree of each region is even. Let C be any cycle dividing the plane into two parts: a part consisting of regions inside the cycle and a part consisting of regions outside the cycle. Suppose the regions inside the cycle are $R_1, R_2, \ldots, R_k$ with degrees $d_1, d_2, \ldots, d_k$. The sum of these k degrees is even. Each edge in any of these regions that is not an edge of C is counted twice in this summation. So the number of edges in C is even. This implies that G is bipartite and hence 2-colorable.

9.88 Show that a map is 2-colorable if and only if it is Eulerian.

Solution. Let G be a map and let G' be its geometric dual. If G is 2-colorable, G' is 2-colorable, which implies that G is bipartite. Hence, G is Eulerian, as shown in Solved Problem 8.62. Conversely, if G is Eulerian, G' is bipartite and 2-colorable. Hence, G is also 2-colorable.

9.89 (*Krol's Theorem*) A plane graph is 3-colorable if and only if it is a subgraph of a triangulation in which the degree of each vertex is even.

Solution. Let G be any 3-colorable graph with four or more vertices. Once a certain 3-coloring using colors, 1, 2, and 3 is defined on G, the nontriangular regions of G can be partitioned into three classes and new vertices and edges can be constructed to obtain a triangulation G' as follows:

(*i*) Regions with an even number of edges in their boundaries in which the vertices are colored using two colors. Construct vertex v in the interior of any such region, and join that to every vertex on the boundary. The new vertex obviously gets the third color.

(*ii*) Regions with four edges in their boundaries such that the vertices in each region are colored using three colors. In each region, there will be two vertices, say u and v, whose colors are not the same. Construct a path that lies inside the region joining these two vertices, and insert two new vertices p and q on this path such that u and p are adjacent. Assign to p the color of v, and assign to q the color of u.

(*iii*) Regions with more than four edges in their boundaries in which the vertices are colored using all three colors. Introduce a new vertex v in the interior, and then join it to all vertices in the boundary that are colored 1 or 2. Then the region gets divided into subregions that are either triangles or with boundaries consisting of four edges. In the latter case, we proceed as in (*ii*). We ultimately have a 3-colorable triangulation G' that contains G as a subgraph. Let v be any vertex of G', and suppose its color is 1. So the vertices adjacent to v are all colored either 2 or 3, and these vertices constitute a cycle. So the degree of v is even. Thus the condition is necessary.

To prove the converse, assume that G is a subgraph of a triangulation T in which the degree of each vertex is even. Thus T is Eulerian. So each region can be colored either red or blue such that no two regions sharing an edge in common have the same color. Orient the edges of T such that each triangle becomes a directed triangle; specifically, the edges of a red triangle have a clockwise orientation, and the edges of a blue region have a counterclockwise orientation. Suppose C is any cycle in the graph enclosing a finite region R consisting of r red triangles and b blue triangles. If there are x edges in C in the clockwise direction, the total number of edges in the interior of R is $3r - x$. Likewise, if there are y edges in the counterclockwise direction in C, the total number of edges in the interior is also equal to $3b - y$. Thus $x - y = 3(r - b) \equiv 0 (\text{mod } 3)$. Let v be any vertex in T, and assign color 1 to it. Let w be any vertex in T, and let P be any path connecting these vertices. Because of the orientation, some arcs in this path will be directed toward w; these are the *forward* arcs in the path. The other arcs are known as the *backward* arcs. Suppose there are p forward arcs and q backward arcs in path P. Define $c(w, P) \equiv (1 + p - q)(\text{mod } 3)$. Let P' be another path between v and w with p' forward arcs and q' backward arcs. So $c(w, P') \equiv (1 + p' - q')(\text{mod } 3)$. If these two paths have no vertex in common, their union is a cycle with $p + q'$ arcs in the clockwise direction and $p' + q$ arcs in the other direction. So $(p + q') - (p' + q) \equiv 0(\text{mod } 3)$, which implies that $(p - q) \equiv (p' - q')(\text{mod } 3)$. Hence, $c(w, P) = c(w, P')$. If paths P and P' have vertices in common, we can partition the union of these two paths into cycles and paths. This once again establishes that $C(w, P)$ is independent of the choice of P.

