INTRO
MEASURE THEORY
SUMMARIZED

(on a set of almost measure-zero).

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A few notes/ acknowledgements:

These are the course notes summarized/explained for MATH 6210 - Measure Theory and Lebesgue Integration taught at Cornell University. The material for these notes comes primarily from Bartle's Book, The Elements of Integration, and instructor notes presented in class by Prof. Mahdi Asgari. A little bit also came from Wolfram/Wikipedia, Rene Schilling's book Measures, Integrals and Martingales, and Ernst Hansen's Measure Theory book.

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MEASURABLE FUNCTIONS:

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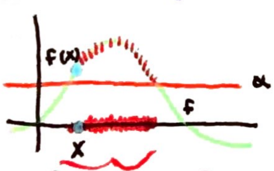
- σ-algebra
  - family of subsets of a set X that meets the following criteria:
    - i) ϕ, X in the family
    - ii) If A in the fam, $X - A = A^c$ in the fam
    - iii) If (A_n) a seq of sets in the fam, then $\bigcup_{n=1}^{\infty} A_n$ in the fam.
  - $(X, \mathcal{A})$, a set X and a σ-algebra $\mathcal{A}$ of subsets of X is a measurable space.
  - "σ-Algebra generated by ..."
    - Smallest σ-alg containing some non-empty collection of subsets of X.
    - Comes from the intersection of all the σ-algs containing this collection.

- Examples of σ-algebras:
  1) A set X w/ all of its subsets
  2) A set X and ϕ
  3) $\mathbb{N}$, ϕ, $\{1, 3, \ldots\}$, and $\{2, 4, \ldots\}$
  4) An uncountable set X and all of the subsets which are countable or have countable complements
  5) If A_1, A_2 are σ-alg of subsets of X, $A_1 \cap A_2 =$ a σ-alg

- Keystone example of a σ-algebra:
  - Borel Algebra
    - σ-alg generated by all open or closed intervals $[a, b]$ in $\mathbb{R}$.
    - Any set in this algebra is a Borel set.
  - Extended Borel Algebra
    - $X = \overline{\mathbb{R}}$ = extended real numbers
    - For any Borel subset E, we include: $E \cup \{-\infty\}$, $E \cup \{+\infty\}$, $E \cup \{-\infty, +\infty\}$

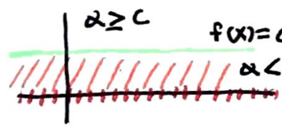
- Measurable ($\mathcal{A}$-Measurable)
  - $f: X \to \mathbb{R}$ measurable if for every $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\}$ is in $\mathcal{A}$.

Example



- Equivalent defs:
  - $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) > \alpha\} \in \mathcal{A}$
  - $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) \leq \alpha\} \in \mathcal{A}$
  - $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) \geq \alpha\} \in \mathcal{A}$
  - $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) < \alpha\} \in \mathcal{A}$
- For f, g measurable and $c \in \mathbb{R}$, cf, f^2, fg, $f+g$, and $|f|$ are measurable.
- For $f: X \to \mathbb{R}$, $f = f^+ - f^-$ where $f^+ = \sup\{f(x), 0\}$, $f^- = \sup\{-f(x), 0\}$
- Extended, real-valued functions: $M(X, \mathcal{A})$
  - $f: X \to \overline{\mathbb{R}}$ measurable if $\{x \in X : f(x) > \alpha\} \in \mathcal{A}$ $\forall \alpha$, and $\{x \in X : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > n\} \in \mathcal{A}$ and $\{x \in X : f(x) = -\infty\} = \left(\bigcup_{n=1}^{\infty} \{x \in X : f(x) > -n\}\right)^c \in \mathcal{A}$
- (f_n) in $M(X, A)$, then $\inf f_n(x)$, $\sup f_n(x)$, $\liminf f_n(x)$, and $\limsup f_n(x)$ in $M(X, A)$.
- If (f_n) in $M(X, A)$ conv. to f on X, then $f \in M(X, A)$.

- Examples of Measurable Functions:
  1) Any constant function
     $\{x \in X : f(x) > \alpha\} = \phi$ for $\alpha \geq c$ and $\{x \in X : f(x) > \alpha\} = X$ for $\alpha < c$



  2) Characteristic Function $\chi_E(x) = \begin{cases} 1, x \in E \\ 0, x \notin E \end{cases}$
     $\{x \in X : \chi_E(x) > \alpha\} = X, E,$ or ϕ, if $E \in \mathcal{A}$.
  3) For $(\mathbb{R}, \mathcal{B})$, any continuous function $f: \mathbb{R} \to \mathbb{R}$.
     Since f cont, $\{x \in X : f(x) > \alpha\}$ is open in $\mathbb{R}$ and the union of a seq of open intervals.
  4) For $(\mathbb{R}, \mathcal{B})$ any monotone function.
     Ex: If f monotone increasing, in that $x \leq x' \Rightarrow f(x) \leq f(x')$, then $\{x \in \mathbb{R} : f(x) > \alpha\}$ is a half-line of form $\{x \in \mathbb{R} : x > \alpha\}$ or $\{x \in \mathbb{R} : x \geq \alpha\}$.

