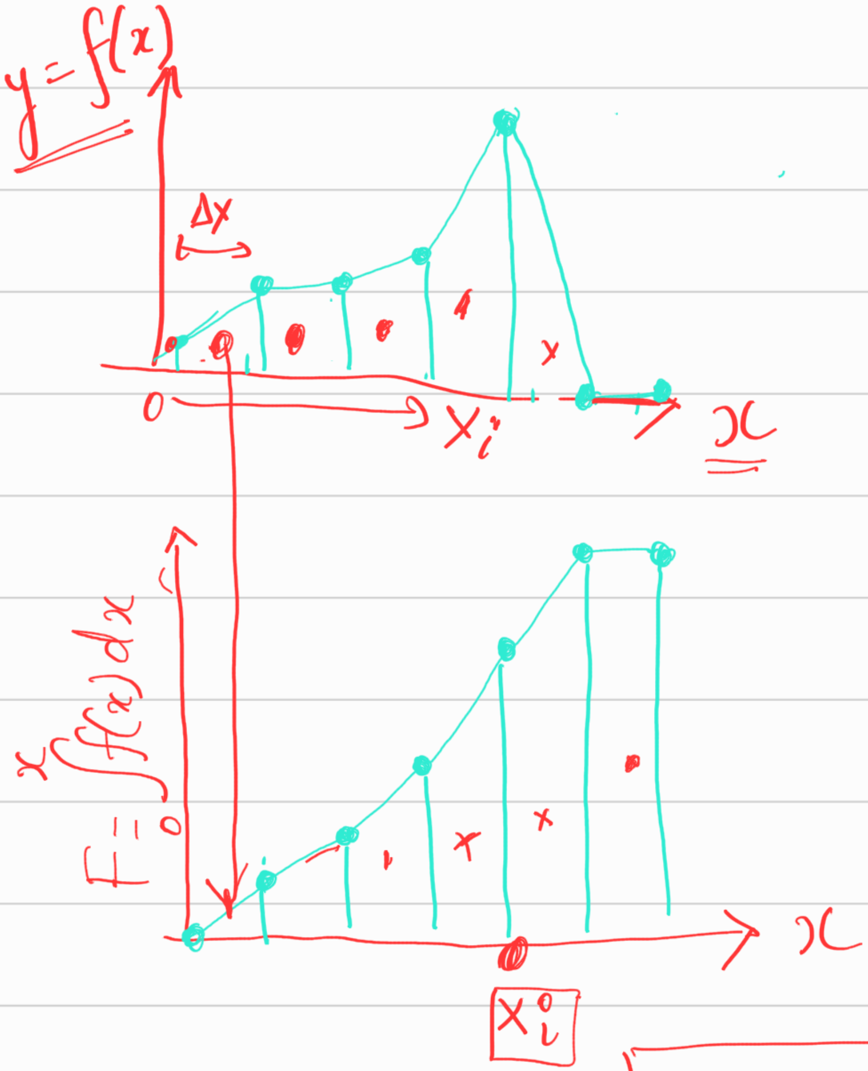
Notes on numerical integration



$$\varepsilon_{zx} = \frac{x}{x^2+y^2}$$

$$\varepsilon_{zx} = \frac{1}{2}\frac{\partial u_z}{\partial x} \to 1,2,3$$

$$\Rightarrow u_z = \int 2\varepsilon_{zx}\,dx$$

$$\downarrow \quad \boxed{0}$$

$$u_z(x) = \int_0^x 2\varepsilon_{zx}\,dx$$

$$u_z(x,y)$$

$$u_z(x) = \int_0^x 2\varepsilon_{zx}\,dx + u_z(\text{left corner})$$

$$\int_0^5\int_0^5 \varepsilon_{xz}\,dx\,dy = [\text{scalar}] \qquad 3.7\ldots$$

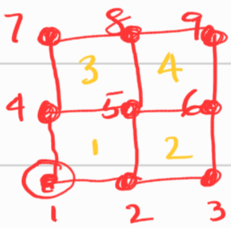
$$\boxed{u_z = \int^x 2\varepsilon_{xz}\,dx} \quad u_z^{(i)} = \sum^N 2\bar{\varepsilon}_{xz}\left[x^{(i)} - x^{(i-1)}\right]$$

0

$n=1$

$$\bar{\varepsilon}_{xz} = \frac{\varepsilon_{xz}^{(i)} + \varepsilon_{xz}^{(i-1)}}{2}$$

$$\varepsilon_{xx}^{(1)} = \frac{u_x^{(2)} - u_x^{(1)}}{\Delta x}$$



$$\varepsilon_{yy}^{(1)} = \frac{u_y^{(4)} - u_y^{(1)}}{\Delta y}$$

$u_x^{(1)} = 0$

$u_y^{(1)} = 0$

$$\varepsilon_{xy}^{(1)} = \frac{1}{2}\left[\frac{u_x^{(4)} - u_x^{(1)}}{\Delta y} + \frac{u_y^{(2)} - u_y^{(1)}}{\Delta x}\right]$$