Thus each vertex w is assigned the unique color $C(w, P)$, which is either 0, 1, or 2, where P is any path between the fixed vertex v (with color 1) and w. It remains to be shown that no two adjacent vertices are of the same color. Let v be the same vertex as before with color 1, and let u and w be two adjacent vertices. Among all the v, u paths and v, w paths, let P be a path with as few arcs as possible, and assume without loss of generality that P is between v and w. Either there is an arc from u to w or from w to u. Let Q be the path consisting of P and the arc that is either from w to u or from u to w. In either case, color $c(u, Q)$ and color $c(w, P)$ cannot be the same.

(An easily verifiable sufficient condition for the 3-colorability of a plane graph is the result known as **Grunbaum's Theorem,** which states that a plane graph is 3-colorable if the number of triangles in G is at most three. The condition is not necessary; consider the wheel with a 4-cycle and a vertex in its interior adjacent to every other vertex.)

Perfect Graphs

9.90 Show that (*a*) a cyclic graph is perfect if and only if it has an even number of vertices, and (*b*) every bipartite graph is perfect.

Solution.

(*a*) Let C be a cyclic graph. If it is perfect, its chromatic number is 2. So its order is even. Conversely, let C be a cyclic graph of even order, and let H be any induced subgraph. If its clique number is 1, its chromatic number also is 1. If its clique number is 2, its chromatic number is also 2. So C is perfect.

(*b*) Let H be an induced subgraph of a bipartite graph G. If the clique number of H is 1, its chromatic number is also 1. If its clique number is 2, its chromatic number is also 2. So G is perfect.

9.91 The **clique covering number** $\boldsymbol{\theta(G)}$ (also known as the **partition** number) of a graph $G = (V, E)$ is the minimum number of pairwise disjoint cliques whose union is the set V. A graph G is **$\boldsymbol{\alpha}$-perfect** if for every induced subgraph H of G, $\theta(H)$ is equal to its internal stability number $\alpha(H)$. Show that a graph is perfect if and only if its complement is α-perfect.

Solution. By definition, the clique number of a graph is the internal stability number of its complement, and the chromatic number of a graph is equal to the clique covering number of its complement. The complement of the complement of G is G.

9.92 Show that the line graph of a bipartite graph G is perfect.

Solution. A clique in the line graph $L(G)$ corresponds to either a triangle in G or a set of edges having a vertex in common. Since G is bipartite, there are no triangles in G. Thus $\omega(L(G)) = \Delta(G)$. But $\Delta(G) = \chi'(G)$ (see Problem 9.52), and $\chi'(G) = \chi(L(G))$. So $\omega(L(G)) = \chi(L(G))$.

9.93 A directed graph is a **transitive digraph** if whenever there is an arc from vertex u to vertex v and an arc from v to vertex w, there is an arc from u to w. A graph G is a **transitively orientable graph** (also known as a **comparability graph**) if it is possible to orient its edges such that the resulting digraph is a transitive digraph. Show that a comparability graph is perfect.

Solution. Observe that if G is transitively orientable, any induced subgraph of G is also. Suppose P is a directed path in a transitive orientation of G with a maximum number of arcs. Let this path be from vertex v to vertex w consisting of k vertices, including u and v. The "length" of P is k. Then the k vertices in P constitute a clique in G, and no clique in G can have more than k vertices. So the clique number $\omega(G)$ is k. Next assign color $c(u)$ to vertex u, where $c(u)$ is the length of a directed path of maximum length that starts from u. Two adjacent vertices cannot have the same color; if there is an arc from vertex x to vertex y, $c(x) > c(y)$. Thus $\chi(G) \leq k = \omega(G)$. But $\omega(G) \leq \chi(G)$. So both numbers are equal to k.

9.94 Let W be a set of vertices in a connected graph $G = (V, E)$ such that the subgraph induced by W is complete and such that $G - W$ is a disconnected graph with components $G_i = (V_i, E_i)$, where $i = 1, 2, \ldots, r$. Let H_i be the subgraph induced by the union of V_i and W for each i. Show that if $\omega(H_i) = \chi(H_i)$ for each i, $\omega(G) = \chi(G)$.

