Governing Equations for elastic bodies

unknown

- → 3 displacements u_i
- → 6 strains ε_{ij}
- → 6 stresses σ_{ij}

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- → density (important when $\underline{a} \neq 0$) ρ

Known

- → 6 displacement-strain relations, $[\varepsilon] = \dfrac{(\nabla u + \nabla u^T)}{2}$
- → 3 equilibrium equations, $\nabla \cdot \sigma + \underline{f} = \rho \underline{a}$
- → 1 continuity equation $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$

<u>10 equations</u>

→ we need constitutive relationships

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{ij}, \underline{\alpha})$$

material/rheological parameters

<u>For elastic bodies,</u>

use $\boxed{\text{Hooke's law}}$ i.e $\sigma_{ij} \propto \varepsilon_{ij}$

<u>linear</u>

$$\sigma_{ij} = \underbrace{C_{ijkl}}_{81 \text{ parameters for generalized Hooke's law}} \varepsilon_{kl}$$

81 parameters for generalized Hooke's law

→ in a homogenous & isotropic medium,

$$C_{ijkl} \longrightarrow \underbrace{[\lambda, G]}_{\text{Lamé's parameters}}$$

Hooke's law : $\boxed{\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}}$ 6 equations

$$\varepsilon_{kk} = (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \qquad \boxed{[\sigma] = \lambda \varepsilon_{kk}[I] + 2G[\varepsilon]}$$

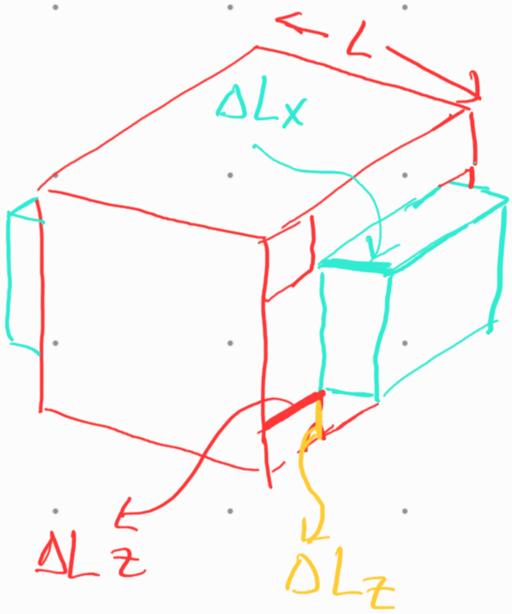
→ λ, G can be measured directly in a solid

→ some related material properties [commonly used]

1. $K = \lambda + \frac{2}{3}G = -V\frac{dP}{dV}$ bulk modulus

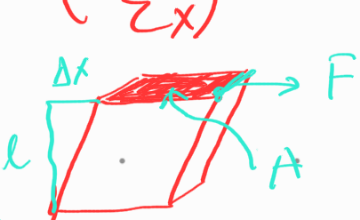
2. $\nu = \dfrac{\lambda}{2(\lambda + G)}$ → Poisson's Ratio

$$= -\left(\frac{\varepsilon_y}{\varepsilon_x}\right) = -\left(\frac{\varepsilon_z}{\varepsilon_x}\right)$$



3. $G = \dfrac{\overset{F/A}{\sigma_{xy}}}{2\underset{(\Delta x/l)}{\varepsilon_{xy}}}$

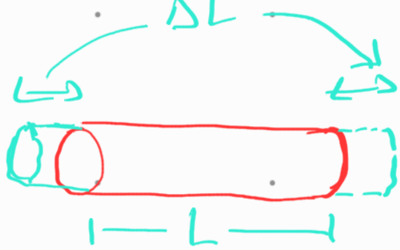
shear modulus / rigidity



4. $E = G\dfrac{(3\lambda + 2G)}{(\lambda + G)}$ → Young's Modulus

$$= 2G(1+\nu)$$

$$= \frac{|\sigma|}{|\varepsilon|} \longrightarrow \frac{\text{axial stress}}{\text{axial strain}}$$



$$\left.\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left(\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right)\\
\varepsilon_{22} &= \frac{1}{E}\left(\sigma_{22} - \nu\left(\sigma_{11}+\sigma_{33}\right)\right)\\
\varepsilon_{33} &= \frac{1}{E}\left(\sigma_{33} - \nu\left(\sigma_{11}+\sigma_{22}\right)\right)
\end{aligned}\right\} \text{tri-axial stress \& strain}$$

Hooke's Law in 3-d

— we generally use (G, ν)

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} = 2G \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ & & \varepsilon_{33} \end{bmatrix} + \lambda \varepsilon_{kk} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{2G}{(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}$$

