Green's functions

(Elastostatic equilibrium in anti-plane)

$$\boxed{G\nabla^2 u + f = 0}$$

In cartesian coordinates, $\dfrac{\partial^2 u_1}{\partial x_2^2} + \dfrac{\partial^2 u_1}{\partial x_3^2} + \dfrac{f}{G} = 0$

In polar coordinates, $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_1}{\partial r}\right) + \dfrac{f}{G} = 0$

Defining ∇ in polar coordinates (r, θ)

$r = \sqrt{x_2^2 + x_3^2}$

$\theta = \tan^{-1}\left(\dfrac{x_3}{x_2}\right)$

$$\boxed{\begin{aligned}\hat{r} &= \cos\theta\,\hat{x}_2 + \sin\theta\,\hat{x}_3\\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{x}_3\end{aligned}}$$

length increment $d\underline{r} = d(r\hat{r})$

$= dr\,\hat{r} + r\,\underbrace{d\hat{r}}$ —— ①

$\hat{r}$ — unit vector [no dependance on r]

using chain rule

$d\hat{r} = \cancel{\dfrac{\partial\hat{r}}{\partial r}}^{\,0}\,dr + \dfrac{\partial\hat{r}}{\partial\theta}\,d\theta \qquad \dfrac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$

$d\hat{r} = \hat{\theta}\,d\theta$ ——→ ②

Apply ② in ①, $\boxed{d\underline{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}}$ —— ③

So now, let's consider any scalar ϕ and its

So now let's consider any scalar ϕ. and its length increment $d\phi$ ← scalar length change.

in cartesian coordinates,

$$\phi(x_2, x_3) \qquad d\phi = \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$$

$$d\phi = (\nabla \phi) \cdot d\underline{x} \quad \text{where } d\underline{x} = \begin{bmatrix} dx_2 \\ dx_3 \end{bmatrix}$$

length increment of $\underline{x}$

in polar coordinates,

$$\phi(r, \theta) \qquad d\phi = \nabla \phi \cdot d\underline{r}$$

← length increment of $\underline{r}$

from chain rule,

$$d\phi(r, \theta) = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$$

$$\frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta = \nabla \phi \cdot d\underline{r} \qquad \text{—} (4)$$

insert (3) into (4) $\left[dr\,\hat{r} + r\,d\theta\,\hat{\theta} \right]$

$$\frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta = A_1\, dr + A_2\, r\, d\theta$$

$$\Rightarrow \boxed{\nabla = \left[\frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right]} \qquad \text{—} (5)$$

Divergence in polar coordinates ($\nabla \cdot \underline{u}$)

Consider a vector field $\underline{u} = u_r \hat{r} + u_\theta \hat{\theta}$

From (5), we know
$$\nabla = \begin{bmatrix} \frac{\partial}{\partial r} \, \hat{r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \, \hat{\theta} \end{bmatrix}$$

So divergence
$$\nabla \cdot \underline{u} = \nabla \cdot \begin{bmatrix} u_r \, \hat{r} \\ u_\theta \, \hat{\theta} \end{bmatrix}$$

$$= \hat{r} \cdot \left(\frac{\partial}{\partial r}\left(u_r \hat{r}\right) + \frac{\partial}{\partial r}\left(u_\theta \hat{\theta}\right) \right)$$

$$+ \frac{\hat{\theta}}{r} \cdot \left(\frac{\partial}{\partial \theta}\left(u_r \hat{r}\right) + \frac{\partial}{\partial \theta}\left(u_\theta \hat{\theta}\right) \right)$$

$$= \hat{r} \cdot \left(\frac{\partial u_r}{\partial r} \hat{r} + u_r \cancel{\frac{\partial \hat{r}}{\partial r}}^{\,0} + \frac{\partial u_\theta}{\partial r} \hat{\theta} + u_\theta \cancel{\frac{\partial \hat{\theta}}{\partial r}}^{\,0} \right)$$