Solution. Let $\omega(G) = k$. A clique in H_i is a clique in G. So $\omega(H_i) < \omega(G) = k$. If u and v are two vertices in two distinct components of $G - W$, there cannot be an edge joining them in G. So every clique in G is a clique in one of these induced subgraphs. In particular, there is an induced subgraph, say H_j, such that $\omega(H_j) = k$. Now $\chi(G) = \max\{\chi(H_i): i = 1, 2, \ldots, r\} = \max\{\omega(H_i)\} = \omega(H_j) = k = \omega(G)$.

9.95 Let W be a set of vertices in a chordal graph G such that W is a minimal separating set. Show that the graph induced by W is complete.

Solution. Suppose the subgraph H induced by W is not complete. So there are two vertices u and v in W that are not adjacent. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two distinct components of $G - W$. If u is not adjacent to any vertex in G_1, set $W - \{u\}$ is a separating set violating the minimality requirement. So there are vertices in both V_1 and V_2 adjacent to u. The same argument goes for v as well. So there is a cycle C in G consisting of an edge joining vertex p in V_1 to u, an edge joining u to vertex q in V_2, a path joining q and vertex r in V_2, the edge joining r and v, then an edge joining v to vertex s in V_1, and finally a path joining s and p. That the graph is chordal implies that there is an edge joining u and v. This shows that H is complete.

9.96 Prove that a chordal graph is perfect.

Solution. Since an induced subgraph of a chordal graph is chordal, it is enough if we prove that $\omega(G) = \chi(G)$ for a chordal graph. The proof is by induction on the order of G. If G is complete, the result is true. Assume that G is not complete. So there exists a minimal separating set W such that the subgraph H induced by W is complete, as proved in Problem 9.95. Let the components of $G - W$ be $G_i = (V_i, E_i)$, and let H_i be the subgraph induced by $V_i \cup W$ for each i. By the induction hypothesis, each H_i is perfect. So $\omega(G) = \chi(G)$, as proved in Problem 9.94.

9.97 Show that a graph is perfect if and only if every induced subgraph H has an independent set W of vertices such that $\omega(H - W) < \omega(H)$.

Solution. Suppose G is perfect and H is any induced subgraph of G. Then $\omega(H) = \chi(H) = k$. Assume that the vertices of H are colored using the colors from the set $\{1, 2, \ldots, k\}$. Let W be set of vertices in H that have color 1. Then W is an independent set in H, and the vertices in $H - W$ can be colored using $k - 1$ colors. So $\chi(H - W) = k - 1$. But $\omega(H - W) \le \chi(H - W)$. Hence, $\omega(H - W) \le k - 1 < k = \omega(H)$. So the condition is necessary. To prove that the condition is sufficient, assume that every induced subgraph H of G contains an independent set W of vertices such that $\omega(H - W) < \omega(H)$. The proof is by induction on the order of H. By the induction hypothesis, the vertices of $H - W$ can be colored using $\omega(H) - 1$ colors since W is independent. Once this is done, the vertices in W can be assigned a new color. Thus the vertices of H can be colored using $\omega(H)$ colors. Hence, $\omega(H) = \chi(H)$ for every induced subgraph H of G. So G is perfect.

9.98 (*Lovasz Replacement Theorem*) Show that if the vertices of a perfect graph are "replaced" by perfect graphs, the resulting graph is perfect.

Solution. It is enough if we consider the case when we "replace" a single vertex v of a perfect graph H by another perfect graph G, where the two graphs have no vertices in common. To do this, construct a graph H' with vertex set $V(H') = V(G) \cup V(H) - \{v\}$. Vertices x and y are adjacent in H' if they satisfy one of the following conditions: (1) x and y are adjacent in G or in H; (2) x is in G, y is in H, and v and y are adjacent; and (3) x is in H, y is in G, and v and x are adjacent. Since G and H are perfect, G has an independent set A of vertices such that $\omega(G - A) < \omega(G)$, and the vertices of H can be colored using $\omega(H)$ colors. Under a specific $\omega(H)$-coloring of H, let B be the set of vertices that have the same color as v. Here B is the **color class** that contains v. Then $D = A \cup B - \{v\}$ is an independent set in H'. Let K be any maximum cardinality clique in H'. There are two cases:

(*i*) K does not intersect G. Then K is a complete subgraph of $H - v$. Now $\omega(H - v) \le \omega(H) = \omega(H')$. So K is a maximum cardinality clique of size $\omega(H)$ in H, and it intersects every color class of every $\omega(H)$-coloring of $H - v$. In particular, K intersects the set $B - v$.

