Mohr's circle (2-d)

For a given stress state $\sigma = \begin{bmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{bmatrix}$, we can compute the tractions by rotating the tensor by an angle $\theta \rightarrow \sigma' = R \sigma R^T$ where $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\textcircled{1} \begin{cases} \sigma_{22}' = \sigma_{22}\cos^2\theta + \sigma_{33}\sin^2\theta + 2\sigma_{23}\sin\theta\cos\theta \\ \sigma_{23}' = -(\sigma_{22} - \sigma_{33})\sin\theta\cos\theta + \sigma_{23}(\cos^2\theta - \sin^2\theta) \end{cases}$$

recall that $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\& \quad \sin 2\theta = 2\sin\theta\cos\theta \quad \text{——} \; \textcircled{2}$$

substituting ② in ①, we get

$$\sigma_{22}' = \frac{1}{2}(\sigma_{22} + \sigma_{33}) + \frac{1}{2}(\sigma_{22} - \sigma_{33})\cos 2\theta + \sigma_{23}\sin 2\theta$$

$$\sigma_{23}' = -\frac{1}{2}(\sigma_{22} - \sigma_{33})\sin 2\theta + \sigma_{23}\cos 2\theta$$

set $\boxed{\dfrac{\sigma_{22}+\sigma_{33}}{2} = \sigma_m \quad \& \quad \dfrac{\sigma_{22}-\sigma_{33}}{2} = \sigma_D}$

$$\sigma_n \rightarrow \sigma_{22}' = \underline{\sigma_m + \sigma_D\cos 2\theta + \sigma_{23}\sin 2\theta} \quad \text{(normal traction)}$$

$$\tau \rightarrow \sigma_{23}' = \underline{-\sigma_D\sin 2\theta + \sigma_{23}\cos 2\theta} \quad \text{(shear traction)}$$

We can try to combine σ_{22}' & σ_{33}' in a way that we eliminate θ & obtain the locus of $\underbrace{\sigma_n \;\&\; \tau}_{\text{scalars}}$

$$\sigma_n - \sigma_m = \sigma_D\cos 2\theta + \sigma_{23}\sin 2\theta \quad \Big\} \; \textcircled{3}$$

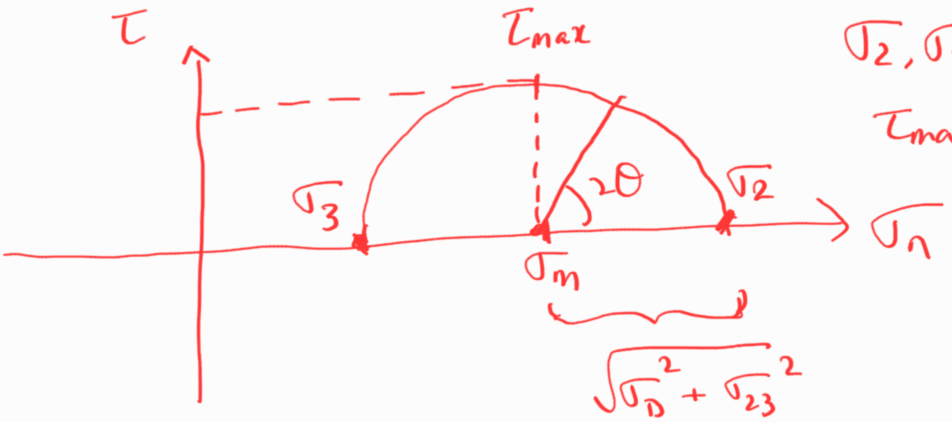
$$\tau = -\sigma_D \sin 2\theta + \sigma_{23} \cos 2\theta$$

to remove θ, it is obvious that we need to square both equations

$$(4)\quad \begin{cases} (\sigma_n - \sigma_m)^2 = \sigma_D^2 \cos^2 2\theta + \sigma_{23}^2 \sin^2 2\theta + 2\sigma_D \sigma_{23} \cos 2\theta \sin 2\theta \\ \tau^2 = \sigma_D^2 \sin^2 2\theta + \sigma_{23}^2 \cos^2 2\theta - 2\sigma_D \sigma_{23} \cos 2\theta \sin 2\theta \end{cases}$$

$$\underbrace{(\sigma_n - \sigma_m)^2 + \tau^2 = (\sigma_D^2 + \sigma_{23}^2)}$$

equation of a circle centred at $(\sigma_m, 0)$ with radius $= \sqrt{\sigma_D^2 + \sigma_{23}^2}$



$$\sigma_2, \sigma_3 = \sigma_m \pm \sqrt{\sigma_D^2 + \sigma_{23}^2}$$

$$\tau_{max} = \sqrt{\sigma_D^2 + \sigma_{23}^2}$$