$$+ \frac{\hat{\theta}}{r} \cdot \left(\frac{\partial u_r}{\partial \theta} \hat{r} + u_r \underbrace{\frac{\partial \hat{r}}{\partial \theta}}_{\hat{\theta}} + \frac{\partial u_\theta}{\partial \theta} \hat{\theta} + u_\theta \underbrace{\frac{\partial \hat{\theta}}{\partial \theta}}_{-\hat{r}} \right)$$

$$= \partial u_r \quad + \quad \frac{1}{\,}\left(u_r\right) \quad + \frac{1}{\,} \partial u_\theta$$

$$\underbrace{\frac{\partial u_r}{\partial r} + \frac{u_r}{r}}_{\frac{1}{r}\frac{\partial}{\partial r}(r u_r)} \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\boxed{\nabla \cdot \underline{u} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_\theta}{\partial \theta}\right]} \quad -(6)$$

$\Rightarrow$ Laplacian in polar coordinates $(\nabla \cdot \nabla \phi)$

— apply (5) & (6),

$$\nabla \cdot \nabla \phi = \nabla \cdot \left(\frac{\partial \phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta} \right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\right]$$

$$\boxed{\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}} \quad -(7)$$

Coming back to elastostatic equilibrium [anti-plane geometry]

$$\underbrace{\underbrace{\nabla^2 u}_{\text{Laplace's equation}} + \frac{f}{G} = 0}_{\text{Poisson's equation}}$$

$\frac{f}{G} \rightarrow$ source term

Poisson's equation

$\nabla^2 u = 0 \longrightarrow (x_2, x_3) \longrightarrow a x_2 + b x_3$

general solutions $\begin{cases} a(x_2^2 - x_3^2) + b x_2 x_3 \\ \exp(u x_3)\left[a \cos u x_2 + b \sin u x_2\right] \end{cases}$

$\longrightarrow (r, \theta) \longrightarrow a \ln r$

general solutions $\Big\} \longrightarrow \left(a r^m + \frac{b}{r^m}\right)\left(c \cos m\theta + d \sin m\theta\right)$

→ These solutions obey linear superposition

→ uniqueness → only when we match boundary conditions.

limit of Gaussian

The source term is a <u>boundary condition</u>

→ point source $\underline{\delta(x_2)\,\delta(x_3)}$ or $\underline{\delta(r)}$

for any location $(y_2, y_3) \rightarrow \delta(x_2 - y_2)\,\delta(x_3 - y_3)$

in polar coordinates, $\delta(\underline{r}) = \frac{1}{2\pi r}\,\delta(r)$

$$\underbrace{\int_{x_2}\int_{x_3} \delta(\underline{x})\, dx_2\, dx_3 = \int_0^{2\pi}\int_0^{\infty} \delta(\underline{r})\, dr\, r d\theta}_{\text{property of Delta-dirac function / point source}} = 1$$

$$\Rightarrow \nabla^2 u + \left(\frac{f}{G}\right) = 0$$

source terms

$\rightarrow$ Delta-dirac function at $r=0$

solution $u(x_2, x_3)$ or $u(r,\theta)$ is a Greens function solution

$\Rightarrow$ response of the full-space to a unit body force at the source.

This has huge implications since any force distribution can be decomposed as

$$f(x_2,x_3) = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x_2-y_2)\,\delta(x_3-y_3)\, f(y_2,y_3)\, dy_2\, dy_3}_{\text{Break } f \text{ into a number of point sources \& sum}}$$

The Greens functions can then be used to compute the weighted sum $\Rightarrow$ <u>displacements</u>

$$\Rightarrow \nabla^2 g(x_2,x_3;\overbrace{y_2,y_3}^{\text{source}}) + \frac{\delta(x_2-y_2)\,\delta(x_3-y_3)}{G} = 0$$

$$\Rightarrow u_1(x_2,x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_2,x_3;y_2,y_3)\, f(y_2,y_3)\, dy_2\, dy_3$$

Because u_1 is a response to a point force, we expect it to decay only radially,

$\Rightarrow \quad u_1 = u_1(r)$ [no θ dependance]

→ makes computing Greens functions easier.