(*ii*) If K intersects G, since A is an independent set and K is a maximum clique, the sets A and K have a nonempty intersection.

So in either case, H' has an independent set D of vertices that intersects every maximum cardinality clique of H'. What is true of H' is true of any induced subgraph of H' in this regard. In view of the result established in Problem 9.97, to show that H' is perfect, it is enough to show that every induced subgraph of H' has an independent set of vertices that intersects every maximum cardinality clique in H''. That is exactly what has been done.

9.99 Show that if a graph G is perfect, it has a clique that intersects with every maximum cardinality independent set in G.

Solution. Suppose that for every clique, there is a maximum cardinality independent set such that the clique and the independent set have no vertices in common. Specifically, let B_i be the set of cliques in G, and let A_i be the corresponding maximum cardinality independent sets such that $B_i \cap A_i$ is empty for $i = 1, 2, \ldots, r$. For each vertex v in G, let $n(v)$ be the number of cliques that contain v. For each clique B_i, we define $n(B_i)$ to be the sum of all $n(v)$, where v runs through the vertices in the clique. Replace each vertex v in G by the perfect graph $K_{n(v)}$, thereby constructing the perfect graph G'. Obviously, every clique in G' is the union of complete graphs that replace the vertices of some clique in G. Consequently,

$$\omega(G') = \max\{n(B_i): 1 \le i \le r\} = \max\left\{\sum_{j=1}^{r} \left|B_i \cap A_j\right|: 1 \le i \le r\right\}$$

But A_i and B_i have no vertices in common, and the cardinality of $(A_i \cap B_j)$ is at most 1 if i and j are not equal. Hence $\omega(G') \le (r - 1)$.

Now the cardinality of each A_i is $\alpha(G)$. So the sum $|A_i| + |A_2| + \cdots + |A_r| = r\alpha(G)$. At the same time, consider the sequence S consisting of the vertices from $A_1, A_2, \ldots, A_r$ arranged one after the other. Each vertex v of G enters this sequence $n(v)$ times. So if n' is the number of vertices in G' we see that $n' = r\alpha(G)$. Now $n'/(\alpha(G')) \le \chi(G') = \omega(G')$. So $r\alpha(G) \le \omega(G')(\alpha(G'))$. But $\alpha(G) = \alpha(G')$. Hence, $r \le \omega(G')$. But $\omega(G') \le (r - 1)$. This contradiction establishes that there is a clique that intersects with every maximum cardinality independent set.

9.100 (*The Perfect Graph Theorem of Lovasz*) Show that the complement of a perfect graph is perfect.

Solution. Let G be a perfect graph, and let H be a nontrivial induced subgraph of the complementary graph G'. Then H' (the complement of H) is an induced subgraph of G. By Problem 9.99, H' has a clique B that intersects every maximum cardinality independent set of H'. Set B is a maximum cardinality independent set in H, and B meets every maximum cardinality clique. So H is perfect, as shown in Problem 9.97.

9.101 Show that a graph is perfect if and only if it is α-perfect.

Solution. This is an immediate consequence of the perfect graph theorem and the definition of α-perfect graphs.

Turan's Theorem

9.102 (*Erdös's Theorem*) If the graph $G = (V, E)$ does not contain K_{k+1} as a subgraph, show that there exists a k-chromatic graph $H = (V, F)$ such that $\deg_H(v) \ge \deg_G(v)$ for every v in V. (Graph G is said to be **degree-majored** by graph H.)