$[\varepsilon_{ij}]$	$[u_i]$
(1) →	1, 2, 4
(2) →	2, 3, 5.
(3) →	4, 5, 7.
(4) →	5, 6, 8

$$\varepsilon_{xy}^{(3)} = \frac{1}{2}\left[\frac{u_x^{(7)} - u_x^{(4)}}{\Delta y} + \frac{u_y^{(5)} - u_y^{(4)}}{\Delta x}\right]$$

$$\varepsilon_{yy}^{(3)} = \frac{u_y^{(7)} - u_y^{(4)}}{\Delta y} \qquad \varepsilon_{xy}^{(3)} = \frac{u_y^{(5)} - u_y^{(4)}}{\Delta x}.$$

Linear equations: u_x, u_y unknown.

- (1) $u_x^{(1)} = 0$

(2) $u_y^{(1)} = 0$

$$\left.\begin{array}{ll} (3) & u_x^{(2)} = \varepsilon_{xx}^{(1)}\Delta x + u_x^{(1)} \\ (4) & u_y^{(4)} = \varepsilon_{yy}^{(1)}\Delta y + u_y^{(1)} \end{array}\right\} \text{explicit}$$

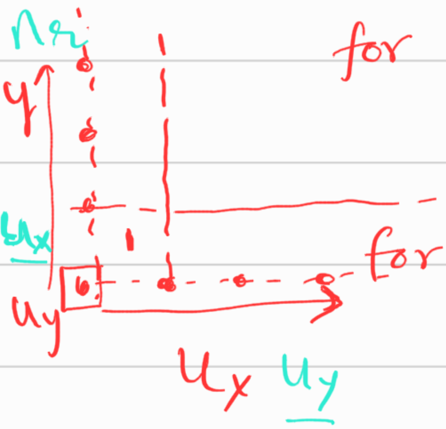
⑤ $$u_x^{(5)} = \left(\frac{\varepsilon_{xx}^{(1)} + \varepsilon_{xx}^{(3)}}{2}\right)\Delta x + \boxed{u_x^{(4)}}$$

⑥ $$u_y^{(5)} = \left(\frac{\varepsilon_{yy}^{(1)} + \varepsilon_{yy}^{(2)}}{2}\right)\Delta y + \boxed{u_y^{(2)}}$$

unknown $\boxed{u_y^{(2)} \ \& \ u_x^{(4)}}$

$$u_x^{(4)}\Delta x + u_y^{(2)}\Delta y = 2\varepsilon_{xy}^{(1)}\Delta x\,\Delta y + u_x^{(1)}\Delta x + u_y^{(1)}\Delta y \quad (7)$$

$$u_x^{(7)}\Delta x = 2\varepsilon_{xy}^{(3)}\Delta x\,\Delta y + u_x^{(4)}\Delta x + u_y^{(4)}\Delta x - u_y^{(5)}\Delta x$$



for $i = 2 : n_c \ \forall\, y$

$$u_x(i) = u_x(i-1) + \frac{\varepsilon_{xx}(i-1) + \varepsilon_{xx}(i)}{2} \cdot \Delta x$$

for $j = 2 : n_r \ \forall\, x$

$$u_y(j) = u_y(j-1) + \frac{\varepsilon_{yy}(j-1) + \varepsilon_{yy}(j)}{2}\Delta y$$

$x : 1 \to n_c$

The only unknowns are

$u_x\,[i = 1,\ j = 2 : n_r]$

$u_y\,[i = 2 : n_c,\ j = 1]$

$$\varepsilon_{xy}(i,j) = \frac{u_x(i, j+1) - u_x(i,j)}{2\,\Delta y} + \frac{u_y(i+1, j) - u_y(i,j)}{2\,\Delta x}$$

$$\varepsilon_{xy}(1,1) = \frac{u_x(1,2) - u_x(1,1)}{2\Delta y} + \frac{u_y(2,1) - u_y(1,1)}{2\Delta x}.$$

$$u_x(1,2)\,\Delta x + u_y(2,1)\,\Delta y = 2\varepsilon_{xy}(1,1)\,\Delta x\,\Delta y + u_x(1,1)\,\Delta x + u_y(1,1)\,\Delta y$$

$$\varepsilon_{xy}(2,1) = \frac{u_x(2,2) - u_x(2,1)}{2\Delta y} + \frac{u_y(3,1) - u_y(2,1)}{2\Delta x}.$$

$$u_y(3,1)\,\Delta y = 2\varepsilon_{xy}(2,1)\,\Delta x\,\Delta y + u_y(2,1)\,\Delta y + u_x(2,1)\,\Delta x - u_x(2,2)\,\Delta x$$

$$\boxed{u_y(i,1)\,\Delta y = 2\varepsilon_{xy}(i-1,1)\,\Delta x\,\Delta y + u_y(i-1,1)\,\Delta y + u_x(i-1,1)\,\Delta x - u_x(i-1,2)\,\Delta x.}$$

known for (1,1)

unknown for $u_x(1,2)$

$$u_x(i-1,2) = u_x(i-2,2) + \frac{\varepsilon_{xx}(i-2,2) + \varepsilon_{xx}(i-1,2)}{2}\,\Delta x$$

