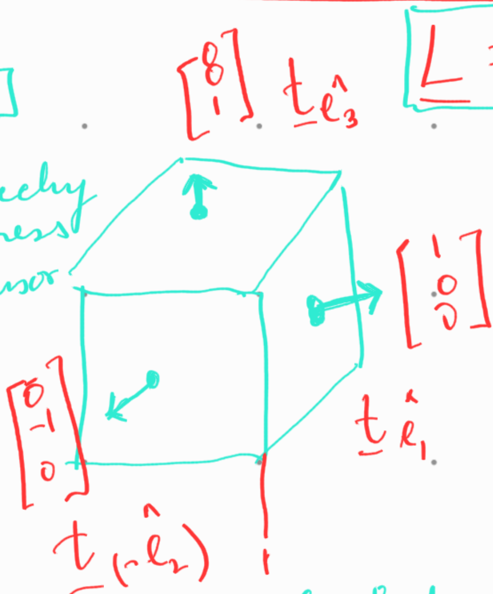
Conservation of (linear) momentum

$$\boxed{\underline{L} = m\underline{v}, \quad \frac{d\underline{L}}{dt} = m\frac{d\underline{v}}{dt} = m\underline{a}}$$ Force balance

$[\sigma]$ ↑ Cauchy stress tensor



$$\underline{t}_{\hat{e}_1}\Big|_{\underline{x}+dx_1\hat{e}_1} dA_1 + \underline{t}_{(-\hat{e}_1)}\Big|_{\underline{x}} dA_1$$

$$+ \underline{t}_{\hat{e}_2}\Big|_{\underline{x}+dx_2\hat{e}_2} dA_2 + \underline{t}_{(-\hat{e}_2)}\Big|_{\underline{x}} dA_2$$

[stress tensor is independent of $\hat{n}$ → Cauchy postulate]

$$+ \underline{t}_{\hat{e}_3}\Big|_{\underline{x}+dx_3\hat{e}_3} dA_3 + \underline{t}_{(-\hat{e}_3)}\Big|_{\underline{x}} dA_3$$

Equilibrium/
Force balance condition $+ \underline{f}\, dV = m\,\underline{a}$ —— (1)

$$\underline{t}_{(-\hat{e}_i)} = -\underline{t}_{(\hat{e}_i)} \quad (2)$$ because $\boxed{\hat{n}\cdot\sigma = \underline{t}}$

$\sigma_{ji} n_j = t_i$

⇒ divide (1) by dV & apply (2),

$$\frac{d}{dx_1}\underline{t}_{\hat{e}_1} + \frac{d}{dx_2}\underline{t}_{\hat{e}_2} + \frac{d}{dx_3}\underline{t}_{\hat{e}_3} + \underline{f} = \rho\,\underline{a} \quad (3)$$

$$\underline{t} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

for $\hat{e}_1$, $\hat{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{t}_{\hat{e}_1} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \ldots \end{bmatrix}$

σ_{13}

for $\hat{e}_2$, $\underline{t}_{\hat{e}_2} = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix}$, for $\hat{e}_3$, $\underline{t}_{\hat{e}_3} = \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix}$

$$\boxed{\nabla \cdot \underline{\sigma} + \underline{f} = \rho \underline{a}} \quad \text{— 3 eqtns}$$

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + f_1 = \rho a_1$$
$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + f_2 = \rho a_2$$
$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} + f_3 = \rho a_3$$

$$\boxed{\sigma_{ji,j} + f_i = \rho a_i}$$

Conservation of angular momentum

$$\underbrace{\underline{\Omega}}_{\text{angular momentum}} = \underline{r} \times \underline{F}, \qquad \underline{F} = \iint \underline{t} \, dA$$

consider the same control volume

- the forces acting on the system will be balanced by the torque on the system

$$\left((\underline{x} + dx_1 \hat{e}_1) \times \underline{t}^{(e_1)}\Big|_{x + dx_1 \hat{e}_1} + \underline{x} \times \underline{t}^{(-e_1)}\Big|_{x} \right) dA_1$$

$(-e_2)$

$$+\left(\left(\underline{x}+dx_2\hat{e}_2\right)\times \underline{t}^{(e_2)}\Big|_{\underline{x}+dx_2\hat{e}_2} + \underline{x}\times \underline{t}\Big|_{\underline{x}}\right)dA_2$$

$$+\left(\left(\underline{x}+dx_3\hat{e}_3\right)\times \underline{t}^{(e_3)}\Big|_{\underline{x}+dx_3\hat{e}_3} + \underline{x}\times \underline{t}^{(-e_3)}\Big|_{\underline{x}}\right)dA_3$$

$$+\left(\underline{x}\times \underline{f}\right)dV = \left[dm\,\underline{x}\times \underline{a}\right]$$

Separate out the LHS $\rightarrow \underline{x}\times \underline{t} + \cdots$

$\rightarrow d\underline{x}\times \underline{t} + \cdots$

the first set of terms looks like -

$$\underline{x}\times\left[\left(\underline{t}^{(e_1)}\Big|_{\underline{x}+dx_1\hat{e}_1} - \underline{t}^{(e_1)}\Big|_{\underline{x}}\right)dA_1 + \left(\underline{t}^{(e_2)}\Big|_{\underline{x}+dx_2\hat{e}_2} - \underline{t}^{(e_2)}\Big|_{\underline{x}}\right)dA_2\right.$$

$$\left.+\left(\underline{t}^{(e_3)}\Big|_{\underline{x}+dx_3\hat{e}_3} - \underline{t}^{(e_3)}\Big|_{\underline{x}}\right)dA_3 + \underline{f}\,dV\right] = \underline{x}\times(dm\,\underline{a})$$

$$\Rightarrow \underline{x}\times\underbrace{\left[\text{Linear momentum balance}\right]}_{\text{This is already 0}} = 0$$

$$\Rightarrow dx_1\hat{e}_1\times \underline{t}^{(e_1)}\Big|_{\underline{x}+dx_1\hat{e}_1} dA_1 + dx_2\hat{e}_2\times \underline{t}^{(e_2)}\Big|_{\underline{x}+dx_2\hat{e}_2} dA_2 + dx_3\hat{e}_3\times \underline{t}^{(e_3)}\Big|_{\underline{x}+dx_3\hat{e}_3} dA_3$$

$$\left[dA_1 = dx_2\,dx_3\right]$$

$= 0$

dividing by dV we get, $\left[\begin{array}{l} A_1 = dx_2 dx_3 \\ A_2 = dx_1 dx_3 \\ A_3 = dx_1 dx_2 \end{array}\right]$

$$\hat{e}_1 \times t^{(e_1)} + \hat{e}_2 \times t^{(e_2)} + \hat{e}_3 \times t^{(e_3)} = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} t_{21} \\ t_{22} \\ t_{23} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{bmatrix} = 0 \Rightarrow \boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\boxed{\sigma = \sigma^T}$$

Cauchy stress tensor is symmetric