Solution. The proof is by induction on k. The result is true if $k = 1$ by letting $G = H$. In this case, the degree (in G and in H) of each vertex is 0. Suppose the theorem is true for all graphs that do not contain K_{k+1} as a subgraph. Consider a graph $G = (V, E)$ that does not contain K_{k+2} as a subgraph. Let v' be a fixed vertex of maximum degree in G, and let W be the set of vertices adjacent to v' in G. Subgraph $G' = (W, F)$ induced by W obviously does not contain K_{k+1} as a subgraph, so by the induction hypothesis, there exists a k-chromatic graph $H' = (W, F')$ such that $\deg_{H'}(v) \ge \deg_{G'}(v)$ for every v in W. Now construct a graph $H = (V, E'')$ as follows. (1) if both u and v are in W and if they are adjacent in H', they are adjacent in H. (2) In H, every vertex in W is adjacent to every vertex in $V - W$. Since H' is k-chromatic, H is $(k + 1)$-chromatic. If v is in W, $\deg_H(v) = |V| - |W| + \deg_{H'}(v) \ge |V| - |W| + \deg_{G'}(v) \ge \deg_G(v)$. Otherwise, $\deg_H(v) = |W| = \deg_G(v') \ge \deg_G(v)$.

Thus there exists a $(k + 1)$-chromatic graph $H = (V, E)$ such that $\deg_H(v) \ge \deg_G(v)$ for every v in V. The result holds for $k + 1$ also.

9.103 (*Turan Number and Turan Graph*) Find a nondecreasing sequence of k positive integers $a_i (i = 1, 2, \ldots, k)$ whose sum is n such that $\Sigma_{1 \le i \le j < k}\, a_i a_j$ is maximum.

Solution. What is true if $k = 2$ holds in the more general case? The desired quantity is a maximum when the numbers are "nearly equal." Thus $|a_i - a_j| \le 1$ for every i and j. Specifically, if k divides n, each a_i is equal to (n/k). Otherwise, let $n = qk + r$. In this case, $a_1 = a_2 = \cdots = a_{k-r} = q$, and the remaining numbers all equal

$q + 1$. For a fixed n and k, with $n \geq k \geq 2$, the number $\Sigma_{1 \leq i \leq k} a_i a_j$ with this choice of $a_i (i = 1, 2, \ldots, k)$ is called the **Turan number $t(n, k)$.** The complete k-partite graph of order n whose vertex set is partitioned into k sets V_i of cardinality $a_i (i = 1, 2, \ldots, k)$ is known as the **Turan graph $T(n, k)$.**

9.104 (*Turan's Theorem*) Show that if a graph $G = (V, E)$ of order n does not contain K_{k+1} as a subgraph. $|E| \leq t(n, k)$.

Solution. According to Problem 9.102, there exists a k-chromatic graph $H = (V, F)$ that degree-majorizes G. Since H is k-chromatic, set V can be partitioned into k nonempty sets V_i each of cardinality $a_i (i = 1, 2, \ldots, k)$ such that $a_1 + a_2 + \cdots + a_k = n$, where we assume that numbers a_i are in nondecreasing order without loss of generality. The maximum number of edges in the k-partite graph H is therefore $t(n, k)$. Hence, $|E| \leq |F| \leq t(n, k)$.

9.105 Find the maximum number of edges in (*a*) a 4-chromatic graph of order 20 and (*b*) a 6-chromatic graph of order 20.

Solution.

(*a*) Here $n = 20$ and $k = 4$. The graph does not contain K_5 as a subgraph. Each part in the 4-partite set contains five vertices. Any two of four can be chosen in six ways. Between any two parts are 25 edges. So the Turan number is 150. The number of edges is at most 150.

(*b*) The graph does not contain K_7 as subgraph. There are six partite sets consisting of 3, 3, 3, 3, 4, and 4 vertices, respectively. The Turan number is $(6)(9) + (8)(12) + 16 = 166$. The number of edges is at most 166.

Supplementary Problems

9.106 Find the chromatic number of a cubic nonbipartite graph.
Ans. If the graph is complete, the chromatic number is 4. Otherwise, it is 3.

9.107 Show that if G is a k-critical graph of order n and size m, $(k - 1)(n) \leq 2m$. [*Hint:* The degree of each vertex is at least $(k - 1)$, and the sum of the degrees is $2m$.]

9.108 Show that if a k-chromatic graph G is uniquely colorable, the subgraph induced by the union of any two or more subsets of a vertex partition defined by a k-coloring is $(k - 1)$-connected. [*Hint:* Such a subgraph is also uniquely colorable. Use Problem 9.31.]

