How to create earthquake cycles on your computer

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1 Introduction

To generate earthquake cycles on a fault we need a few key ingredients. A description of elastic interactions between various sections of the fault, the variation of friction at different stages of the earthquake cycle and a set of mechanical constraints which need to be satisfied at all times.

Here I describe the simplest kind of spontaneous earthquake cycles I know to simulate: rate-and-state friction earthquake cycles in a quasi-dynamic framework. This means that I use rate-state friction to describe the variation of friction with velocity and time while the elastic interactions neglect the dynamic effects of seismic waves. I estimate the energy lost from a slipping section of a fault in an earthquake using the radiation damping approximation to account for inertial effects, which is what makes it quasi-dynamic. Earthquakes, fault creep and slow slip events emerge naturally in this framework which is what makes it very exciting.

2 Earthquake Cycles with the Ageing Law

Rate-and-state friction models on-fault friction (μ) using the a parameter to describe the velocity/direct effect (v) while the parameter b describes the evolution effect (θ). It is generally written in the following form:

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$$\mu = \mu_0 + a \log \frac{v}{v_0} + b \log \frac{v_0 \theta}{L} \tag{1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = f(v, \theta, L) \tag{2}$$

There are two popularly used functional forms to model the evolution of frictional asperities on the fault θ.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 - \frac{v\theta}{L} \qquad \text{Ageing Law} \tag{3a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{v\theta}{L} \log \frac{v\theta}{L} \qquad \text{Slip Law} \tag{3b}$$

Here I will discuss the use of the ageing law formulation to model earthquake cycles and in the following section I will use the slip law. It is not clear which law is a more accurate description of the underlying mechanics, but they seem to produce similar results when we use them in a simple earthquake cycles.

2.1 Forces, Momentum and Acceleration

There are a few quantities known to us. These are: $a, b, v_0, L, \bar{\sigma}, v_s$ which describe the properties and confining stresses on the fault. We use these quantities to develop a set of equations which need to be solved to describe the stress state on the fault.

First consider the force balance on the fault:

$$\underbrace{\tau^{\infty}}_{\text{loading stress}} + \overbrace{\tau}^{\text{elastic interactions}} + \underbrace{\mu\bar{\sigma}}_{\text{frictional stress}} + \frac{G}{2v_s}v = 0 \tag{4}$$

The last term is the radiation damping approximation to dynamic stresses, which is significant only as fault slip velocities approach earthquake velocities. At most other times, this term contributes minimally and the fault is in a quasi-static state.

We use the following variable transformations to ease numerical computation since at very low slip velocities we might encounter floating point errors on the computer.

$$\phi = \log \frac{v_0 \theta}{L}; \qquad \theta = \frac{L}{v} \exp \phi \tag{5a}$$

$$\zeta = \log \frac{v}{v_0}; \qquad v = v_0 \exp \zeta \tag{5b}$$

The force balance is now,

$$-\left(\tau^{\infty} + \tau\right) = \left(\mu_0 + a\zeta + b\phi\right) \bar{\sigma} + \frac{G}{2v_s} v_0 \exp \zeta \tag{6}$$

The time derivative of Equation 6 gives us the momentum balance. We now assume that the elastic tractions and their time-derivative can be described by a linear traction kernel K. This is defensible in the quasi-static limit because K is computed as an analytic solution to the equation of motion in the infinitesimal limit. If we choose a backslip model for interseismic loading, there is another simplification we can use, $\tau^{\infty} = -K(v^{\infty} t)$. This allows us to combine the loading and elastic interaction term as $K\left((v^{\infty} - v)t\right)$.

$$-\left(\dot{\tau}^{\infty} + K v_0 \exp \zeta\right) = \left(a \frac{\mathrm{d}\zeta}{\mathrm{d}t} + b \frac{\mathrm{d}\phi}{\mathrm{d}t}\right) \bar{\sigma} + \frac{G}{2v_s} v_0 \exp \zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} \tag{7}$$

To solve the momentum balance we need to know ζ, $\frac{\mathrm{d}\zeta}{\mathrm{d}t}$, ϕ, $\frac{\mathrm{d}\phi}{\mathrm{d}t}$.

2.1.1 $\frac{\mathrm{d}\phi}{\mathrm{d}t}$

Using the rules of variable transformations we get the following relations for ϕ and its time derivative.

$$\mathrm{d}\phi = \frac{\mathrm{d}\theta}{\theta} \tag{8a}$$

$$\mathrm{d}\theta = \frac{L}{v_0} \exp{(\phi)} \mathrm{d}\phi \tag{8b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{v_0} \exp{(\phi)} \frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{8c}$$

Comparing these equations with the ageing law we get

$$\frac{L}{v_0} \exp{(\phi)} \frac{\mathrm{d}\phi}{\mathrm{d}t} = 1 - \frac{v}{v_0} \exp{(\phi)} \tag{9}$$

Rearranging this gives us the rate of change of the transformed state variable with time.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v_0}{L} \exp(-\phi) - \frac{v}{L} \tag{10}$$

2.1.2 $\frac{\mathrm{d}\zeta}{\mathrm{d}t}$

Now we carry out a similar analysis with ζ in Equation 5.

$$\mathrm{d}v = v\mathrm{d}\zeta \tag{11a}$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = v_0 \exp(\zeta) \frac{\mathrm{d}\zeta}{\mathrm{d}t} \tag{11b}$$

Substituting these relations in Equation 7 give us,

$$-\left(\dot{\tau}^{\infty} + K v_0 \exp\zeta\right) = \left(a\bar{\sigma} + \frac{G}{2v_s} v_0 \exp\zeta\right) \frac{\mathrm{d}\zeta}{\mathrm{d}t} + b\bar{\sigma}\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{12}$$

Rearranging this gives us the rate of change of the transformed velocity with time i.e. the instantaneous fault acceleration.