So the governing PDE in polar coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \underbrace{\frac{\delta(r)}{2\pi r}}_{-\frac{f(r)}{G}}$$

The solution is found by integrating twice, $\boxed{u_1(r) = \frac{1}{2\pi}\ln r}$

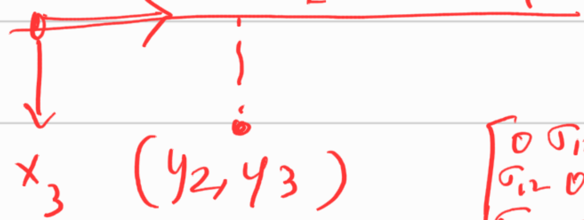
in (x_2, x_3) $\boxed{u_1(x_2, x_3) = \frac{1}{4\pi}\ln(x_2^2 + x_3^2)}$

Greens Function for a point source at $(y_2, y_3) = (0,0)$

$$\xi(x_2, x_3; y_2, y_3) = \frac{1}{4\pi}\ln\left((x_2-y_2)^2 + (x_3-y_3)^2\right)$$

These Greens Functions are defined in a full-space, if we want to introduce a free-surface, we need to match the boundary condition.

x_2 traction-free



$$\sigma \cdot n = 0 \Rightarrow \begin{bmatrix} -\sigma_{13} \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix}, \quad n = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Boundary condition is $\sigma_{13}(x_2, x_3 = 0) = 0$.

$$\sigma_{13} = \frac{G}{2} \frac{\partial u_1}{\partial x_3}\Big|_{x_3=0} = 0$$

This is solved by introducing image sources at $(y_2, -y_3)$

$$\Rightarrow \boxed{\mathcal{G}(x_2, x_3; y_2, y_3) = \frac{1}{4\pi} \left\{ \ln\left((x_2 - y_2)^2 + (x_3 - y_3)^2\right) + \ln\left((x_2 - y_2)^2 + (x_3 + y_3)^2\right) \right\}}$$

Half-space Greens functions

Equivalent body-forces & BCs

- We used the source term in Poisson's eqtn to describe a force applied at a location (y_2, y_3), and it only existed as $\delta(y_2)\delta(y_3)$

- Now when we need a finite length source, we build on the same principles.

The equivalent body force density $\underline{f}$ is given from the divergence of the moment density tensor.

$$m_{ij} = \underbrace{G\,\varepsilon_{ij}^{(i)}}_{\text{stress tensor}} \leftarrow \text{inelastic strain tensor}$$

stress tensor [only defined on the fault plane]

Equilibrium equation applied at the fault,

$$\nabla \cdot m + f = 0.$$

$$\Rightarrow \underline{f} = -\nabla \cdot m = -\nabla \cdot \left(G\,\varepsilon^{(i)}\right)$$

$$f_i = -G\,\varepsilon_{ji,j}^{(i)}$$

slip vector
not a unit vector

$$\varepsilon^{(i)} = \frac{1}{2}\left(\hat{n} \otimes \underline{\gamma} + \underline{\gamma} \otimes \hat{n}\right)$$

for a vertical fault $\hat{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{\gamma} = \begin{bmatrix} 0 \\ S \\ 0 \end{bmatrix}$

only $\varepsilon_{12}^{(i)} = \varepsilon_{21}^{(i)} \neq 0$

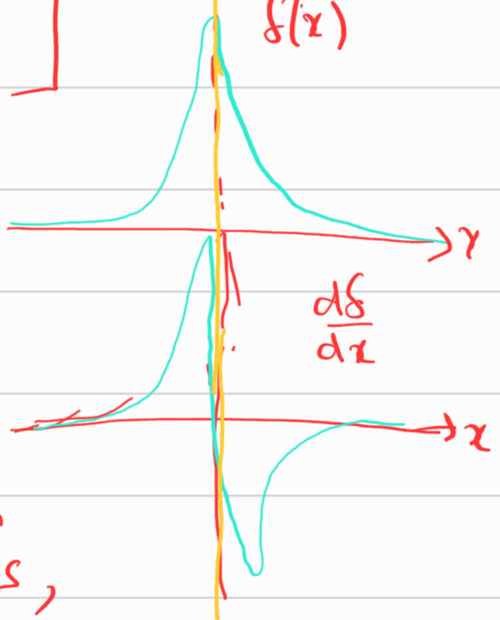
$$= \frac{S}{2}\,\delta(x_2)\,\Pi\left(\frac{x_3}{W}\right)$$

