Deformations, Strains and Rotations

Deformation gradient (for coordinates)

Displacement gradient

$$\mathbf{F} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)' = (\nabla \mathbf{U})' + \mathbf{I}$$

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$$

What does it mean when **F** = **I**?

Deformations, Strains and Rotations

Deformation gradient (for coordinates)

$$\mathbf{F} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)' = (\nabla \mathbf{U})' + \mathbf{I}$$

Strain tensors

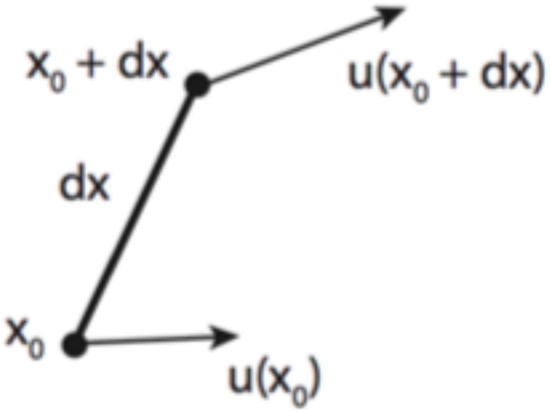
$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T.\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\nabla \mathbf{U}' + \nabla \mathbf{U} + (\nabla \mathbf{U}).(\nabla \mathbf{U})'\right)$$

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T}.\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\nabla \mathbf{u}' + \nabla \mathbf{u} - (\nabla \mathbf{u}).(\nabla \mathbf{u})'\right)$$

Deformations, Strains and Rotations

Infinitesimal deformation

$$\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right| << 1$$



Displacement field and its gradient
(chain rule)

$$u_i(x) = u_i(x_0) + \left.\frac{\partial u_i}{\partial x_j}\right|_{x_0} dx_j$$

Displacement
gradient

Deformations, Strains and Rotations

Infinitesimal strain tensor

$$\epsilon = \frac{1}{2}\left((\nabla \mathbf{u})^T + (\nabla \mathbf{u})\right)$$

For a 3 component displacement field, write out all components of the strain tensor

Split into an isotropic and deviatoric tensor

Deformations, Strains and Rotations

Rotation tensor (again)

$$\nabla \mathbf{u} = \frac{1}{2}\left((\nabla \mathbf{u})^T + (\nabla \mathbf{u})\right) + \frac{1}{2}\left((\nabla \mathbf{u}) - (\nabla \mathbf{u})^T\right)$$

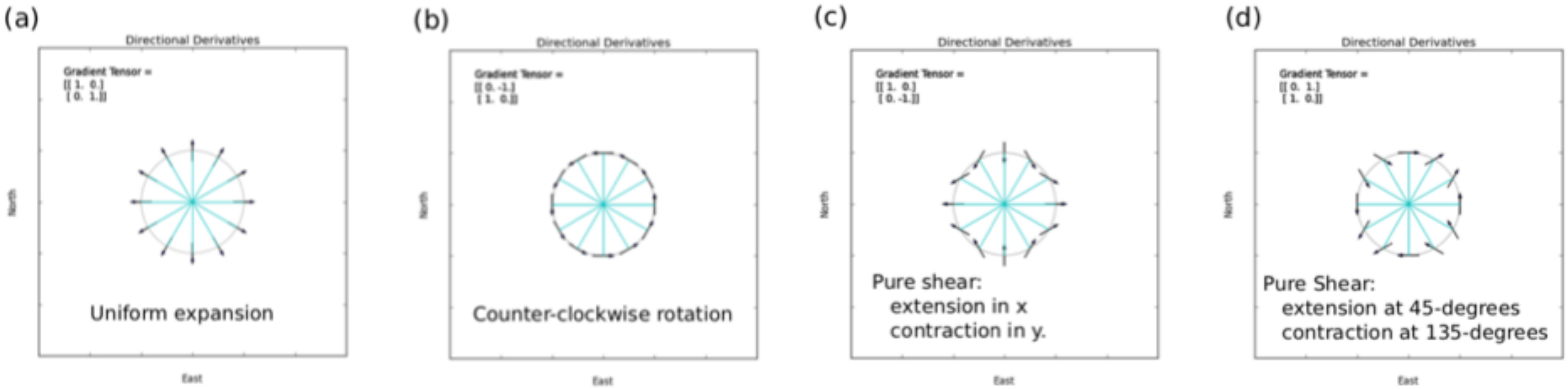
strain tensor
(symmetric)

rotation tensor
(skew-symmetric)

For a 3 component displacement field, write out all components of the rotation tensor.
Can you represent this as a cross product?

