# THREE DIMENSIONAL SYSTEMS

# Lecture 6: The Lorenz Equations

# 6. The Lorenz (1963) Equations

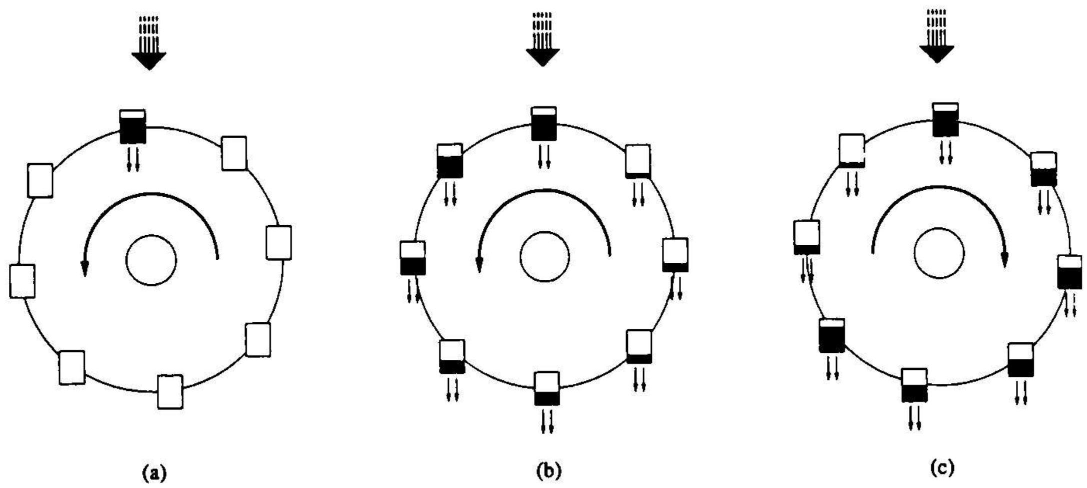
The Lorenz equations were originally derived by Saltzman (1962) as a 'minimalist' model of thermal convection in a box

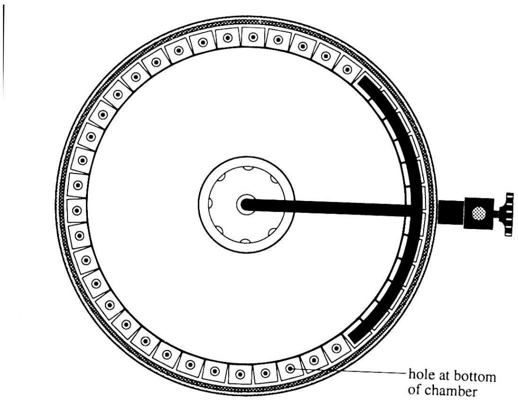
$$\dot{x} = \sigma(y - x) \quad (1)$$

$$\dot{y} = rx - y - xz \quad (2)$$

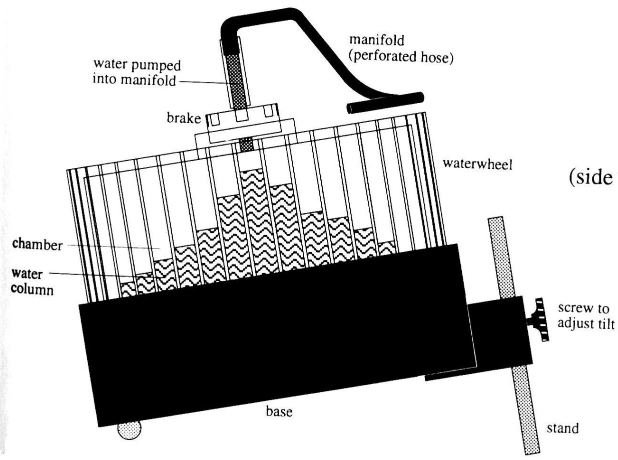
$$\dot{z} = xy - bz \quad (3)$$

where $\sigma$ ("Prandtl number"), $r$ ("Rayleigh number") and $b$ are parameters ($> 0$). These equations also arise in studies of convection and instability in planetary atmospheres, models of lasers and dynamos etc. Willem Malkus also devised a water-wheel demonstration...

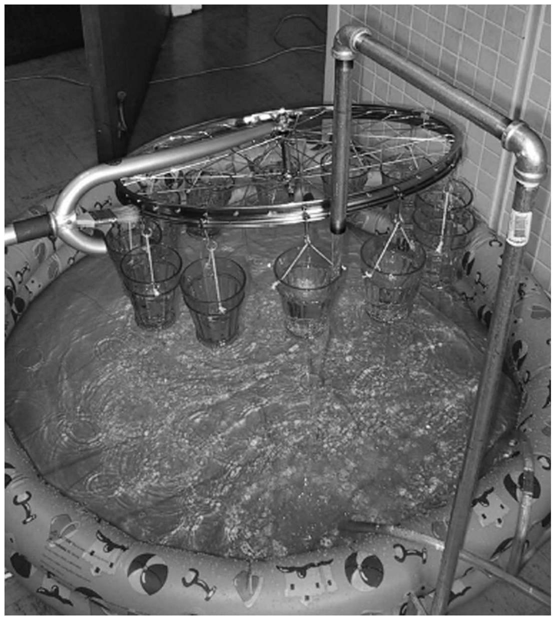





(top view)




(side view)



Malkus' water wheel

This simple-looking deterministic system turns out to have “interesting” erratic dynamics!

