Vector Calculus

1. Scalars and vectors
2. Dot products, cross products and dyadics
3. Tensors and matrices
4. Rotations and coordinate transformations

Vectors

- In physical space, they have magnitude and direction
- Not limited to physical space – you need a basis or coordinate system to describe vectors

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 + \ldots = \Sigma_{i=1}^{n} x_i\hat{\mathbf{e}}_i$$

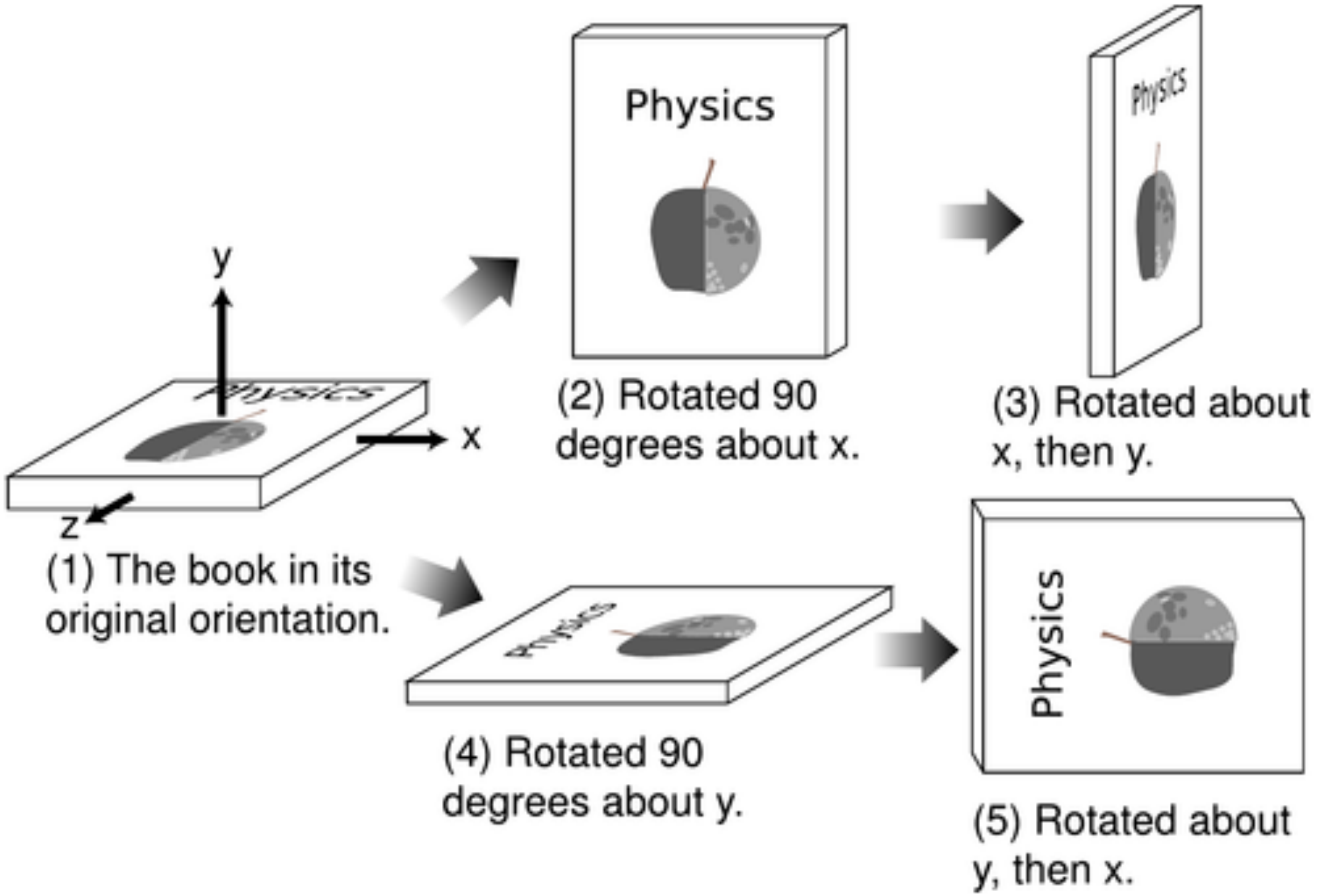
Can be orthonormal if?
Are they always orthogonal?

