Derivation of constitutive relation for Burger's body

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1 Maxwell material

Consider a Maxwell body, an elastic spring and a viscous dashpot are connected in series. This implies that both elements share a common stress while the strains are additive.

$$\varepsilon_T = \varepsilon_e + \varepsilon_M \tag{1}$$

Here ε_T is the total strain while the subscripts e and M refer to the strains in the elastic and viscous elements respectively.
This additive relationship holds even when we consider time derivatives.

$$\dot{\varepsilon}_T = \dot{\varepsilon}_e + \dot{\varepsilon}_M \tag{2}$$

We can substitute the component strain rates using the shared stress and constitutive relations for each element to give us a relationship between stress σ and the total strain rate $\dot{\varepsilon}_T$ at any time in a Maxwell body.

$$\dot{\varepsilon}_T = \frac{\dot{\sigma}}{G} + \frac{\sigma}{\eta_M} \tag{3}$$

G is the spring's elastic modulus while η_M is the viscosity of the dashpot element.

Now we will rewrite the governing equations such that it can be used in boundary element simulations. First, instead of working with the total strain rate, we will consider the elastic and viscous components separately.

We stated that both elements share a common stress σ. We use this to derive a relationship between the elastic strain and the viscous strain rate as follows.

$$\begin{aligned} \sigma_e = \sigma_M &= \sigma \\ G\varepsilon_e &= \eta_M \dot{\varepsilon}_M \end{aligned} \tag{4}$$

Since the strains are additive, we can substitute for ε_e in terms of $\varepsilon_T, \varepsilon_M$.

$$\begin{aligned} G(\varepsilon_T - \varepsilon_M) &= \eta_M \dot{\varepsilon}_M \\ G(\dot{\varepsilon}^\infty t) - G(\varepsilon_M) &= \eta_M \dot{\varepsilon}_M \end{aligned} \tag{5}$$

For earthquake sequence simulations, the total strain in the viscoelastic medium is a linear function in time, whose spatial variation is given by solving a viscous boundary value problem. Here we simply call it $\dot{\varepsilon}^\infty$.

1.1 Earthquake sequence simulations

So now when we want to study the response of a number of these Maxwell bodies connected together, we must first calculate the elastic transfer function i.e. a matrix that defines the relationship between inelastic strain in an element to the stress transferred everywhere else in the medium due to elasticity ($\sigma \propto \varepsilon_M$). Since this is the elastic stress kernel, it is the same for any time derivative too. We will denote this as $K_{a,b}$ where a is the source strain component and b is stress component at the observation point. K is negative for self-stress so we do not have to add a negative sign as we typically do for single degree-of-freedom models (e.g. $G(\dot{\varepsilon}^\infty t) - G(\varepsilon_M) = \eta_M \dot{\varepsilon}_M$).

For an antiplane strain system where there are two strain components $\varepsilon_{12}, \varepsilon_{13}$, we present the stress balance over the entire viscoelastic domain as a system of coupled ODEs -

$$\begin{bmatrix} \dot{\sigma}_{12}^{\infty} t \\ \dot{\sigma}_{13}^{\infty} t \end{bmatrix} + \begin{bmatrix} K_{12,12} & K_{13,12} \\ K_{12,13} & K_{13,13} \end{bmatrix} \begin{bmatrix} \varepsilon_{M_{12}} \\ \varepsilon_{M_{13}} \end{bmatrix} = \eta_M \begin{bmatrix} \dot{\varepsilon}_{M_{12}} \\ \dot{\varepsilon}_{M_{13}} \end{bmatrix} \tag{6}$$

This gives a system of coupled ODEs where $\frac{d\varepsilon_M}{dt}$ is given as a function of ε_M, which we can integrate using any standard numerical integration scheme. To be consistent with the approach we follow for the Burger's body, we can simply rewrite the above coupled system of ODEs for $\frac{d^2\varepsilon_M}{dt^2} = f\left(\frac{d\varepsilon_M}{dt}\right)$.

2 Burger's body

In a Burger's body, there is a Maxwell element in series with a Kelvin element. The Maxwell and Kelvin elements share a common stress σ while the total strain is additive.

$$\varepsilon_T = \varepsilon_1 + \varepsilon_2 \tag{7}$$

We know from earlier that the total strain in the Maxwell element itself is a sum of an elastic and viscous component, while the entire Kelvin element shares a common strain ε_K. This means we can decompose the total strain or strain rate into three different parts.

$$\begin{aligned} \varepsilon_T &= \varepsilon_K + \varepsilon_e + \varepsilon_M \\ \dot{\varepsilon}_T &= \dot{\varepsilon}_K + \dot{\varepsilon}_e + \dot{\varepsilon}_M \end{aligned} \tag{8}$$

Using the constitutive relations for each individual material, we can describe the total strain rate in terms of shared stress σ.