- One other Lemma...
  - If f non-neg. in $M(X, A)$, $\exists\ (\phi_n) \in M$ with
    - a) $0 \leq \phi_n(x) \leq \phi_{n+1}(x)$
    - b) $f(x) = \lim \phi_n(x)\ \forall x \in X$
    - c) Each ϕ_n has only finitely many real vals.

MEASURES:

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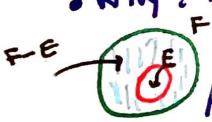
- **What is a measure?**
  - extended, real valued function μ defined on a σ-algebra A of subsets of X that satisfies the following properties:
    - i) $\mu(\phi) = 0$
    - ii) $\mu(E) \geq 0 \ \forall E \in A$
    - iii) Countable additivity:
      $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$
      if (E_n) is any disjoint seq of sets in A.
      - If $\mu(E_n) = +\infty$, or series diverges, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = +\infty$
  - μ finite if $\mu < +\infty$
  - μ σ-finite if $\exists$ a seq (E_n) in A with $\bigcup_{n=1}^{\infty} E_n = X$ and $\mu(E_n) < +\infty$
    - In words: "μ σ-finite if the measure of the countable union that makes up X is finite."

- **Examples of Measures:**
  1) X is nonempty, and A is σ-alg of all subsets of X.
     $\mu_1(E) = 0 \ \forall E$ in A.
     $\mu_2(E) = +\infty$ if $E \neq \phi$, otherwise, $\mu_2(E) = 0$.
     μ_1 is finite, μ_2 is not.
  2) X is nonempty, and A is σ-alg of all subsets of X. For E in A,
     $$\mu(E) = \begin{cases} 0 & \text{if point } p \notin E \\ 1, & \text{otherwise.} \end{cases}$$
     for p in X. μ finite and the "unit measure at p"
  3) $X = \mathbb{N}$, and $A = \sigma$-alg of all subsets of $\mathbb{N}$. If E in A, let $\mu(E) = \#$ of elements in E. If E infinite $\mu(E) = +\infty$. Measure is σ-finite and called "counting measure"
  4) $(\mathbb{R}, B)$, Lebesgue Measure for intervals (a,b), $(a,b]$, $[a,b)$, $[a,b]$
     $\lambda = b - a$
  5) $(\mathbb{R}, B)$, Borel-Stieltjes Measure λ_f. If f monotone increasing, and $E = (a,b)$, then $\lambda_f(E) = f(b) - f(a)$

- **Some useful results about Measures:**
  - **Subsets:**
    - If μ defined on σ-alg A, and E and $F \in A$, with $E \subseteq F$, $\mu(E) \leq \mu(F)$.
  - **Subtraction:**
    - If μ defined on A, and $E, F \in A$ and if $\mu(E) < +\infty$, $\mu(F - E) = \mu(F) - \mu(E)$
    - Why? Works when $E \subseteq F$ and $F = E \cup (F-E)$ and $E \cap (F-E) = \phi$.

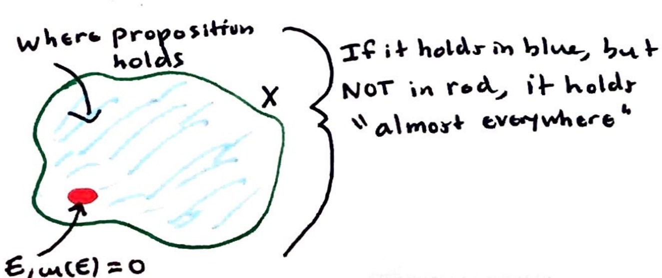


So $\mu(F) = \mu(E) + \mu(F-E)$.
Also, since $\mu(F-E) \geq 0$, $\Rightarrow \mu(E) \leq \mu(F)$.

  - **Sequences and Limits:**
    - If μ defined on σ-alg X, then:
      - i) If (E_n) increasing in X,
        $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$$
      - ii) If (F_n) decreasing in X,
        $$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n)$$

- **Charges / Signed Measure.**
  - If A is a σ-alg, and A consists of all subsets of X, then the real-valued function λ defined on A is a charge if:
    - i) $\lambda(\phi) = 0$
    - ii) Countable additivity holds for disjoint seq in A.
  - What's the difference?
    - λ can be negative!

- **Measure Space and "Almost Everywhere"**
  - Measure space: (X, A, μ), set, σ-alg, and measure.
  - "Holds almost everywhere" – if proposition true everywhere BUT a set of measure zero.