## 8.1 Simple properties of the Lorenz Equations

- *Nonlinearity* - the two nonlinearities are $xy$ and $xz$

- *Symmetry* - Equations are invariant under $(x, y) \to (-x, -y)$. Hence if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$

- *Volume contraction* - The Lorenz system is *dissipative* i.e. volumes in phase-space contract under the flow

- *Fixed points* - $(x^*, y^*, z^*) = (0, 0, 0)$ is a fixed point for *all* values of the parameters. For $r > 1$ there is also a pair of fixed points $C^{\pm}$ at $x^* = y^* = \pm\sqrt{b(r-1)}$, $z^* = r-1$. These coalesce with the origin as $r \to 1^+$ in a *pitchfork bifurcation*

## Linear stability of the origin

Linearization of the original equations about the origin yields

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y \\ \dot{z} &= -bz \end{aligned}$$

Hence, the $z$-motion decouples, leaving

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with trace $\tau = -\sigma - 1 < 0$ and determinant $\Delta = \sigma(1 - r)$.

For $r > 1$, origin is a *saddle point* since $\Delta < 0$ (see Lecture 4)

For $r < 1$, origin is a *sink* since $\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - r) = (\sigma - 1)^2 + 4\sigma r > 0 \rightarrow$ a stable node.

Actually for $r < 1$ it can be shown that *every* trajectory approaches the origin as $t \rightarrow \infty$ the origin is *globally stable*, hence there can be *no limit cycles or chaos for* $r < 1$.

Subcritical Hopf bifurcation occurs at

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \equiv r_H > 1$$

assuming that $\sigma - b - 1 > 0$.

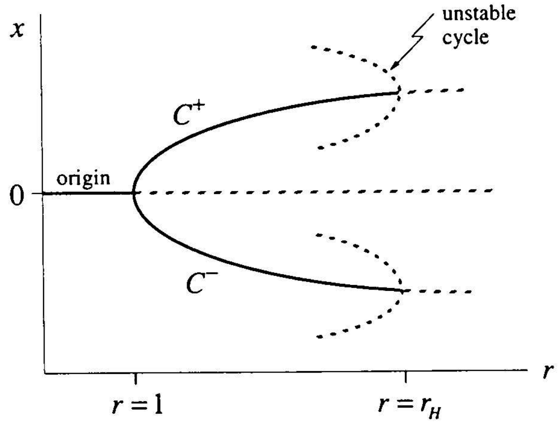



Fig. 6.1.1 - (Partial bifurcation diagram)

Recall from Lecture 5 that at a subcritical Hopf bifurcation trajectories must fly off to a *distant attractor*. This 2nd attractor must have some *strange properties*, since any limit cycles for $r > r_H$ are unstable (cf "proof" by Lorenz). The trajectories for $r > r_H$ are therefore continually being repelled from one unstable object to another. At the same time, they are confined to a *bounded set of zero volume*, yet manage to move in this set forever *without intersecting* - a **strange attractor!**

## 6.2 Chaos on a Strange Attractor

Lorenz considered the case $\sigma = 10, b = 8/3, r = 28$ with $(x_0, y_0, z_0) = (0, 1, 0)$.

NB $r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1) \simeq 24.74$, hence $r > r_H$. The resulting solution $y(t)$ looks like...

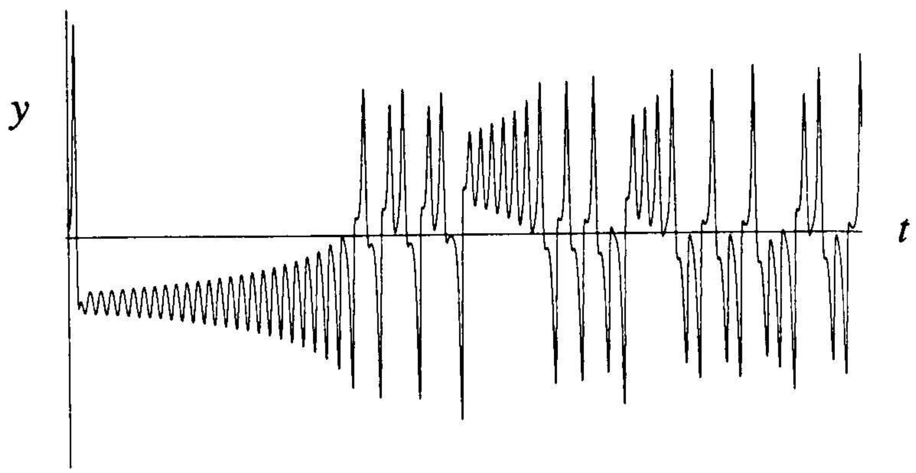



Fig. 6.2.1

After an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$ but never repeats exactly. The motion is *aperiodic*.