Decomposing the displacement gradient

$$\nabla \mathbf{u} = \frac{\Delta}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \omega\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \epsilon_1\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + \epsilon_2\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$



Compatibility conditions

For a 3 component displacement field, we can easily compute its spatial derivatives => strains and rotations. But does it work vice versa?

Compatibility conditions

For 6 independent strain components, there are 9 strain-displacement equations. How to solve?

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_1 \partial X_3} + \frac{\partial^2 \varepsilon_{13}}{\partial X_2^2} = \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_2 \partial X_3},$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_3^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_3}.$$

This guarantees that the strains can be integrated to give a single-valued **u**, but it does not guarantee a unique **u**. Why?

Displacements from strains

- Given a strain tensor (symmetric), first verify if it indeed represents a compatible strain tensor and then integrate it to compute displacements analytically (and numerically)

  - 1-d problem, $0 < x < 5$: $\varepsilon_{xx} = \log(x)$, compute u_x

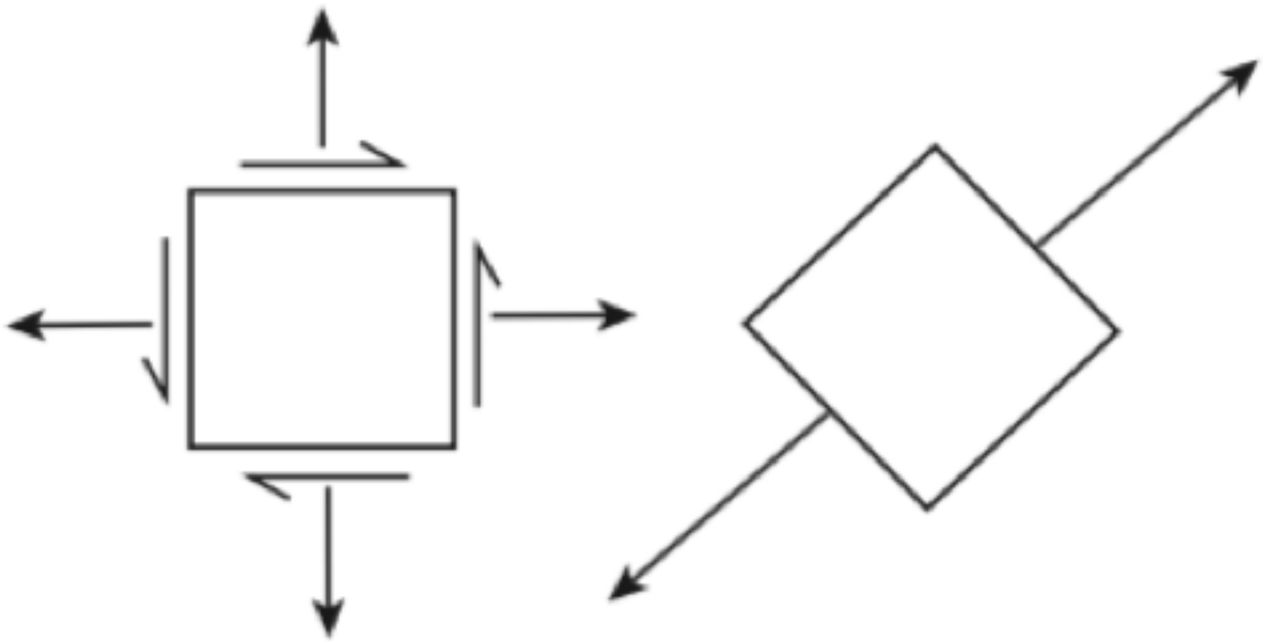
  - 2-d problem, $0 < x < 1$, $0 < y < 1$: $e_{xx}, e_{yy}, e_{zz}, e_{xy} = 0$,
    $e_{zx} = x/(x^2 + y^2)$
    $e_{zy} = y/(x^2 + y^2)$, compute u_z