THE LEBESGUE INTEGRAL: (For nonnegative functions) Pt 1:

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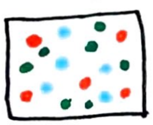
- **Simple Functions and the Integral:**
  - Simple function is real valued and finite. Has form: $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$
    - $a_j \in \mathbb{R}$, χ_{E_j} = Char. function of $E_j \in A$.
  - Standard Representation of ϕ occurs when E_j are disjoint and a_j's distinct.
    - If $a_1, a_2, \ldots, a_n$ distinct and if $E_j = \{x \in X : \phi(x) = a_j\}$ then E_j's disjoint
  - **The Integral:**
    - Integral of a simple function ϕ in $M^+(X, A)$ with respect to measure μ:

$$\boxed{\int \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j)}$$

  - Another way to conceptualize the Lebesgue Integral:
    - Consider X as a bucket of values. In bucket X,

X 

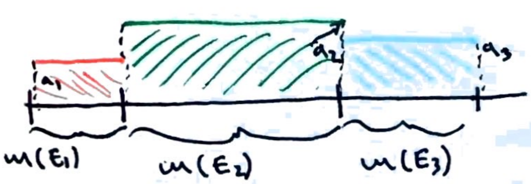
Some values give $\phi(x) = a_1$, some give $\phi(x) = a_2$, and others give $\phi(x) = a_3$. In our bucket, • gives $\phi(\bullet) = a_1$, • gives $\phi(\bullet) = a_2$, and • $\phi(\bullet) = a_3$.

- We go through bucket X, and sort • • • into their own buckets: E_1, E_2, and E_3.

- We see that not every bucket is full / they each have varying degrees of "fullness".
- This measure of "fullness" of each bucket E_j, $j = 1, 2, 3$. will be $\mu(E_j)$.
- So, $\int \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j)$, or "in words,"

  "take the "fullness" of each bucket, multiply the bucket's "fullness" by the "amount $\phi(x)$" associated w/ all the items in the bucket, and add it all up."

- Visual:



$= \int \phi \, d\mu$

- Some helpful properties with integrals:
  - If ϕ and ψ simple functions in $M^+(X, A)$ and $c \geq 0$,

$$\int c\phi \, d\mu = c \int \phi \, d\mu$$ • multiplication by a constant

$$\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$$ • linearity

THE INTEGRAL Pt. 2 (still nonnegative functions)

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- The integral of f in $M^+(X, A)$:
  - $\boxed{\int f \, d\mu = \sup \int \phi \, d\mu}$
  - Supremum is extended over all simple functions ϕ in $M^+(X, A)$.
  - These simple functions must also satisfy $0 \leq \phi(x) \leq f(x) \; \forall x \in X$.
- Integral of f over E, $E \in A$.
  $\int_E f \, d\mu = \int f \chi_E \, d\mu$
  - Somewhat analagous to integrating over some region a to b in Riemann case.
- What if f, g in $M^+(X,A)$ and $f \leq g$?
  - $\int f \, d\mu \leq \int g \, d\mu$.
- What if f in $M^+(X,A)$ and $E, F \in A$, $E \subseteq F$?
  $\int_E f \, d\mu \leq \int_F f \, d\mu$

- Some consequences of the convergence thms below:
  1) If $f \in M^+(X, A)$ and $\lambda(E) = \int_E f \, d\mu$, then λ is a measure
  2) If $f \in M^+(X, A)$, then $f(x) = 0$ μ.a.e on X iff $\int f \, d\mu = 0$.
  3) Absolute Continuity w/r/t a measure.
     - If f in $M^+(X, A)$ and $\lambda(E) = \int_E f \, d\mu$, then λ absolutely continuous w/r/t μ if E in A and $\mu(E) = 0 \Rightarrow \lambda(E) = 0$.
  4) If (f_n) is monotone increasing seq of functions in $M^+(X, A)$ which conv. μ-a.e. on X to a function f in M^+, then $\int f \, d\mu = \lim \int f_n \, d\mu$.
  5) Interchange of sum and integral
     - If (g_n) a seq. in M^+, then $\int \left(\sum_{n=1}^{\infty} g_n\right) d\mu = \sum_{n=1}^{\infty} \int g_n \, d\mu$

- Convergence Theorems:
  - Monotone Convergence Theorem
    - If (f_n) a monotone seq of functions that is increasing, and (f_n) in $M^+(X,A)$, and converges to f, then $\boxed{\int f \, d\mu = \lim \int f_n \, d\mu.}$
    - $\int f \, d\mu = \sup_{\phi} \int \phi \, d\mu \leq \lim \int f_n \, d\mu$
    - Do we need to assume that $\int f \, d\mu$ or $\lim \int f_n \, d\mu$ finite?
      - No
  - From MCT we get the following:
    - $\int c f \, d\mu = c \int f \, d\mu$
    - $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

    for $c \geq 0$, f, g in $M^+(X, A)$

  - Fatou's Lemma
    - If (f_n) belongs to $M^+(X,A)$, then $\boxed{\int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu}$
    - Can Fatou fail?
      - Yes, if $f_n \ngeq 0$.

INTEGRABLE FUNCTIONS (f can be negative!)