$$\begin{aligned} \dot{\varepsilon}_T &= \dot{\varepsilon}_K + \frac{\dot{\sigma}}{G} + \frac{\sigma}{\eta_M} \\ \dot{\varepsilon}_T &= \frac{\sigma - G\varepsilon_K}{\eta_K} + \frac{\dot{\sigma}}{G} + \frac{\sigma}{\eta_M} \end{aligned} \tag{9}$$

The above equation features ε_K, which is a hidden under a time integral. To overcome this we take the second derivative of the above equation,

$$\ddot{\varepsilon}_T = \frac{\dot{\sigma} - G\dot{\varepsilon}_K}{\eta_K} + \frac{\ddot{\sigma}}{G} + \frac{\dot{\sigma}}{\eta_M} \tag{10}$$

And we substitute in $\dot{\varepsilon}_K = \dot{\varepsilon}_T - \dot{\varepsilon}_e - \dot{\varepsilon}_M = \dot{\varepsilon}_T - \frac{\dot{\sigma}}{G} - \frac{\sigma}{\eta_M}$, to get the constitutive equation for a Burger's body,

$$\sigma + \frac{(\eta_K + 2\eta_M)}{G}\dot{\sigma} + \frac{\eta_M\eta_K}{G^2}\ddot{\sigma} = \eta_M\dot{\varepsilon}_T + \frac{\eta_M\eta_K}{G}\ddot{\varepsilon}_T \tag{11}$$

2.1 Earthquake sequence simulations

Now when we want to use a Burger's body in earthquake sequence simulations, we need to discretize the domain using the same elastic stress kernel as described earlier. Then we write the stress balance for each spatially discretized element in terms of the kelvin strain rate and the total stress as follows,

$$\begin{aligned} \dot{\varepsilon}_{K_{12}} &= \frac{\sigma_{12} + \begin{bmatrix} K_{12,12} & K_{13,12} \end{bmatrix} \begin{bmatrix} \varepsilon_{K_{12}} \\ \varepsilon_{K_{13}} \end{bmatrix}}{\eta_K} \\ \dot{\varepsilon}_{K_{13}} &= \frac{\sigma_{13} + \begin{bmatrix} K_{12,13} & K_{13,13} \end{bmatrix} \begin{bmatrix} \varepsilon_{K_{12}} \\ \varepsilon_{K_{13}} \end{bmatrix}}{\eta_K} \end{aligned} \tag{12}$$

We have an expression for the strain rate in the Kelvin element as a function of the stress and Kelvin strains. We then proceed to obtain the total non-elastic strain rate ε_v. This is the component of strain rate that results in an 'elastic' stressing rate (when multiplied with the elastic stress kernel).

$$\begin{aligned}\dot{\varepsilon}_{v_{12}} &= \dot{\varepsilon}_{K_{12}} + \frac{\sigma_{12}}{\eta_M}\\ \dot{\varepsilon}_{v_{13}} &= \dot{\varepsilon}_{K_{13}} + \frac{\sigma_{13}}{\eta_M}\end{aligned} \tag{13}$$

This non-elastic strain rate combined with the driving strain rate (computed by solving the appropriate viscous boundary value problem) gives us the instantaneous stressing rate.

$$\begin{aligned}\dot{\sigma}_{12} &= \dot{\sigma}_{12}^{\infty} + \begin{bmatrix} K_{12,12} & K_{13,12}\end{bmatrix}\begin{bmatrix}\dot{\varepsilon}_{v_{12}}\\ \dot{\varepsilon}_{v_{13}}\end{bmatrix}\\ \dot{\sigma}_{13} &= \dot{\sigma}_{13}^{\infty} + \begin{bmatrix} K_{12,13} & K_{13,13}\end{bmatrix}\begin{bmatrix}\dot{\varepsilon}_{v_{12}}\\ \dot{\varepsilon}_{v_{13}}\end{bmatrix}\end{aligned} \tag{14}$$

The total strain rate at any point in the Burger's body can be calculated by summing the elastic and non-elastic components together.

$$\begin{bmatrix}\dot{\varepsilon}_{T_{12}}\\ \dot{\varepsilon}_{T_{13}}\end{bmatrix} = \begin{bmatrix}\dot{\varepsilon}_{K_{12}} + \frac{\sigma_{12}}{\eta_M}\\ \dot{\varepsilon}_{K_{13}} + \frac{\sigma_{13}}{\eta_M}\end{bmatrix} + \begin{bmatrix}K_{12,12} & K_{13,12}\\ K_{12,13} & K_{13,13}\end{bmatrix}^{-1}\begin{bmatrix}\dot{\sigma}_{12}\\ \dot{\sigma}_{13}\end{bmatrix} \tag{15}$$

Thus, we have derived coupled system of ODEs, $\left(\frac{d\varepsilon_K}{dt}, \frac{d\sigma}{dt}\right)$ as a function of (ε_K, σ), that can be numerically integrated to give the strain and stress evolution in a Burger's body.