Problem Set 4

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CSE 317 Design and Analysis of Algorithms
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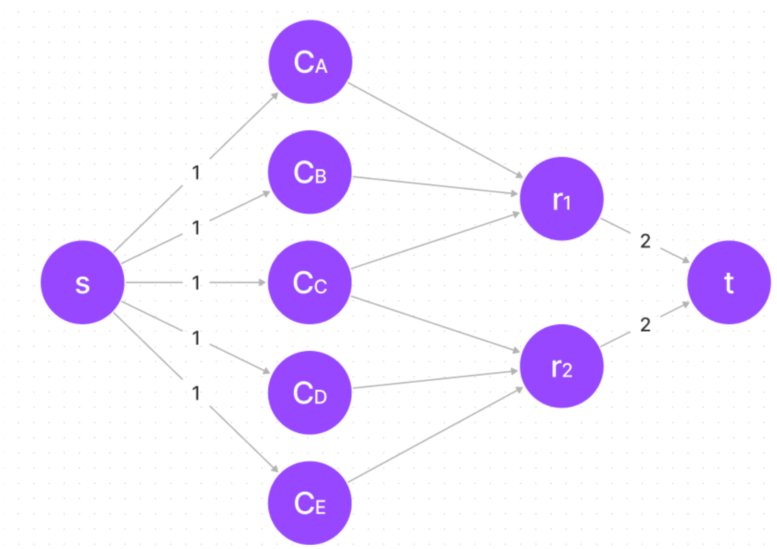
1 So Many Courses, So Little Time

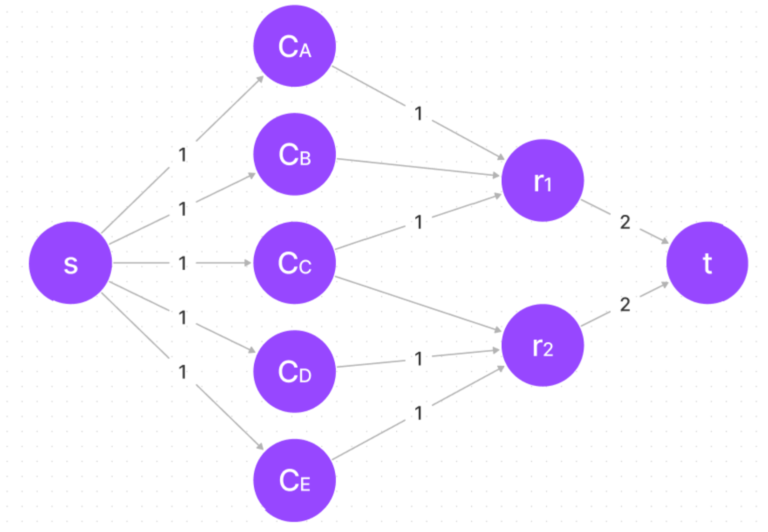
To reduce the Course Requirement problem to the Network Flow problem, we will create a network flow graph with a source node s and a sink node t and represent each course C and requirement r as individual nodes.

- Each course node Ci is connected to the source node s with an edge of capacity 1.
- Each requirement node r_i is connected to the sink node t with an edge of capacity k_i; the number of courses needed to fulfil this requirement.
- Each r_i is connected to each of the respective Ci in the subset Si are connected with an edge of capacity 1, this is to ensure the rule that any given course cannot be used towards satisfying multiple requirements.



To find the maximum flow from s to t, apply the Edmonds-Karp algorithm to this network flow graph representation for computing the maximum flow. This runs in $O(VE^2)$ time, which is polynomial.

If the maximum flow value equals the number of courses that can be taken (i.e., the sum of all k_i), then the student can graduate.



When the Edmonds-Karp is applied onto this problem, we see that the max flow is 4, which is equivalent to the sum of k_i in each r_i.

This reduction fulfills all the requirements of the Course Requirement problem because, as given in the question, if a student has taken courses [B, C, D], they would not be able to graduate because the graph is not fully saturated yet.