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- **What is an integrable function?**
  - $f = f^+ - f^-$ is integrable, if $\int f^+ d\mu < +\infty$ and $\int f^- d\mu < +\infty$
  - $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$
  - If this is the case, $f \in L = L(X, A, \mu)$
  - **Can we express f in another way?**
    - If f_1 and f_2 are nonnegative, measurable and have finite integrals then $\int f \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu$
  - **Is there a relationship to measure?**
    - Since $\lambda^+(E) = \int_E f^+ d\mu$ and $\lambda^-(E) = \int_E f^- d\mu$ both define measures by def, $\lambda(E) = \int_E f \, d\mu = \lambda^+ - \lambda^-$, is a charge.
    - If (E_n) disjoint seq, and $\cup E_n = E$, then $\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$

- **Absolute integrability and belonging to L:**
  - $f \in L$ iff $|f| \in L$, and so $\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$

- **Measurability and Integrability:**
  - What happens if one function f is measurable and another g is integrable? with $|f| \leq |g|$
    - $\int |f| \, d\mu \leq \int |g| \, d\mu$
    - In words: integral of a measurable function is bounded by integral of an integrable function.

- **Sum and Constant Multiple of $f \in L$:**
  - a constant multiple of a function in L and a sum of two functions in L both belong to L and:
  - $\int \alpha f \, d\mu = \alpha \int f \, d\mu$
  - $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

- **Lebesgue Dominated Convergence Theorem:**
  - Let (f_n) be a seq of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g \; \forall n$, then f is integrable, and $$\int f \, d\mu = \lim \int f_n \, d\mu$$
  - Big idea: In words – so long as (f_n) is a seq of functions in L that converges to f in M, and this seq is dominated at every point by some g in L, then we know f has to be in L, and the consequence of MCT holds.

- **How do big 3 conv. thms compare?**
  - All 3 state something about $\int f \, d\mu$ in terms of $\int f_n \, d\mu$.

• Fatou	• MCT	• DCT
• Weakest hypothesis • Weakest conclusion • Just assumes f_n bounded below by 0	• good hybrid • f_n bounded below by 0 and above by f itself • Holds even if f not in L	• Strongest hyp. • Strongest concl. • f_n bounded below and above by fixed integrable functions

THE LEBESGUE SPACES - Part One:

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- **What is a Banach Space and why do we care?**
  - functions need a place to live and work too!
  - What about norms and such?
    - Equivalence classes and normed linear spaces will give us a nice framework!

- **Vector spaces and Norms:**
  - V is a vector space if the following hold $\forall X, Y, Z \in V$ and any scalars a, b
    1) commutativity, 2) associativity of vector add.
    3) Additive I.D, 4) existence of an add. inverse
    5) associativity of scalar mult, 6) distributive property of scalar sums, 7) distributive prop. of vector sums, 8) scalar mult. identity
  - N is a norm on V if it satisfies:
    i) $N(v) \geq 0$ $\forall$ elements v in V
    ii) $N(v) = 0$ iff $v = 0$
    iii) $N(\alpha v) = |\alpha| N(v)$ $\forall v \in V$ and $\alpha \in \mathbb{R}$
    iv) $N(u+v) \leq N(u) + N(v)$ $\forall u, v \in V$ (triangle ineq.)

  - **What happens if one of those conditions is dropped?**
    - **Semi (pseudo) norm**
      - $N(v)$ can $= 0$ even if $v = 0$
      - If f in $L(X, A, \mu)$, $N_\mu(f) = \int |f| d\mu$
        - Semi-norm on L

  - **Is $L(X, A, \mu)$ a linear space?**
    - Yes, under $(f+g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$, $x \in X$
    - $N_\mu(f)$ is a semi-norm, and $N_\mu(f) = 0$ iff $f(x) = 0$ for μ-almost all x in X.
    - But how do we make L into a normed space?

- **Equivalence Classes and μ-equivalent:**
  - What can we say about two functions in L?
    - Two functions belonging to L are μ-equivalent if they're equal a.e.
  - We can norm L if we consider equivalence classes of functions instead of just functions
  - So we can "pair up" those functions that are essentially the same except on a measure-zero set and they have an equivalence rln. between them
  - **Equivalence class determined by f in L**
    - $[f]$ is the set of all functions in [redacted] L that are μ-equiv. to f.
    - Can use this to motivate/define a norm.

- **Examples of Norms:**
  - Absolute value function on $\mathbb{R}$
  - Norms on $\mathbb{R}^n$:
    $N_1 = |u_1| + \cdots + |u_n|$
    $N_p = \{|u_1| + \cdots + |u_n|\}^{1/p}$
    $N_\infty = \sup\{|u_1|, \cdots, |u_n|\}$
  - All real-valued sequences with $N_1(u) = \Sigma |u_n| < +\infty$ is a normed linear space
  - Collection of all bounded, real-valued functions on X: $N(f) = \sup\{|f(x)| : x \in X\}$

- **Examples of Semi-norms:**
  - On the linear space of functions on $[a,b]$ to $\mathbb{R}$, which have cont. derivs: $N_0(f) = \sup\{|f'(x)| : a \leq x \leq b\}$
  - Why is this a semi-norm?
    - $N_0(f) = 0$ iff f constant on $[a,b]$

- **The Lebesgue Spaces:**
  - What's in a Lebesgue Space?
    - equivalence classes of functions in L
  - **L_p Space**
    - for $1 \leq p < \infty$, $L_p(X, A, \mu)$ holds all μ-equivalence classes of A-measurable, real-valued functions for which $\int |f|^p d\mu < +\infty$.
  - Norm: $\|f\|_p = \left\{ \int |f|^p d\mu \right\}^{1/p}$
  - L_p is complete under the norm.

