Skype A Scientist - Mallory Gaspard, Cornell University
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Powers, Logs, and Googols, Oh My!

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1 Building Blocks of Logs: Understanding Powers

Before we can fully understand what a logarithm does, we need to understand powers and what they mean.

1. To start, let's look at dividing a shape into smaller shapes. But here's the catch...not only do we want to divide the shape into smaller pieces, but we want to do it in such a way that we use *all* of the space that was given to us inside of the shape at the start.

   Question 1. Let's start by drawing a square below. We want to try to divide this square into 4 pieces while using all of the area available to us. Draw how you would divide this shape into four pieces.

   Question 2. Now that you've divided up your square into four pieces, let's group the smaller shapes together that make up the square. Each group should consist only of shapes of the same type, and each group should have the same number of members in it. *Hint:* How many times do you need to have a group of two to put the big square back together? Write your answer below!

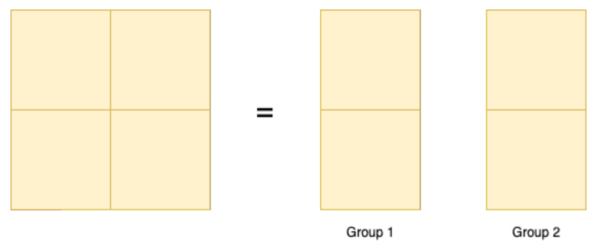
Question 3. Now, let us repeat Question 1, but this time, we want to divide our square into 9 pieces using all of the area that is given to us in the large square. Think about how to divide this square up, and draw your method below.

Question 4. After you've divided up the square into 9 pieces, we want to answer the same question that we answered in Question 4, but this time, we want to do it with our 9 squares instead of 4. Write your answer below!

Question 5. Look back at your answers to the questions above, especially questions 2 and 4. Do you notice any sort of pattern? If so, write your ideas in the space below. *Hint:* Think about multiplication.

After we divided up the squares and separated the smaller shapes into groups, we notice something interesting. In the first example, we needed two groups of two smaller squares to rebuild the larger square. So we needed $2 \times 2 = 2^2 = 4$ smaller squares to make the big square. Similarly, in the second example, we needed three groups of three smaller squares to rebuild the larger square. This means that we needed $3 \times 3 = 3^2 = 9$ smaller squares to make the big square.

When n copies of a number a are multiplied together, we say that number a is raised to the n^{th} power. Instead of writing $a \times a \times a \times a \times \cdots \times a$ which takes up a lot of space, we use the shorter notation a^n to say that "n copies of a are being multiplied." The number n is called an *exponent* or a *power*.



2 Logs and Powers - A Powerful Friendship

While it is helpful to be able to break a shape down into smaller shapes, sometimes, we may not be able to break it down so easily or have enough information to start with. We're going to introduce a special mathematical tool that will allow us to figure out how many of a certain number we need to multiply together to get a larger number, but before we do that, let's think about our example from Question 3.

Question 6. Instead of trying to divide a large square up into 9 pieces, suppose we know the following to start:

- We want the square to be divided up into a total of 9 smaller pieces
- We only want to multiply two of the same number together to get our total of 9 pieces.

With this information, what number would you choose to multiply together so that we have 9 total pieces in the square? Write your answer below.

Question 7. Once you figure out Question 6, let's start to move away from squares and shapes and look more closely at numbers and patterns. Let's try to see if we can figure out a special relationship in the next three questions.

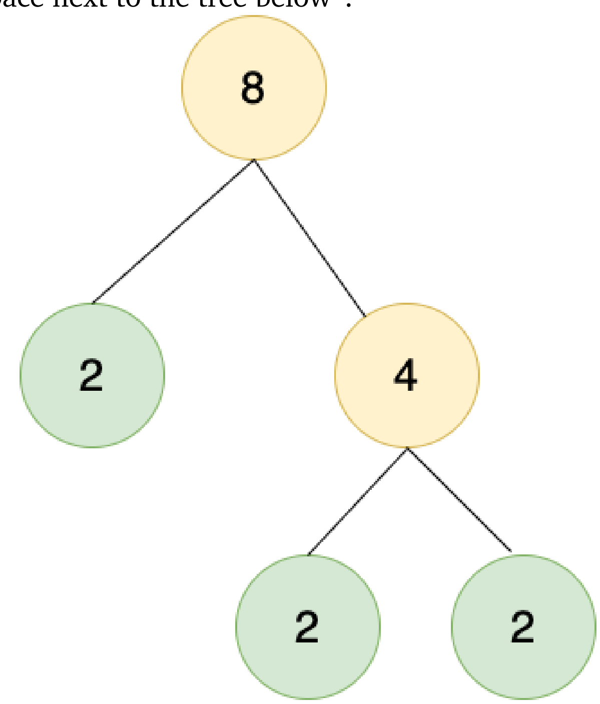
1. Suppose we have 8 objects. How many 2's do we need to multiply together to get 8?

2. Suppose we have 25 objects. How many 5's do we need to multiply together to get 25?

3. Suppose we have 27 objects. How many 3's do we need to multiply together to get 27?

Question 8. Based on your answers above, do you notice any special connection to another mathematical concept? If so, write your idea below. *Hint:* Think about multiplication and exponents.

