Week 6 Topics

(1) Unicycle library
- Reading fault geometry
- Computing displacements & stresses in 2-d & 3-d
- Stress transfer b/w 2 faults

(2) Friction on faults
- Coulomb friction (static)
- Dynamic friction coefficient (velocity dependance)
  - linear
  - power-law
  - logarithmic
- Rate-state dependant friction

discuss FD integration

(3) Governing equations for fault-slip in an elastic medium

- $\nabla \cdot \sigma + \underline{f} = 0$ ← static equilibrium

subject to frictional BCs.

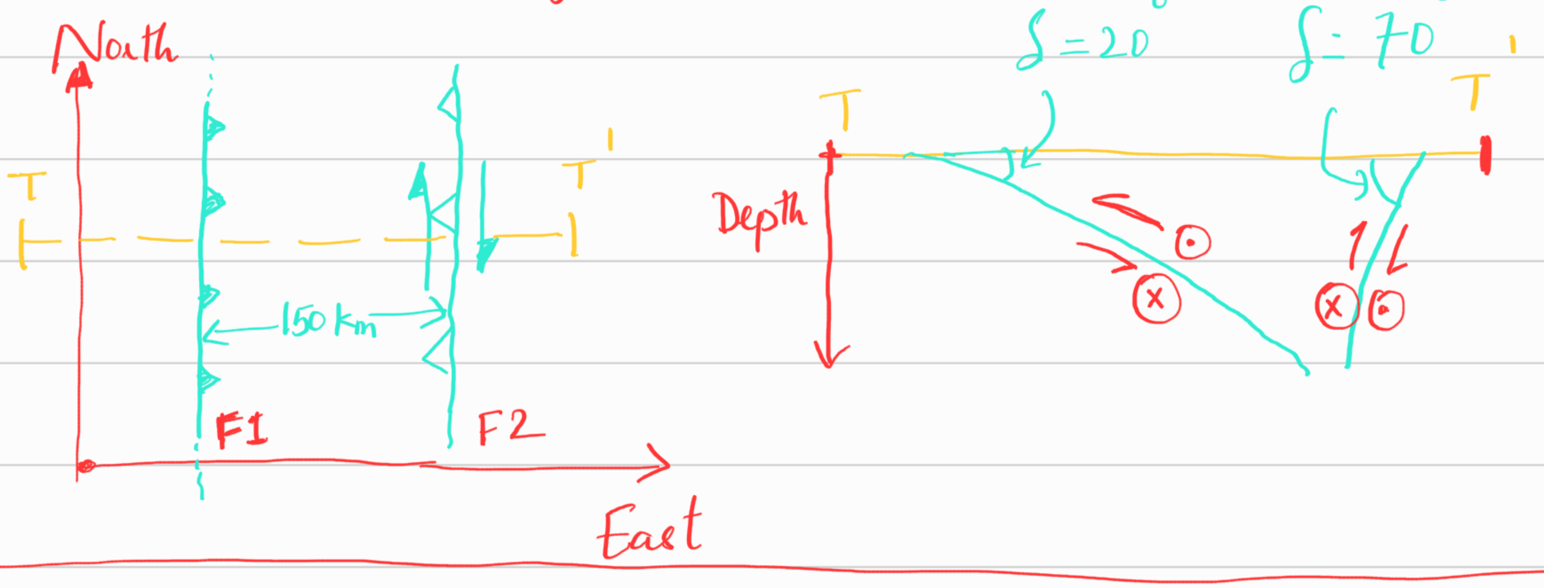
taking $\frac{\partial}{\partial t}(\,)$, $\nabla \cdot \dot{\sigma} + \underline{\dot{f}} = 0$ ← quasi-static equilibrium

equilibrium

- solutions using ODE solvers
  - Implicit / Explicit Euler
  - Adams-Bashforth / Adams-Moulton
  - Crank-Nicholson
  - Runge-Kutta

In-class exercise,

You know there are 2 parallel faults, both of whose surface trace are aligned with the longitudes that pass through them.



Each fault is 120 km along-strike & extends to a depth of 20 km. Mesh the surfaces using 10 km × 10 km square patches.

