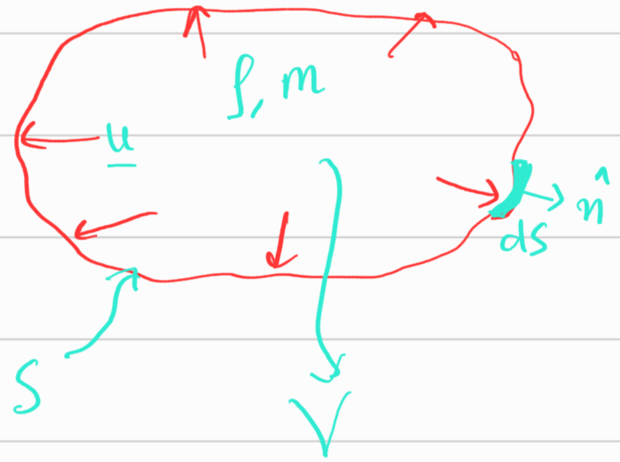
Mass Conservation



Total mass of the body is <u>constant</u>

$$M = \int_V dm = \int_V \rho\, dV + V d\rho$$

Consider the mass flux in/out of the CV,

$\boxed{\text{flux} = (\rho \underline{v})}$ → advective flux.

→ can also have diffusive flux [not important here]

This flux travels through some surface ($\hat{n}$) with area dS

$= \rho \underline{v} \cdot \hat{n}\, dS$

Total flux out of the CV (integrate over surface S) $= \int_S \rho \underline{v} \cdot \hat{n}\, dS$. —— ①

This flux is balanced by an associated change in the density of the body, $\frac{\partial \rho}{\partial t} dV$

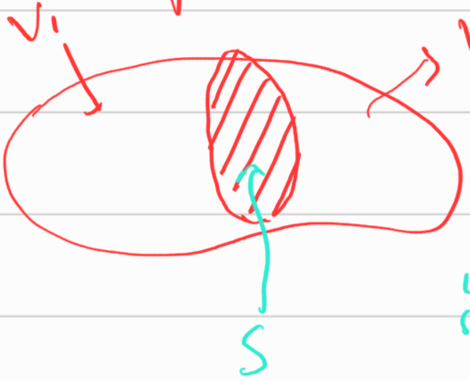
The total mass change (integrated over the volume V) $= \int_V \frac{\partial \rho}{\partial t} dV$ → ②

① + ② $= 0$ because $\frac{dM}{dt} = 0$.

$$\Rightarrow \iint \rho \underline{v} \cdot \hat{n}\, dS + \iiint \frac{\partial \rho}{\partial t} dV = 0 \rightarrow ③$$

S

Divergence Theorem (for any vector field $\underline{F}$)



$[\phi(V_1) + \phi(V_2)] \rightarrow$ flux out of V

at S, $\iint \underline{F}\cdot\hat{n}\, dS$ for ①

inner surface $= -\iint \underline{F}\cdot\hat{n}\, dS$ for ②

$\phi(V_1) + \phi(V_2) =$ flux only out of outer surface

$= \phi(V)$

$\Rightarrow$ if the body was broken up into n volumes.

$$\phi(V) = \sum_{i=1}^{n} \phi(V_i) \leftarrow \text{only outer surface}$$

$$\iint_{S_0} \underline{F}\cdot\hat{n}\, dS$$

$$\sum_i \underline{F}_i \hat{n}_i dA_i$$

as $n \to \infty$, $\phi(V) = \sum_{i=1}^{\infty} \iint_{S_0} \underline{F}\cdot\hat{n}\, dS_i$

$((\nabla\cdot\underline{F})dV$

$$\phi(V) = \iiint_V \nabla\cdot\underline{F}\, dV$$

$$\Rightarrow \boxed{\iint_S \underline{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\underline{F}\, dV} \rightarrow ④$$

Applying ④ in ③, we get

$$\iiint\left(\nabla\cdot(\rho\underline{v}) + \frac{\partial\rho}{\partial t}\right)dV = 0$$

$\iiint_V \Big(\qquad \partial t \Big)$

Differential form : $$\boxed{\nabla \cdot (\rho \underline{v}) + \frac{\partial \rho}{\partial t} = 0}$$

Continuity Equation / mass conservation

If the material is incompressible i.e ρ is constant,

$$\boxed{\nabla \cdot \underline{v} = 0}$$