Lorenz discovered that a wonderful structure emerges if the solution is visualized as a *trajectory in phase space*. For instance, when $x(t)$ is plotted against $z(t)$, the famous *butterfly wing pattern* appears

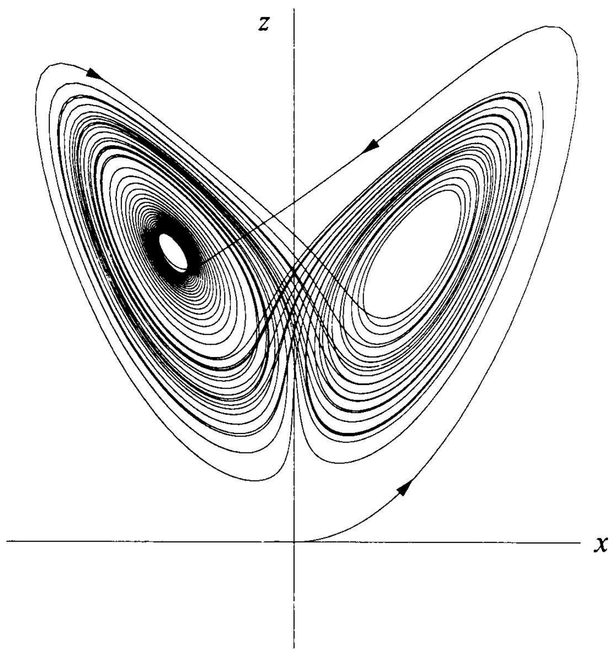



Fig. 6.2.2

- The trajectory appears to cross itself repeatedly, but that's just an artifact of projecting the 3-dimensional trajectory onto a 2-dimensional plane. In 3-D *no crossings occur!*

- The number of circuits made on either side varies *unpredictably* from one cycle to the next. The sequence of the number of circuits in each lobe has many of the characteristics of a *random sequence!*

- When the trajectory is viewed in all 3 dimensions, it appears to settle onto a thin set that looks like a pair of butterfly wings. We call this attractor a *strange attractor* and it can be shown schematically as...

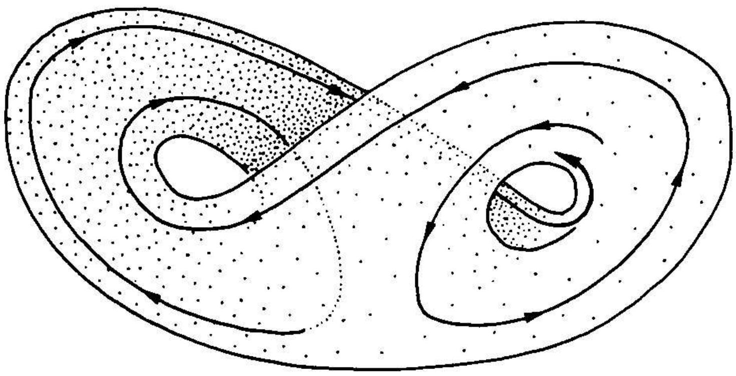


Fig. 6.2.3

What is the *geometric structure* of the strange attractor?

The uniqueness theorem means that trajectories *cannot cross* or merge, hence the two surfaces of the strange attractor can only *appear* to merge.

Lorenz concluded that "there is an infinite complex of surfaces" where they *appear* to merge. Today this "infinite complex of surfaces" would be called a **FRACTAL**.

A *fractal* is a set of points with zero volume but infinite surface area.

Fractals will be discussed later after a closer look at chaos...

## Exponential divergence of nearby trajectories

The motion on the attractor exhibits *sensitive dependence on initial conditions*. Two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. The practical implication is that long-term prediction becomes impossible in a system like this, where small uncertainties are amplified enormously fast.

Suppose we let transients decay so that the trajectory is "on" the attractor. Suppose $\mathbf{x}(t)$ is a point on the attractor at time $t$, and consider a nearby point, say $\mathbf{x}(t) + \delta(t)$, where $\delta$ is a tiny separation vector of initial length $||\delta_0|| = 10^{-15}$, say