- The fault F1 slipped in a recent earthquake

The slip patch was at the centre of the fault and extended uniformly radially with a radius of 30 km.

$$[S] = \begin{cases} \text{dip-slip} - 2\,\text{m},\ \text{strike-slip} - 1\,\text{m} & r \leq 30\ \text{km} \\ 0 & r > 30\ \text{km} \end{cases}$$

- compute the stress change $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix}$ at all patch centers for both faults F1 & F2.
- use the 6 stresses to compute the traction vector $[\underline{t}]$ at each patch.
- compute & plot each traction scalar (τ_s, τ_d, σ_n) for each patch.
- It appears that fault F2 has a slip-averaged rake angle of 140° (mostly right-lateral strike-slip). Compute the Coulomb stress change on F2 due to an earthquake on F1.

Friction on faults

If fault-plane is not at failure

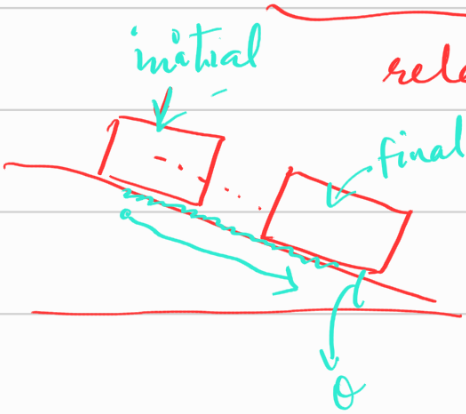
σ_3 σ_m σ_1 = σ_n

for sliding to occur,
Frictional Stress/strength = Internal stress

- if $\tau_f = f\sigma_n$, & we know earthquakes or frictional slip causes $\Delta\tau \downarrow$, what causes τ_f to change?

→ when sliding started/nucleated,

$\underbrace{\text{Internal elastic stress}}_{\text{released}}$ = Frictional strength



→ to continue sliding, τ_f decreases faster than elastic stress release.

Coulomb friction is only 'static' → insufficient to describe sliding effects.

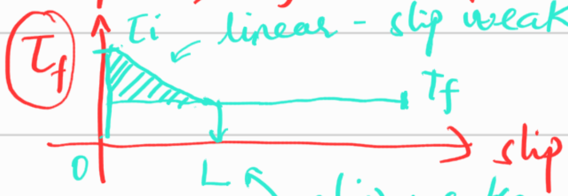
Dynamic friction

↓ slip-dependant

↳ slip-weakening

$\Delta\tau_f \propto -S$

as slip ↑, $\tau_f \downarrow$ → positive feedback



→ $\Delta\tau_f$ due to Δf, minimal $\Delta\sigma_n$ effects

Rate-dependant.

↳ velocity-dep friction.

$\boxed{f \propto v \quad \text{linear}}$

$f \propto v^m$ power-law

$f \propto \log v$ log-form

Rate-state dependant friction

$$f = f_0 + \underbrace{a \log\left(\frac{v}{v_0}\right)}_{\substack{\text{velocity-dep} \\ \text{rate-effect}}} + \underbrace{b \log\left(\frac{v_0 \theta}{d_c}\right)}_{\text{related to contact-time}}$$

another way of slip-dependence

state-effect

$$\frac{d\theta}{dt} = f(v, \theta)$$

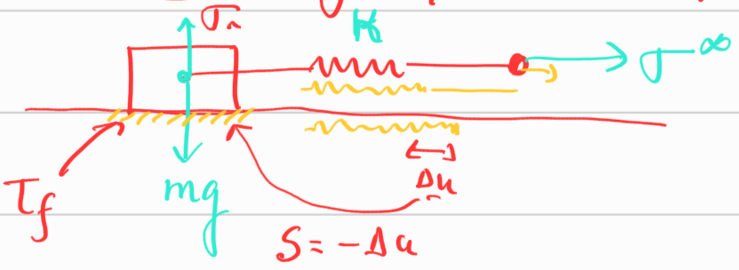
When we did GFs derivations, we solved

$\nabla \cdot \sigma + f = 0$, come from imposed slip on a fault patch.

$GF(x_1, x_2, x_3; y_1, y_2, y_3, \underbrace{\phi, \delta, \lambda}_{\hat{s},\, \hat{d}}, L, W)$ ← linear b/w $u_i \propto S_j$

but how much should a fault slip?

