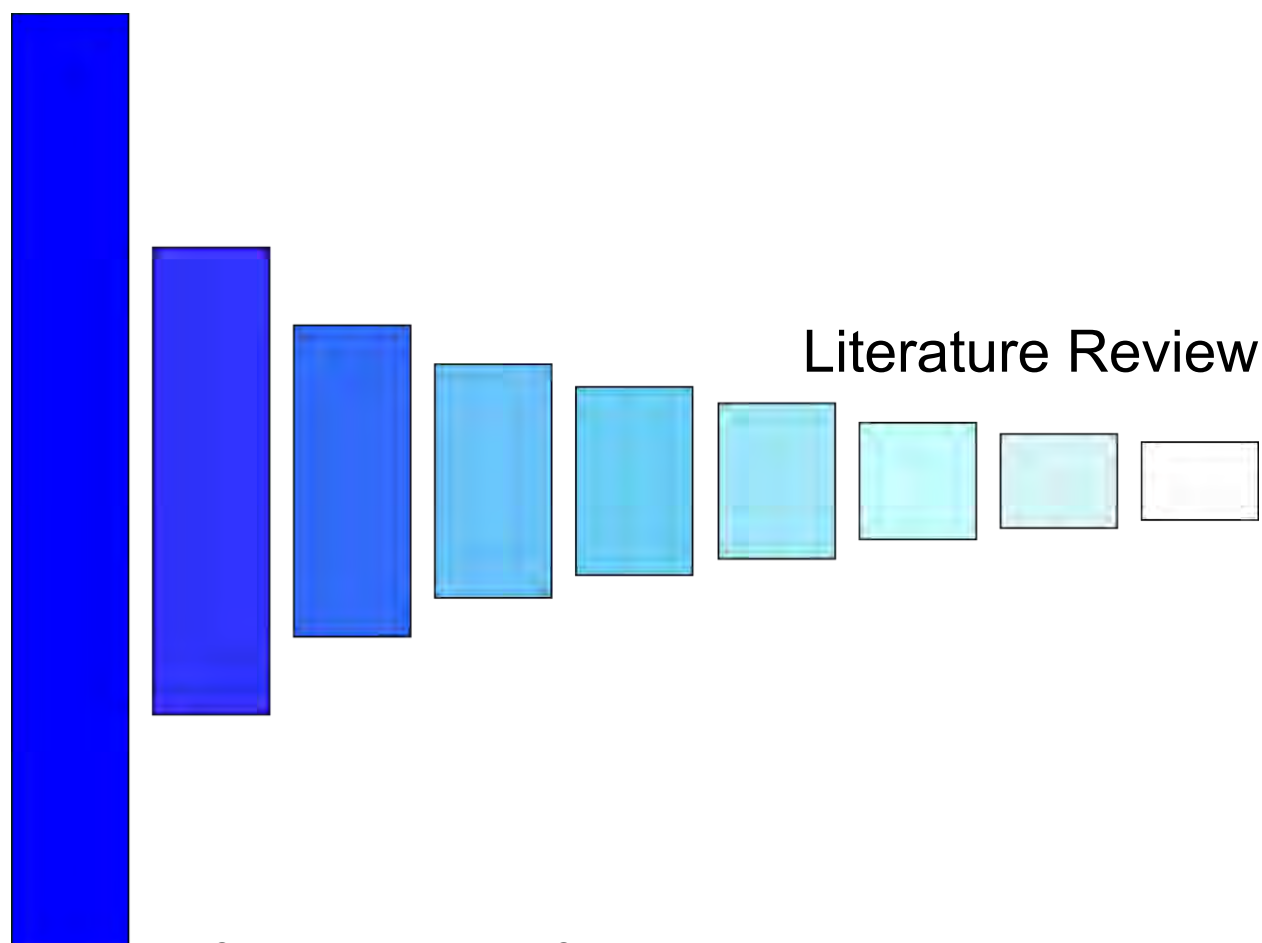# Foundations to Learning and Teaching Fractions: Addition and Subtraction

Literature Review

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Prior to engaging in collaborative action research with teacher teams and students in Ontario classrooms, this literature review was undertaken to provide a synthesis of existing knowledge from the field of educational research focused on the learning and teaching of fractions. The authors of this study are educational researchers in Ontario, Canada. They have experience as practicing teachers, teacher educators and academic researchers with ongoing intensive work in classrooms with teachers and students. This literature review provides the foundation for continued Canadian research in the learning and teaching of the addition and subtraction of fractions.

The literature review begins with an extensive examination of foundational concepts of fractions as the groundwork for understanding operations of fractions. It is clear from the research that issues and challenges in the learning and teaching of fractions operations (adding and subtracting) stem from fragile and superficial understandings of these foundational concepts of fractions.

This literature review is organized under the following headings: I) research on the foundations of fractions, II) research on learning fractions, and III) research on teaching fractions.

Finally, the literature review ends with a discussion of the implications for Canadian research directions.

Introduction

A collaborative action research project focused on the learning and teaching of fractions in the junior grades was undertaken in 2011-2012. This project focused on representing, comparing and ordering fractions, and engaged teacher teams in examining student thinking over a three-month period. The findings of this project were shared with nearly 300 Ontario educators through Math CAMPPP 2012 and are also available in a digital paper on www.edugains.ca as well as through video studies on www.tmerc.ca.

This comprehensive literature review, which examines research from around the world, was completed in anticipation of continuing the collaborative action research with a focus on addition and subtraction of fractions in the 2012-2013 school year. Both the literature and the findings from the work undertaken last year underscore the importance of thoughtful selection of learning tasks and representations, as well as the long-term benefits of building a strong conceptual understanding of fractions. Additionally, a number of researchers and Ontario educators participated in a Think Tank focused on fractions operations, in October 2012 where both research and instructional resources were shared and discussed. This literature review provides a synthesis of the research to date, and highlights areas of agreement as well as areas for further research.

I. RESEARCH ON THE FOUNDATIONS OF FRACTIONS

Section 1. The Importance of Fractions

> *The mathematics education literature is resounding in its findings that understanding fractions is a challenging area of mathematics for North American students to grasp (National Assessment of Educational Progress, 2005). Students also seem to have difficulty retaining fractions concepts (Groff, 1996). Adults continue to struggle with fractions concepts (Lipkus, Samsa, & Rimer, 2001; Reyna & Brainerd, 2007) even when fractions are important to daily work related tasks.* (Bruce & Ross, 2009)

Fractions involve difficult-to-learn and difficult-to-teach concepts that present ongoing pedagogical challenges to the mathematics education community. These difficulties begin early in the primary years (Empson & Levi, 2011; Moss & Case, 1999) and persist through middle school (Armstrong and Larson, 1995; Kamii and Clark, 1995), then into secondary and even tertiary education (see Orpwood, Schollen, Leek, Marinelli-Henriques, & Assiri, 2011). The challenges and misunderstandings students face in understanding fractions (Gould, Outhred, & Mitchelmore, 2006; Hiebert 1988; NAEP, 2005) persist into adult life and pose problems in such wide-ranging fields as medicine and health care, construction and computer programming. The fields of science, technology, engineering and mathematics (STEM) demand considerable fractions knowledge; a shaky grounding in fractions can prevent individuals from pursuing advanced mathematics and shut students off from a significant number of career opportunities in later life. In medicine, the implications of inadequate fractions understanding can be severe; for example, "pediatricians, nurses, and pharmacists…were tested for errors resulting from the calculation of drug doses for neonatal intensive care infants… Of the calculation errors identified, 38.5% of pediatricians' errors, 56% of nurses' errors, and 1% of pharmacists' errors would have resulted in administration of 10 times the prescribed dose" (Grillo, Latif, & Stolte, 2001, p.168). Helping students to achieve a solid grounding in mathematics in general and in fractions in particular has long-term high-stakes ramifications, suggesting that it is worth spending the time and effort to enhance student understanding in the elementary years in order to ensure student success in later mathematics, career and life.

The mathematics education and research communities have much more work ahead to begin to resolve the challenges presented by the learning and teaching of fractions. The implications are broad (touching on, for example, a wide range of career fields), but they are also deep, effecting foundational understandings that help or hinder the learning of other areas of mathematics. Behr, Harel, Post & Lesh (1993), for example, have insisted that "learning fractions is probably one of the most serious obstacles to the mathematical maturation of children" (in Charalambous & Pitta-Pantazi, 2007, 293). Fractions understandings are underpinned by larger mathematics cognitive processes including proportional

reasoning (Moss & Case, 1999) and spatial reasoning (Mamolo, Sinclair, Whitely, 2011). Additionally fractions themselves underpin probability (Clarke & Roche, 2009) and algebraic reasoning (Brown & Quinn, 2007; Empson & Levi, 2011). Empson and Levi (2011) view "the study of fractions as foundational to the study of algebra in particular because it offers students the opportunity to grapple with the fundamental mathematical relationships that constitute the core of algebra…[that] govern how addition, subtraction, multiplication, and division work in algebra as well as arithmetic" (xxiii). In addition, limited understanding of particular aspects of the different meanings of fractions (e.g., fraction as operator, as in $\frac{1}{5}$ of 3) affects the ability of students to generalize and to work with unknowns, both of which are fundamental to algebra (Hackenberg & Lee, 2012).

It is clear that a weak foundation in fractions can eventually cut students off from higher mathematics and we must make strides through mathematics educational research and classroom practice to ameliorate this situation. However, the problem is complex and requires a long-term commitment to gaining a greater understanding of how to support students in building that solid foundation.

Section 2. Multiple Interpretations of Fractions

Clarke and Roche (2011) encourage educators to place a greater emphasis on the various meanings of fractions during instruction in order to improve students' understanding of fraction. Within North American education, fraction learning is often focused on one "type" of fraction, namely that which represents a part-whole relationship. This singular interpretation, along with an overuse of proper fractions, prohibits students from deeply understanding fractions greater than one (Charalambous, Delaney, Hsu & Mesa, 2010). Post, Cramer, Behr, Lesh and Harel (1993) claim that "ratio, measure and operator constructs are not given nearly enough emphasis in the school curriculum" (328).

A fraction is a number which can tell us about the relationship between two quantities. These two quantities provide information about the parts, the units we are considering and the whole. Determining the whole is important when working with fractions (Ontario Ministry of Education, in publication). There is general agreement amongst researchers about the various interpretations of a fraction (Clark & Roche, 2011; Empson & Levi, 2011; Petit, Laird & Marsden, 2010; Steffe & Olive, 2010, Marshall, 1993; Kieren, 1980). The following, adapted from *Math for Teaching: Ways We Use Fractions* (Ontario Ministry of Education, in publication), is a summary of these constructs:

A *linear interpretation*, also referred to as measure, is based upon the fraction's distance from zero and allows for the numerical value of the fraction to be located relative to the unit of 1. $\frac{2}{6}$ can be represented on a number line in the following manners:

Figure 1: Number Line Models
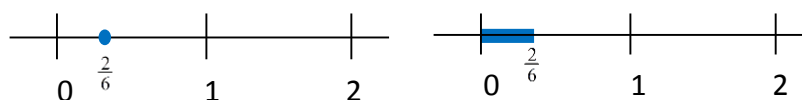
The *part-whole interpretation* is based upon either a continuous model (such as an area or a volume) or a discrete model (such as a set). For continuous models, the whole is partitioned into equal-sized parts while for discrete models, the whole is partitioned into sub-sets which share a common attribute.   Each of the following are examples of appropriately partitioned continuous models showing $\frac{2}{6}$ with the shaded regions:

Figure 2: Continuous Models

Note that in Figure 2, the areas must be the same size but not necessarily the same shape.