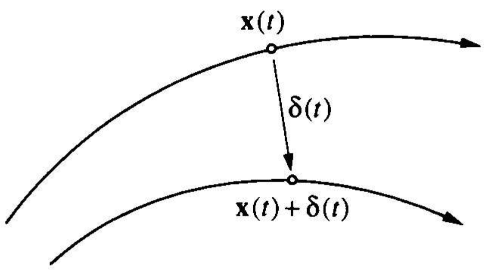
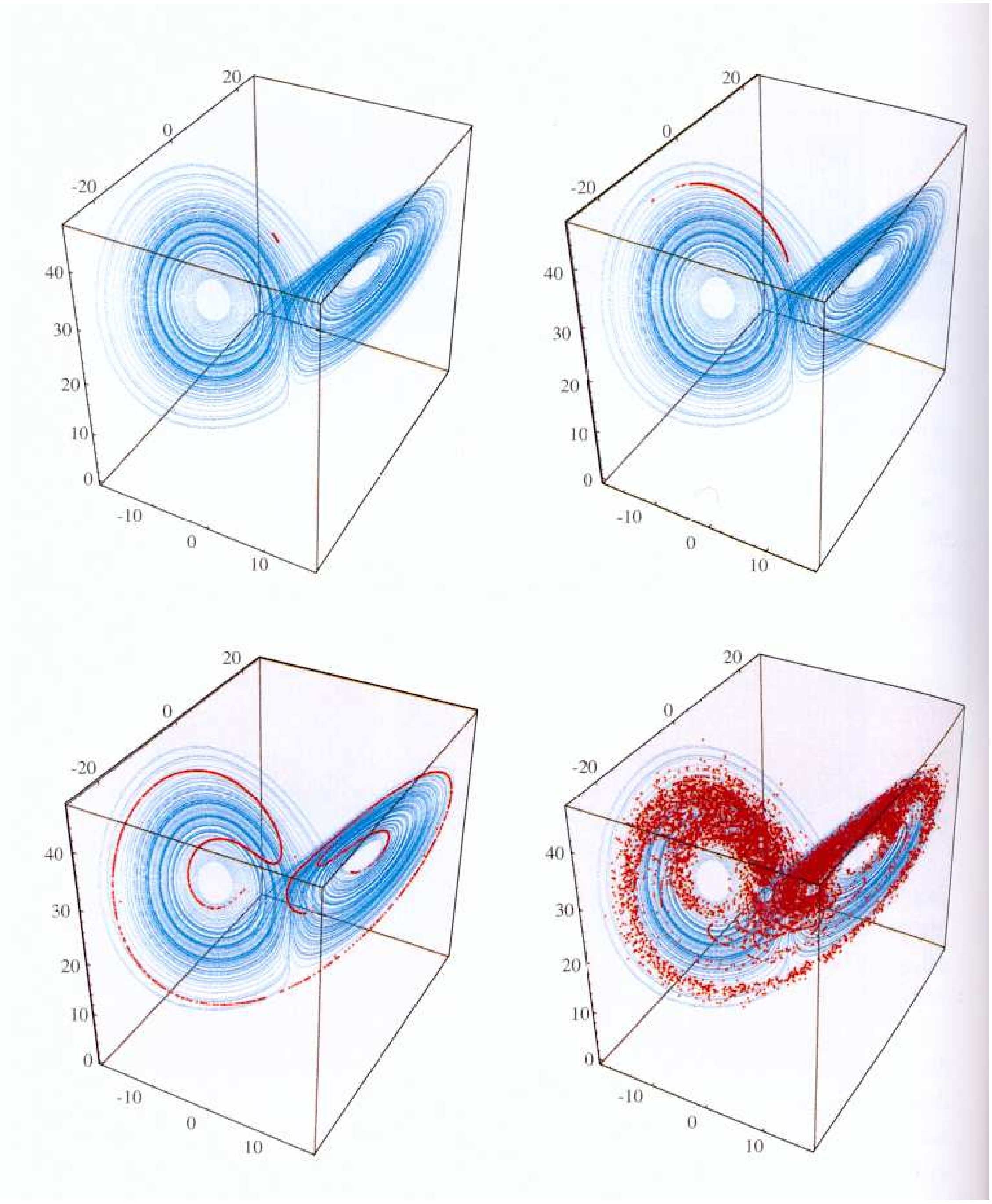



Fig. 6.2.4




Sensitivity to initial conditions on Lorenz attractor

In numerical studies of the Lorenz attractor, one finds that $||\delta(t)|| \sim ||\delta_{\mathbf{0}}||e^{\lambda t}$, where $\lambda \simeq 0.9$.

Hence *neighbouring trajectories separate exponentially fast!*

Equivalently, if we plot $\ln ||\delta(t)||$ versus $t$, we find a curve that is close to a straight line with a positive slope $\lambda$.

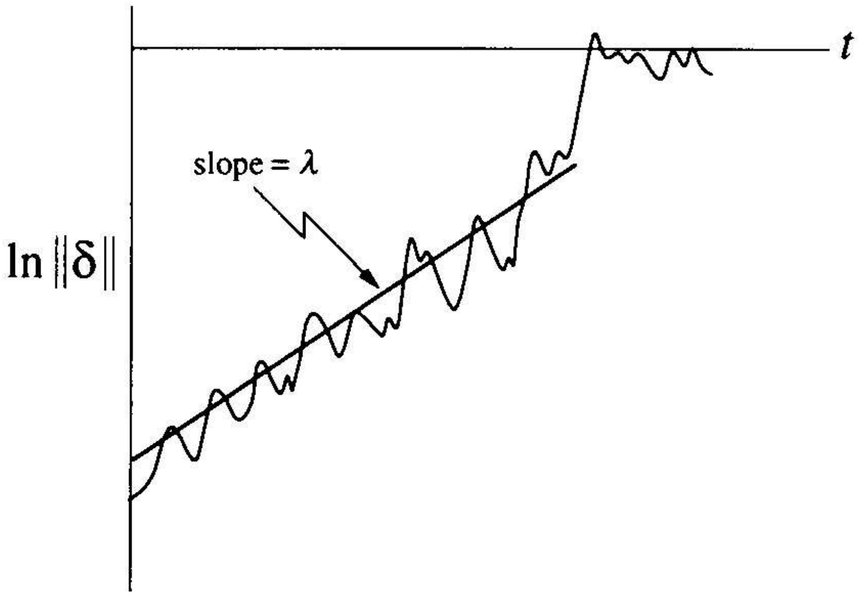



Fig. 6.2.5

Note...

