9.9 Curl of a vector field

Let $\mathbf{v}(x,y,z) = v_1(x,y,z)\,\mathbf{i} + v_2(x,y,z)\,\mathbf{j} + v_3(x,y,z)\,\mathbf{k}$ be a differentiable vector function, where x, y, z are Cartesian coordinates. Then the *curl* of *the vector function* $\mathbf{v}$ or *of the vector field given by* $\mathbf{v}$ is defined by,

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}\,. \tag{9.9.1}$$

This is the formula when x, y, z are right-handed. If x, y, z are left-handed, the determinant has a minus sign in front (just as in eqn (9.3.2*) in Sec. 9.3).

Note: Instead of $\operatorname{curl}\mathbf{v}$, the notation $\operatorname{rot}\mathbf{v}$ is also sometimes used (suggested by "rotation", see Example 2).

Example 1: Curl of a vector function

Let $\mathbf{v} = yz\,\mathbf{i} + 3zx\,\mathbf{j} + z\,\mathbf{k}$ (with right-handed x, y, z), then eqn (9.9.1) gives,

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}\,.$$

The curl plays an important role in many applications. See Example 2. More about the nature and significance of the curl in Sec. 10.9 on Stokes's theorem.

Example 2: Rotation of a rigid body. Relation to the curl

As seen in Example 5 of Sec. 9.3, a rotation of a rigid body B about a fixed axis in space can be described by a vector $\mathbf{w}$ of magnitude ω in the direction of the axis of rotation, where $\omega\,(>0)$ is the angular speed of the rotation, and $\mathbf{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\mathbf{w}$. According to eqn (9.3.9), the velocity field of the rotation can be represented in the form,

$$\mathbf{v} = \mathbf{w}\times\mathbf{r},$$

where $\mathbf{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system *having the origin on the axis of rotation*. Let us choose right-handed Cartesian coordinates such that the axis of rotation is the z-axis. Then (see Example 2 in Sec. 9.4),

$$\mathbf{w} = \omega\,\mathbf{k}, \qquad \mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}\,.$$

Hence,

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\,\mathbf{k} = 2\mathbf{w}.$$

This proves the following theorem.

Theorem 1: Rotating body and curl

The curl of the velocity field of a rotating rigid body has the direction of the axis of rotation, and its magnitude equals twice the angular speed of the rotation.

The following two relations among grad, div, and curl are basic and shed further light on the nature of the curl.

Theorem 2: Grad, div and curl

Gradient fields are *irrotational*. That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,

$$\boxed{\text{curl}\,(\text{grad } f) = \mathbf{0}.} \tag{9.9.2}$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function $\mathbf{v}$ is zero,

$$\boxed{\text{div}\,(\text{curl } \mathbf{v}) = 0.} \tag{9.9.3}$$

Proof: Both eqns (9.9.2) and (9.9.3) follow directly from the definitions by straightforward calculation. In the proof of eqn (9.9.3) the six terms cancel in pairs.

- $\text{grad } f = \nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$,

$$\text{curl}\,(\text{grad } f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\mathbf{k},$$

$$= 0.$$

- $$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k},$$

$$\operatorname{div}(\operatorname{curl}\mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = 0.$$

Example 3: Rotational and irrotational fields

The field in Example 2 is not irrotational. A similar velocity field is obtained by stirring tea or coffee in a cup. The gravitational field in Theorem 3 of Sec. 9.7 has $\operatorname{curl}\mathbf{p} = \mathbf{0}$. It is an irrotational gradient field.

The term "irrotational" for $\operatorname{curl}\mathbf{v} = \mathbf{0}$ is suggested by the use of the curl for characterizing the rotation in a field. If a gradient field occurs elsewhere, not as a velocity field, it is usually called *conservative* (see Sec. 9.7). Relation (9.9.3) is plausible because of the interpretation of the curl as a rotation and of the divergence as a flux (see Example 2 in Sec. 9.8).

Since the curl is defined in terms of coordinates, it is necessary to find out whether the curl is a vector. This is true as stated in the next theorem.

Theorem 3: Invariance of the curl

The curl of the vector function $\mathbf{v}$ indicated by $\operatorname{curl}\mathbf{v}$ is a vector. That is, it has a length and direction that are independent of the particular choice of a Cartesian coordinate system in space. (Proof in Appendix 4 of EK (2011).)