For discrete models, attributes other than size may be considered. In the following examples, attributes include colour, shape, and number of items respectively to represent $\frac{2}{6}$.
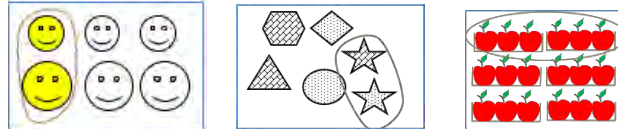
A fraction can represent a *part-part relationship*, in which case it is comparing the size of two measures. In a part-part relationship, the whole is the sum of the parts. Part-part relationships can be represented using linear, continuous or discrete models. In Figure 2, a part-part example is 2 equal areas shaded to 4 equal areas unshaded. The whole has 6 parts.

A fraction is also a *quotient*, or a division statement. For example, $\frac{2}{6}$ is the same as 2 ÷ 6 or 2 partitioned into 6 equal parts, which can be visualized as follows:

Figure 4: Quotient Model
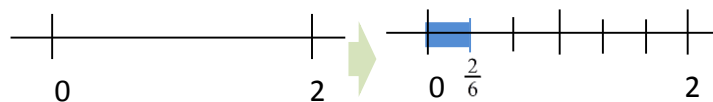
Finally, a fraction is an *operator*, in that it acts as a transformer by either enlarging or shrinking the operand. Examples of this include $\frac{2}{6}$ of the area of the floor or $\frac{3}{2}$ of the recipe.

Moseley and Okamoto (2008) found that, unlike top achievers, average and high achieving students are not developing these multiple meanings of rational numbers, resulting in a student focus on surface similarities of the representations rather than the numerical meaning. Furthermore, Moseley (2005) demonstrated that students who were familiar with both the part-part and part-whole interpretations had a deeper understanding of rational numbers. These results highlight the need to expand students' fractions understanding beyond the typical meaning that is focused on in mathematics programs in North America – fractions as part-whole relationships – in order to increase both the breadth and depth of student understanding of fractions, and to prepare students for a more seamless and coherent transition to operations with fractions. The recently

released report *The Mathematical Education of Teachers II* identifies these multiple constructs of fractions as essential learning for classroom educators (Conference Board of the Mathematical Sciences, 2012).

II. RESEARCH ON LEARNING FRACTIONS

Section 3. Fractions as Challenging Math Content

Fractions are difficult to learn because they require deep conceptual knowledge of part-whole relationships (how much of an object or set is represented by the fraction symbol), measurement (fractions are made up of numbers that can be ordered on a number line) and ratios (Hecht, Close & Santisi, 2003; Moss & Case, 1999).

The following specific challenges faced by learners are discussed in this section:
- difficulties understanding and representing fraction relationships;
- confusion about the roles of the numerator and the denominator and the relationship between them;
- use of a 'gap thinking' approach; and
- lack of attention to equivalence and equi-partitioning.

The section concludes with an examination of these challenges as evidenced within Ontario classrooms.

i) Difficulties Understanding and Representing Fraction Relationships

Student gaps and misconceptions are powerfully revealed through their drawn representations of fractions, and studies in this area provide evidence to suggest that the multitude of representations used, some of which are potentially distracting representations, do not help students build deep understanding (Kilpatrick, Swafford, & Findell, 2001). Young children in North America use an abundance of circular representations of fractions (e.g. the classic pizza party problems of sharing circular pizzas). Circular representations are problematic because partitioning circles equally is more difficult for odd or large numbers. For example, Gould, Outhred and Mitchelmore (2006) asked young children to represent one half, one third and one sixth using circle area diagrams. In this study, the researchers found that most students were accurate when shading in one half of the region of a circle, using either a horizontal or a vertical line to partition the circle into two equal parts. However, when children were asked to represent one third and one sixth, there were a wide range of incorrect responses where the partitioning of circles was uneven (non-equal parts) and the students relied on a count-wise 'number-of-pieces' approach (where the number of pieces in total and the number of pieces shaded was more important than size of pieces). For example, one student illustrated one sixth by partitioning a circle into eight sections and then shaded in six of these sections, and inserted a numeral (1 through 6) in each of the shaded pieces. This was an effort to illustrate one sixth by highlighting each piece of the circle with a counting number (rather than a fractional unit). In this example, the student is treating the numerator of 1 and the denominator of 6 as two independent numbers, rather than as a single number relationship (see Jigyel & Afamasaga-Fuata'i, 2007 for

similar findings), while simultaneously ignoring the 'extra two sections.' Prior to this study, Hart (1988) had similar results when working with 12- and 13-year-olds who were able to correctly shade in two thirds of a regional model but the students almost always used the strategy of counting the number of sections in a figure (3 sections in total) and then counting the sections that required shading (2 sections require shading), rather than interpreting the fraction as part of a whole region, or understanding that each of the equally partitioned regions show one one-third of the whole region. Two promising representations, as noted in the research to date, are bar or rectangular area models and number lines as a linear representation because both are more readily and accurately partitioned evenly for odd and large numbers of partitions (Watanabe, 2012).

Hackenberg & Lee (2012) also emphasize that students who demonstrate 'pre-fractional understanding' are not able to unitize or iterate; that is, to see a fraction such as three fifths as three one-fifths rather than as a region of three whole shaded parts embedded within a larger whole. The authors of the study also insist that to truly understand part-whole fractional relationships, students must also be able to "disembed" those parts within the whole – to see them as separate from "the whole while keeping the whole mentally intact" (943) (while embedding itself allows the student to see the part within the whole in the first place). Even when students are able to accurately partition an area model to show a fraction (correctly showing the number of parts within the whole), they may actually then ignore the whole as an essential piece of information in understanding the fractional relationship.

As an example of the cognitive processes required to disembed (to see parts while keeping track of the whole), suppose that a student is asked to identify $\frac{1}{6}$ of the following figure, and then use that $\frac{1}{6}$ unit to create a shape that is $\frac{5}{6}$ of the original whole.

Figure 5: Embedding in the Context of Partitioning and Iterating
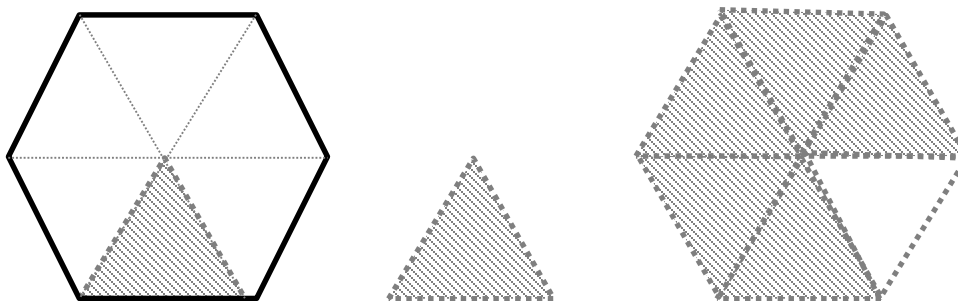
A student would first partition the figure into six equal pieces. They would then 'disembed' the one-sixth whilst simultaneously holding the mental image of the whole. The student would then iterate the one sixth to make $\frac{5}{6}$ of a whole.

Student gaps and misconceptions also are revealed in studies where students are asked to order and/or compare fractions. In 1995, for example, a representative national US study found that only one third of the sample of 13-year-old students were able to correctly place a simple fraction on a number line – a learning objective for 11-year-olds (Kamii & Clark, 1995). And there appears to be little progress when twenty years later, on the 2004 National Assessment of Educational Progress (NAEP), 50% of 8th graders were not able to order three fractions from least to greatest with accuracy (Siegler et al., 2010).

ii. Confusing the Roles of – and the Relationship Between – the Numerator and the Denominator

Students frequently conceive a fraction as being two separate whole numbers (Jigyel & Afamasaga-Fuata'i, 2007) and consequently apply whole number reasoning when working with fractions. For example, the majority of Grade 9 students, when asked to estimate the sum of $\frac{11}{12}$ + $\frac{7}{8}$, choose 19 or 20 as the answer in a multiple-choice format (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Additionally, students will reason that $\frac{1}{3}$ is larger than $\frac{1}{2}$ since the number 3 is larger than 2. Huinker (2002), as cited in Petit et al., (2010) states that 'students who can translate between various fraction representations "are more likely to reason with fraction symbols as quantities and not as two whole numbers" when solving problems' (p. 146)

When transitioning from whole number thinking to working with fractions, students need to develop a strong understanding of the multiple constructs of fractions. Without this, students may not understand the possible meanings of the numerator and of the denominator, and of the distinctions between them (Empson & Levi, 2011; Petit et al., 2010; Jigyel et al., 2007). For example, when considering a fraction as representing a part-whole relationship, the numerator represents the count and the denominator represents the unit. When working with a fraction as a quotient the numerator is a quantity (dividend) which is being divided by the denominator (divisor). Students must also understand that the numerator and denominator have different roles within the fraction and that the interpretations vary depending on the role. For example, when comparing fractions, if students do not understand that the numerator's role is different than the denominator's then they will struggle with the following two statements.

$$\frac{3}{4} > \frac{3}{6} \qquad\qquad \frac{3}{4} > \frac{1}{4}$$

In the first, the fraction with the smaller denominator (4) actually represents a larger fraction since the pieces are larger. In the second, the fraction with the smaller numerator (1) represents a smaller fraction since the pieces are the same size (Behr et al., 1993). The importance of developing this *quantitative notion* of fractions is that it helps students to evaluate the relative quantity or 'bigness' when comparing fractions (Bezuk & Bieck, 1993).

Further confusion about the role of the numerator and denominator arises with a premature introduction to fraction notation and/or the inadvertent use of imprecise language. Describing two-thirds as 'two over three' or 'two out of three' leads to students conceptualizing each as a separate whole number, rather than recognizing the multiplicative relationship that is inherent in the notation (that is to say that two-thirds is two one-third units or that it is referencing two one-thirds of a whole) (Mack, 1995; see also Lamon, 1999; Brown, 1993).

iii. Use of a 'Gap Thinking' Approach

Gap thinking (Pearn & Stephens, 2004) is another common but inappropriate reasoning strategy students demonstrate. For example, when a student is comparing four fifths and eight ninths, she or he might argue that these are equal because both of these fractions require 'one more' to make a 'whole amount'. These students are considering the numerical sequence of numbers, where the gap between four and five is the same amount as the gap between eight and nine; students are considering an absolute numeric difference, rather than the actual size or area of the pieces in question (which is the ratio of the numerator to the denominator). Rather, students need to be able to compare the *size* of the named fractional amount to the whole, as in the following example comparing four fifths and eight ninths.