- The curve is never straight, but has wiggles since the strength of exponential divergence varies somewhat along the attractor.

- The exponential divergence must stop when the separation is comparable to the "diameter" of the attractor - the trajectories cannot get any further apart! (curve saturates for large $t$)

- The number $\lambda$ is often called the *Lyapunov exponent*, though this is somewhat sloppy terminology....

This is because...

(i) There are actually $n$ different Lyapunov exponents for an $n$-dimensional system, defined as follows...

Consider the evolution of an infinitesimal sphere (in phase space) of perturbed initial conditions. During its evolution the sphere becomes distorted into an infinitesimal ellipsoid. Let $\delta_k(t), k = 1, 2, 3...n$ denote the length of the $k$th principal axis of the ellipsoid. Then

$$\delta_k(t) \sim \delta_k(0)e^{\lambda_k t},$$

where the $\lambda_k$ are the Lyapunov exponents. For large $t$, the diameter of the ellipsoid is controlled by the most positive $\lambda_k$. Thus our $\lambda$ above is actually the *largest Lyapunov exponent*.

(ii) $\lambda$ depends (slightly) on which trajectory we study. We should really *average* over many different points on the same trajectory to get the true value of $\lambda$.

When a system has a positive Lyapunov exponent, there is a time horizon beyond which prediction will break down....

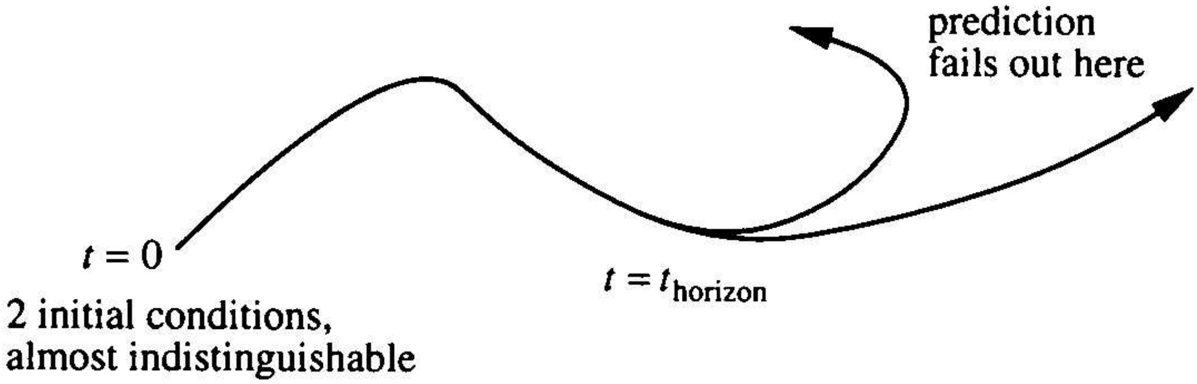



Fig. 6.2.6

Suppose we measure the initial conditions of an experimental system very accurately. Of course no measurement is perfect - there is always some error $||\delta_0||$ between our estimate and the true initial state. After a time $t$ the discrepancy grows to $||\delta(t)|| \sim ||\delta_0||e^{\lambda t}$.

Let $a$ be a measure of our tolerance, i.e. if a prediction is within $a$ of the true state, we consider it acceptable. Then our prediction becomes intolerable when $||\delta(t)|| \geq a$, i.e. after a time

$$t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{||\delta||}\right)$$

The logarithmic dependence on $||\delta_0||$ is what hurts....!

Example Suppose $a = 10^{-3}$, $||\delta_0|| = 10^{-7}$.

$$\Rightarrow t_{horizon} = \frac{4 \ln 10}{\lambda}$$

If we improve the initial error to $||\delta_0|| = 10^{-13}$,

$$\Rightarrow t_{horizon} = \frac{10 \ln 10}{\lambda}$$

i.e. only $10/4 = 2.5$ times longer!

## Defining Chaos

No definition of the term "chaos" is universally accepted - even now! - but almost everyone would agree on the three ingredients used in the following working definition:

*Chaos is* **aperiodic long-term behaviour** *in a* **deterministic** *system that exhibits* **sensitive dependence on initial conditions**

1. *Aperiodic long-term behaviour* means that there are trajectories which do not settle down to fixed points, periodic or quasi-periodic orbits as $t \to \infty$.

2. *Deterministic* means that the system has no random or noisy inputs or parameters. Irregular behaviour arises solely from the system's nonlinearity.

3. *Sensitive dependence on initial conditions* means that nearby trajectories diverge exponentially fast, i.e. the system has at least one positive Lyapunov exponent.

Some people think that chaos is just a fancy word for instability. For example, the system $\dot{x} = x$ is deterministic and shows exponential separation of nearby trajectories. However, we should not consider this system to be chaotic!

Trajectories are repelled to infinity, and never return. Hence infinity is *a fixed point* of the system, and ingredient 1. above specifically excludes fixed points!

## Defining "attractor" and "strange attractor"

The term *attractor* is also difficult to define in a rigorous way. Loosely, an attractor is a set of points to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples.

More precisely, we define an attractor to be a closed set $A$ with the following properties:

1. $A$ is an *invariant set*: any trajectory $\mathbf{x}(t)$ that starts in $A$ stays in $A$ for all time.