- **The Inequalities:**
  - **Hölder**: for $f, g \in L_p, L_q$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|f\|_q$
  - **Cauchy-Schwarz**: for $f, g \in L_2$, [redacted] then fg integrable and $\left|\int fg \, d\mu\right| \leq \int |fg| d\mu \leq \|f\|_2 \|g\|_2$
  - **Triangle (Minkowski)**: for $f, h \in L_p$, $p \geq 1$, $f+h \in L_p$ and $\|f+h\|_p \leq \|f\|_p + \|h\|_p$

- **Cauchy & Convergent in L_p:**
  - (f_n) in L_p Cauchy in L_p if $\forall \varepsilon > 0$ $\exists M \ni$ if $m, n \geq M$, then $\|f_m - f_n\|_p < \varepsilon$
  - (f_n) conv. to f in L_p if $\forall \varepsilon > 0$, $\exists N \ni$ if $n \geq N$, $\|f_n - f\|_p < \varepsilon$
  - If (f_n) conv. to f in L_p, then it's Cauchy.
  - If every cauchy seq conv. in L_p, then L_p complete.

- **L_∞:**
  - Consists of all equiv. classes of A-meas. functs, that are bounded a.e. If $f \in L_\infty$, and $N \in A$ w/ $\mu(N) = 0$,
  - $S(N) = \sup\{|f(x)| : x \notin N\}$, $\|f\|_\infty = \inf\{S(N) : N \in A, \mu(N) = 0\}$
  - L_∞ complete.

MODES OF CONVERGENCE: Part One: Definitions/concepts.

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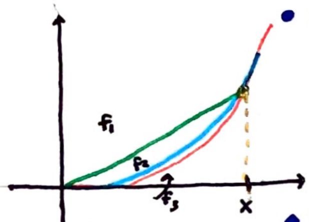
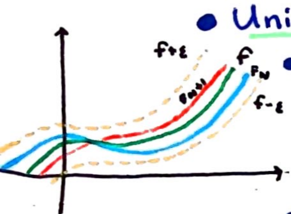
- **Basic types of convergence that do not involve measure-theory:**
  - **Pointwise Convergence:**



Pointwise conv.

  - (f_n) conv. p.w to f if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ that depends on x and ε s.t. if $n \geq N$, $x \in X$, $|f_n - f| < \varepsilon$.
  - $\lim_{n\to\infty} f_n(x) = f(x)$

- **Uniform Convergence:**



Uniform conv.

  - (f_n) conv. uniformly to f if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ that depends on ε only s.t. if $n \geq N$, $x \in X$, $|f_n - f| < \varepsilon$
  - Much stronger than pw and preserves continuity

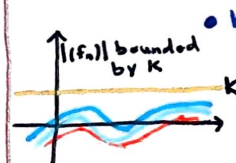
- **Convergence and Measures:**
  - What if the domain is a measure space and we are concerned with measurable functions?
  - **Convergence almost everywhere**
    - Basic idea in words: "pw conv. of f_n to f everywhere but some measure-0 set.

    - Formal Def: (f_n) conv to f a.e. if $\exists$ a set $\bar{A}$ in A $\ni$ $\mu(\bar{A}) = 0$ and $\forall \varepsilon > 0$, $x \in X - \bar{A}$, $\exists N \in \mathbb{N}$ that depends on x and ε $\ni$ if $n \geq N$, $|f_n - f| < \varepsilon$.
    - Notice: pw conv. $\to$ a.e. conv. but it doesn't mean that a.e. conv. $\to$ p.w. conv.
  - **Almost uniform Convergence:**

    - Basic idea in words: "consider some set E_δ. The measure of E_δ must be less than δ, so we can make E_δ "arbitrarily small." If f_n is uniformly conv. to f on the complement of these E_δ sets, then f_n is almost uniformly conv.

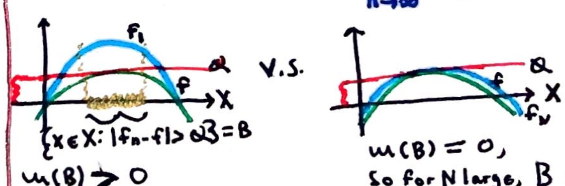
    - Formal Def: (f_n) conv. to f a.u. if $\forall \delta > 0$, $\exists E_\delta$ in A $\ni \mu(E_\delta) < \delta$ and that (f_n) conv. unif. to f on $X - E_\delta$.
    - Almost uniformly Cauchy: (f_n) a.u.c. seq. if $\forall \delta > 0$, $\exists E_\delta$ in A with $\mu(E_\delta) < \delta$ $\ni$ (f_n) conv. uniformly on $X - E_\delta$.