Figure 6: Comparing Named Fractional Amounts (four fifths and eight ninths)

Four-fifths green   Eight-ninths green
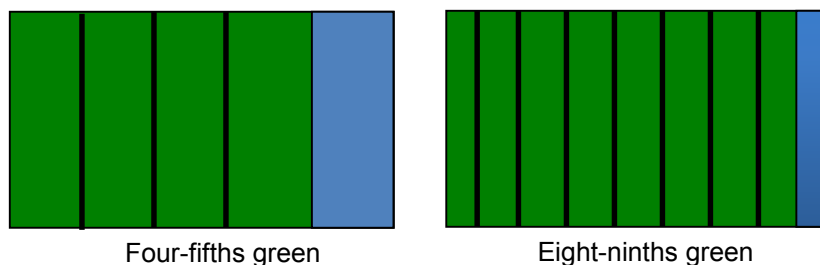
iv. Lack of Attention to Equivalence and Equi-partitioning

In a 1986 study, Post, Behr & Lesh asked fourth graders to compare rational numbers in pairs of like ($\frac{1}{4}$, $\frac{3}{4}$) or unlike denominators (e.g., $\frac{1}{4}$, $\frac{1}{5}$) as well as like numerators (e.g., $\frac{2}{5}$, $\frac{2}{4}$). This study found that the most important feature of children's thinking enabling them to accurately compare the fractions in all three types of pairs was the consideration of the fraction notation as a conceptual unit, or *quantity*, rather than two discrete numbers separated by a horizontal bar (a common student misconception). Comparing fractional values requires a strong sense of equal partitions as well as equivalence. Unfortunately, minimal time is allocated to understanding the general concept of equivalence throughout the

primary grades, including foundational understandings that connect with later fractions concepts; for example, in primary, there is a lack of attention to establishing the concept of "same as" represented by the equals sign. In both primary and junior levels, minimal time is devoted to equi-partitioning (e.g., splitting an area into equal regions) to establish the importance of equivalent partitions in fractions situations. These are foundational ideas required when comparing fractions and understanding fractional units (Confrey, 2012). Confrey, J., Maloney, A. P., Wilson, P. H., & Nguyen, K. H. (2010) identify three criteria that must be coordinated (or considered simultaneously) when equi-partitioning. : (1) the creation of equal sized groups or parts, (2) the organization of the correct number of groups or parts, and (3) the exhaustion of the entire collection or whole. They state that through these acts, students "gain proficiency in mathematical reasoning practices such as justification and naming (e.g., as a count, fraction, or ratio) and begin to develop understandings of fundamental mathematical properties that later influence the ways that they fairly share multiple wholes" (Confrey et al., 2010).

Similar challenges in the Ontario context

The challenges identified in this literature review were confirmed in a fractions study conducted by its authors in Ontario, Canada in 2011. In this study, students in grades 4 through 7 and across three school boards consistently demonstrated significant struggles with:

- recognizing the importance of defining the whole. When asked to place a number of fractions on the number line, students would place $\frac{1}{2}$ at the half-way point relative to the length of the line and then place other numbers, such as $2\frac{5}{6}$ at the appropriate position between 2 and 3. Frequently students did not revisit the placement of $\frac{1}{2}$ to revise their thinking.

Figure 7: Placing Fractions on a Number Line
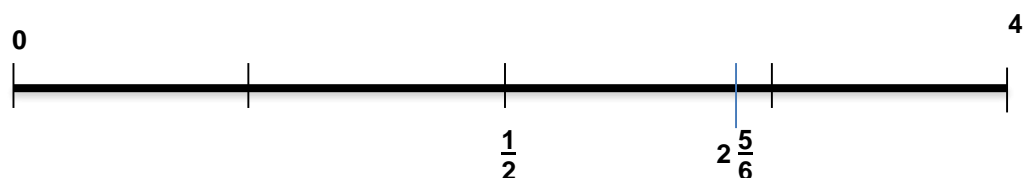
- partitioning circles into the appropriate number of congruent segments, particularly when working with denominators such as 5 or 10

- recognizing that the procedural solution should align with a representation. In Figure 8, students correctly state at the top of their work that the fractions are equivalent but none of the representations confirm this statement.
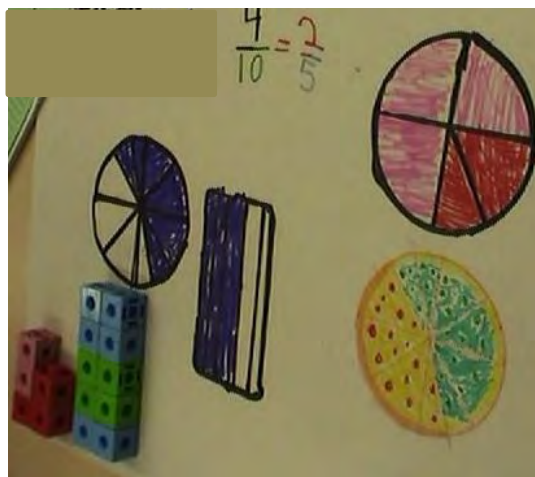
- distinguishing between the different meanings of fractions (part-whole, part-part, measure) and when to use which meaning. This same sample shows students using both part-whole and part-part representations for comparing fractions.
- understanding the role of the numerator and the denominator as well as their relationship to each other. Students consistently partitioned the figure and then shaded in a number of sections to represent a fraction rather than considering the portion of the area that should be shaded.
- understanding fraction as number, possibly due to confusion introduced through imprecise language used to describe fractions, such as 'over' and 'out of'. In the following examples, the first shows a student's interpretation of 'four tenths' as 'four tens'. The second shows a student literally representing $\frac{4}{10}$ as 'four over ten'.

Figure 9: Student Interpretations of Four-tenths
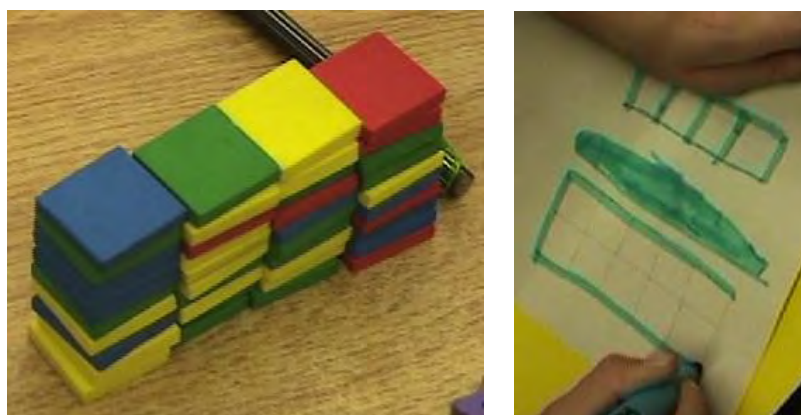
Section 4. Related Difficulties with Learning Fractions Operations

Without the requisite conceptual understanding such as the importance of equivalence, estimation, unit fractions, and part-whole relationships, students struggle to complete calculations with fractions. As referenced in Kong (2008, p. 246), Huinker (1998), Niemi (1996b) and Pitkethly & Hunting (1996) all confirm that "…learners seldom understand the procedural knowledge associated with fractional operations such as addition and subtraction" and this is strongly connected to their lack of foundational understanding of the meaning and ways of thinking about fractions. For example, students who have difficulty understanding ratio relationships run into difficulty when considering the following scenario, adapted from Empson & Levi (2011): Imagine we are examining two bowls of candy. In the first bowl 50% ($\frac{50}{100}$) of the candies are green. In the second bowl, 20% ($\frac{20}{100}$) of the candies are green. If we put the two bowls together, are 70% green? Does it make sense to simply add all of the green candies without considering how many candies are in the bowls in total? This type of fractions addition problem seems to perplex students. Students who lack an understanding of fractions as representing ratio relationships fail to understand that there are not necessarily 100 candies in each bowl. They may also add the percentages together, applying whole number reasoning to rational number situations. Knowing an algorithm for adding fractions will likely not support a student in correctly responding to this question.

III. RESEARCH ON TEACHING OF FRACTIONS

Section 5: Challenges Teaching Fractions

The challenges teaching fractions run parallel to and overlap with many of the challenges learning fractions. There are a number of factors contributing to the widespread challenges associated with teaching fractions, some of which are addressed in greater detail here due to their practical relevance and for consideration in the development of new learning tools and supports for student learning of fractions. In this section we will discuss the current resources available to teachers and students and related common instructional practices that promote limited fractions understanding.

Hasemann (1981) provided several possible explanations for why children find simple fractions so challenging, including: 1) fractions are not obviated in daily life, but instead are hidden in contexts that children do not recognize as fractions situations; 2) the written notation of fractions is relatively complicated; and, 3) there are many rules associated with the procedures of fractions, and these rules are more complex than those of natural numbers. Moss and Case (1999) agree that notation is a challenge for students, but they also suggest several other pedagogical complications; to begin, when rational numbers are first introduced to students they may not be sufficiently differentiated from whole numbers, neglecting the importance of the relationship that a fraction names (Kieren, 1995). Later on, the importance of procedural manipulations of fractions may be privileged over the development of conceptual understanding. A focus on procedural manipulations, without any understanding of why the procedures work, may contribute to a student's perspective of the senselessness of mathematics. Consider the following summary of the procedures for fraction operations:

> When we add or subtract fractions, we have to find a
> common denominator, but not when we multiply or divide.
> And once we get a common denominator, we add or
> subtract the numerators, but not the denominators, despite
> the fact that when we multiply, we multiply both the
> numerators and denominators, and when we divide, we
> divide neither the numerators nor the denominators.
> (Siebert & Gaskin 2006, p. 394)

These "rules" might make sense to those who already conceptually understand fractions operations, but they do not help to support students who are just learning how to work with operations that include fractions. Unfortunately, students are often presented with wordy rules for procedures, such as the example above, that are difficult to understand and get conflated with definitions of what it *means* to perform an operation. To further complicate matters,

spontaneous or invented strategies for adding and subtracting fractions are typically discouraged, inadvertently discouraging students from sense making (see Confrey, 1994; Kieren, 1995; Mack, 1993; Sophian & Wood, 1997).

i. Current Resources

The textbook resources available to educators in North America consistently treat fractions as a discrete topic or 'unit' of mathematics learning each year. In these discrete units of study, students are shown a vast range of visual representations of fractions, perhaps due to a widespread belief that in showing many different representations, something will make sense to the student. Provincial curriculum documents also present learning expectations as discrete outcomes that are focused on precise skills, such as representing a fraction, but with no explicit connection across expectations beyond fractions. This discrete approach to fractions learning has come into question in the mathematics educational research community. In 1999, for example, Moss and Case encouraged curriculum developers to shift focus from the "attainment of individual tasks toward the development of more global cognitive processes" (123). They came to this recommendation based on the intensive study of children's learning of fractions. Similarly, Watanabe (2012), a mathematics researcher who has been studying fractions for over two decades, suggests that the focus of mathematics programs should be on fractions as quantity, allowing students to make a strong connection to their existing knowledge of whole numbers as quantity. Although the Ontario curriculum is strong, students would benefit from further resources including a revised curriculum based upon known development progressions: one that strongly supports connections among number systems, and between fractions, estimation and proportional reasoning.