Rules for vectors

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Is (finite) rotation a vector?

Rules for vectors

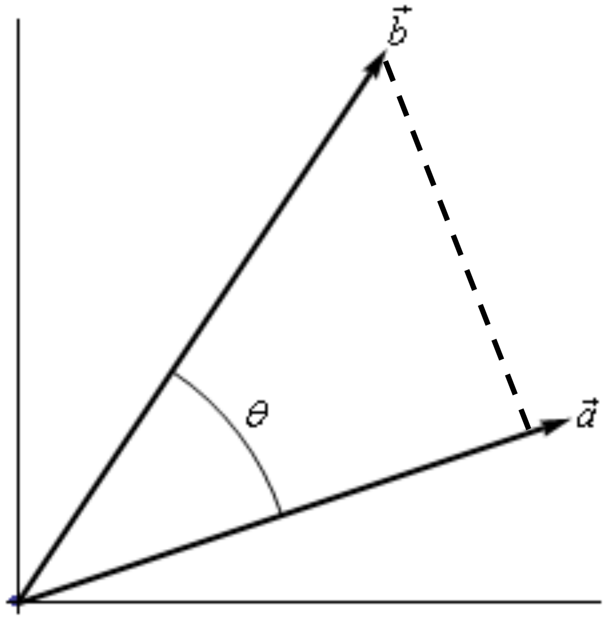


(Picture source: Benjamin Crowell, *General Relativity*, p. 256.)

Vector Multiplication

Dot product – a projection

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 + \ldots = \Sigma_{i=1}^{n}(\mathbf{x}.\hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i$$



$$\mathbf{a}.\mathbf{b} = a_i b_j \delta_{ij}$$
$$\mathbf{a}.\mathbf{b} = ||a||\,||b||\cos\theta$$

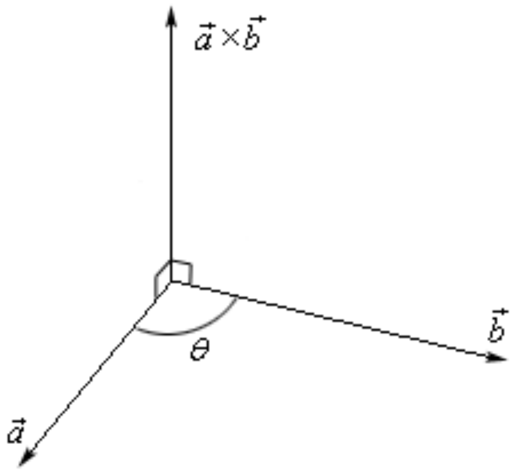
Tells us if vectors are parallel or at an angle

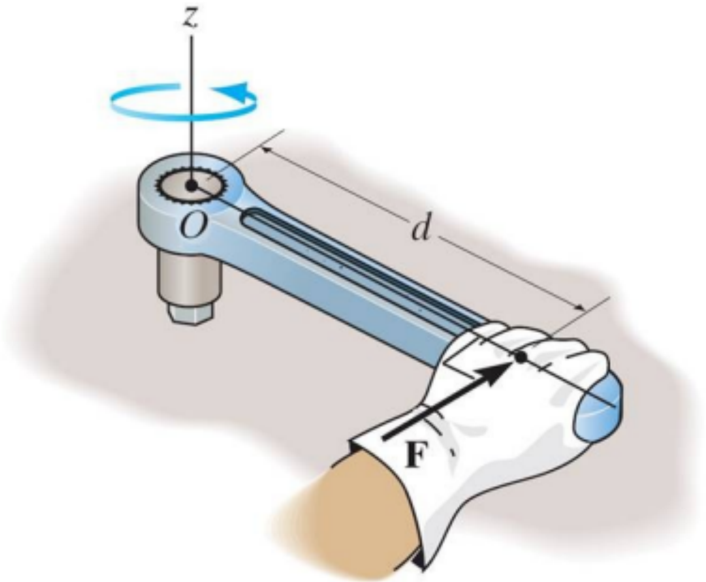
Dot, cross and dyadic products

Cross product – a rotation

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$





Also tells us if vectors are parallel

Dot, cross and dyadic products

Dyadic product – make tensors

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Write this out in matrix form