6 equations

Apply this to momentum conservation

$$\underbrace{\nabla\cdot\sigma}_{} + \underline{f} = \rho \underline{a}$$

$$(\sigma_{ji,j} \text{ or } \sigma_{ij,j}) \leftarrow \sigma = \sigma^T$$

insert Hooke's Law to $[\sigma]$

$$\underbrace{(\lambda \varepsilon_{kk}\delta_{ji} + 2G\,\varepsilon_{ji})_{,j}}_{}$$

$$\varepsilon_{kk} \leftarrow (u_{k,k}) \qquad \varepsilon_{ji} \leftarrow \tfrac{1}{2}(u_{j,i} + u_{i,j})$$

$$\left(\lambda u_{k,k}\delta_{ji} + G(u_{j,i} + u_{i,j})\right)_{,j}$$

$$\lambda u_{j,ij} + G(u_{j,ij} + u_{i,jj}) + f_i = \rho a_i$$

$$(\lambda + G)\, u_{j,ij} + G\, u_{i,jj} + f_i = \rho a_i$$

$$\boxed{\nabla \underline{u} = \begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{bmatrix}}$$

– vector/tensor notations

$$\nabla\cdot\sigma + \underline{f} = \rho \underline{a}$$

$$\nabla\cdot\left(\lambda \varepsilon_{kk} \underline{I} + 2G[\varepsilon]\right) + \underline{f} = \rho \underline{a}$$

$$\lambda\, \nabla\cdot \begin{bmatrix} \nabla\cdot\underline{u} & 0 & 0 \\ 0 & \nabla\cdot\underline{u} & 0 \\ 0 & 0 & \nabla\cdot\underline{u} \end{bmatrix} + G\nabla\cdot(\nabla\underline{u} + \nabla\underline{u}^T) + \underline{f} = \rho\underline{a}$$

$$\lambda \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1}(\nabla\cdot\underline{u}) \\ \frac{\partial}{\partial x_2}(\nabla\cdot\underline{u}) \\ \frac{\partial}{\partial x_3}(\nabla\cdot\underline{u}) \end{bmatrix}}_{\nabla(\nabla\cdot\underline{u})} + G \underbrace{\nabla^2 \underline{u}}_{\begin{bmatrix} \nabla^2 u_1 \\ \nabla^2 u_2 \\ \nabla^2 u_3 \end{bmatrix}} + G \underbrace{\nabla\cdot\nabla\underline{u}^T}_{} + \underline{f} = \rho\underline{a}$$

$$\nabla\cdot\nabla\underline{u}^T = \begin{bmatrix} u_{1,11} + u_{2,12} + u_{3,13} \\ u_{1,21} + u_{2,22} + u_{3,23} \\ u_{1,31} + u_{2,32} + u_{3,33} \end{bmatrix} = \nabla(\nabla\cdot\underline{u})$$

$$\boxed{(\lambda+G)\,\nabla(\nabla\cdot\underline{u}) + G\nabla^2\underline{u} + \underline{f} = \rho\underline{a}}$$

$$\underline{a} = \frac{\partial^2 \underline{u}}{\partial t^2}$$

Elastic-wave equation

i.e $$\boxed{(\lambda+G)\nabla(\nabla\cdot\underline{u}) + G\nabla^2\underline{u} + \underline{f} = \rho\frac{\partial^2 \underline{u}}{\partial t^2}}$$

$\underline{u} \longrightarrow$ $\boxed{\text{3 unknowns}}$, $\boxed{\text{3 equations}}$

$(\lambda, G, \rho) \longrightarrow$ known for material

Elasto-statics $\longrightarrow$ $\underline{a} \longrightarrow 0$

$$\boxed{(\lambda+G)\left[\nabla\cdot(\nabla\underline{u})\right] + G(\nabla^2\underline{u}) + \underline{f} = 0}$$

If we take the time-derivative

$$\boxed{(\lambda+G)\left[\nabla\cdot(\nabla\underline{v})\right] + G(\nabla^2\underline{v}) + \underline{f}(t) = 0}$$

Quasi-static equilibrium [elastic body]

Simplifying geometries

Anti-plane

$$\underline{u} = [u_1, 0, 0]$$

$$\frac{\partial(\underline{u})}{\partial x_1} = 0$$

in-plane

$$\underline{u} = [0, u_2, u_3]$$

$$\frac{\partial(\underline{u})}{\partial x_1} = 0$$

3 equilibrium equations

$$(\lambda+G)\left[u_{1,11}+u_{2,12}+u_{3,13}\right]+G\left[u_{1,11}+u_{1,22}+u_{1,33}\right]+f_1=0$$

$$(\lambda+G)\left[u_{1,21}+u_{2,22}+u_{3,23}\right]+G\left[u_{2,11}+u_{2,22}+u_{2,33}\right]+f_2=0$$

$$(\lambda+G)\left[u_{1,31}+u_{2,32}+u_{3,33}\right]+G\left[u_{3,11}+u_{3,22}+u_{3,33}\right]+f_3=0$$

Anti-plane strain : $G\left(u_{1,22}+u_{1,33}\right)+f_1=0$

Plane-strain :

$$(\lambda+G)\left[u_{2,22}+u_{3,23}\right]+G\left[u_{2,22}+u_{2,33}\right]+f_2=0$$

$$(\lambda+G)\left[u_{2,32}+u_{3,33}\right]+G\left[u_{3,22}+u_{3,33}\right]+f_3=0$$