Of considerable interest is the use of multiple representations in current resources. As early as 1994, Pirie and Kieran found that student understanding is significantly influenced by "strong attachments to initial particular images" (Pitta-Pantazi, Gray & Christou, 2004, 42). For example, if a student is exposed to circle representations as the first fractions illustrations, this likely becomes the 'go to' representation for that student when working with fractions. Numerical representations of numbers also go through two key stages related to a child's cognitive development, beginning with a semiotic stage where meaning is established by relying on previously constructed representations, and then moving to an autonomous stage where new systems of representations become independent of their precursor (Thomas, Mulligan & Goldin, 2002). Again this signals the importance of the first 'images' or representations of fractions and the value in a deliberate selection of representations to build upon as the learning about fractions deepens over time. High achievers have been shown to have much greater facility in thinking flexibly between representations and in making a mental map of the web of connections between representations (Pitta-Pantazi et al., 2004). Unfortunately, in analyzing North American print resources for students, it is unclear which representations are most helpful to struggling

students nor is it clear how the representations are constructed to build meaning over time.

ii. Instructional Practices in Understanding Fractions

Over-emphasis of fractions as part-whole relationships

Research has identified an overemphasis of fractions as exclusively part-whole relationships in North American classroom instruction, which limits student understanding of fraction as quantity, leading to a number of consistent misunderstandings with respect to fractions. For example, there is general agreement that this singular interpretation of fractions as part-whole interpretation results in students' struggling to build an understanding of and work with improper fractions (Lamon, 2001; Smith 2002; Thompson & Saldanha, 2003; Charalambous & Pitta-Pantazi 2005; Watanabe, 2006). This is further reinforced when students are provided with pre-partitioned figures and count first the number of partitions and then the number of shaded sections, generating two numbers which are combined into a fraction. According to Simon (2002), when students don't understand the equivalency of pieces of congruent figures that have been partitioned in half, this indicates an understanding of fractions as an arrangement rather than a quantity.

Mathematics education researchers emphasize that students must have a deep understanding of fraction as measure, in addition to part-whole, in order to support their development of an understanding of addition and subtraction of fractions (Lamon 2001; Keijzer & Terwel 2001; Watanabe, 2006). Rather than overemphasizing part-whole relationships, and building on the work of Moseley and Okamoto (2008), the 2011-2012 KNAER project highlighted five potential ways of thinking about fractions: as linear measures, part-whole relationships, part-part relationships, quotients and operators. Watanabe (2006) further amplifies the different ways of understanding fractions in his discussion of the Japanese emphasis of the unit fraction to define a fraction. That is to say that a fraction may also be considered to be a multiple of a unit fraction. For example, "$\frac{2}{3}$ would be considered as 2 times the unit obtained by partitioning 1 into 3 equal parts" (Watanabe, 2006). We can name this as "two one-third units". If we are adding fractions such as $\frac{1}{3}$ and $\frac{2}{3}$ we can say "one one-third and two more one-thirds gives us three one-third units."

Imprecise language

A lack of precise language also contributes to the perception of a fraction as two numbers rather than one quantity. The fraction $\frac{2}{5}$ can be read as 'two fifths'. However, frequently it is also read as 'two over five' or 'two out of five'. Siebert & Gaskin (2006) explain that the use of 'out of' leads students to see themselves

as having five items and then removing two of those items. The students see both the numerator and denominator as whole numbers and do not develop an understanding of the underlying act of partitioning a whole into five parts. They also see the denominator as being the whole rather than the number of equal partitions within the whole; students may see five as the whole, rather than considering the actual whole, such as the area of a figure (which is partitioned into fifths). Furthermore, students are not able to extend this 'out of' definition to improper fractions, for how do we 'take' 4 items 'out of' 3 when presented with $\frac{4}{3}$? Students exposed to this imprecise language may likewise struggle with adding and subtracting fractions as they can easily overlook or lack an appreciation for the need for common denominators.

Similarly, the use of 'two over five' does not contribute to students' understanding of fraction as a number. Jigyel and Afamasaga-Fuata'I (2007) found in an Australian study of 56 students that 63.6% of Year 8 students (approximately ages 12-13) and 66.7% of Year 6 students (approximately ages 10-11) chose 'two over five' as one of the correct ways of saying the fraction $\frac{2}{5}$. Some of them reasoned that this was correct because the two and the five are unrelated quantities stacked on top of one another:

"Two is on a line above 5 so you can say 2 over 5 or two fifths." (Year 8)
"There is a two over a five." (Year 6)

Imprecise representations

Circle representations are difficult to partition equally, leading students to focus more on the number of partitions and less on the congruency of partitions, resulting in student confusion about whether partitions must be congruent or not. According to Moss and Case (1999), this count-wise approach to pieces of a circle where each piece counts as a whole number (one piece) does not account for the importance of equal area nor the importance of the whole in relation to the pieces. In Ontario, it is particularly perplexing where, although students in primary grades use circle representations when studying fractions, the concept of area of a circle is not formally addressed until intermediate grades. This creates an interesting situation in which students are required to use the construct of area of a circle to create equal partitions yet have not been formally exposed to the properties of area of the circle (Watanabe, 2012). There is substantive documentation of students failing when they attempt to partition circles evenly unless they are considering halves and fourths. Fractions other than halves and fourths including thirds, fifths, sixths, ninths, etc. appear to be highly problematic (Ontario Ministry of Education, in publication). Additionally, Watanabe (2007) emphasizes that there is currently an overemphasis on pre-partitioned fractions in North American textbooks, which limits the opportunities for children to engage in "direct and active partitioning as an exploration of the creation and meaning of fractions"(4) and that as a result, students use a count method to solve rather than seeing the partitions as fractions of the whole.

In American resources, students from kindergarten through Grade 8 are exposed to up to 25 different representations of fractions, compared to only four in Japanese resources (Murata, 2012). The four representations are:
- number lines
- rectangular area models
- volume models
- flats and rods

These representations are used consistently and with the purpose of developing students' understanding of *fraction as quantity* and to emphasize the underlying concepts of (i) expressing all fractions as a multiple of a unit fraction, (ii) making comparisons based on like-units, and (iii) identification of the whole (Watanabe, Murata, Okamoto, 2012). This set of representations strongly supports moving from understanding the different meanings of fractions to operations with fractions relatively seamlessly because these representations are extremely flexible in their use as the curriculum builds. The effectiveness of the consistent use of representations is supported by the findings of researchers Pirie and Kieran (1994), who found that students hold on to the representations they are initially exposed to as the grounding for their conceptual understanding. Since this is the case, it makes sense to select precise representations that have longevity and power.

Premature privileging of numeric-symbolic procedures

Kiernan, as cited in Huinker (2002) and referenced in Petit et al. (2010), found that "premature experiences with formal procedures (algorithms) may lead to symbolic knowledge that is not based on understanding, or connected to the real world" (148). This is further compounded by the progressive removal of the use of models of fractions to privilege symbol notation, which has the potential to impede students in developing fluency across the different representations of fractions. Jigyel and Afamasaga-Fuata'i (2007) found in their research that many of the Year 8 students (approximately ages 12-13) could not explain how a fraction wall (bars) demonstrated equivalency. This lack of fractions understanding results in students relying on memorized algorithms and making frequent errors in the application of these algorithms (Brown & Quinn 2006). Saxe, Taylor, McIntosh & Gearhart (2005) suggest monitoring understanding of fraction notation separately from the understanding of fraction concepts as students develop these two domains somewhat independently.

Moss and Case (1999) found similar evidence of two independent processes: a) a global structure for proportional evaluation and b) a numeric structure for splitting or doubling. In their study, coordination of these two structures did not occur until approximately ages 11 and 12, leading the child to be able to understand semi-abstract concepts of relative proportion and simple fractions and percentages such as one half (or 50 percent) and three fourths (or 75 percent). Based on these observations, Moss & Case developed an innovative instructional lesson sequence, beginning with a beaker of water. The students

began using general terms to describe the beaker as nearly full, nearly empty, etc. The lessons then introduced percents such as "100% full," linking to children's pre-existing knowledge and schema, as well as their familiarity with real contexts and familiar representations. Next, the lesson sequence introduced decimals, and finally connected these forms of describing amounts to fractions. The study used a pre-post control and treatment group design. Both the control and the treatment groups showed improvement from pre to post; however, the treatment group who had experienced the innovative lesson sequence showed statistically significant gains. The children in the control group were able to perform standard procedures with simple numbers, however when confronted with novel problems, these students were less successful. The treatment group children demonstrated flexibility in their thinking and approaches to the problems presented, and were more accurate with their solutions. The results of this study suggest that reconceptualizing the order of tasks and concepts, as well as the representations used, hold promise for building on students' existing knowledge and tackling the significant challenges presented by fractions learning and teaching.

The artifice of word problems

Research indicates that the superficial inclusion of fractions in traditional word or story problems is also problematic. Word problems have typically been used in an attempt to make mathematics more meaningful or relevant to students, however students (and teachers) tend to treat word problems as situations where the procedures are simply hidden in words and the challenge is to decipher what steps need to be taken. (To see a meta-analysis on the effects of word problems, go to http://nichcy.org/research/summaries/abstract9.) Boaler (1993), who studied schools with different pedagogical orientations, found that 12- to 13-year-old students experiencing a teacher-directed math program (with an emphasis on procedures, repetition and traditional word problems) had difficulty translating these experiences in mathematics to inquiry-oriented context-rich situations. In the case of fractions, when students from teacher-directed programs were asked to compare fractions in a more context-rich form, they were unsuccessful. On the other hand, student participants from a school committed to teaching for deep understanding through inquiry approaches were more successful both with traditional word problems and with inquiry-oriented novel problems. As Petit, Laird & Marsden (2010) explain, "premature experiences with formal procedures (algorithms) may lead to symbolic knowledge that is not based on understanding, or connected to the real world (Kieren, as cited in Huinker, 2002)" (148).

The implications here suggest that students with strong procedural abilities, even with fractions, may have weak conceptual foundations and/or the ability to apply understanding, depending on the type of mathematics classroom programming.

Section 6: Different Cultural Approaches to Fractions Instruction

Numerous cross-cultural comparative studies over the last two decades have demonstrated that mathematics performance in East Asian countries far surpasses that of the United States (Son, 2011; Charalambous et al., 2010; Watanabe, 2007; Zhou, Peverly, & Xin, 2006; Stigler & Perry, 1988), whereas Canada ranks near the top in international comparisons (OECD, 2009; Mullis, I., Martin, M., Foy, P., & Arora, A., 2011). Asian-American differences in mathematics achievement have been discovered as early as kindergarten and international assessments have shown these differences to be pervasive across almost all mathematical categories, including fractions. And although Canadian students achieve well on these assessments, fractions understanding is weak. In this section, we ask: What are those countries who excel with fractions doing right?