2. $A$ *attracts an open set of initial conditions*: there is an open set $U$ containing $A$ such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to $A$ tends to zero as $t \to \infty$. Hence $A$ attracts all trajectories that start sufficiently close to it. The largest such $U$ is called the *basin of attraction* of $A$.

3. $A$ is *minimal*: there is no proper subset of $A$ that satisfies conditions 1. and 2.

Example $\begin{cases} \dot{x} &= \mu x - x^3 \\ \dot{y} &= -y \end{cases}$

Let $I$ denote the interval $-1 \leq x \leq 1, y = 0$.
Is $I$ an attractor?

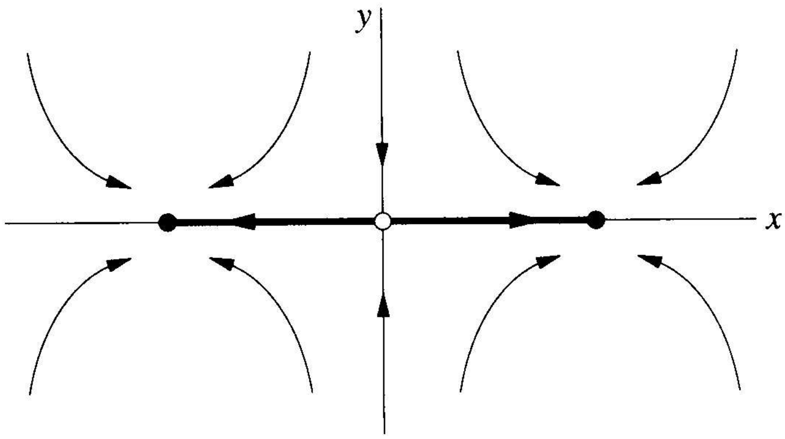



Fig. 6.2.7

So, $I$ is an invariant set (condition 1.). Also, $I$ attracts an open set of initial conditions - it attracts *all* trajectories in the $xy$-plane. But $I$ is *not* an attractor because it is not minimal. The stable fixed points $(\pm 1, 0)$ are proper subsets of $I$ that also satisfy conditions 1. and 2. These points are the *only* attractors for the system.

Maybe the same is true for the Lorenz Equations? - Nobody has yet proved that the Lorenz attractor is *truly* an attractor!!

Finally we define a *strange attractor* to be *an attractor that exhibits sensitive dependence on initial conditions*.

Strange attractors were originally called strange because they are often (but not always...) fractal sets. Nowadays, this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions. The terms "chaotic attractor" and "fractal attractor" are used when one wishes to emphasize one or other of these aspects.

## The Lorenz Map

Consider the following view of the Lorenz strange attractor...

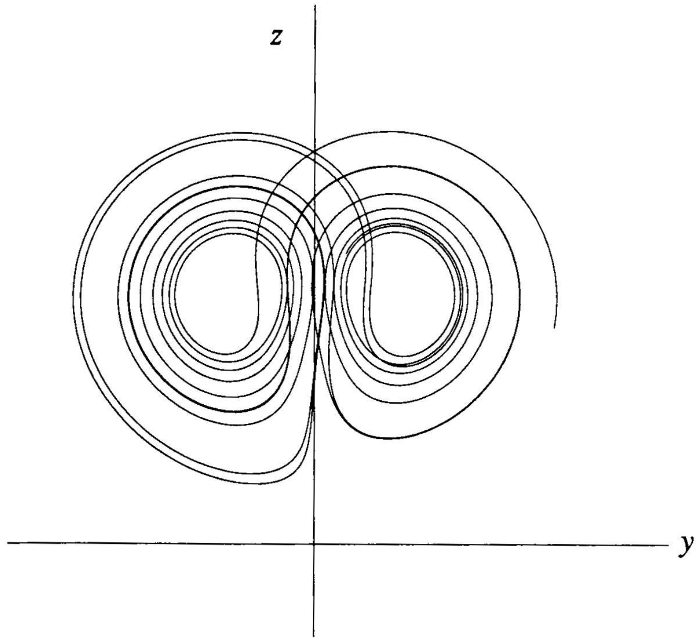



Fig. 6.3.1

Lorenz wrote that "the trajectory apparently leaves one spiral only after exceeding some critical distance from the centre... It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit."

The "single feature" he focused on was $z_n$, the $n$th local maximum of $z(t)$.

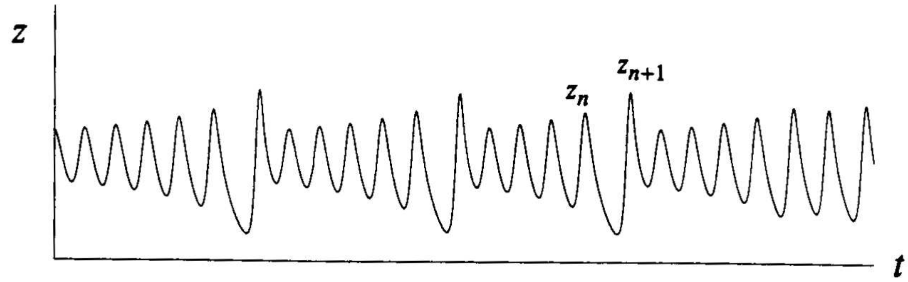



Fig. 6.3.2

Lorenz's idea is that $z_n$ should predict $z_{n+1}$. He checked this by numerical integration. The plot of $z_{n+1}$ vs $z_n$ looks like....