- **Convergence and L_p:**
  - How do sequences converge in an L_p space?
  - **Convergence in L_p:**
    - Big idea in words: f_n conv. to f in L_p if the p-norm of $f_n - f$ is less than ε.
    - Formal Def: (f_n) in L_p converges in L_p to f in L_p if $\forall \varepsilon > 0$, $\|f_n - f\|_p = \left\{\int |f_n - f|^p d\mu\right\}^{1/p} < \varepsilon$. When $n \geq N \in \mathbb{N}$ provided that $N \in \mathbb{N}$ exists and depends only on ε.
  - **What about Cauchy Seq. in L_p?**
    - (f_n) is cauchy in L_p if $\exists N \in \mathbb{N}$ dependent only on ε such that $\forall \varepsilon > 0$, $\|f_m - f_n\|_p < \varepsilon$.
    - What does this imply about convergence?
      - If (f_n) cauchy in L_p, then $\exists f$ in L_p and (f_n) converges to this particular f.
  - **What happens if $\mu(X)$ is finite?**
    - If $\mu(X) < +\infty$ and (f_n) is a seq. in L_p that conv. uniformly on X to f, then f itself also belongs to L_p and (f_n) conv. in L_p to that f.
    - What can we say if (f_n) bounded?



      - If $\mu(X) < +\infty$ and (f_n) in L_p that conv. p.w a.e to a measurable function f, and if $\exists K$ such that $|f_n(x)| \leq K$ for $x \in X$ and $N \in \mathbb{N}$, then $f \in L_p$ and f_n conv. to f in L_p.
  - **What if (f_n) bounded by some function in L_p?**
    - If (f_n) in L_p and converges a.e. to a measurable function f and $\exists g \in L_p \ni |f_n(x)| \leq g(x)$, $x \in X$, $n \in \mathbb{N}$, then f belongs to L_p and (f_n) conv. in L_p to f.

- **Convergence in Measure:**
  - Big idea in words: "The measure of the set of x-values that satisfy $|f_n(x) - f(x)| \geq \alpha$ for $\alpha > 0$ goes to zero as $n \to \infty$."
  - Formal Def: (f_n) of measurable, real-valued functions is said to converge in measure to a measurable, real-valued function f when: $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f| \geq \alpha\}) = 0$



  - **What can we say about subsequences?**
    - If (f_n) a seq. of measurable, real-valued functions, which is Cauchy in measure, then $\exists$ a subsequence which conv. a.e. and in measure to a measurable, real-valued function f.
  - **What about functions and conv. in L_p?**
    - If (f_n) a seq. of measurable real-valued functions that is Cauchy in measure, then $\exists$ meas. real valued f that (f_n) conv in measure to, and f is unique a.e.
    - If (f_n) in L_p, conv. in meas. to f, and $g \in L_p$ w/ $|f_n| \leq g$ a.e., then, $f \in L_p$ and (f_n) conv in L_p to f.

MODES OF CONVERGENCE: Part Two: Relationships

M. GASPARD

- **Two Named Theorems and How they Relate:**

1) **Egoroff's Theorem:**
   - If $\mu(X) < +\infty$ and (f_n) is a seq of measurable, real-valued functions that conv. a.e. on X to a measurable, real-valued function f, then (f_n) conv. almost uniformly and conv. in measure to f.

2) **Vitali Convergence Theorem:**
   - If (f_n) in $L_p(X, A, \mu)$ with $1 \le p < \infty$, then the following 3 conditions are necessary and sufficient for L_p conv. of (f_n) to f.
     - i) (f_n) conv. to f in measure.
     - ii) For every $\varepsilon > 0$, $\exists\, E_\varepsilon$ in A that has finite measure where if $F \subseteq A$ and $F \cap E_\varepsilon = \emptyset$, then $\int_F |f_n|^p \, d\mu < \varepsilon^p \ \forall n \in \mathbb{N}$.
     - iii) For every $\varepsilon > 0$, $\exists\, \delta > 0 \ni$ if E in A and $\mu(E) < \delta$, then $\int_E |f_n|^p \, d\mu < \varepsilon^p \ \forall n \in \mathbb{N}$.

• Tables of Implications •

• General Measure Space •

	pw	u	a.e.	Lp	M	AU
pw		✓				
u						
a.e	✓			X		✓
Lp						
M			X	✓		✓
AU				X	X	

• Finite Measure Space •

	pw	u	a.e	Lp	M	AU
pw		✓				
u						
a.e.	✓				X	✓
Lp						
M			✓	✓		✓
Au			✓	X		

Key:

Red implies green.

"✓" = holds for a sequence.

"X" = holds for a subsequence only

Gold indicates that red implies green AND green implies red

Terms:

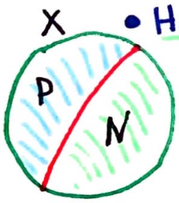
- pw = pointwise
- u = uniform
- a.e. = almost everywhere
- Lp = conv. in Lp
- M = conv. in measure
- au = almost uniform

DECOMPOSITION OF MEASURES:

M. GASPARD.

- **Big idea:** What if we could decompose/break up charges and measures?