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\, dx = x + C$
2. $\int a\, dx = ax + C$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\, dx = -\cos x + C$
5. $\int \cos x\, dx = \sin x + C$
6. $\int \sec^2 x\, dx = \tan x + C$
7. $\int \csc^2 x\, dx = -\cot x + C$
8. $\int \sec x(\tan x)\, dx = \sec x + C$
9. $\int \csc x(\cot x)\, dx = -\csc x + C$
10. $\int \frac{1}{x} dx = \ln|x| + C$
11. $\int e^x dx = e^x + C$
12. $\int a^x dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$

Vector calculus

Chain rule for differentiation

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Vector calculus

Problem set for differentiation of a scalar field (analytically and numerically)

1. Compute the derivative of $f(x) = 2x^2 - 6x + 3$ for $-5<x<5$ and plot f, f'
2. Compute the derivative of $f(x) = (5x - 8)^{0.5}$ for $2<x<5$ and plot f, f'
3. Compute the derivative of $f(x) = (x^{2/3})(1-x^2)$ for $-1<x<1$ and plot f, f'
4. The timeseries of a tide gauge reading can be described by the function $z(t) = (6t^{1/3})/(4t+1)$. Determine when sea level is rising, and when is it falling?

Gradients of vector fields

We look at taking derivatives of scalar fields. Now we will look at how to work with gradients of vector fields.

$$d\mathbf{u}_i = \frac{\partial u_i}{\partial x_j} dx_j \mathbf{e}_i$$

We do this with the help of the del or nabla operator

$$\nabla u, \nabla \mathbf{u}, \nabla . \mathbf{u}, \nabla \times \mathbf{u}$$

Harps, p. 984.

In addition to gradients, del can also be used to find out if vector fields are convergent/divergent or rotating/circulating

Gradients of scalar and vector fields

Problem set for differentiation of 2-d fields (numerically)

1. Compute the partial derivatives of $f(x,y) = \exp(-(x^2+y^2))$ for $-5<x,y<5$. Plot f(x,y) in the given domain, and the gradient vector in a grid. Comment on the pattern the gradient makes.

2. Given a vector field, here it is the horizontal velocities of deforming blocks given as

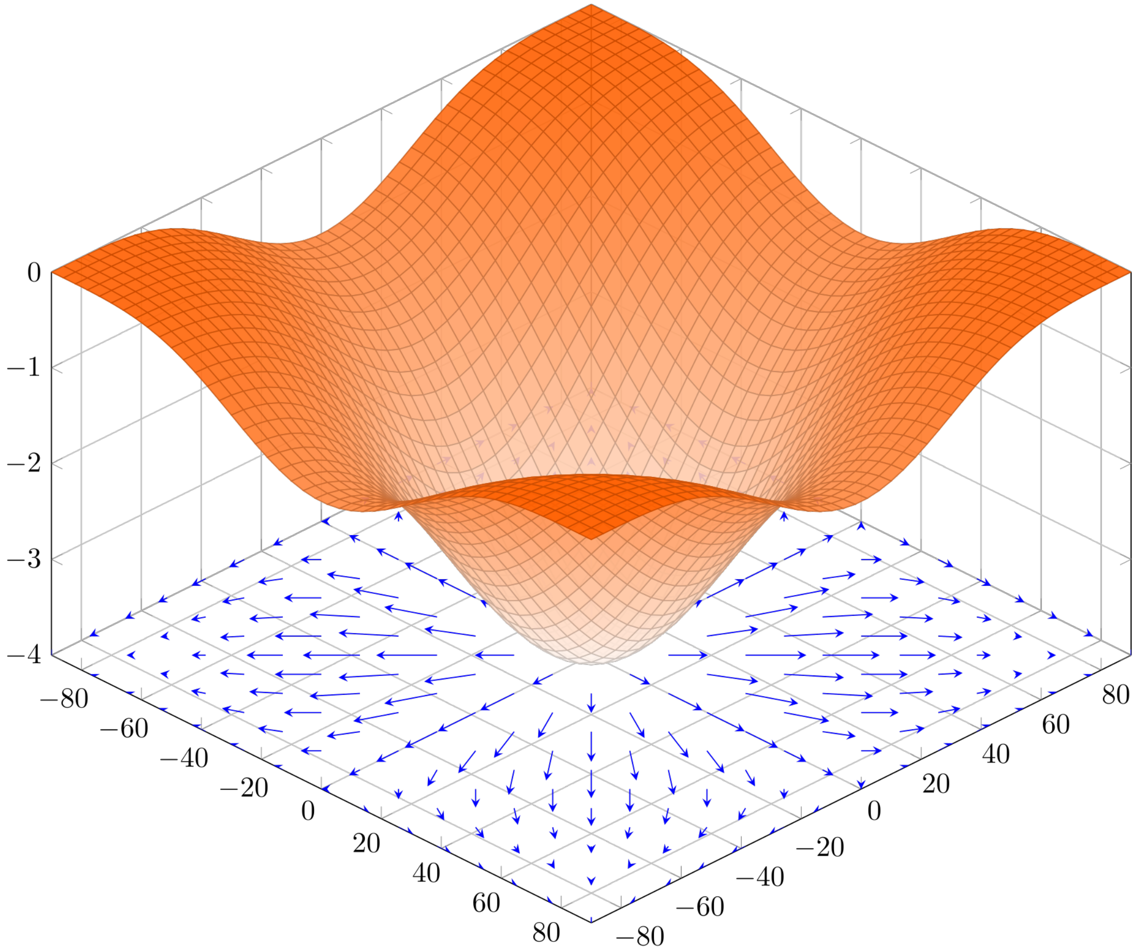
   $v_x(x,y) = mx/r^2 + 2ay*\cos(a*xy)$, $v_y(x,y) = my/r^2 + 2ax*\cos(a*xy)$, where $[0 < r^2 = x^2+y^2 < 3]$ and $a = 0.7$, $m = 3$