Question 9. To help make this a little clearer, we can view the numbers that we multiplied together to solve each problem in question 7 as leaves on a tree. The tree for problem 1 in question 7 is displayed below. What do you notice about the number of 2's in this tree? As another example, can you figure out the tree for problem 3 in question 7? Draw your tree for problem 3 in the empty space next to the tree below[1].



[1] This tree gives the "prime factorization" of the top number. The prime-factorization of the number at the top of the tree breaks that number down into the prime numbers (numbers that cannot be divided by any other counting number but one and itself) that we must multiply together to get the top number. This type of break-down is very important in an area of math called Number Theory. There are many cool topics to explore in Number Theory, and I encourage you to go check it out!

2.1 The Logarithm

What did all three of the problems we solved in Question 7 have in common? If you said that we used the *exponent* we wanted and the final answer we wanted to figure out what number we should multiply together to get that final answer, then you're right! Basically, we were trying to *undo* the exponent to figure out *which* number we had to multiply together to get the number we wanted. *Undoing* raising a number to an *exponent* or *power* is called taking the *logarithm* of a number.

Suppose that when we multiply some number a by itself n times, we get that the result of the multiplication is b. Written in *exponential* form with *exponents* this means

$$a^n = b$$

But, we can also write this in *logarithmic* form with a *logarithm*. If we do that, we write

$$\log_a(b) = n$$

This translates to *"We need to multiply n number of a's together to have a final answer of b."*

Question 10. Awesome! Now that we know about the *log*, let's rework an example that we did above by looking at it in logarithmic form. Solve the problems below:

1. $\log_2(8) =$

2. $\log_5(25) =$

3. $\log_3(27) =$

Now, that we know how to compute with logarithms, you're probably wondering, where do logarithms come up in the real-world? Many scientists use logarithms and exponents to describe how certain things behave or to look at the outcome of some event. For example, scientists can measure the strength of an earthquake using a special scale called the Richter Scale. The Richter Scale uses logarithms to tell us how powerful an earthquake is by taking what would likely be a very large number and rewriting it in a way that is easier to understand.

Scientists use a special machine called a *seismometer* that records the ground shaking during an earthquake. As the ground shakes, this machine records a graph of the ground's *vibrations* called a *seismograph*, and this graph of the movement looks like a wave. From the seismograph, the scientist reads off a number telling us how high the wave goes, called the *amplitude*, and uses this number to compare it with a standard earthquake's amplitude. Then, the scientist takes the logarithm of this number to determine the Richter Scale measurement, R, of the earthquake. The general formula to find R is given as

$$R = \log_{10}\left(\frac{\text{Amplitude of Measured Earthquake}}{\text{Amplitude of Standard Earthquake}}\right) \tag{1}$$

Question 11. Suppose that we just felt two earthquakes. We unfortunately do not have the seismograph reading, so we do not know their exact amplitudes, but we do know that the first earthquake has a Richter Scale measure of $R = 5$ and the second earthquake has a Richter Scale measure of $R = 8$. With this in mind, how much stronger was the second earthquake than the first? Can you generalize this relationship? For an increase of 1 unit on the Richter Scale, how much stronger is that earthquake than an earthquake with the previous Richter Scale value? *Hint:* Look at the formula for R and think about some of the topics we've discussed above.

3 How big is BIG? - Discovering the Googolplex

We saw above that we can use our knowledge of logs and exponents to help us understand very very *very* large numbers and make them easier to understand and work with.

Many mathematicians have wondered about how large some numbers can get, and in the 1940's, a mathematician named Edward Kasner introduced a number called the *googol*. So what is a googol you might ask? A googol is a number made by putting one-hundred zeros after the number 1. If you feel up to it, try writing it out sometime! As you can imagine, googol is very large, but perhaps we can figure out a way to write this number in a more compact or condensed form.

Question 12. Let's start looking for a pattern by investigating some smaller numbers first. Look at the numbers below, and in the space provided, write the number of zeros that follow the one. Do you notice a way to write each number in terms of exponents? If so, write each number in its exponential form as well.

- 10

- 100

- 1000

- 10000

- 100000

At this point, you may notice a pattern beginning to emerge. In this case, since each number above is a multiple of 10, we can relate the number of zeros that follow the one with the number of copies of 10 that we must multiply together to get the final number.

Question 13. Write the relationship between the number of zeros that follow the 1 in the numbers above and the number of copies of ten we need to multiply together to get the final answer. Can you now write what a googol should be using this relationship?

As large as a googol is, believe it or not, there's a number out there even *larger* than a googol. It's called a *googolplex*, and it's made by putting a googol zeros behind the number one.

Question 14. Using the approach that we took to express the numbers in Question 12 and Question 13, how would you write a googolplex? Write your answer below!

Throughout this sheet, we've seen how exponents and logs can help us understand and work with very large numbers. With that in mind, here are a few other questions to think about:

Question 15. What is $\log_{10}(\text{googolplex})$

Question 16. Do you think a googolplex is finite or infinite? Why?

Question 17. If you spent the rest of your life trying to write out every zero in a googolplex, do you think you could do it? Why or why not?