- **Positive, Negative and Null Sets:**
  - Consider any set E in A.
  - If λ is a charge on A, then.
    - P in A Positive w/r/t λ if $\lambda(E \cap P) \geq 0$
    - N in A is Negative w/r/t λ if $\lambda(E \cap N) \leq 0$.
    - M in A is Null if $\lambda(E \cap M) = 0$.



- **Hahn Decompositions:**
  - If λ a charge on A ~~and~~ then, $\exists$ P, N in A with $X = P \cup N$ and $P \cap N = \emptyset$. Also, P is positive and N is negative.
  - Big idea: Under a charge it is possible to divide X into disjoint sets where the charge is negative on one part and positive on the other.
  - P and N form a Hahn Decomp.
  - Hahn Decomps are not unique.
  - Can we relate two H.D's?
    - Yes! If P_1, N_1 and P_2, N_2 are H.D's, then for E in A, $\lambda(E \cap P_1) = \lambda(E \cap P_2)$ and $\lambda(E \cap N_2) = \lambda(E \cap N_1)$.

- **Absolute Continuity**
  - Can we relate measures to each other and say something about their implications?
    - Yes! Consider λ, and any set E in A. Suppose we also have measure m. If λ a measure, and $m(E) = 0$ implies that $\lambda(E) = 0$, then λ is Absolutely cont. w/r/t m.
  - For a tighter, more formal def.
    - If λ, m, finite measures on A, then λ abs cont. w/r/t m iff $\forall \varepsilon > 0$, $\exists \delta > 0 \ni E$ in A and $m(E) < \delta \Rightarrow \lambda(E) < \varepsilon$.

- **Radon-Nikodým Theorem:**
  - If λ, m σ-finite measures defined on A and λ is abs. cont. w/r/t m, then $\exists$ $f \in M^+(X, A) \ni$
    $$\lambda(E) = \int_E f \, dm$$
    and f is unique a.e.
  - What's so special about f?
    - f is the Radon-Nikodým derivative.
    - f not necessarily integrable
    - f m-equivalent to an integrable function iff λ finite
    - f denoted by $\frac{d\lambda}{dm}$
  - So what's the big deal?
    - Two measures are absolutely continuous (or more precisely, one measure is absolutely cont. w/r/t another) if the sets which have small measure w/r/t one measure also have small measure w/r/t the other.

- **Singular Measures:**
  - Idea: Can we split X into two parts where one part has measure 0 w/r/t one measure, and the other part has measure 0 w/r/t the other part?
  - Yes! Consider measures λ and m on A. λ and m are singular if $\exists \bar{A}, \bar{B} \in A \ni \bar{A} \cap \bar{B} = 0$, and $X = \bar{A} \cup \bar{B}$ and $\lambda(\bar{A}) = m(\bar{B}) = 0$.

- **Lebesgue Decomposition Thm:**
  - If λ, m, σ-finite measures on A, $\exists$ a measure λ_1 sing w/r/t m and measure λ_2 abs. cont. w/r/t m $\ni \lambda = \lambda_1 + \lambda_2$ and λ_1, λ_2 unique.

- **Variations and Total Variation:**
  - If we can express λ in terms of a "+" and "−" part, what are λ^+ and λ^-?
  - Given a H.D, and any set E in A,
    - $\underbrace{\lambda^+(E) = \lambda(E \cap P)}_{\text{Positive variation}}$ • $\underbrace{\lambda^-(E) = -\lambda(E \cap N)}_{\text{Negative Variation}}$
  - Total Variation: $|\lambda|(E) = \lambda^+(E) - \lambda^-(E)$
  - So what does this mean for a charge?
    - Jordan Decomposition Thm:
      - If λ a charge, then $\lambda = \lambda^+ - \lambda^-$, which is the diff. of two finite measures, λ^+ and λ^-.
      - So if $\lambda = m - \nu$ and m, ν are both finite measures, then $m(E) \geq \lambda^+(E)$ and $\nu(E) \geq \lambda^-(E)$
  - Since λ^+, λ^- are measures, can we express them in integral form?
    - Yes! For $f \in L(X, A, m)$,
      $$\lambda^+(E) = \int_E f^+ \, dm, \quad \lambda^-(E) = \int_E f^- \, dm$$
      $$|\lambda|(E) = \int_E |f| \, dm$$

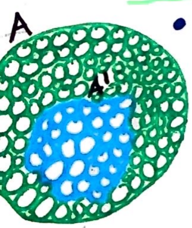
- **Riesz Representation Theorem:**
  - What's a Linear functional?
    - A mapping G of L_p to $\mathbb{R}$ such that $G(af + bg) = aG(f) + bG(g)$, $\forall a, b \in \mathbb{R}$, $f, g \in L_p$.
    - Can we bound this?
      - G bounded if $\exists M \ni |G(f)| \leq M\|f\|_p$ $\forall f \in L_p$.
      - $\|G\| = \sup\{|G(f)| : f \in L_p, \|f\|_p \leq 1\}$
      - If $g \in L_q$, then $G(f) = \int fg \, dm$ and $\|G\| = \|g\|_q$ *
    - Any G bounded can be written as difference of two positive linear functionals. $G(f) = G^+(f) - G^-(f)$ $\forall f \in L_p$.
  - R.R.T in σ-finite measure space:
    - If (X, A, m) σ-finite measure space, and G bounded linear functional on L_1, then $\exists$ $g \in L_\infty \ni$ * holds $\forall f \in L_1$, and $\|G\| = \|g\|_\infty$, and $g \geq 0$ if G positive.
  - R.R.T. in any measure space:
    - If (X, A, m) arbitrary measure space, and G bounded linear funct. on L_p, $1 < p < \infty$, $\exists$ $g \in L_q$, $q = p/(p-1) \ni$ * holds $\forall f \in L_p$ and $\|G\| = \|g\|_q$.

