(r, θ) polar coordinates [2-d]

$$r = \sqrt{x^2 + y^2} \qquad \hat{r} = \cos\theta \, \hat{x} + \sin\theta \, \hat{y}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \underbrace{\hat{\theta} = -\sin\theta \, \hat{x} + \cos\theta \, \hat{y}}_{\text{in cartesian co-ordinates}}$$

Length increments in (r, θ) system

$$\underline{dr} = d(r \hat{r})$$

$$= dr \, \hat{r} + r \, \underline{d\hat{r}} \quad \text{—— (1)}$$

use chain-rule to expand $d\hat{r}$,

$$d\hat{r} = \frac{\partial \hat{r}}{\partial r} dr + \frac{\partial \hat{r}}{\partial \theta} d\theta$$

$\hat{r}$ is a unit vector $\Rightarrow$ does not depend on r

$$\Rightarrow d\hat{r} = \frac{\partial \hat{r}}{\partial \theta} d\theta \quad \text{—— (2)}$$

apply (2) in (1),

$$\underline{dr} = dr \, \hat{r} + r \, d\theta \frac{\partial \hat{r}}{\partial \theta} \quad \text{—— (3)}$$

$d\underline{r} = \ldots \frac{\partial}{\partial\theta}$

consider $\frac{\partial \hat{r}}{\partial\theta}$, we know $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$

$$\Rightarrow \frac{\partial \hat{r}}{\partial\theta} = \underbrace{-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}}_{\hat{\theta}} \quad \text{--- } (4)$$

Apply (4) in (3),

$$\underbrace{d\underline{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}}_{(5)} = \begin{bmatrix} dr \\ r\,d\theta \end{bmatrix}$$

∇ in polar coordinates

For some scalar u,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{in } [\hat{x}, \hat{y}] \text{ system}$$

$$= (\nabla u)\cdot d\underline{x} \leftarrow \text{increment in length}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

This scalar u in a polar coordinates is,

$u = u(r, \theta)$

$\Rightarrow du = (\nabla u) \cdot d\underline{r} \leftarrow$ length increment

$= \nabla u \cdot \begin{bmatrix} dr \\ r\,d\theta \end{bmatrix} \leftarrow$ from 5

we expect $du(r,\theta) = \dfrac{\partial u}{\partial r} dr + \dfrac{\partial u}{\partial \theta} d\theta$

$$\left.\begin{aligned} \Rightarrow (\nabla u)_r\, dr + (\nabla u)_\theta\, r\,d\theta \\ = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta \end{aligned}\right\}$$ comparing the 2 equations,

$$\boxed{\nabla_{(r,\theta)} = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right]}$$

Divergence $(\nabla \cdot \underline{u})$

$(\underline{u} = u_r \hat{r} + u_\theta \hat{\theta})$

$\left(\nabla = \dfrac{\partial}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\hat{\theta}\right)$

$\nabla \cdot \underline{u} = \left[\dfrac{\partial}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\hat{\theta}\right] \cdot \left[u_r \hat{r}\right.$

$$\nabla\cdot\underline{u} = \left[\frac{\partial}{\partial r} \qquad \frac{1}{r}\frac{\partial}{\partial\theta}\right] \cdot \left[\; u_\theta\,\hat{\theta}\right]$$

$$= \hat{r}\cdot\left(\frac{\partial \underline{u}}{\partial r}\right) + \frac{\hat{\theta}}{r}\left(\frac{\partial \underline{u}}{\partial \theta}\right)$$

$$= \hat{r}\cdot\left[\frac{\partial}{\partial r}\left(u_r\hat{r} + u_\theta\hat{\theta}\right)\right] + \frac{\hat{\theta}}{r}\cdot\left[\frac{\partial}{\partial\theta}\left(u_r\hat{r} + u_\theta\hat{\theta}\right)\right]$$

$$= \hat{r}\cdot\left[\frac{\partial u_r}{\partial r}\hat{r} + \cancel{u_r\frac{\partial\hat{r}}{\partial r}}^{\,0} + \frac{\partial u_\theta}{\partial r}\cancel{\hat{\theta}} + \cancel{u_\theta\frac{\partial\hat{\theta}}{\partial r}}^{\,0}\right] + \frac{\hat{\theta}}{r}\cdot\left[\frac{\partial u_r}{\partial\theta}\cancel{\hat{r}} + u_r\frac{\partial\hat{r}}{\partial\theta} + \frac{\partial u_\theta}{\partial\theta}\hat{\theta} + \cancel{u_\theta\frac{\partial\hat{\theta}}{\partial\theta}}\right]$$

$$\boxed{\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \quad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \quad \hat{r}\cdot\hat{r} = 1, \quad \hat{r}\cdot\hat{\theta} = 0, \quad \hat{\theta}\cdot\hat{\theta} = 1}$$

$$= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\Rightarrow \nabla_{(r,\theta)}\cdot\underline{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\boxed{\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}$$

Laplacian $(\nabla \cdot \nabla \phi)$

let $\underline{u} = \nabla \phi$

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\underline{u} = \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{bmatrix} \Rightarrow \nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}$$