When we get to the last node $[n_c, 1]$, we have 2 conditions

$$\varepsilon_{xy}(n_c-1,1) = \frac{u_y(n_c,1) - u_y(n_c-1,1)}{2\Delta x} + \frac{u_x(n_c-1,2) - u_x(n_c-1,1)}{2\Delta y}$$

$$\varepsilon_{xy}(n_c,1) = u_y(n_c,1) - u_y(n_c-1,1) + u_x(n_c,2) - u_x(n_c,1)$$

$$\varepsilon_{xy}(n_c, 1) = \frac{\ldots}{2\Delta x} + \frac{\ldots}{2\Delta y}$$

$$\Rightarrow u_y(n_c,1) = 2\varepsilon_{xy}(n_c-1,1)\Delta x\Delta y + u_y(n_c-1,1)\Delta y$$
$$+ u_x(n_c-1,1)\Delta x - u_x(n_c-1,2)\Delta x$$

&

$$u_y(n_c,1) = 2\varepsilon_{xy}(n_c,1)\Delta x\Delta y + u_y(n_c-1,1)\Delta y$$
$$+ u_x(n_c,1)\Delta x - u_x(n_c,2)\Delta x.$$

Now let's consider $u_y(1,j)$

$$\varepsilon_{xy}(i,j) = \frac{u_x(i,j+1) - u_x(i,j)}{2\Delta y} + \frac{u_y(i+1,j) - u_y(i,j)}{2\Delta x}.$$

$j = 2 : n_r$

$$u_x(1,j)\Delta x = 2\varepsilon_{xy}(1,j-1)\Delta x\Delta y$$
$$+ u_x(1,j-1)\Delta x$$
$$+ u_y(1,j-1)\Delta y - u_y(2,j-1)\Delta y$$

known for (1,1)

unknown for (2,1)

$i = 2 : n_c$

$$u_y(i,1)\Delta y = 2\varepsilon_{xy}(i-1,1)\Delta x\Delta y$$
$$+ u_y(i-1,1)\Delta y$$
$$+ u_x(i-1,1)\Delta x - u_x(i-1,2)\Delta x.$$

known for (1,1)

unknown for $u_x(1,2)$

$$u_y(2,1)\,\Delta y + u_x(1,2)\,\Delta x = 2\varepsilon_{xy}(1,1)\,\Delta x\,\Delta y + u_y(1,1)\,\Delta y + u_x(1,1)\,\Delta x$$

simplified problem (4 nodes)

4 nodes, $(u_x, u_y) \rightarrow 8$ unknowns

$\rightarrow$ 2 BCs : $u_x(1) = 0$

$u_y(1) = 0$

2 direct $\rightarrow$ 4 equations from ε_{xx} & ε_{yy}

$\hookrightarrow u_x(2),\ u_y(3)$

ε_{xx}: 1,2 ; 3,4 $\quad$ ε_{yy}: 1,3 ; 2,4

4 $\rightarrow$
$$\left\{\begin{aligned}
\varepsilon_{xy}(1) &= \frac{1}{2}\left(\frac{u_x(3)-u_x(1)}{\Delta y} + \frac{u_y(2)-u_y(1)}{\Delta x}\right)\\
\varepsilon_{xy}(3) &= \frac{1}{2}\left(\frac{u_x(3)-u_x(1)}{\Delta y} + \frac{u_y(4)-u_y(3)}{\Delta x}\right)\\
\varepsilon_{xy}(2) &= \frac{1}{2}\left(\frac{u_x(4)-u_x(2)}{\Delta y} + \frac{u_y(2)-u_y(1)}{\Delta x}\right)\\
\varepsilon_{xy}(4) &= \frac{1}{2}\left(\frac{u_x(4)-u_x(2)}{\Delta y} + \frac{u_y(4)-u_y(3)}{\Delta x}\right)
\end{aligned}\right.$$

$u_x(3),\ u_y(2)$

$u_x(4),\ u_y(4)$

$$u_x(4) = u_x(3) + \frac{\varepsilon_{xx}(3)+\varepsilon_{xx}(4)}{2}\,\Delta x$$