  - 2-d problem, $0 < x < 1$, $0 < y < 1$: e_{zz}, e_{zx}, $e_{zy} = 0$,
    $e_{xx} = ((k+1)x^2y + (k-3)y^3)/(x^2 + y^2)^2$
    $e_{yy} = ((k-3)x^2y + (k+1)y^3) /(x^2 + y^2)^2$
    $e_{xy} = (4xy^2)/(x^2 + y^2)^2$, compute u_x and u_y (for $k = 4$)

Principal Strains

The strain tensor can be rotated such that all the off-diagonal components vanish and we are left only with diagonal terms – principal strains

For any square matrix, we can compute ‘eigenvalues’ and ‘eigenvectors’ – characteristic polynomial



Principal Strains

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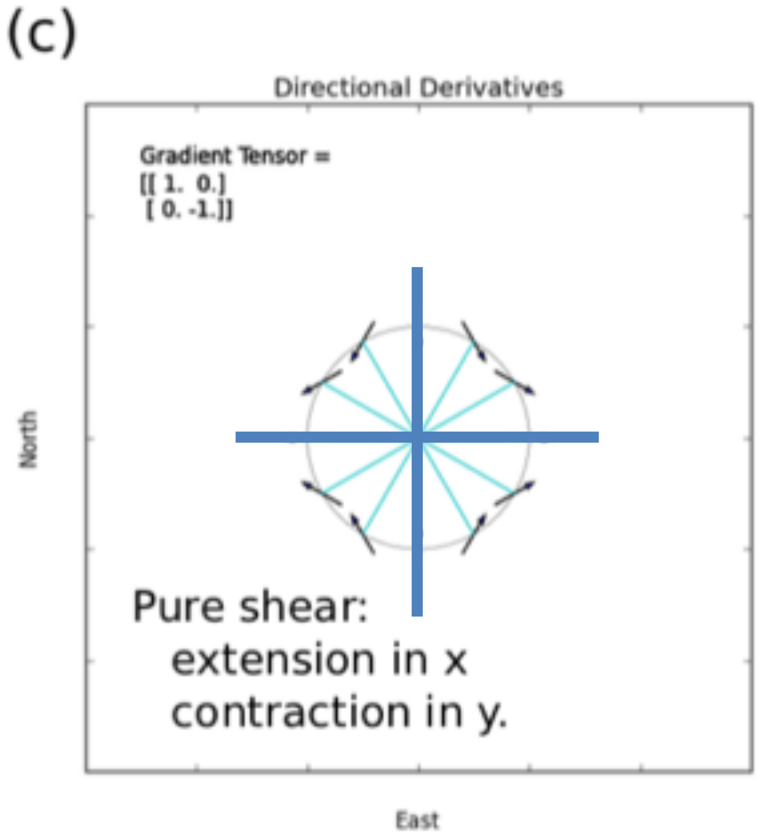
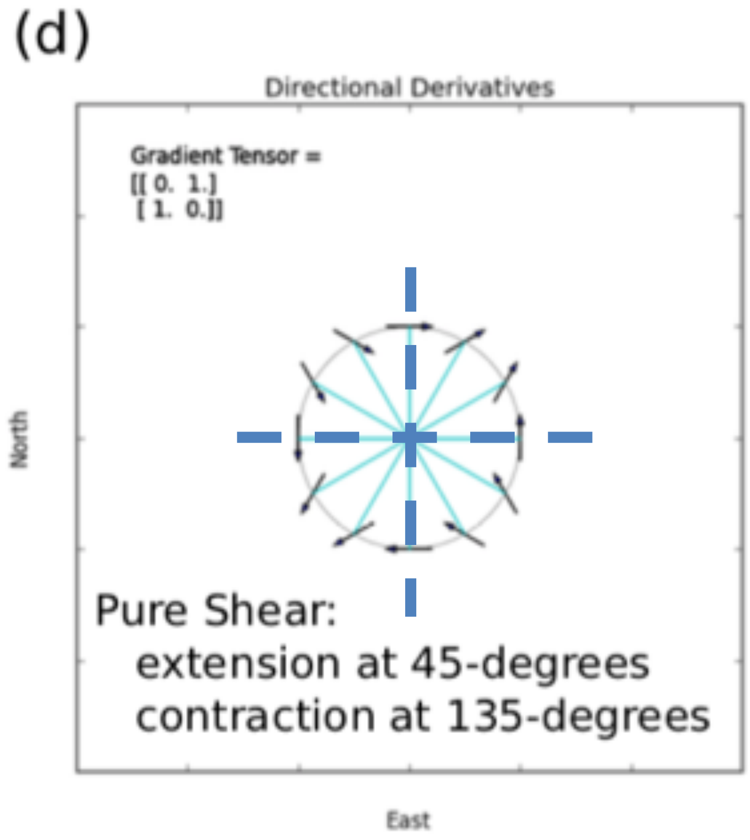
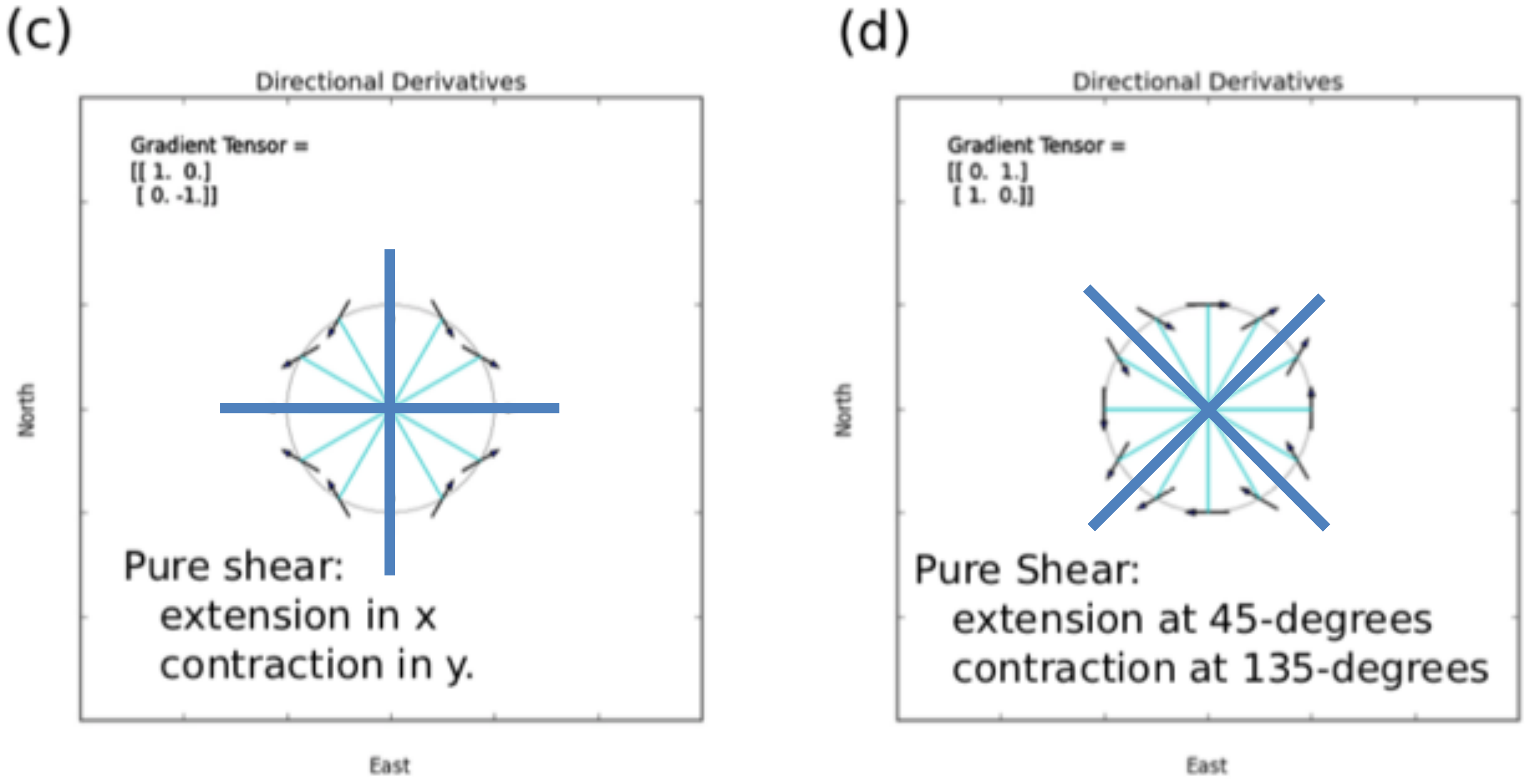