9.109 Show that $\chi(G) \leq 1 + n - \alpha(G)$.
Ans. Assign the same color to each vertex in a maximum independent set. The number of uncolored (out of n vertices) vertices is $n - \alpha(G)$.

9.110 Find the chromatic polynomial of $K_{1,n}$. *Ans.* $x(x - 1)^n$

9.111 If G is the connected graph obtained by linking a triangle and a cyclic graph of order 4 so that they share one vertex in common, find the chromatic polynomial of G.
Ans. $(x - 1)(x - 2)(x^4 - 4x^3 + 6x^2 - 3x)$

9.112 If G is a connected graph of order n, prove that $P(G, x) \leq x(x - 1)^{n-1}$. [*Hint:* Any coloring of G is a coloring of any spanning tree in G.]

9.113 Show that Vizing's theorem need not be true if the graph under consideration is not simple. [*Hint:* Consider the multigraph G consisting of three vertices such that joining each pair of vertices are p edges, where $p > 1$.]

9.114 Show that the four-color theorem is true if and only if the following condition holds: Every planar graph has an orientation such that the flow ratio of any cycle in the orientation is at most 3. [*Hint:* Use Minty's theorem (Problem 9.81).]

9.115 Show that Hadwiger's conjecture is true for $k = 5$. [*Hint:* Use the four-color theorem and Problem 9.85.]

9.116 Show that any outerplanar graph is 3-colorable. [*Hint:* Use the definition of outerplanarity.]

9.117 Show that a triangulation is 3-colorable if and only if the degree of each vertex in it is even. [*Hint:* Use Problem 9.89.]

9.118 The complement of a comparability graph is known as an **incomparability graph.** Show that an incomparability graph is both perfect and α-perfect. [*Hint:* Use Problem 9.96.]

9.119 Show that the Petersen graph is not perfect. [*Hint:* It has an odd hole with five vertices.]

9.120 Show that an interval graph is perfect. [*Hint:* The complement of an interval graph G is a comparability graph (see Solved Problem 2.35.]

9.121 Find the size of the largest (*a*) 7-chromatic graph of order 21 and (*b*) 7-chromatic graph order 22.
Ans. (*a*) $49 + 49 + 49 = 147$; (*b*) $49 + 56 + 56 = 161$

Important Symbols

Symbol	Meaning	Page
$\alpha(G)$	Independence number (internal stability number)	13
$\alpha_1(G)$	Edge-independence number	14
$\beta(G)$	Vertex-covering number	13
$\beta_1(G)$	Edge-covering number	14
$\chi(G)$	Chromatic number	39, 244
$\chi'(G)$	Chromatic index	247
$\Delta(G)$	Maximum degree	4
$\delta(G)$	Minimum degree	4
$\delta(P)$	Flow capacity of the semipath P	133
$\kappa(G)$	Connectivity number	30
$\lambda(G)$	Edge-connectivity number	30
$\nu(G)$	Crossing number	225
$\theta(G)$	Thickness	227
$\theta'(G)$	Clique covering number (partition number)	281
$\sigma(G)$	Vertex domination number (external stability number)	13
$\omega(G)$	Clique number	245
(S, T)	Cut (source-sink cut)	131
$C(G)$	Center	31, 128
$c(G)$	Closure	80
$c(S, T)$	Capacity of the cut (S, T)	131
C_n	Cyclic graph	3
$d(G)$	Diameter	128
$e(v)$ or $e(i)$	Eccentricity of a vertex	31, 124
$f(G)$	Value of the feasible flow f	131
$f(S, T)$	Flow along the cut (S, T)	131
$\overline{G}$	Complement	9
G'	Geometric dual	201
G^*	Abstract dual	202
$G.e$	Contraction	52
$K_{m,n}$	Complete bipartite graph	1
K_n	Complete graph	1
$L(G)$	Line graph	44
$P(G, x)$	Chromatic polynomial	246
Q_k	Hypercube (k-cube)	19
$r(A)$	Rank of A in a matroid	104
$r(G)$	Radius	31, 128
$R(p, q)$	Ramsey number	19
$T(n, k)$	Turan graph	284
$t(n, k)$	Turan number	284

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