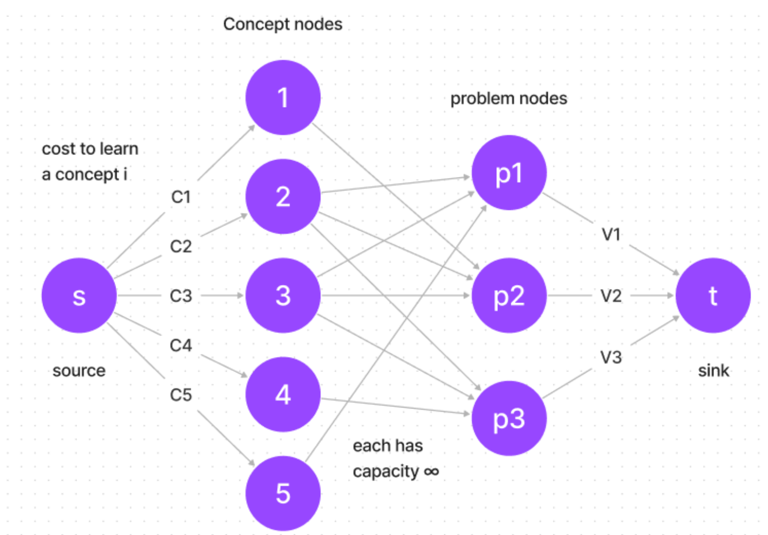
Conversely, a student taking [B, C, D] will not fulfill the requirements since Cc will only have a capacity of 1, so only the edge from C_c to r_1 or the edge from C_c to r_2 will be saturated.

This gives a polynomial-time reduction of the Course Requiremment problem to the Network Flow problem.

2 Algorithms For Zombies

To solve this problem in polynomial time using an s-t-min-cut algorithm, we will create a network flow graph with a source node s and a sink node t and represent each concept i and problem p_j as individual nodes.

- Each concept node i is connected to the source node s with an edge of capacity equal to the cost of learning that concept.
- Each problem node p_j is connected to the sink node t with an edge of capacity V_j i.e. the value of the corresponding problem (equal to the utility of solving p_j.).
- Each concept node i is connected to each problem node p_j with an edge with infinite capacity since learning a concept allows a problem to be solved. Once a concept is learned, there shouldn't be anymore additional flow on that edge.



To find a subset $R \subseteq \{1, \dots, n\}$ of the concepts to understand, to maximize the net utility (values of problems solved - cost of learning the concepts needed), we need to calculate the minimum s-t cut in the network we have created.
We will run Edmonds-Karp on this to find the max flow value which is equivalent to the min-cut. The min-cut obtained divides the vertices into subsets

of those that are reachable from s and those that are reachable to t. The edges crossing the cut tell us which concepts should be learned and which problems will contribute to the utility.

If a problem p_j is reachable from s, it means that problem will not contribute to the utility (losing value). If a concept i is reachable to t it means the concept will not be learned (reducing cost).

The solution provides which concepts to learn (those connected directly to s across the cut) and which problems will be solved (those connected directly to t across the cut), maximizing the net utility.

The max-flow min-cut theorem ensures that the minimum cut corresponds to the optimal selection of concepts to maximize our net utility. Since Edmonds-Karp runs in polynomial time, we can solve this in polynomial time as well.

3 Dual Nature

3.1

$$\mathbf{max}\ (x_1 + 3x_2 - 2x_3)$$
$$\textbf{such that:}$$
$$x_1 + x_2 + 2x_3 \leq 2$$
$$\mathbf{7x}_1 + 2x_2 + 5x_3 \leq 6$$
$$\mathbf{2x}_1 + x_2 - x_3 \leq 1$$
$$\mathbf{x}_1, x_2, x_3 \geq 0$$

Adding variables on both sides of the equations

$$y_1(x_1 + x_2 + 2x_3) \leq 2y_1$$
$$\mathrm{y}_2(7x_1 + 2x_2 + 5x_3) \leq 6y_2$$
$$\mathrm{y}_3(2x_1 + x_2 - x_3) \leq 1y_3$$

Combine all equations

$$(y_1 + 7y_2 + 2y_3)x_1 + (y_1 + 2y_2 + y_3)x_2 + (2y_1 + 5y_2 - y_3)x_3 \leq 2y_1 + 6y_2 + y_3$$

New objective function, variables and constraints for the dual:

$$\min(2y_1 + 6y_2 + y_3)$$
$$(y_1 + 7y_2 + 2y_3) \geq 1$$
$$(\mathrm{y}_1 + 2y_2 + y_3) \geq 3$$
$$(2\mathrm{y}_1 + 5y_2 - y_3) \leq -2$$
$$\mathrm{y}_1, y_2, y_3 \geq 0$$

3.2

$$\mathbf{max}(x_1 - 3x_2 + 2x_3)$$
$$\textbf{such that:}$$
$$3x_1 + 2x_3 \geq 2$$
$$\mathbf{2x}_2 - x_3 \leq 5$$
$$\mathbf{x}_1, x_2, x_3 \geq 0$$