2

$$u_y(4) = u_y(2) + \frac{\varepsilon_{yy}(2) + \varepsilon_{yy}(4)}{2} \Delta y$$

(1)

$$2\varepsilon_{xy}(1)\Delta x\Delta y + u_x(1)\Delta x + u_y(1)\Delta y = u_x(3)\Delta x + u_y(2)\Delta y$$

(2)

$$2\varepsilon_{xy}(2)\Delta x\Delta y + u_x(2)\Delta x + u_y(1)\Delta y = u_x(4)\Delta x + u_y(2)\Delta y$$

(3)

$$2\varepsilon_{xy}(3)\Delta x\Delta y + u_x(1)\Delta x + u_y(3)\Delta y = u_x(3)\Delta x + u_y(4)\Delta y$$

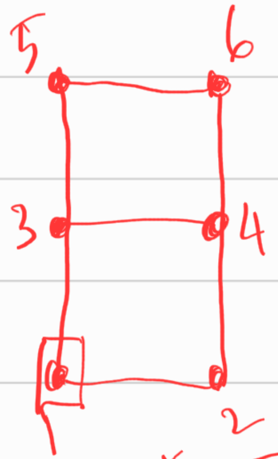
(4)

$$2\varepsilon_{xy}(4)\Delta x\Delta y + u_x(2)\Delta x + u_y(3)\Delta y = u_x(4)\Delta x + u_y(4)\Delta y$$

$$\overbrace{\begin{bmatrix} \Delta x & \Delta y & 0 & 0 \\ 0 & \Delta y & \Delta x & 0 \\ \Delta x & 0 & 0 & \Delta y \\ 0 & 0 & \Delta x & \Delta y \end{bmatrix}}^{G} \begin{bmatrix} u_x(3) \\ u_y(2) \\ u_x(4) \\ u_y(4) \end{bmatrix} = \begin{bmatrix} (1) \\ (2) \\ (3) \\ (4) \end{bmatrix}$$

$$\begin{bmatrix} u_x(3) \\ u_y(2) \\ u_x(4) \\ u_y(4) \end{bmatrix} = G^{-1} \begin{bmatrix} (1) \\ (2) \\ (3) \\ (4) \end{bmatrix}$$

6 nodes → 12 unknowns



5 direct solutions

$$u_x(1) = 0$$

$$u_y(1) = 0$$

$$u_x(2) = u_x(1) + \frac{\varepsilon_{xx}(1)+\varepsilon_{xx}(2)}{2}\Delta x$$

$$u_y(3) = u_y(1) + \frac{\bar{\varepsilon}_{yy}}{2}\Delta y$$

$$u_y(5) = u_y(3) + \frac{\bar{\varepsilon}_{yy}}{2}\Delta y$$

remaining unknowns (7): $u_x(3), u_x(4), u_x(5), u_x(6)$, $u_y(2), u_y(4), u_y(5), u_y(6)$ $\Big]\ \underline{u}$

① $$2\varepsilon_{xy}(1)\Delta x\Delta y + u_x(1)\Delta x + u_y(1)\Delta y = u_x(3)\Delta x + u_y(2)\Delta y$$

② $$2\varepsilon_{xy}(2)\Delta x\Delta y + u_x(2)\Delta x + u_y(1)\Delta y = u_x(4)\Delta x + u_y(2)\Delta y$$

③ $$2\varepsilon_{xy}(3)\Delta x\Delta y + u_x(1)\Delta x + u_y(3)\Delta y = u_x(3)\Delta x + u_y(4)\Delta y$$

④ $$2\varepsilon_{xy}(4)\Delta x\Delta y + u_x(2)\Delta x + u_y(3)\Delta y = u_x(4)\Delta x + u_y(4)\Delta y$$

⑤ $$2\varepsilon_{xy}(3)\Delta x\Delta y + u_y(3)\Delta y = u_x(5)\Delta x + u_y(4)\Delta y - u_x(3)\Delta x$$

⑥ $$\frac{\varepsilon_{xx}(3)+\varepsilon_{xx}(4)}{2} = u_x(4) - u_x(3)$$

$$2\varepsilon_{xy}(5)\,\Delta x\,\Delta y + u_x(3)\,\Delta x = u_x(5)\,\Delta x + u_y(6)\,\Delta y - u_y(5)\,\Delta y$$

7

$$\frac{\varepsilon_{yy}(2)+\varepsilon_{yy}(4)}{2} = u_y(4) - u_y(2)$$

8

$$\frac{\varepsilon_{xx}(5)+\varepsilon_{xx}(6)}{2} = u_x(6) - u_x(5)$$

10

$$\frac{\varepsilon_{yy}(4)+\varepsilon_{yy}(6)}{2} = u_y(6) - u_y(4)$$

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