Governing equations for fault slip



if block is stationary, and σ^∞ is applied then spring will extend

⇒ elastic/spring stress ↑

→ if the block slides, then spring will contract ⇒ elastic stress ↓

$$\sigma^\infty + \underbrace{K(\Delta u)}_{(-KS)} = f(\sigma_n)$$

elastic stress

this is time-invariant

$\frac{\partial}{\partial t}(\sigma^\infty) = c$

f is v-dependent

$\frac{\partial}{\partial t}()$

$$\dot{\sigma}^\infty - K v = \frac{d}{dt}(f)\,\sigma_n \quad —— \quad ①$$

$v \to$ unknown, 1 differential equation.

$\to$ we need an initial condition $\Rightarrow$ IVP

$f = Av \longrightarrow$ apply in 1,

$$\dot{\sigma}^{\infty} - Kv = A\sigma_n \frac{dv}{dt}$$

$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} \to$ given

you wanted u_i

$$\boxed{\frac{dv}{dt} + \frac{K}{A\sigma_n} v = \dot{\sigma}^{\infty}} \longrightarrow \text{subject to } \underbrace{v = v_i \text{ at } t = 0.}_{IC}$$

$$\frac{dv}{dt} = \left(\dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v\right)$$

$$\frac{dv}{\left[\dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v\right]} = dt.$$

$$\int \frac{dx}{(a-x)} = \int dt$$

$$-\ln(a-x) = t + c.$$

$$\frac{A\sigma_n}{K} \log\left(\dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v\right) = t + c.$$

$\longrightarrow$ insert IC to calculate C.

$\Rightarrow$ solution for $v(t)$.

Numerical solution

$$\frac{dv}{dt} + \frac{K}{A\sigma_n}v = \dot{\sigma}^{\infty}, \quad \underset{IC}{v(t=0) = v_i}$$

$v_0 \quad v_1 \quad v_2 \quad v_3 \quad v_4 \quad \ldots$

$t_0 \quad t_1 \quad t_2 \quad t_3$

Δt

$$\frac{dv}{dt} = \dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v$$

$$\frac{v_{i+1} - v_i}{\Delta t} = \dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v_i$$

$$v_{i+1} = \Delta t\left(\dot{\sigma}^{\infty} - \frac{K}{A\sigma_n}v_i\right) + v_i \leftarrow \text{Finite difference approximation.}$$

Known

Cheryl $\to$ Adams-Bashforth / Adams-Moulton methods

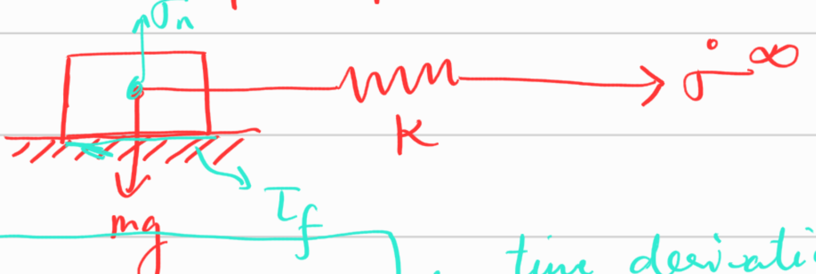
Grace $\to$ Implicit & Explicit Euler method

Iatan → $\underbrace{\text{Runge-Kutta methods}}$

Method to solve ODEs → $\frac{d^n y}{dt^n}$ ← vector

- Friction on faults → v - dependant
  → slip - dependant. $\dot{\sigma}^{\infty} = K V_{pl}$

↳ spring-slider



$$\boxed{\dot{\sigma}^{\infty} - Kv = \frac{d}{dt}(\tau_f)}$$ ← time derivative of stress

τ_f → velocity - dependent (linear) $\sim f\sigma_n$

$\tau_f = (A\sigma_n)v$ $\qquad f = Av$

$\frac{d\tau_f}{dt} = A\sigma_n \frac{dv}{dt}$

unknown $v(t)$

$$\boxed{\dot{\sigma}^{\infty} - Kv = A\sigma_n \frac{dv}{dt}}$$

$$\frac{dv}{dt} = \left[\frac{\dot{\sigma}^{\infty} - Kv}{A\sigma_n}\right], \quad \frac{dv}{\dot{\sigma}^{\infty} - Kv} = \frac{1}{A\sigma_n} dt$$