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{-\left(\dot{\tau}^{\infty} + K v_0 \exp\zeta\right) - b\bar{\sigma}\frac{\mathrm{d}\phi}{\mathrm{d}t}}{a\bar{\sigma} + \frac{G}{2v_s} v_0 \exp\zeta} \tag{13}$$

3 Earthquake Cycles with the Slip Law

Following the details of the previous section I will now use the slip law to derive the equations that must be solved to simulate earthquake cycles.

3.1 Forces, Momentum and Acceleration

In this section too we are trying to solve the momentum equation in Equation 7. We use the same variable transformations for v, θ as in the previous case (Equation 5). However, because the slip law has a different functional form compared to the ageing law, the corresponding transformed variables also differ.

3.1.1 $\frac{\mathrm{d}\phi}{\mathrm{d}t}$

We use the chain rule to obtain the derivative of the transformed variable as follows.

$$\mathrm{d}\phi = \frac{\mathrm{d}\theta}{\theta} \tag{14a}$$

$$\mathrm{d}\theta = \frac{L}{v_0}\exp{(\phi)}\mathrm{d}\phi \tag{14b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{v_0}\exp{(\phi)}\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{14c}$$

Comparing these equations with the slip law we get

$$\begin{aligned}\frac{L}{v_0}\exp{(\phi)}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= -\frac{v}{v_0}\exp{(\phi)}\log\left(\frac{v}{v_0}\exp{(\phi)}\right)\\ &= -\frac{v}{v_0}\exp{(\phi)}\left(\log\frac{v}{v_0} + \phi\right)\end{aligned} \tag{15}$$

Rearranging this and using Equation 5 gives us the rate of change of the transformed state variable with time.

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= -\frac{v}{L}\log\frac{v}{v_0} - \frac{v}{L}\phi\\ \frac{\mathrm{d}\phi}{\mathrm{d}t} &= -\frac{v_0}{L}\left(\zeta\exp{(\zeta)} + \phi\exp{(\zeta)}\right)\end{aligned} \tag{16}$$

3.1.2 $\frac{\mathrm{d}\zeta}{\mathrm{d}t}$

Since $\frac{\mathrm{d}\zeta}{\mathrm{d}t}$ is derived from the momentum balance in Equation 7, it is identical to the ageing-law case (Equation 13).

4 Solution

Now we have all the ingredients we need to solve the problem. We have 2 unknowns ζ, ϕ and two coupled equations involving their time derivatives

$\frac{\mathrm{d}\zeta}{\mathrm{d}t}(\zeta,\phi)$, $\frac{\mathrm{d}\phi}{\mathrm{d}t}(\zeta,\phi)$.

$$\begin{bmatrix} \dfrac{\mathrm{d}\zeta}{\mathrm{d}t} \\ \\ \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} \dfrac{-\left(\dot{\tau}^{\infty} + K v_0 \exp\zeta\right) - b\bar{\sigma}\dfrac{\mathrm{d}\phi}{\mathrm{d}t}}{a\bar{\sigma} + \frac{G}{2v_s} v_0 \exp\zeta} \\ \\ \frac{v_0}{L}\left(\exp\left(-\phi\right) - \exp\left(\zeta\right)\right) \quad - \text{ Ageing-Law} \\ \text{or} \\ -\frac{v_0}{L}\left(\zeta\exp\left(\zeta\right) + \phi\exp\left(\zeta\right)\right) \quad - \text{ Slip-Law} \end{bmatrix} \tag{17}$$

This is a typical initial value problem (Ordinary Differential Equation) that can be solved with MATLAB's ode45 numerical solver.

For a given initial value $\begin{bmatrix} \zeta_0 \\ \phi_0 \end{bmatrix}$ at $t = t_0$, we can numerically integrate the instantaneous time derivatives to obtain the time series for $\zeta(t), \phi(t)$.

Quantities we generally want for visualization and discussion are s, v, τ (slip,velocity,stress). While velocity is readily available as $v = v_0 \exp\zeta$, the other two quantities (s, τ) need to solved for by integrating $v, \dot{\tau}$. The stressing rate is calculated from the momentum balance in Equation 7.

$$\dot{\tau} = -\left(\dot{\tau}^{\infty} + K\left(v_0 \exp\zeta\right)\right) - \frac{G}{2v_s} v_0 \exp\zeta \tag{18}$$