The focus of this section is to provide a picture of effective fractions instruction across Asian countries. In particular, fractions instruction in Japan, Korea, and Taiwan are discussed and then compared and contrasted with that of North America. At the end of the section, a summary of key ideas central to an effective and coherent fractions program is outlined that draws upon the similarities in programming across the countries. We begin with Japan, a country often noted for its sound mathematics programming (Watanabe, 2007; Stigler & Perry, 1988).

i. Japan

In Japan, fractions are formally introduced in Grade 4 (Watanabe, 2007). According to the teachers' manual that accompanies Japanese textbooks, teachers are responsible for communicating two main ideas when teaching fractions: 1) fractions are used to denote quantities less than 1; and 2) fractions are numbers just like whole numbers. Both of these are key concepts emphasized throughout fractions instructions from its introduction in the fourth grade to the end of elementary education and beyond.

In the fourth grade, fractions instruction focuses on developing the meaning of fractions and also introduces the concept of mixed number (Watanabe, 2006). Although the Japanese curriculum also emphasizes part-whole relationships, exposing students to mixed numbers and improper fractions early prevents students from developing the misconception that all kinds of fractions must be parts of one whole. Furthermore, decimal numbers are also introduced alongside fractions in the fourth grade. In the fifth grade, the relationships between fractions, decimals and whole numbers are further consolidated. Finally, in the sixth grade, students engage in an in-depth investigation of arithmetic with fractions.

Five fraction constructs comprise the core of fractions instruction in the Japanese elementary mathematics curriculum (Watanabe, 2007). They are: 1) part-whole

relations, 2) unit and non-unit fractions, 3) fractions as operators, 4) fractions as quotients, and 4) fractions as ratios. The first three are introduced in fourth grade and the remaining two in the fifth and sixth grades, once students have built a foundational understanding of fractions. In this manner, the Japanese curriculum places almost equal emphasis on all five constructs for the purpose of familiarizing students with the different interpretations of fractions.

Of particular notability is the construct of unit and non-unit fractions. This interpretation of fractions is almost absent in the North American context. In Japanese fractions instruction, the unit fraction – $\frac{1}{n}$ – (e.g., $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, etc.) is introduced early on and all other fractions – $\frac{m}{n}$ – (e.g., $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{6}$, etc.) could be considered "m times $\frac{1}{n}$". Therefore, $\frac{3}{5}$ is interpreted as "three one-fifth units", emphasizing the fact that $\frac{3}{5}$ represents a quantity that is three times that of $\frac{1}{5}$. In Japanese textbooks, a measurement context is often used when discussing the treatment of unit and non-unit fractions.

Analyses of Japanese textbooks have revealed that most fractions problems are framed within a measurement context where linear representations of fractions are used (Watanabe, 2007). The predominance of linear representations in the form of rulers (when integrating fractions and decimals) and number lines is due to an effort by the Japanese curriculum to emphasize that fractions are numbers. In Japanese textbooks area models are not often used due to the fact that fractions are generally introduced before area measurement. Japanese educators reason that using a representation about which students do not have deep conceptual understanding would not help them when solving problems about fractions (Watanabe, 2012). On the other hand, linear models such as tape diagrams are used throughout early Japanese elementary education in the study of whole numbers. Therefore, students already have some familiarity with linear representations by the time they begin to study fractions. The transition from tape diagram representations to number lines during the study of fractions is both purposeful and intentional to ensure a natural progression in learning for students.

The following excerpts from the Japanese Textbook Share with Your Friends: Mathematics for Elementary School (translated to English) (Hitotumatu, 2011) allow for an examination of the structure and sequence of the mathematics learned in Grade 4. It is important to note that such text resources are supported by teacher resources and a robust curriculum document which allows for teachers to use the textbooks as a supplement following active learning (Watanabee, 2012).

An excerpt from the table of contents is shown in Figure 10.   Note that decimal numbers (including how to represent decimal numbers and the structure of

decimal numbers) are learned before the fractions module, which focuses on fractions larger than 1, equivalent fractions and addition and subtraction of fractions.

Figure 11 shows the beginning pages of the addition and subtraction of fractions section. Notice that volume models are used and students are immediately reminded of the unit fraction. This allows them to connect the addition and subtraction of fractions appropriately with their prior learning of whole number operations. In the second question, this use of unit fractions is clearly represented pictorially. Since students have already worked with a variety of tasks to build their understanding of the concepts, the textbook provides an opportunity to consolidate their learning.

Figure 11: Japanese Textbook: Excerpt on Addition and Subtraction of Fractions

(Hitotumatu, 2011)

ii. Korea

Like Japan, students in Korea consistently achieve top rankings in mathematics performance across different international studies (Son, 2011). Fractions instruction in Korea bears some similarities to both the North American and Japanese curriculum, although some of its unique features may have further contributed to its effectiveness. Fractions are introduced in the third grade, a year earlier than in Japan. In the introduction of fractions, the focus is on fractions as representing parts of a whole, parts of a set, and points on a line. Furthermore,

as in Japan, decimals are introduced alongside fractions to help solidify the relationship between them early on.

In the third grade, the Korean curriculum focuses on introducing fractions as parts of a whole and parts of a set (Son, 2011). Similar to Japan, unit and non-unit fractions are also introduced in a measurement context. Finally, students also learn to compare fractions with like denominators while learning about the relationship between fractions and decimals. In the fourth grade, different types of fractions such as mixed numbers and improper fractions are introduced. Students learn basic arithmetic with fractions and are taught to interpret fractions as quotients or as ratios. In the fifth and sixth grades, students continue to explore arithmetic with fractions, learning how to complete operations such as multiplication and division with different types of fractions. Furthermore, the relationship between fractions and decimals is further emphasized and investigated.

Like the Japanese curriculum, the Korean curriculum also focuses on five fractions constructs (Son, 2011). They are: 1) part-whole relationship, 2) measurement (the treatment of unit and non-unit fractions, 3) fractions as quotients, 4) fractions as ratios, and 5) fractions as operators. Most of these constructs are introduced early on and then revisited throughout the later elementary years. Fractions as operators, for example, are introduced alongside part-whole relationships in Grade 3 and then revisited again in the fifth and sixth grades.

In terms of fractions representations, the Korean curriculum introduces fractions with area models to emphasize fractional amounts as parts of a whole and then progresses into using discrete models for lessons on fractions as parts of a set (Son, 2011). When dealing with unit and non-unit fractions, linear models in the form of fraction bars are used and problems are often set in a measurement context. Fraction bars are also used when decimals are treated alongside fractions. Therefore, the Korean curriculum focuses on balanced usage of a few carefully selected models in different problem contexts (fraction bars being a favoured representation because of their flexibility and longevity).

Finally, analyses of Korean textbooks reveal that they provide more problem types than American or Japanese textbooks (Son, 2011). Each fractions lesson in a Korean elementary mathematics textbook is divided into four or five different activities, with some focused on solidifying conceptual understanding and others set in real-life contexts. Most activities are accompanied by questions that encourage students to explain their thinking process when providing a solution.

iii. Taiwan

Charalambous and colleagues (2010) completed a comparative study examining textbooks and their treatment of addition and subtraction of fractions in Taiwan, Cyprus, and Ireland. The results of the analysis demonstrated that by comparison, Taiwanese textbooks have the most comprehensive and effective

fractions instruction program. This does not come as a surprise as Chinese students from Taiwan consistently rank highly in international mathematics assessments.

Like the US and Korea, the Taiwanese mathematics curriculum introduces fractions in the third grade with a focus on the composition and decomposition of fractions (Charalambous et al., 2010). The meaning of fractions is developed in Grade 3, while addition and subtraction of fractions are introduced in the fourth grade. At this time, students learn to add and subtract proper and improper fractions and mixed numbers with like denominators. The addition and subtraction of fractions with unlike denominators is not addressed until the later elementary grades.

Although in the introduction of fractions the focus is primarily on solidifying student understanding of fractions as representing part-whole relationships, Taiwanese textbooks also emphasize the treatment of unit and non-unit fractions in measurement contexts through representations such as line segments, weight, or the volume of liquids in volumetric glasses (Charalambous et al., 2010). Overall, Taiwanese textbooks use a combination of area model, set, and linear representations for fractions. The usage of different fractions representations depends on the particular problem context and/or fraction construct. In particular, the Taiwanese curriculum emphasizes the importance of unit fractions as the link between different fractions constructs as well as the idea that fractions are relative quantities.

In terms of textbook organization, Taiwanese textbooks have many graphical displays including cartoon figures explaining steps in a procedure (Charalambous, et al., 2010). Similar to Korean textbooks, students are expected to write mathematical sentences and explanations to clarify their thinking as they investigate the solution to different fractions problems.

iv. Key Elements in Effective Fractions Instruction Across Asian Countries

Although fractions instruction highlighted in each of the countries above differs in terms of when and how they present fractions content, there are key similarities. In the summary below, these similarities are captured and their importance discussed.

There is greater attention paid to fraction constructs beyond part-whole, helping students recognize that fractions have different meanings and representations and understand the relationships between these (Son, 2011; Charalambous et al., 2010, Watanabe, 2007).

Importance: In North America, the preoccupation with the understanding of fractions as parts of a whole, along with an overemphasis on proper fractions, is

troublesome as students often have difficulties comprehending fractions greater than one (Charalambous et al., 2010). Furthermore, all constructs of fractions are important and ones such as fractions as operator, quotient, and ratio become more so in advanced mathematics. Neglecting these constructs and/or failing to clearly communicate the relationship between them will present students with difficulties as they continue to pursue mathematics.

The treatment of unit and non-unit fractions help students see fractions as a single quantity rather than two whole numbers stacked together (Son, 2011; Charalambous et al., 2010, Watanabe, 2007).

Importance: When thinking of fractions $\frac{m}{n}$ as being "m out of n", students often fail to perceive the fraction as a singular quantity, and instead focus on counting parts out of a whole (Watanabe, 2007). This makes it difficult for students to quantify unfamiliar fractions such as $\frac{5}{6}$ or $\frac{7}{18}$ and thus comparing these fractions becomes extremely challenging. Comparatively, a focus on the unit fraction emphasizes fractions as quantities; for example, when these fractions are described as "five $\frac{1}{6}$ units" or "seven $\frac{1}{18}$ units," not only are they seen as a singular quantity, they can also be easily compared since visualizing the unit fraction becomes easier.

Representations are chosen to fit the problem context, and the fact that fractions are singular quantities is emphasized (Son, 2011; Charalambous et al., 2010, Watanabe, 2007).

Importance: Unlike the North American preoccupation with the 'pizza model' or other circular area models, East Asian countries use a combination of carefully selected models that have longevity in terms of their application in representing fractions and that reflect the notion of fraction as a quantity. Perhaps the most notable of these are the linear model and the related bar model, most heavily emphasized in Japanese fractions instruction, but also used in both Korean and Taiwanese textbooks (Son, 2011; Charalambous et al., 2010, Watanabe, 2007).

Decimals are introduced alongside fractions and the relationship between the two are emphasized throughout fractions instruction (Son, 2011; Watanabe, 2007).