$$\Rightarrow m = \begin{bmatrix} 0 & 2G\,\varepsilon_{12}^{(i)} & 0 \\ 2G\,\varepsilon_{21}^{(i)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } \underline{f} = -\nabla \cdot m = \begin{bmatrix} -2G\,\varepsilon_{21,2}^{(i)} \\ 0 \\ \end{bmatrix}$$

0

$$f_1 = -2G s \underbrace{\frac{\partial \delta(x_2)}{\partial x_2}}_{\text{delta-derivative}} \Pi\left(\frac{x_3}{w}\right)$$



Use this with our full-space GFs,

$\frac{f}{G} \leftarrow$ source

$$u_1(x_2,x_3) = -\frac{s}{4\pi}\iint_{-\infty}^{\infty} f_1(y_2,y_3)\ln\left((x_2-y_2)^2+(x_3-y_3)^2\right) dy_2\, dy_3$$

$$= -\frac{s}{4\pi}\iint_{-\infty}^{\infty} \frac{\partial \delta(y_2)}{\partial y_2} \ln\left((x_2-y_2)^2+(x_3-y_3)^2\right)\Pi\left(\frac{y_3}{w}\right) dy_2\, dy_3$$

- use δ-identity, $\int_{-\infty}^{\infty}\delta'(x-a)f(x)\,dx = -f'(a)$
- use $\int_{-\infty}^{\infty}\Pi\left(\frac{x}{w}\right) f(x)\,dx = \int_{-w/2}^{w/2} f(x)\,dx$.

$$= -\frac{s}{4\pi}\int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{2(x_2-0)}{x_2^2+(x_3-y_3)^2}\, dy_3$$

$$= -\frac{s}{2\pi}\int^{w/2} \frac{x_2}{\ldots}\, dy_3 = -\frac{s}{\ldots}\left[\tan^{-1}\frac{x_3-y_3}{\ldots}\right]^{y_3=\frac{w}{2}}$$

$$\dots \int_{-W/2} \frac{\dots}{x_2^2+(x_3-y_3)^2} \dots = \frac{\dots}{2\pi}\left[\dots \frac{\dots}{x_2} \right]_{y_3=-\frac{W}{2}}$$

$$\boxed{u_1(x_2,x_3) = \frac{S}{2\pi}\left[\tan^{-1}\frac{x_3+\frac{W}{2}}{x_2} - \tan^{-1}\frac{x_3-\frac{W}{2}}{x_2}\right]}$$

Full-space solution for finite fault located at $\underline{(0,0)}$ with width $\underline{W}$

To move the source location to (y_2, y_3)

$$u_1(x_2,x_3) = \frac{S}{2\pi}\left[\tan^{-1}\frac{x_3-y_3+\frac{W}{2}}{x_2-y_2} - \tan^{-1}\frac{x_3-y_3-\frac{W}{2}}{x_2-y_2}\right]$$

To account for free-surface i.e half-space, add image solution at $(y_2, -y_3)$

$$u_1(x_2,x_3) = \frac{S}{2\pi}\left[\tan^{-1}\frac{x_3-y_3}{x_2-y_2}\right]_{y_3+\frac{W}{2}}^{y_3-\frac{W}{2}} + \frac{S}{2\pi}\left[\tan^{-1}\frac{x_3+y_3}{x_2-y_2}\right]_{y_3+\frac{W}{2}}^{y_3-\frac{W}{2}}$$

$$\boxed{u_1(x_2, x_3) = \frac{S}{2\pi}\left[\tan \frac{x_3 - y_3}{x_2 - y_2}\right]}$$

← Cumbersome Notation!