   Compute the gradients of each component of this field.

   Plot the vector field and the gradient of each component separately.

The del (nabla) operator

Gradient (scalar field) $\nabla u, \nabla \mathbf{u}$ Gradient (vector field)



$$\begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathsf{T}} f_1 \\ \vdots \\ \nabla^{\mathsf{T}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

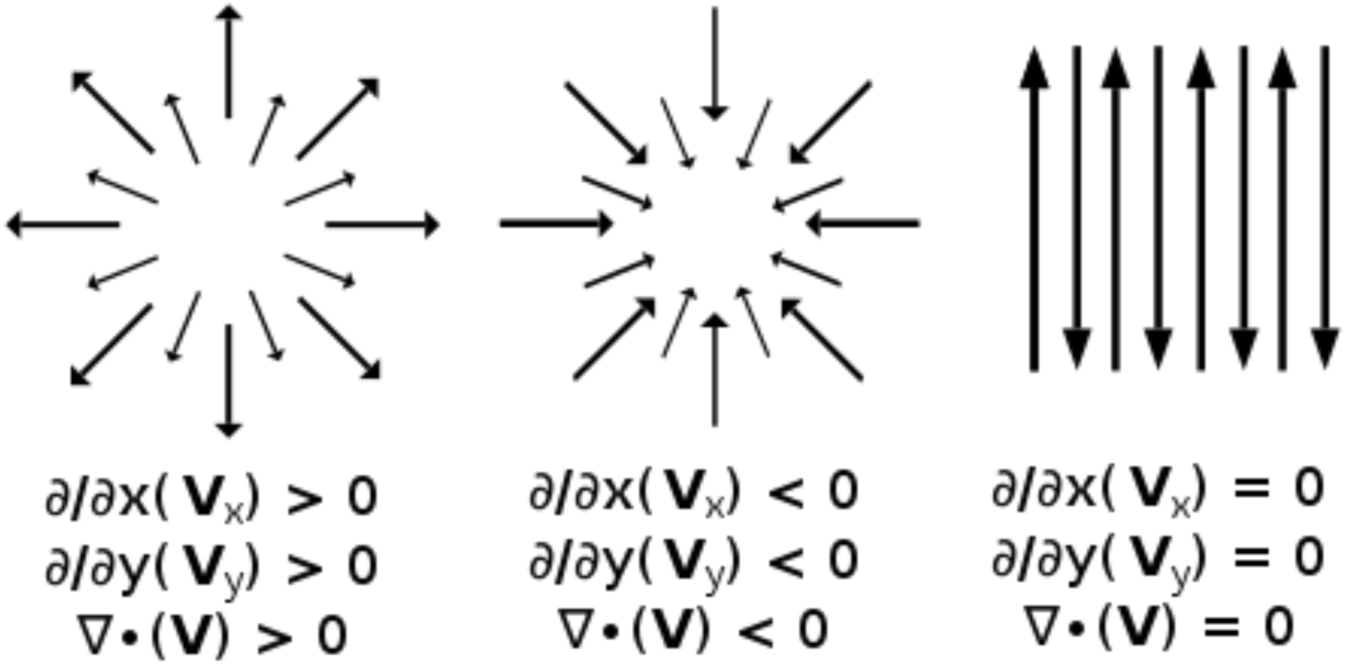
- An example is the Jacobian or the design matrix in inverse problems
- Also think of strain tensors