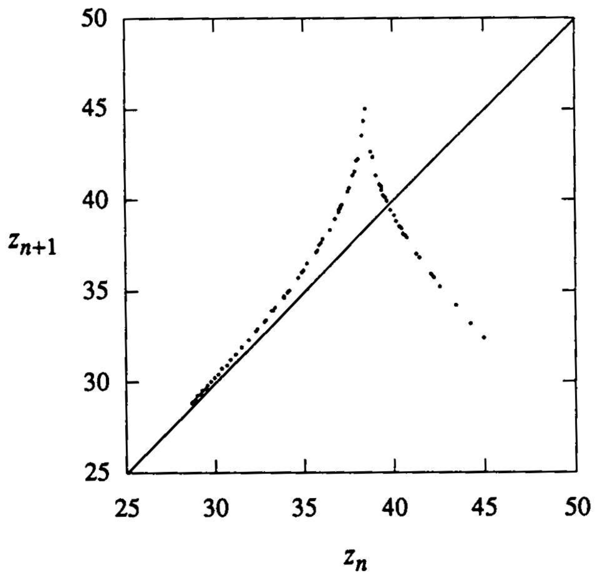



Fig. 6.3.3

The data from the chaotic time series appears to fall neatly on a curve - there is no "thickness" to the graph. Hence *Lorenz was able to extract order from chaos!*

The function $z_{n+1} = f(z_n)$ is now called the *Lorenz Map*. Note...

- The graph is not actually a curve - it does have *some* thickness so, strictly speaking, $f(z)$ is not a well-defined function.

- The Lorenz Map reminds us of the Poincaré map, *but* there is a distinction - the Poincaré map tells us how the *two* coordinates of a point on a surface change after first return to the surface, while the Lorenz map characterises the trajectory by only *one* number. This approach only works if the attractor is very "flat" (close to 2-D).

## Exploring Parameter Space

So far we have concentrated mainly on the case $\sigma = 10, b = 8/3, r = 28$ as in Lorenz's original work published in 1963.

Changing parameters is like a walk through the jungle! One finds many sorts of exotic behaviours, such as exotic limit cycles tied in knots, pairs of limit cycles linked through each other, intermittent chaos, noisy periodicity.....as well as strange attractors! In fact much remains to be discovered....

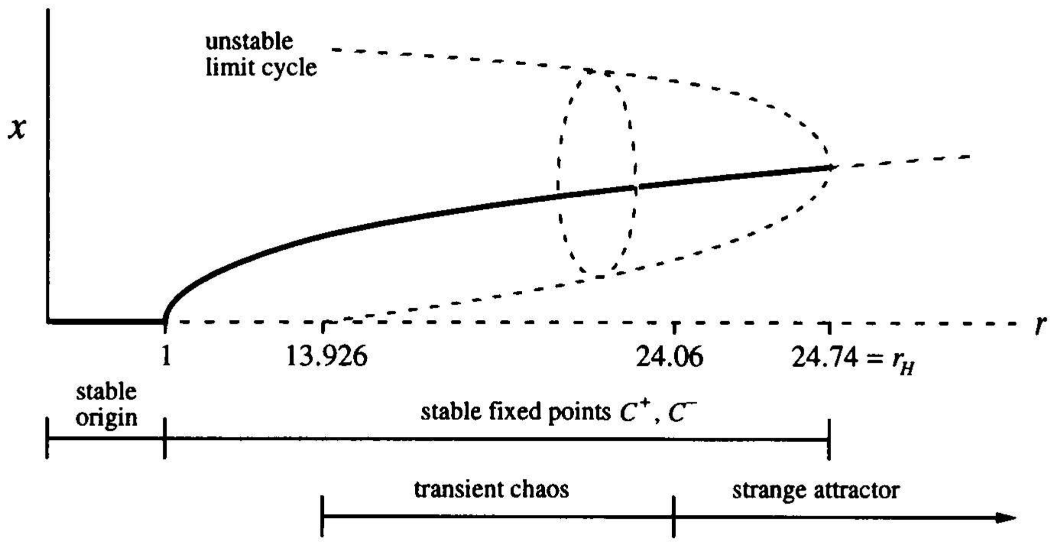



Fig. 6.4.1

Example Consider $r = 21$, with $\sigma = 10$ and $b = 8/3$ as usual. The solution exhibits *transient chaos*, in which the trajectory seems to be tracing a strange attractor, but eventually spirals towards $C^+$ or $C^-$...