Principal Strains

The strain tensor can be rotated such that all the off-diagonal components vanish and we are left only with diagonal terms – principal strains



Principal Strains

$$\epsilon'_{11} = \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + 2\epsilon_{12}\sin\theta\cos\theta$$
$$\epsilon'_{22} = \epsilon_{11}\sin^2\theta + \epsilon_{22}\cos^2\theta - 2\epsilon_{12}\sin\theta\cos\theta$$
$$\epsilon'_{12} = (\epsilon_{22} - \epsilon_{11})\sin\theta\cos\theta + \epsilon_{12}(\cos^2\theta - \sin^2\theta)$$

Find the rotation angle that makes shear strain 0 and then what are the other components?

What is the maximum shear strain for this system?

Strain invariants

$$I_1 = \operatorname{tr}[\epsilon]$$

$$I_2 = \frac{1}{2}\left(\operatorname{tr}[\epsilon]^2 + \operatorname{tr}[\epsilon^2]\right)$$

$$I_3 = \det[\epsilon]$$

Assignments

1. Using the data given to you, first plot the elevation grid and overlay it with the displacement vectors. Make sure the vectors are visible.
2. Project the displacements on to the elevation gradient direction (unit vector) – to do this you will have to resample either the displacements or the elevation data. Plot transects of this projected quantity and discuss what you observe.
3. Estimate the elevation change due to materials moving. Hint: to do this you will have to use div(**u**) = 0 in 3-d. Think about and discuss what this assumption implies. You are free to come up with any other method of estimating elevation change as well.
4. Strains
    1. Calculate the horizontal strain components from the displacement field
    2. Plot the trace of the strain tensor and overlay it with displacement field. What does the trace represent?
    3. Bonus: plot the principal strains at each point (maybe not each point because that will look overcrowded)

Assignments

Reading Assignment:

Delbridge, B. G., R. Bürgmann, E. Fielding, S. Hensley, and W. H. Schulz (2016), Three-dimensional surface deformation derived from airborne interferometric UAVSAR: Application to the Slumgullion Landslide, J. Geophys. Res. Solid Earth, 121, 3951–3977, doi:10.1002/2015JB012559.

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Journal of Geophysical Research: Solid Earth

RESEARCH ARTICLE

10.1002/2015JB012559

Key Points:
- Provides reliable and accurate measurements of 3-D surface deformation
- Inversion procedure to invert for landslide depth from 3-D surface velocity measurements
- Viscoplastic flow provides tighter theoretical constraints on the rheological parameter

Three-dimensional surface deformation derived from airborne interferometric UAVSAR: Application to the Slumgullion Landslide

Brent G. Delbridge[1], Roland Bürgmann[1], Eric Fielding[2], Scott Hensley[2], and William H. Schulz[3]

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Abstract In order to provide surface geodetic measurements with "landslide-wide" spatial coverage,