Importance: In the North American context, fractions and decimals are rarely introduced together and their relationship is not always clearly communicated or understood by students (Moss & Case, 1999). Considering the numerous circumstances where fractions and decimals can be used interchangeably, a solid understanding of how the two are related is critical to facilitate the understanding of both concepts.

Mixed numbers and improper fractions are introduced early in the study of fractions (Son, 2011; Charalambous et al., 2010, Watanabe, 2007).

Importance: Due to the overreliance on part-whole relationships and area models, North American students often have difficulty understanding fractions greater than one in the form of mixed numbers and improper fractions (Charalambous et al., 2010). Introducing them alongside proper fractions and using different types of representations to show students how these fractions are applicable in real life and measurement contexts is important to prevent difficulties later on.

Section 7: Research-Informed Effective Instructional Practices

The research suggests that explicit and precise changes to learning and teaching practices can have a substantial impact on children's understanding of fractions and future mathematical success. Instructional decisions have a significant bearing upon students' ability to understand the concept of fraction, including the ability to represent fractions appropriately, compare the relative magnitude of two fractions, and complete calculations accurately. Research has been documenting what students understand about fractions and how this knowledge is impacted by instruction for over thirty years (Steffe & Olive, 2010; Jigyel & Afamasaga-Fuata'i, 2007; Mack 2004; Lamon, 2001; Kamii & Warrington, 1999; Hart, 1988; Carpenter et al., 1980; Kieren, 1980). In this section, we will discuss the importance of i) building on student intuition, ii) identifying content foci for instruction, iii) sequencing of instruction for fractions and iv) transitioning to addition and subtraction of fractions.

i) Building on Student Intuitions (and Benchmarking)

Moss and Case (1999) explored the benefits of building learning activities from students' intuitions and prior knowledge when supporting development of proportional reasoning. As previously discussed, children develop their initial understanding of relative proportion and simple fractions by the ages of 11 or 12 by coordinating the two rational number schema. When student learning is expanded to include different types of fractions, as well as connections between decimals and fractions, students are able to construct a generalized notion of rational numbers, usually near the end of high school. In comparison to a control group, the students exposed to the student-directed instructional sequence demonstrated increased proportional reasoning skills and decreased misapplication of whole number thinking, with no loss of accuracy on standard procedures (Moss & Case, 1999).

In *Developing Effective Fractions Instruction for Kindergarten through 8th Grade*, Siegler et al. (2010) make five recommendations regarding the learning and teaching of fractions, the first of which is to "build on students' informal understanding of sharing and proportionality to develop initial fraction concepts" (1). Children develop an understanding of sharing and proportionality prior to formal instruction in school and as a result can equally share a set of objects among a group of people and, using common shapes, identify equivalent proportions. Teachers of early grades can purposefully build upon students' intuitive knowledge when formal fraction concepts are introduced. This allows students to connect multiple concepts addressed through fair-sharing activities, including ratio, division, and fractions. As students engage in this learning they also develop informal understandings of the magnitude of fractions and equivalence concepts (Siegler et al., 2010).

ii) Identifying Content Foci for Instruction

Curriculum documents and classroom resources focus on the skills related to proficiency with fractions, including representing, comparing and ordering along with accurately completing calculations. Within these broad clusters of mathematics content are a number of implicit skills and knowledge that have considerable positive impact on students' abilities with fractions when embedded effectively in learning. Three main ones – developing proficiency with iterating and partitioning, developing fraction sense, and building understanding of unit fractions – have traditionally been underemphasized in North American instruction.

Developing Proficiency with Iterating and Partitioning

Iterating and partitioning are of particular importance in building conceptual understanding of fractions. Partitioning involves "creating smaller, equal-sized amounts from a larger amount" while iterating involves "making copies of a smaller amount and combining them to create a larger amount" (Siebert & Gaskin, 2006, 395). The actions of partitioning and iterating enable students to understand by doing. In this way, students use their fractions representation as the site of the problem solving where they are required to physically act upon the object thereby kinesthetically and visually creating fractions. Students engage in the actions that create a fraction – partitioning – as well as the action that can be used to verify the fraction – iterating (Siebert & Gaskin 2006). For example, partitioning an area model to represent one-fifths involves dividing a whole into five equal parts, such that each part is an equal amount of area, and showing one of those parts. To show one one-fifth might involve shading of that one one-fifth.

Figure 12: Partitioning a Whole into Fifths

Through the act of partitioning, students create unit fractions. They may count '1 one-fifth, 2 one-fifths, 3 one-fifths, 4 one-fifths, 5 one-fifths' when creating the whole from the unit fraction. This reinforces the notion of a fraction as a number and the meaning of the numerator and denominator as well as the connection between the unit fraction and the whole (Empson & Levi, 2011; Petit et al., 2010). Petit et al. (2010) suggest that students should begin with partitioning that involves repeated halving and then engage in creating partitions of even numbers, such as ten. Students should then create odd number partitions, such as 9, and then work with composite number partitions, such as 12, which can be constructed using a rectangle partitioned into 3 rows and 4 columns. This engages students in multiplicative reasoning to partition the wholes.

Iterating an area model to represent a whole involves copying the unit of $\frac{1}{5}$ (one-fifth) five times.

Figure 13: Iterating Fifths to create a Whole

| $\frac{1}{5}$ | $\Rightarrow$ | | | | | | $\Rightarrow$ | Whole |

As with partitioning, students can be encouraged to count during iteration to create the whole. Knowledge of partitioning and iterating supports students in understanding fractional relationships as well as operating on them (Siebert & Gaskin, 2006, p. 395).

One distinction from whole numbers is the density of fractions, in that it is always possible to identify a fraction that lies between two other fractions on the number line (e.g., the fraction $\frac{5}{14}$ lies between $\frac{2}{7}$ and $\frac{3}{7}$). This is not true for whole numbers, as there is no whole number between one and two.

Developing Fraction Number Sense

Students who have a strong understanding of fraction as number can use their whole-number strategies, including estimation and benchmarking, when working with fraction operations. The knowledge of benchmark fractions supports students in estimating sums and differences (Johanning, 2011). Rather than emphasizing procedure, students should be expected to demonstrate an understanding of the appropriate use of the operations as well as the different effects that operations have on numbers. This can be developed by asking students to explain their reasoning and provide a solution without calculating an exact answer (Johanning, 2011).   Johanning (2011) states that "providing simple contextual problems that require estimation and that can be solved using models or diagrams will encourage the use of number sense when operating with fractions" (100; also see Cramer, Monson, Whitney, Leavitt, & Wyberg. 2010, Van de Walle, Karp, & Bay-Williams, 2010).

Charalambous et al. (2010) found that "engaging students with multiple fraction constructs catalyses learning, not only because each fraction construct captures part of the broader notion of fractions but also because moving from one construct to another reinforces understanding" (142). Students must have an understanding of fraction across multiple constructs in order to have flexibility when using fractions in context. The multiple meanings are underdeveloped when the instruction is focused almost exclusively on the part-whole relationship of fractions, reducing the importance of fractions as operators, or as a value by which a quantity is enlarged or shrunk, a concept utilized in the algebraic application of fractions.

Students engaged in reasoning, proving and discussing their mathematics will develop a deeper understanding of the concepts being examined. The intentional creation of a math talk learning community (Bruce, 2007) allows students to reflect upon their thinking and determine the reasonableness of an answer, which involves the evaluation of reasonableness of the algorithm used (Johanning, 2011).

Building Understanding of Unit Fractions

When students add two whole numbers, the common unit – one whole number – is implicit. That is to say, students would not define '5' as '5 whole number units of 1' even though this is true, because the unit is embedded and, therefore, a 'given.' However, when students add fractions, they must attend to the unit before adding quantities together. As students transition from whole numbers to other number systems, including fractions, explicit attention must be paid to the unit so that students develop this understanding. When representing fractions, students should think of $\frac{2}{5}$ as '2 one-fifth units' (Watanabe 2006, Mack 2004).

As discussed in Section 6, this is a common practice in other cultures, such as Asian countries including Taiwan, Korea, and Japan. This use of unit fractions allows students to make connections between the different constructs of numbers and "provides a catalyst for students' transition from whole to rationale numbers (Charalambous et al., 2010, 142; see also Carpenter, Fennema, & Romberg, 1993; Mack, 2001). However, in North America the need for a common unit is more commonly addressed by determining a common denominator using a procedural approach. A number of researchers have highlighted the need for a continued focus on the common unit during instruction of fractions, including Mack (2004) and Watanabe (2006).

When students understand that the need for a common unit is universal for all addition and subtraction, they can more readily connect their understanding of whole number addition to other number systems, such as decimals and fractions, as well as algebraic operations. This allows students to develop a common understanding of addition and subtraction across all number systems. Consider the following examples:

Whole Numbers			Decimal Numbers		
2 + 3	2	1 units	0.2 + 0.3	2	0.1 units
	+ 3	1 units		+ 3	0.1 units
	5	1 units		5	0.1 units
	= 5			= 0.5	

Fractions				Algebra		
$\frac{2}{7}+\frac{3}{7}$	2	$\frac{1}{7}$ units		2a + 3a	2	a units
	3	$\frac{1}{7}$ units			+ 3	a units
					5	a units
	5	$\frac{1}{7}$ units			= 5a	
	$=\frac{5}{7}$					

When students transition to addition and subtraction of fractions, they must first understand that adding and subtracting always involves combining like units. (This focus on the unit also helps with addition and subtraction of larger numbers and the understanding of place value.) When fluency with equivalent fractions is developed, students are better able to consider addition of unlike fractional units by first relating each quantity to a common unit (common denominator) (Empson & Levi, 2011). This also aids in distinguishing the fractional unit, such as thirds, and the whole unit, such as cups, within a contextual problem (Empson & Levi, 2011).

Consider how the following representation can be used to productively show unit fractions in the context of the whole. Number lines (linear measures) can effectively show proper and mixed fractions simultaneously and promote attention to the relationship of the numerator to the denominator. Consider this representation of $\frac{2}{6}$ and $\frac{8}{6}$, adapted from Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn, & Gearhart (ND):

Figure 14: Number Line Model
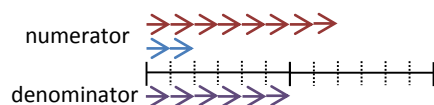
We can see from the diagram that the distance to one is 6 one sixth units. Also, since the red arrows show $\frac{8}{6}$ and the blue arrows show $\frac{2}{6}$, students can also compare the two fractions. They may also see that the difference between the two fractions is $\frac{6}{6}$ or that $\frac{8}{6}$ is four times $\frac{2}{6}$.

iii) Sequencing of Instruction for Fractions

The topic of rational numbers in mathematics has an ample research base that illustrates, in some cases meticulously, how children's thinking about fractions

could progress (Behr, Harel, Post, & Lesh, 1993; Davydov & Tsvetkovich, 1991; Empson & Levi, 2011; Hackenberg, 2010; Steffe & Olive, 2010; Streefland, 1991; Tzur, 1999). Taken collectively this research does not appear to converge on a single trajectory of learning.