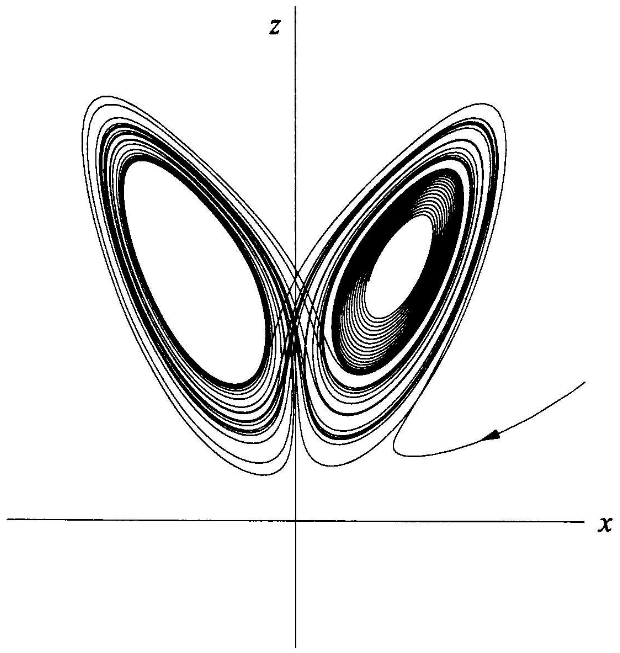



Fig. 6.4.2

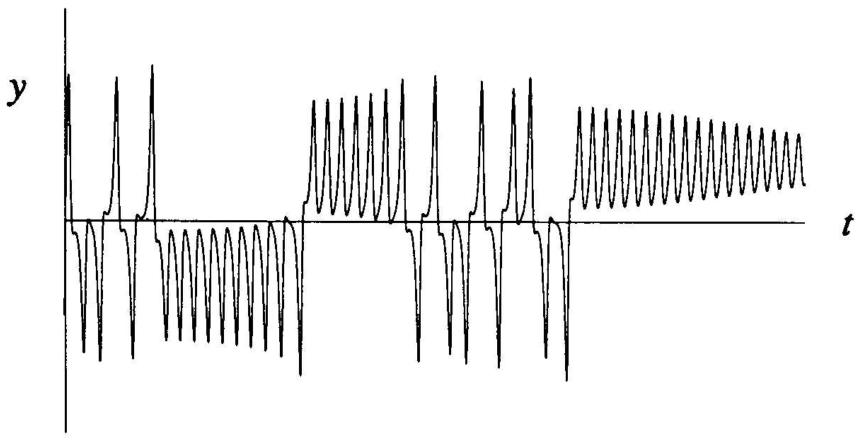



Fig. 6.4.3

The long-term behaviour is not aperiodic - hence the dynamics are not chaotic. However, the dynamics do exhibit sensitive dependence on initial conditions - the trajectory could end up on either $C^+$ or $C^-$ - hence the system's behaviour is unpredictable.

Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple. This is familiar from everyday experience - many games of "chance" used in gambling are essentially demonstrations of transient chaos....e.g. a rolling dice...

*You don't need strange attractors to generate effectively random behaviour!*

It turns out that for large $r$ the dynamics become simple again - the system has a global attracting limit cycle for $r > 313$.

Example Consider $r = 350$, with $\sigma = 10$ and $b = 8/3$ as usual.

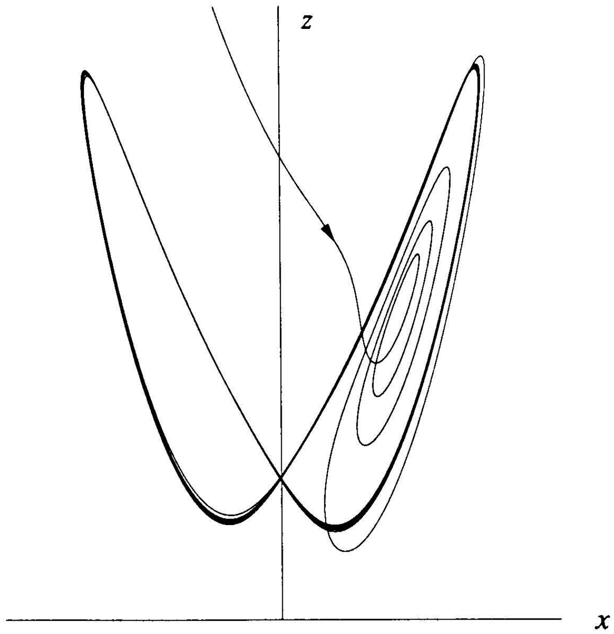



Fig. 6.4.4

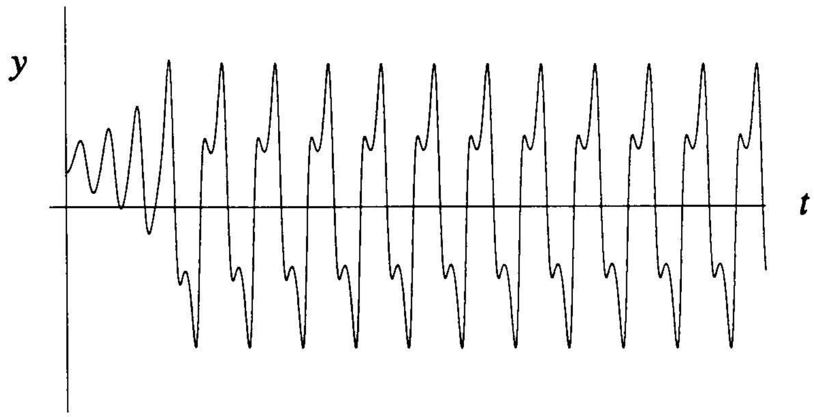



Fig. 6.4.5