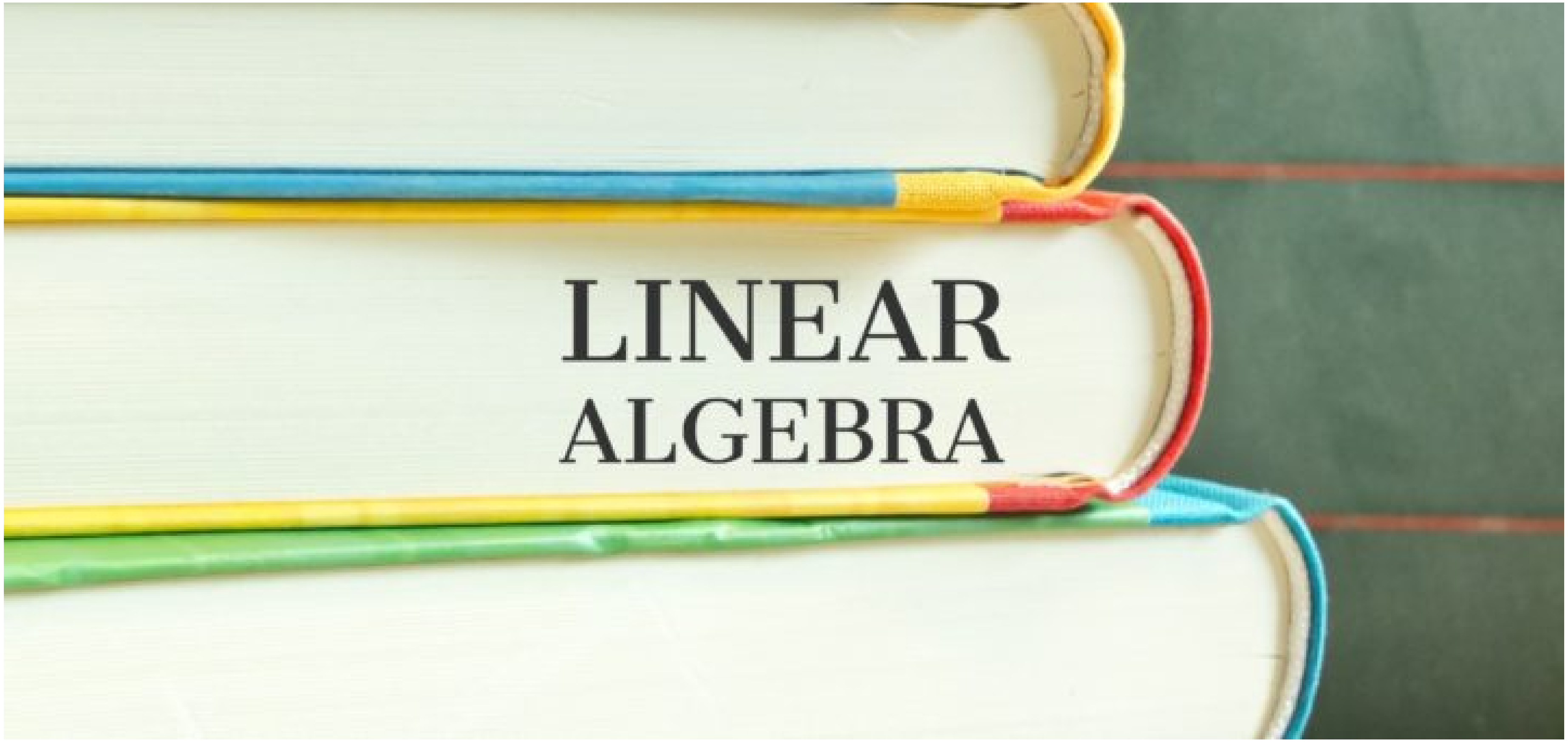
Eigenvalues of a Stochastic Matrix is Always Less than or Equal to 1

BY YU · PUBLISHED 11/17/2016 · UPDATED 07/06/2017



Problem 185

Let $A = (a_{ij})$ be an $n \times n$ matrix.

We say that $A = (a_{ij})$ is a **right stochastic matrix** if each entry a_{ij} is nonnegative and the sum of the entries of each row is 1. That is, we have

$$a_{ij} \geq 0 \quad \text{and} \quad a_{i1} + a_{i2} + \cdots + a_{in} = 1$$

for $1 \leq i, j \leq n$.

Let $A = (a_{ij})$ be an $n \times n$ right stochastic matrix. Then show the following statements.