Rewrite in the less than equal to form

$$-3x_1 - 2x_3 \leq -2$$

Adding variables on both sides

$$y_1(-3x_1 - 2x_3) \leq -2y_1$$
$$\mathrm{y}_2(2x_2 - x_3) \leq 5y_2$$

Adding all equations

$$(-3y_1)x_1 + (2y_2)x_2 + (-2y_1 - y_2)x_3 \leq 5y_2 - 2y_1$$

New objective function, variables and constraints for the dual:

$$\min(5y_2 - 2y_1)$$
$$(-3y_1) \geq 1$$
$$(2\mathrm{y}_2) \leq -3$$
$$(\text{-}2\mathrm{y}_1 - y_2) \geq 2$$
$$\mathrm{y}_1, y_2 \geq 0$$

4 Enumerate This

To find the vertices of the convex region or polyhederon represented by the constraints:

$$\begin{aligned}
p_{ab|xy} &\geq 0 \quad \textit{Positivity} \\
\sum_{a,b} p_{ab|xy} &= 1 \quad \textit{Normalisation} \\
\sum_{a} p_{ab|xy} &= \sum_{a} p_{ab|x'y} \text{ for all } b, x, x', y X - Consistent \\
\sum_{b} p_{ab|xy} &= \sum_{b} p_{ab|xy'} \text{ for all } a, x, y, y' Y - Consistent
\end{aligned}$$

Using these constraints, we can construct inequalities and qualities that bound that region, when we enumerate values of a, b,x,y. There are 16 variables that we have to consider when creating these qualities and inequalities since each of a, b, x, y can have the value either 0 or 1.

We will come up with our equalities and inequalities using the given constraints:
Positivity: We'll write for every possible value of a, b, x, y where each variable can be either 0 or 1, so we will have $2^4 = 16$ inequalities.
Normalization: Only x and y will be changing will have $2^2 = 4$ inequalities.
X Consistent: Only b, x, y will be changing will have $2^3 - 8$ inequalities.
Y Consistent: Only a, x, y will be changing will have $2^3 = 8$ inequalities.

We obtain a total of $16+4+8+8 = 36$ inequalities, which we will convert into two cdd compatible matrices. One of equalities and one of inequalities. Using cdd we can convert the inequality and equality matrices and obtain the vertices of this convex region.
Using our cdd output, we can identify the total number of vertices of the region which is 24 since it has 24 rows. Each vertex (row) has 17 digits. The first digit (1) tells us that that row is a vertex and the rest of the 16 digits tell us the conditional probabilities $p_{ab|xy}$ in the following ordering:

$p_{00|00}\, p_{01|00}\, p_{10|00}\, p_{11|00}\, p_{00|01}\, p_{01|01}\, p_{10|01}\, p_{11|01}\, p_{00|10}\, p_{01|10}\, p_{10|10}\, p_{11|10}\, p_{00|11}\, p_{01|11}\, p_{10|11}\, p_{11|11}$

cdd Output

V-representation
begin
24 17 rational
1 1 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0
1 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1
1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
1 1 0 0 0 1 0 0 0 0 0 1 0 0 0 1 0
1 1/2 0 0 1/2 1/2 0 0 1/2 0 1/2 1/2 0 1/2 0 0 1/2
1 1/2 0 0 1/2 1/2 0 0 1/2 1/2 0 0 1/2 0 1/2 1/2 0
1 1/2 0 0 1/2 0 1/2 1/2 0 0 1/2 1/2 0 0 1/2 1/2 0
1 1/2 0 0 1/2 0 1/2 1/2 0 1/2 0 0 1/2 1/2 0 0 1/2
1 0 1/2 1/2 0 1/2 0 0 1/2 0 1/2 1/2 0 0 1/2 1/2 0
1 0 1/2 1/2 0 1/2 0 0 1/2 1/2 0 0 1/2 1/2 0 0 1/2
1 0 1/2 1/2 0 0 1/2 1/2 0 0 1/2 1/2 0 1/2 0 0 1/2
1 0 1/2 1/2 0 0 1/2 1/2 0 1/2 0 0 1/2 0 1/2 1/2 0
1 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
1 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 1
1 0 1 0 0 1 0 0 0 0 1 0 0 1 0 0 0
1 0 1 0 0 1 0 0 0 0 0 0 1 0 0 1 0
1 0 0 1 0 0 0 0 1 0 0 1 0 0 0 0 1
1 0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0
1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
1 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0
1 0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0
1 0 0 0 1 0 0 1 0 0 0 0 1 0 0 1 0
1 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
end

5 Collaborators

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