$$\int \frac{dx}{(b-ax)} = \frac{-\log(ax)}{a}$$

$$\frac{-\log(\dot{\sigma}^{\infty} - Kv)}{K} = \frac{t}{A\sigma_n} + C$$

variable separable method

IC: at $t=0$, $v = v_i$,

$$(\dot{\sigma}^{\infty} - Kv) = \exp\left(-\frac{tK}{A\sigma_n} + C\right)$$

$$\dot{\sigma}^{\infty} - Kv = e^{-\frac{tK}{A\sigma_n}} \exp(C)$$

IC : $\dot{\sigma}^{\infty} - KV_i = \exp(C)$ $\qquad t_R = \dfrac{A\sigma_n}{K}$

$$\Rightarrow \dot{\sigma}^{\infty} - KV = e^{-t/t_R}\left(\dot{\sigma}^{\infty} - KV_i\right)$$

$$\boxed{V(t) = \frac{\dot{\sigma}^{\infty} - e^{-t/t_R}\left(\dot{\sigma}^{\infty} - KV_i\right)}{K}}$$

$V(t) \propto e^{-t/t_R}$

$t_R = \dfrac{A\sigma_n}{K}$

$\dot{\sigma}^{\infty} = KV_{pl}$

$$V(t) = \frac{KV_{pl} - e^{-t/t_R}\left(K(V_{pl} - V_i)\right)}{K}$$

$\exp\left(-\dfrac{t}{t_R}\right) = \dfrac{1}{\exp\left(\frac{t}{t_R}\right)}$

$$\boxed{V(t) = V_{pl} - e^{-t/t_R}\left(V_{pl} - V_i\right)}$$

at $t=0$, $V(t) = V_i$

at $t \to \infty$, $V(t \to \infty) = V_{pl}$

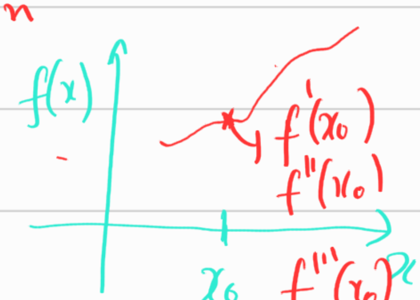
$$\boxed{\frac{dV}{dt} = \frac{K}{A\sigma_n}\left(V_{pl} - V\right)} \leftarrow \text{solve this numerically}$$

Taylor Series

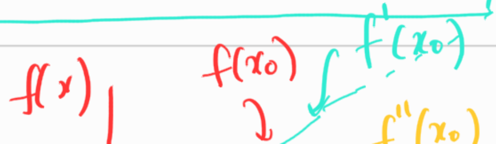
calculated at x_0

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!}\frac{df}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2f}{dx^2} + \cdots \frac{(x-x_0)^n}{n!}\frac{d^nf}{dx^n}$$

$$\boxed{\begin{aligned} f(x) &= f(x_0) + (x - x_0)\frac{df}{dx} \\ \frac{f(x) - f(x_0)}{(x - x_0)} &= \frac{df}{dx} \end{aligned}}$$



predicted $f(x)$ (1st order)



using taylor

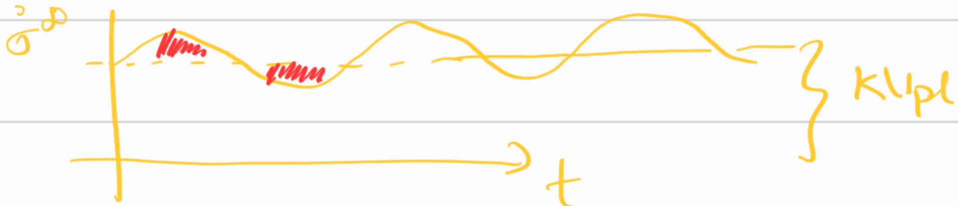
$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \left[a_5 x^5 \ldots \right.$$

error $\rightarrow [\Delta f] \rightarrow (\Delta x)^5$

$$\frac{dv}{dt} = \frac{\dot{\sigma}^\infty - Kv}{A\sigma_n}, \quad \text{we said } \dot{\sigma}^\infty = K V_{pl}$$

when loading was constant

$$\dot{\sigma}^\infty(t) = K V_{pl} + \alpha \sin \omega t$$



$$\frac{dv}{dt} = \frac{K V_{pl} + \alpha \sin \omega t - Kv}{A\sigma_n}$$

$$\frac{dv}{dt} = \frac{K V_{pl} - Kv + \alpha \sin \omega t}{A\sigma_n}$$

$$= \frac{K V_{pl}}{A\sigma_n} - \frac{Kv}{A\sigma_n} + \frac{\alpha}{A\sigma_n} \sin \omega t$$