GENERATION OF MEASURES:

M. GASPARD

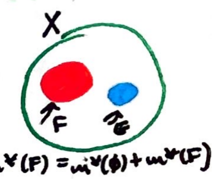
- **Big idea: How can we build a measure?**
  - **Algebras / Fields:**
    - What if we have a collection of sets but it's not necessarily a σ-algebra?
  - **An Algebra/Field:**
    - A family of subsets of a set X, called A' that satisfies the following 3 properties:
      i) $\emptyset, X$ are in A'
      ii) If E in A', $X-E$ in A'
      iii) If $E_1, \dots, E_n$ in A', $\bigcup_{n=1}^{N} E_n$ in A'.
    - How does this differ from a σ-alg?
      - The countable union not necessarily in A'!
  - **Can we assign a measure on A'?**
    - Yes! A measure μ on A' is an extended, Real-valued function μ defined on A' that satisfies:
      i) $\mu(\emptyset)=0$; ii) $\mu(E)\geq 0$, $\forall E$ in A',
      iii) If (E_n) disjoint seq of sets in A' with $\bigcup_{n=1}^{\infty} E_n$ in A', then
      $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right)=\sum_{n=1}^{\infty}\mu(E_n).$$
  - But how can we relate this back to σ-algebras and everything else we'd like to do with them?

- **Extension of Measures: Idea**
  - What do we want to show?

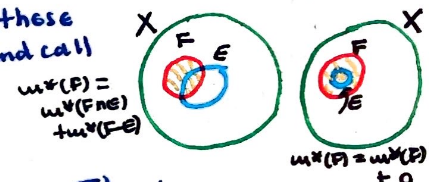


    - If we have an algebra of subsets of X and a measure μ on that algebra, we want to show that A' is contained in a sigma-algebra A and that there is a measure μ^* such that $\mu^*(E)=\mu(E)$ for E in A'.
    - We want to extend a measure on A' to the σ-algebra A.
  - What way / How do we do this?
    - Use the measure μ on A' to get a new function that is defined for every subset of X.
    - Then, look at the collection of subsets where additivity property holds.

- **Extension of Measures: The Execution**
  - **Outer Measure.**
    - a function that is not generally a measure, but is generated by μ.
    - $\mu^*(B)=\inf\sum_{j=1}^{\infty}\mu(E_j)$ for $B, (E_j)$ in A'.
      - $B\subseteq\bigcup_{j=1}^{\infty}E_j$
    - Properties of μ^*:
      i) $\mu^*(\emptyset)=0$ ii) $\mu^*(B)\geq 0$ for $B\subseteq X$
      iii) If $\bar{A}\subseteq B$, then $\mu^*(\bar{A})\leq\mu^*(B)$
      iv) If B in A', $\mu^*(B)=\mu(B)$
      v) If (B_n) seq. of subsets of X, then $\mu^*\left(\bigcup_{n=1}^{\infty}B_n\right)\leq\sum_{n=1}^{\infty}\mu^*(B_n)$. = countable subadditivity
  - **What does it mean to be μ^* measurable?**
    - $E\subseteq X$ is μ^* measurable if $\mu^*(F)=\mu^*(F\cap E)+\mu^*(F-E)$ $\forall$ subsets F of X.



    - We can collect all of these μ^* measurable sets and call the collection A^*.



  - **Carathéodory Extension Thm:**
    - The collection of all μ^*-measurable sets is a σ-algebra containing A'.
    - If (E_n) is a disjoint sequence in A^*, then
      $$\mu^*\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\mu^*(E_n)$$
    - This allows us to extend μ on A' to μ^* on a σ-algebra! So we can obtain A^* with this.
  - **Hahn Extension Thm:**
    - Suppose that μ σ-finite measure on A^*. Then, the extension of μ to a measure on A^* is unique.

- **Lebesgue Measure:**
  - How do we generate a measure on the real line?
    - Look at the algebra F of all finite unions of the form: $(a,b]$, $(-\infty,b]$, $(a,+\infty)$, $(-\infty,+\infty)$.
    - Let l be length of an interval. = Lebesgue Measure
    - Let F^* be σ-alg of all l^*-meas. sets.
      - Smallest F^* = Borel Alg.
      - Then, $l^*((a,b])=b-a$.
  - **Borel-Stieltjes / Lebesgue-Stieltjes Measure**
    - Measure defined as $\mu((a,b])=g(b)-g(a)$
      - Think of function g like "the integral" in a more traditional sense.