Velocity solutions for

$$K(V_{pl} - v) = \frac{df}{dt}\sigma_n$$

(1) Linear v-dependent friction: $f = Av$

$$\frac{K}{A\sigma_n}(V_{pl} - v) = \frac{dv}{dt} \qquad \frac{1}{t_R}$$

$$v(t) = e^{-t/t_R}\left(V_{pl}\left(e^{t/t_R} - 1\right) + V_i\right)$$

$$v(t) = V_{pl}\left(1 - e^{-t/t_R}\right) + V_i e^{-t/t_R}$$

$$\boxed{v(t) = V_{pl} + e^{-t/t_R}\left(V_i - V_{pl}\right)}$$

(2) Power-law : $f = A\sqrt{v}$

$$\frac{2K}{A\sigma_n}(v - V_{pl}) = \frac{1}{\sqrt{v}}\frac{dv}{dt}.$$

$$\frac{2K}{A\sigma_n}\sqrt{v}\,(v - V_{pl}) = \frac{dv}{dt}.$$

$$\frac{1}{t_R}$$

$$\boxed{v(t) = V_{pl} \tanh^2\left[\tanh^{-1}\sqrt{\frac{V_i}{V_{pl}}} + \frac{\sqrt{V_{pl}}\,K}{A\sigma_n} t\right]}$$

$$v(t) = V_{pl} \tanh^2\left\{\tanh^{-1}\sqrt{\frac{V_i}{V_{pl}}} + \frac{t}{t_R}\right\}$$

(3) log-form : $f = A \log v$

$$\frac{K}{A\sigma_n}(V_{pl} - v) = \frac{1}{v}\frac{dv}{dt}.$$

$$\frac{K}{A\sigma_n} v(V_{pl} - v) = \frac{dv}{dt}.$$

$$v(t) = \frac{V_{pl} V_i\, e^{\frac{V_{pl}K}{A\sigma_n} t}}{V_i\left[e^{\frac{V_{pl}K}{A\sigma_n} t} - 1\right] + V_{pl}}.$$

$\frac{V_{pl}K}{A\sigma_n} \to \frac{1}{t_R}$

$$= \frac{V_i\, e^{t/t_R}}{\frac{V_i}{V_{pl}}\left(e^{t/t_R} - 1\right) + 1} = \frac{V_i}{\frac{V_i}{V_{pl}}\left(1 - e^{-t/t_R}\right) + e^{-t/t_R}}$$

$v(t) = \quad V_i$

$$\boxed{\left(\frac{V_i}{V_{pl}}\right) + e^{-t/t_R}\left(1-\frac{V_i}{V_{pl}}\right)}$$

For a fixed $\frac{V_i}{V_{pl}}$ & $\frac{K}{A\sigma_n}$ & V_{pl}

$\Rightarrow \frac{V_i}{V_{pl}}$ & t_R are fixed. , $\xi = \frac{K}{A\sigma_n}$

The 3 functional forms are :

$$\frac{v(t)}{V_{pl}} = \left[1+e^{-Kt}\left(\frac{V_i}{V_{pl}}-1\right)\right] \quad \text{—} (1)$$

$$\frac{v(t)}{V_{pl}} = \tanh^2\left\{\tanh^{-1}\sqrt{\frac{V_i}{V_{pl}}} + K\sqrt{V_{pl}}\,t\right\} \quad \text{—} (2)$$

$$\frac{v(t)}{V_{pl}} = \frac{\left(\frac{V_i}{V_{pl}}\right)}{\left(\frac{V_i}{V_{pl}}\right) + e^{-V_{pl}\xi t}\left(1-\frac{V_i}{V_{pl}}\right)} \quad \text{—} (3)$$

$$\frac{v(t)}{V_{pl}} = \begin{cases} 1+e^{-at}(b-1) \\ \tanh^2\left\{\tanh^{-1}\sqrt{b} + a\sqrt{V_{pl}}\,t\right\} \\ \frac{b}{\;\cdots\;} \end{cases}$$

$$b + e^{-V_{pl} a t}(1-b)$$

depends on V_{pl}

for $V_{pl} = 1$,

$$\frac{v(t)}{V_{pl}} = \left\{ \frac{\left[1 + e^{-at}(b-1)\right] \tanh^2\left\{\tanh^{-1}\sqrt{b} + at\right\} b}{b + e^{-at}(1-b)} \right\}$$