Sequence Considerations for Fraction Constructs

In keeping with Moss and Case's suggestion to focus on global cognitive practices, much of the research suggests broad learning clusters when discussing fractions knowledge acquisition in children. Moss and Case (1999) also suggest that when considering scope and sequence for instruction, "order is more arbitrary and … what matters is that the general sequence of coordinations remains progressive and closely in tune with children's original understandings" (124).

There is general agreement that in the early primary grades students should be developing whole number concepts and operations, with considerable and consistent links to concrete materials. Fraction concepts should be developed informally through the use of fair share questions and with the same considerable and consistent use of concrete materials (Brown & Quinn, 2006; Petit et al., 2010; Empson & Levi, 2011).

Likewise, research indicates that in the later primary grades students require opportunities to extend their understanding of whole number to fractions. These learning opportunities should focus on the use of concrete materials and unit rectangles and number lines as pictorial representations. Students should also be required to partition and iterate in order to solve problems involving fractions (Lamon 1999, Huinker 1998, as referenced in Brown & Quinn, 2006). It is important to note that in some countries, fractions and decimals are learned simultaneously (see Section 6 for more discussion on this approach).

Some research indicates that students have greater success when instructions begins with percents and then extends into decimals and finally fractions within the same learning unit. As well, the use of one model in depth as a starting point, such as graduated cylinders, rather than multiple representations is preferable. In the literature described in Section 6, East Asian countries also tend to introduce decimals simultaneously with fractions and even in the introduction of fractions, mixed numbers and fractions greater than one are included to prevent confusion with them later on.

Research indicates that learning in the junior grades should focus on the comparison and ordering of fractions, including the determination of equivalent fractions using a variety of strategies (Petit et al., 2010). Students should have ample opportunity to consolidate their understanding of the meaning of a fraction prior to finding an equivalent fraction (Petit et al., 2010). For any given fraction, the number of equivalent fractions that can be identified is infinite. Students should be encouraged to consider a variety of strategies for determining

equivalent fractions, beyond a simple doubling strategy upon which they frequently rely (although this in itself is a good starting point for establishing the concept of equivalence) (Charalambous & Pitta-Pantazi, 2005 ; Empson & Levi, 2011; Meagher, 2002; Petit et al., 2010; Kamii & Warrington, 1999). When students determine an equivalent fraction they are changing the unit of measure by either splitting or merging the partitions of the original fraction. The following illustration demonstrates these concepts using a continuous (area) model:

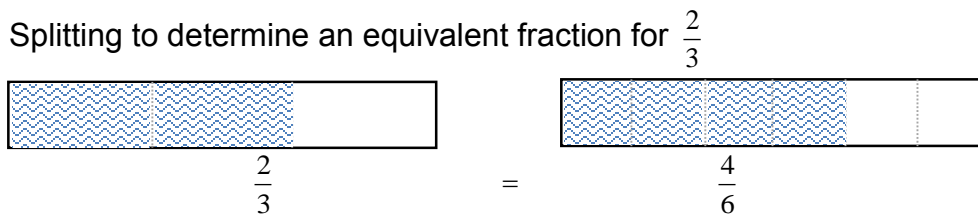**Figure 15: Splitting to Determine Equivalence**

Splitting to determine an equivalent fraction for $\frac{2}{3}$

$$\frac{2}{3} \qquad = \qquad \frac{4}{6}$$

Figure 16: Merging to Determine Equivalence

Merging to determine an equivalent fraction for $\frac{6}{8}$

$$\frac{6}{8} \qquad = \qquad \frac{3}{4}$$
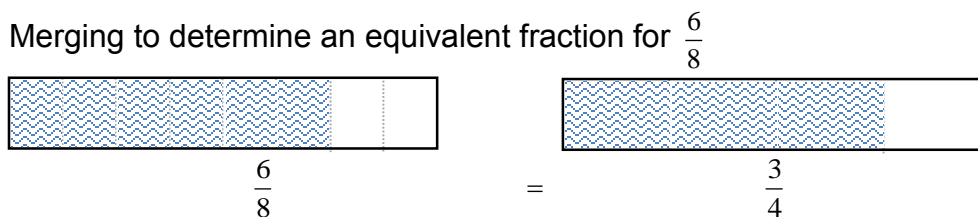
The exploration of equivalence allows students to develop an understanding of equivalent fractions as simply being a different way of naming the same quantity; it also supports them in viewing the fraction as a numeric value. Although underutilized in North American instruction, linear representations such as the number line support the study of equivalent fractions, as any point on the line can represent an infinite number of equivalent fractions. A physical area model can only be further partitioned into a certain amount of parts before it is visually difficult to see, whereas this is not the case on a number line. The standard algorithm for finding equivalence by multiplying the numerator and denominator by the same number reinforces the idea that a fraction is comprised of two whole numbers rather than representing a single value (Charalambous & Pitta-Pantazi, 2005 ; Empson & Levi, 2011; Meagher, 2002; Petit et al., 2010; Kamii & Warrington, 1999).

Comparing and ordering fractions allows students to develop a sense of fraction as quantity, as well as a sense of the size of a fraction, both necessary prior knowledge components for understanding fraction operations (Johanning, 2011). All students must develop facility with the use of and connections between multiple representations. However, instruction must extend beyond encouraging students to draw visual models of their thinking after-the-fact, and instead use visual and concrete models as *the site* of problem solving and reasoning

mathematically. This instruction must extend beyond the construction of pictures or use of rules for comparison of fractions. Johanning (2011) cautions that

> using visual models such as fraction strips and number lines support students' ability to visualize fractions and develop a sense of relative size. However, visual models are not enough. During instruction, students should routinely be asked to use their understanding of relative size to make sense of situations in which fractions are used operationally. (99)

Additionally, junior grade students should be presented with tasks that allow them to understand fraction operations as they connect to whole number operations, including provision of ample time to allow students to construct their own algorithms for the operations (Huinker, 1998; Brown & Quinn, 2006; see also Lappan & Bouck, 1998; Sharp, 1998). By focusing on sense-making rather than memorization of an algorithm, students will be able to extend this learning into algebraic contexts in secondary and post-secondary studies (Brown & Quinn, 2006; see also Wu, 2001). Lamon (1999), as referenced in Brown and Quinn (2006) states that:

> studies have shown that if children are given the time to develop their own reasoning for at least three years without being taught standard algorithms for operations with fractions and ratios, then a dramatic increase in their reasoning abilities occurred, including their proportional thinking (5).

Although the learning progression is not linear, there are some strongly interconnected components which support students understanding of subsequent concepts.

Figure 17: Components of Fraction Number Sense
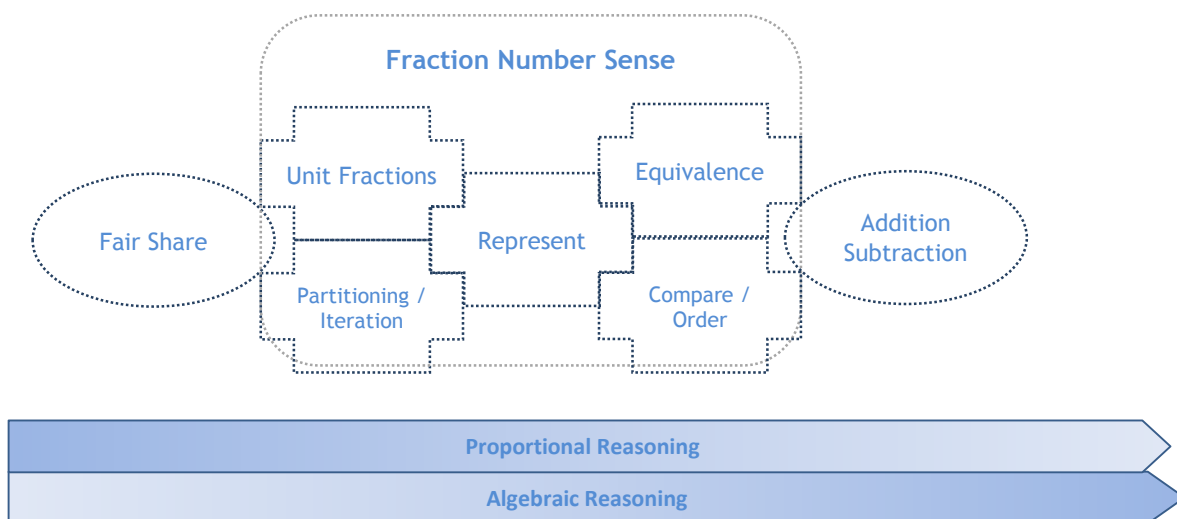
iv) Transitioning to Addition and Subtraction of Fractions

As students transition from representing, comparing, and ordering fractions into more formal approaches for adding and subtracting fractions there are critical pieces of understanding that they must possess. In addition, there appear to be a number of considerations for the learning sequence for adding and subtracting.

Essential Prior Knowledge

The research is clear that a student's procedural fluency and conceptual understanding combine to deepen understanding of fraction operations (Petit et al., 2010). Students must have a solid understanding of equivalence as well as part-whole and quantity constructs of fractions prior to a more formal exposure to fraction addition and subtraction (Petit et al., 2010). Increased fraction number sense allows students to accurately judge the relative quantity of a fraction (Bezuk & Bieck as cited in Empson & Levi, 2011). This allows students to understand that the sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to 1 by reasoning that $\frac{1}{12}$ is very close to 0, while $\frac{7}{8}$ is very close to 1. Huinker (2002), as referenced in Petit et al. (2010), cites flexible use of fraction representations as contributing to increased ability to reason about fraction operations.

Empson and Levi (2011) state that "as a rule of thumb, if students do not have *intuitive strategies* for solving [the following problems], then they are not ready to learn standardized procedures for adding and subtracting fractions" (187; emphasis added). These problems include:

$$1-\frac{1}{3} \qquad 4-1.5 \qquad 60-\frac{5}{8} \qquad 2-0.1$$

$$12\frac{7}{8}+10\frac{1}{2} \qquad 10-0.28 \qquad 7\frac{1}{8}-3\frac{3}{4} \qquad 3.2-1.01$$

(Empson & Levi, 2011, 186)

Sequencing Learning of Addition and Subtraction of Fractions

There are multiple proposed instructional sequences for adding and subtracting fractions. Although each sequence has common elements, such as addition and subtraction of proper fractions, there is great variation on other elements, including the starting point for instruction and the blending or separating of addition and subtraction. In each sequence, the emphasis is on developing conceptual understanding as well as procedural strategies in a balanced and purposeful manner. Careful selection of models for the addition and subtraction of fractions will support the development of a deeper understanding (Petit et al., 2010).