Integrating factor

$$\boxed{\frac{dv}{dt} + \frac{Kv}{A\sigma_n} = \underbrace{\frac{K V_{pl}}{A\sigma_n} + \frac{\alpha}{A\sigma_n} \sin \omega t}_{\text{only a function of } t}}$$

$M(t)$

$M(t) \rightarrow$

$$\underbrace{M\frac{dv}{dt} + \frac{MK}{A\sigma_n}v} = M(t)\left[\frac{KV_{pl}}{A\sigma_n} + \frac{\alpha}{A\sigma_n}\sin\omega t\right]$$

$$\frac{d}{dt}\left(v \cdot M(t)\right) = M(t) \cdot Q(t)$$

$$v\frac{dM}{dt} + M\frac{dv}{dt}.$$

$\rightarrow$ integrating both sides

$$v(t)M(t) = \int M(t)\,Q(t)\,dt$$

method of Integrating factor

$$\Rightarrow \cancel{v}\frac{dM}{dt} = \frac{MK\cancel{v}}{A\sigma_n}.$$

$$\boxed{\frac{dM}{dt} = \frac{K}{A\sigma_n}M.}$$

$$\frac{dM}{M} = \frac{K}{A\sigma_n}dt.$$

$$\log M = \frac{K}{A\sigma_n}t.$$

$$M = \exp\left(\frac{K}{A\sigma_n}t\right)$$

$$v.\exp\left(\frac{K}{A\sigma_n}t\right) = \int \exp\left(\frac{K}{A\sigma_n}t\right)Q(t)\,dt.$$

$$= \int e^{\frac{t}{t_R}}\left[\frac{V_{pl}}{t_R} + \frac{\alpha}{A\sigma_n}\sin\omega t\right]dt.$$

Find integral.

$$\int e^{\frac{t}{t_R}}\left(\frac{V_{pl}}{t_R}\right)dt + \frac{\alpha}{A\sigma_n}\int e^{\frac{t}{t_R}}\sin\omega t\,dt$$

$$= V_{pl}\,e^{t/t_R} + \frac{\alpha}{A\sigma_n}\left[\frac{-t_R\exp\left(\frac{t}{t_R}\right)\left[t_R\omega\cos\omega t - \sin\omega t\right]}{t_R^2\omega^2 + 1}\right]$$

$$v(t) = \frac{K^2V_{pl} + (A\sigma_n\omega)^2V_{pl} + \alpha K\sin\omega t - A\sigma_n\alpha\omega\cos\omega t}{K^2 + (A\sigma_n\omega)^2}$$

$$\exp\left(-\frac{Kt}{\ldots}\right) \qquad \boxed{t_R = \frac{A\sigma_n}{\ldots}}$$

$$+ C_1 \exp\left(- \frac{\quad}{A\sigma n}\right) \qquad \boxed{t_R = \frac{A\sigma n}{K}}$$

$$\vartheta(t) = \frac{V_{pl} + t_R^2 \omega^2 V_{pl} + \frac{\alpha}{k} \sin \omega t - \frac{\alpha t_R \omega}{K} \cos \omega t}{1 + (t_R \omega)^2}$$

$$+ C_1 \exp\left(-t/t_R\right)$$

at $t = 0$, ϑ_i — Initial Condition.

$$\vartheta(t) = \exp\left(-\frac{t}{t_R}\right)\left[\vartheta_i - \frac{V_{pl} t_R^2 \omega^2 - \frac{\alpha}{k} t_R \omega + V_{pl}}{(t_R \omega)^2 + 1}\right]$$

$$+ \frac{V_{pl} + V_{pl} t_R^2 \omega^2 + \frac{\alpha}{K} \sin(\omega t) - \frac{\alpha}{K} t_R \omega \cos(\omega t)}{1 + (t_R \omega)^2}$$