(a) The stochastic matrix A has an eigenvalue 1.

(b) The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.

♡ Add to solve later

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Proof.

(a) The stochastic matrix A has an eigenvalue 1.

We compute that

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} \cdots + a_{nn} \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Here the second equality follows from the definition of a right stochastic matrix.

(Each row sums up to 1.)

This computation shows that 1 is an eigenvector of A and $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

(b) The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.

Let λ be an eigenvalue of the stochastic matrix A and let $\mathbf{v}$ be a corresponding eigenvector.

That is, we have

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Comparing the i-th row of the both sides, we obtain

$$a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n = \lambda v_i \tag{*}$$

for $i = 1, \ldots, n$.

Let

$$|v_k| = \max\{|v_1|, |v_2|, \ldots, |v_n|\},$$

namely v_k is the entry of $\mathbf{v}$ that has the maximal absolute value.

Note that $|v_k| > 0$ since otherwise we have $\mathbf{v} = \mathbf{0}$ and this contradicts that an eigenvector is a nonzero vector.

Then from (*) with $i = k$, we have

$$\begin{aligned} |\lambda| \cdot |v_k| &= |a_{k1}v_1 + a_{k2}v_2 + \cdots + a_{kn}v_n| \\ &\leq a_{k1}|v_1| + a_2|v_2| + \cdots + a_{kn}|v_n| && \text{(by the triangle inequality and } a_{ij} \geq 0\text{)} \\ &\leq a_{k1}|v_k| + a_2|v_k| + \cdots + a_{kn}|v_k| && \text{(since } |v_k| \text{ is maximal)} \\ &= (a_{k1} + a_{k2} + \cdots + a_{kn})|v_k| = |v_k|. \end{aligned}$$

Since $|v_k| > 0$, it follows that

$$\lambda \leq 1$$

as required.

Remark.

A stochastic matrix is also called probability matrix, transition matrix, substitution matrix, or Markov matrix.

More from my site

Stochastic Matrix (Markov Matrix) and its Eigenvalues and Eigenvectors (a) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a matrix such that $a_{11} + a_{12} = 1$ and $a_{21} + a_{22} = 1$. Namely, the sum of the entries in each row is 1. (Such a matrix is called (right) stochastic matrix (also termed [...]

Find the Limit of a Matrix Let

$$A =$$

be 3×3 matrix. Find

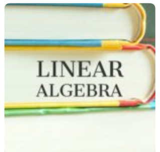
.

(Nagoya University Linear [...]

Basis with Respect to Which the Matrix for Linear Transformation is Diagonal Let be the vector space of all real polynomials of degree 1 or less. Consider the linear transformation $T:$ $\to$ defined by

$$T(ax+b)=(3a+b)x+a+3,$$

for any $ax+b \in$. (a) With respect to the basis $B=\{1, x\}$, find the matrix of the linear transformation [...]



If Every Vector is Eigenvector, then Matrix is a Multiple of Identity Matrix Let A be an $n \times n$ matrix. Assume that every vector $\mathbf{x}$ in is an eigenvector for some eigenvalue of A. Prove that there exists $\lambda \in \mathbb{R}$ such that $A=\lambda I$, where I is the $n \times n$ identity matrix.   Proof. Let us write [...]

Given All Eigenvalues and Eigenspaces, Compute a Matrix Product Let C be a 4×4 matrix with all eigenvalues $\lambda=2, -1$ and eigensapces \[E_2=\Span\left \{\quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad\right \} \text{ and } E_{-1}=\Span\left \{ \quad\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 [...]

Eigenvalues of a Hermitian Matrix are Real Numbers Show that eigenvalues of a Hermitian matrix A are real numbers. (The Ohio State University Linear Algebra Exam Problem) We give two proofs. These two proofs are essentially the same. The second proof is a bit simpler and concise compared to the first one. [...]

[Determine Eigenvalues, Eigenvectors, Diagonalizable From a Partial Information of a Matrix] Suppose the following information is known about a 3×3 matrix A. \[A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}=6\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ -1 \\ 1 [...]

[Given Eigenvectors and Eigenvalues, Compute a Matrix Product (Stanford University Exam)] Suppose that $\begin{bmatrix} \\ \end{bmatrix}$ is an eigenvector of a matrix A corresponding to the eigenvalue 3 and that $\begin{bmatrix} \\ \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2. Compute $A^2\begin{bmatrix} 4 [...]

Tags: eigenvalue eigenvector linear algebra Markov matrix matrix probability matrix stochastic matrix substitution matrix transition matrix triangle inequality vector

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