As previously discussed, it is important to first define addition as the combination of two or more *like-unit* quantities. Empson and Levi (2011) suggest that as students become fluent with equivalent fractions they are better able to consider addition of unlike fractional units by first relating each quantity to a common unit. Contrary to a more traditional instructional sequence, Empson and Levi suggest that students should not be solving problems presented as equations before they have had ample opportunities to solve and discuss different kinds of problems that are carefully designed and/or selected to make particular aspects fractions understanding explicit; for example, Chapter 5 of their 2011 book on cognitively guided instruction is entitled "Making Relational Thinking Explicit" and includes a host of problems focused just on this one aspect of fractional thinking. Students' ability to look at the entire problem and identify the relationships that will be useful is aided by presenting them with a blend of addition and subtraction problems, involving both mixed numbers and only proper fractions. Their research indicates that students find problems with familiar fractions, such as halves, fourths, eighths, thirds and sixths, easiest followed by problems where one denominator is twice the other, such as tenths and twentieths. Problems where the denominators have a common factor, such as fifths and twentieths, are easier than ones with no common factor. Empson and Levi (2011) further suggest that if students engage in problems that involve both proper fractions and mixed fractions then they will be better able to determine the relationship that can be used to solve the problem. The instructional sequence they suggest for grades 2 through 6 is as follows:

Grade 2: addition and subtraction problems involving like denominators or a whole number and a fraction, with a focus on $\frac{1}{2}$ and $\frac{1}{4}$, built out of fair-sharing problems and solved by reasoning with concrete materials

Grade 3: addition and subtraction problems involving like denominators, with unit and non-unit fractions having denominators of 2, 4, 3, 8 and 6

Grade 4: addition and subtraction problems involving like denominators with unit and non-unit fractions with denominators 2, 4, 3, 8, 6, 12, 10 and 100 and including whole-number amounts; addition and subtraction word problems involving unlike denominators of 2 and 4; addition and subtraction estimation problems with like and unlike denominators; all solved by students reasoning rather than a procedural approach to determining a common denominator

Grade 5: addition and subtraction problems with unlike denominators but where the denominators have a common factor

Grade 6: addition and subtraction problems and equations involving unlike denominators, including problems where one denominator is not a factor of the other

Improper fractions are incorporated throughout the instruction from fair-share problems through to addition and subtraction.

(Empson & Levi, 2011, 217-222)

Mack (2004) developed an instructional sequence which considered not only the type of fractions being added but the type of fraction that resulted. Mack built the instructional sequence on students' prior knowledge and found that focusing on like-size units avoided development of some common misconceptions, such as adding denominators together. Asking students to reflect upon the attributes of the question and solution strategy encouraged students to identify similarities between solution strategies and similarities between increasing complex problems.

The sequence suggested is as follows:

Figure 18: Mack Suggested Instruction Sequence

Generalized Mathematical Form of Problem	Problem Example
$a/b \pm c/b = d/b$, where $d < b$	$3/8 + 2/8 = 5/8$, $3/8 - 2/8 = 1/8$
$a/b + c/b = n\, d/b$, where $n = 1$	$5/8 + 7/8 = 1\ 4/8$
$1 - c/b = d/b$	$1 - 3/8 = 5/8$
$n - c/b = m\, d/b$, where $n > 1$ and $m \geq 1$	$4 - 3/8 = 3\ 5/8$
$n\ 1/b - c/b = m\, d/b$, where $n > 1$, $m \geq 1$ and $c > 1$	$4\ 1/8 - 3/8 = 3\ 6/8$
$n\ 1/b - y\, c/b = m\, d/b$, where n & $y > 1$, $m \geq 1$ and $c > 1$	$4\ 1/8 - 2\ 3/8 = 1\ 6/8$
$n\ a/b - y\, c/b = m\, d/b$, where n & $y > 1$, $m \geq 1$ and $a < c$	$4\ 3/8 - 2\ 5/8 = 1\ 6/8$
$1/b \pm 1/g = a/g$, where g is a multiple of b	$1/4 + 1/8 = 3/8$, $1/4 - 1/8 = 1/8$
$1/b \pm 1/g = a/z$, where b and g are relatively prime	$1/2 + 1/3 = 5/6$, $1/2 - 1/3 = 1/6$
$a/b \pm c/g = x/z$, where a & $c > 1$ and b and g are relatively prime	$3/4 + 2/3 = 1\ 5/12$, $3/4 - 2/3 = 1/12$
$n\ a/b \pm y\, c/g = m\, x/z$, where n & $y \geq 1$, a & $c \geq 1$, and b and g are relatively prime	$4\ 1/3 + 1\ 1/2 = 5\ 5/6$ $4\ 1/3 - 1\ 1/2 = 2\ 5/6$

(Mack, 2004)

Section 8: Implications for Canadian Research

The research summarized in this review has highlighted some of the primary challenges faced by students and teachers in the learning of fractions and related operations. Based on this review it is clear that in the Canadian context an emphasis on the selection, generation and use of efficient, accurate and appropriate fractions representations as well as focused attention on developing deep understanding of what a fraction is (including understanding the multiple meanings of fractions based on context) sit at the heart of fractions learning.

Issues that require further attention in classroom-based research with Canadian teachers and students include:
i. Developing practices and lessons that begin by exposing and working with children's intuitions about fractions and benchmark fractions such as one-half, one-fourth and one-tenth
ii. Increasing the precision of fractions language to promote greater understanding of fraction units, fractional relationships, and fractions operations
iii. Taking advantage of those representations that have longevity and offer a smoother transition between fractions meanings and operations with fractions
iv. Selecting and/or creating fractions problems that explicitly highlight underpinning fractions concepts (when paying attention to simple fractions addition, emphasis on problems that concentrate on fractional units and combining one-sixths of coloured ribbons, for example) rather than focusing solely on out-of-context procedures and algorithms (such as 'add the numerators, but not the denominators when you have like denominators').

Research questions of importance include:
- How does increased understanding of unit fractions support students in becoming proficient with fractions operations?
- How can instructional trajectories support educators in seeing (and teaching) mathematics in a meaningful and interconnected manner (rather than a set of discrete units of study)?
- What resources exist that support the effective teaching and learning of fractions? What resources still need to be developed?

There is a limited amount of fractions research in Canada, and indeed in Ontario. In order to investigate research questions and topics such as those outlined above, researchers and practitioners require opportunities to work together in forms of collaborative action research and through mixed-methods field trials that build deep connections and overlap between theory and practice, research and teaching.

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Appendix A: Fractions Addition and Subtraction Learning Trajectory (Draft)

The learning trajectory of Appendix A was drafted by the authors of this literature review based on evidence from 18 months of primary and secondary research related to the KNAER fractions project. A learning trajectory (see Daro, Mosher & Corcoran, 2011) is a suggested or hypothesized sequencing of ideas or concepts that should engender growth, and build deep and broad student understanding - in this case the trajectory focuses on fractions understanding leading to understanding of fractions addition and subtraction. In practice, through 'field testing' of the learning trajectory, the suggested order is checked against evidence of student progress. The draft trajectory in this document builds on the work of Jere Confrey and her research team (see Confrey, 2012) but is customized to the Ontario context thanks to the findings from this literature review, the Ontario Think Tank on Fractions (October, 2012), and findings from collaborative action research with local teachers and their students in junior and intermediate grades classrooms.

The trajectory is chunked into three broad content areas: Working with unit fractions; Equivalence and comparing fractions; and, Addition and subtraction operations with fractions. The trajectory works both horizontally, to show the flow from sub-topic to sub-topic, as well as vertically, to show parallel areas of focus that will likely emerge and require simultaneous attention.

It is intended that in future research activity, this trajectory will be field-tested and populated with lessons and greater detail in newly developed lower layers that sit beneath this skeletal top layer.

Fractions Addition and Subtraction Learning Trajectory
(Draft)
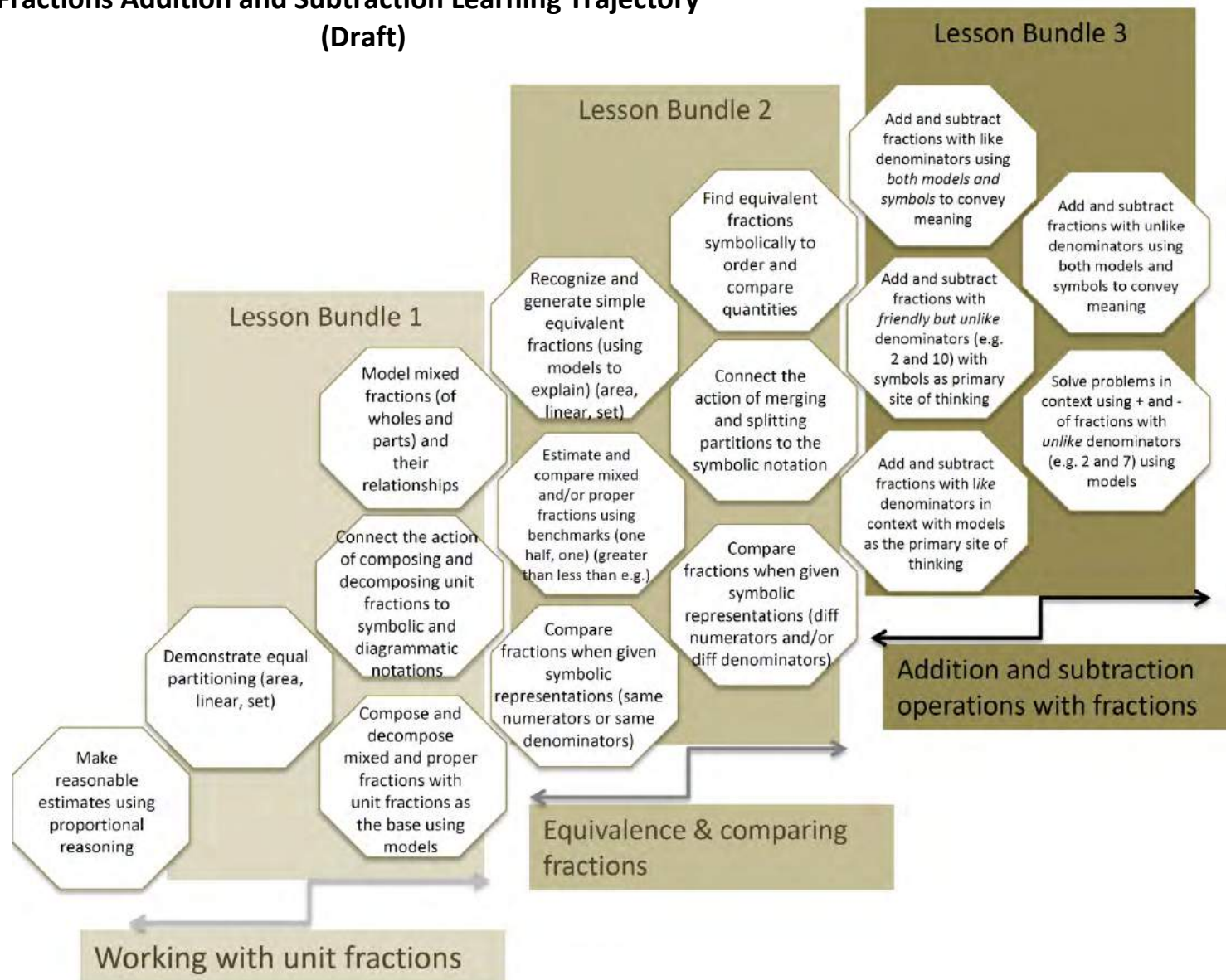
This trajectory is based upon and inspired by the work of Jere Confrey and her team at North Carolina State University.