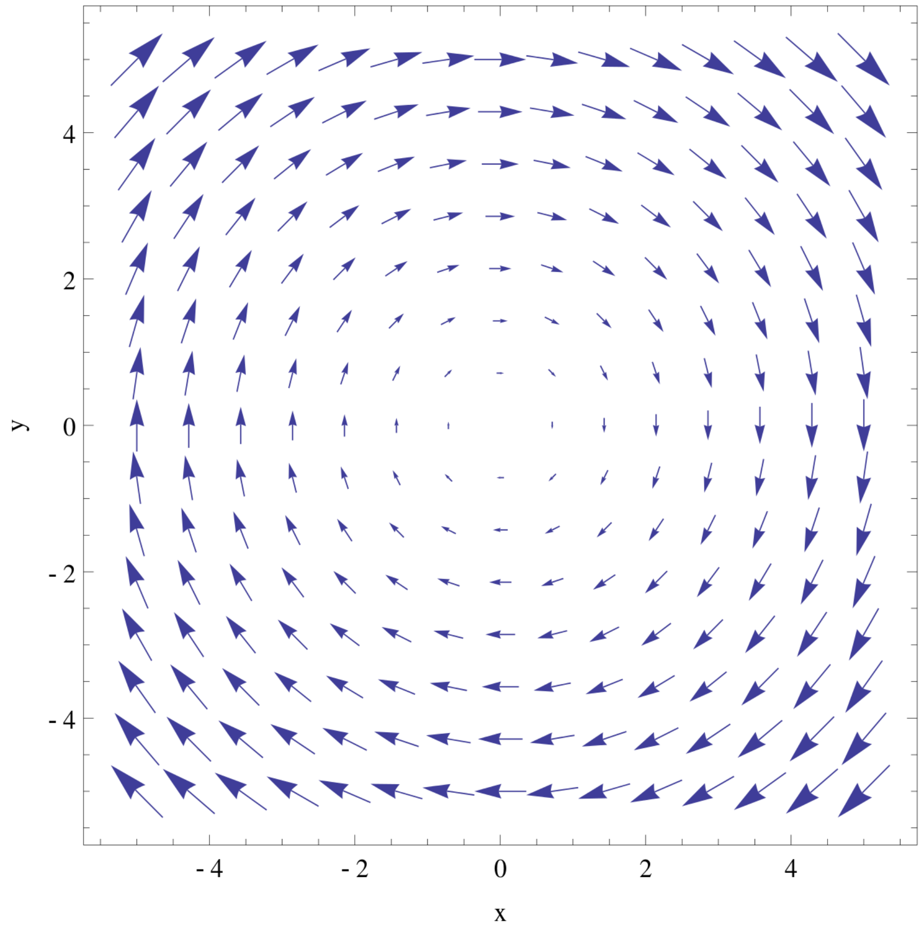
The del (nabla) operator

Divergence

$$\nabla . \mathbf{u}, \nabla \times \mathbf{u}$$

Curl





Vector calculus

Problem set (solve analytically and numerically)

1. The temperature field at any point in a lake is given by $T(x,y,z) = \sin(yz) + \log(1+x^2)$ for all x,y,z. Plot this scalar field at the lake surface ($z=0$) and at 1 unit below the surface (z=1).
2. Compute the gradient of $T(x,y,z)$. Plot this vector field at the lake surface.
3. At the point $(1,1,0)$ on the lake surface, a boat is moving in the $\langle 3,1,0 \rangle$ direction. First calculate the unit vector for the boat's motion, and then compute the directional directive of the temperature field as seen by the boat. Hint: directional derivatives are projections of the gradient vector along some unit vector.