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MICRO-ECONOMIC THEORY

Springer-Verlag

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Microeconomic Theory

With 88 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo

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ISBN 3-540-17994-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-17994-1 Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging-in-Publication Data.
Otani, Yoshihiko Microeconomic theory. Bibliography: p. 1. Microeconomics. I. El-Hodiri, Mohamed A., 1938– . II. Title. HB172.E396 1987 338.5 87-20628

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Printed in Germany

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Typesetting: K+V Fotosatz GmbH, Beerfelden
Printing: Kiliandruck Grünstadt
Bookbinding: G. Schäffer GmbH u. Co. KG., Grünstadt
2142/7130-543210

Preface

Contrary to widely held beliefs, microeconomic theory bears no relation to the size of the product under consideration; indeed a micro theorist can just as easily discuss the sale of a whale as he would discuss a whale of a sale in amoebae.

In fact, it possibly is true that a theorist, and a microeconomic theorist in particular, does not have any specific products in mind when he bandies his propositions about. Nor does he have to.

For these in the final analysis are just that; propositions. They are propositions that are motivated by economic reality as observable, not to mention controllable, as that may be, but they are no more and no less than comments about that economic reality and they emphatically are not descriptive assays of it. They are more or less, caricatures of economic reality or metaphors where bold distortions are pressed to the task of describing preconceived visions of that reality. These visions, given their fundamentally qualitative nature, are hardly fit to be put to the test of statistical verification. Perhaps only the judgement and "intuitive feel" of practicing economists over the years are the only true tests of the viability and robustness of these propositional comments on economic reality which make up the body of economic theory.

It is not the abstractions that make the difference, all science is that way; metaphoric. It is the particular set of axioms or "first principles" and the specific intellectual environment that makes for the particular fuzziness in the nature of truth in all of the social sciences.

Innocent or intentional, there is bound to be a degree of covering up various distortions and premeditated policy implications. For instance one could be eloquent; making good use of words, symbols, numbers or even historical narration. One could then claim for the pronouncements the exactitude of the tools, be they numbers, words, logical coding schemes, or flights of fanciful phraseology.

In microeconomic theory, however, the above considerations are less relevant though certainly not insignificant. A typical textbook in microeconomics does not normally advocate.

Having dispensed with generalities we now go into the specifics. But first just one more generality; this book shares with all economists, Marx included, the axiom that individuals act out of choice and not out of coercion or, heaven forbids, necessity.

We tried to present an honest statement of microecomic theory as we learned from our teachers and our colleagues. Obviously it is not complete. In particular we regret that we have not dealt with uncertainty and with externalities and that our presentation of general equilibrium and welfare economics is rather perfunctory because of its introductory nature. Nor have we tried to be most general, in technique, nor the most encompassing.

Our aim was, rather, to capture the essence of intermediate microeconomics and attempt to recast it in a more rigorous mold. Indeed, there currently appears to be a gap between microeconomic theory as presented in intermediate level textbooks and that presented in more advanced textbooks. In particular, we find it rather unfortunate and distressing that fundamental insights and analysis inherited from the past of economic thought make it through intermediate textbooks but simply get cast aside in advanced treatments of microeconomics. We have attempted to rectify this situation at least in part.

We, naturally, reflect our cultural upbringing in our native eastern cultures and our acquired 1960's-Minnesota education. This will be clear to the reader, we hope, through reading the book but, just in case, let us make sure.

The father of them all is Alfred Marshall (Marshall 1920), second Wrangler in the Cambridge Tripos (apparently Marshall thought that such facts were important), and then Hicks (1948) and on the new world side one meets Evans (1930) and Samuelson (1947).

The book is divided into four parts. The first part deals with the theory of consumer's demand where we study the relationship between preferences and demand and present various approaches to explaining the effect of price changes on that demand. Part two is devoted to the study of production sets and functions as well as to costs and profit functions of price taking firms. In the third part we present different market structures; from monopoly to competition. Part four provides a brief introduction to general equilibrium and welfare economics. Finally there is a mathematical appendix which presents, in a chatty way, a narrative description of the principal mathematical concepts that the text makes use of.

On the distribution of credits for contributions between the authors, El-Hodiri would like to clearly state that the bulk of the substantive work was done by Otani. Specifically, all of Part II and all of Part III except for Chapter 10 were written by Otani with minor editorial changes and asides by El-Hodiri. Part I is essentially jointly written with Otani providing roughly 2/3 of the material. In Part IV, Chapter 11 is due to Otani and Chapter 12 was written by El-Hodiri. The mathematical appendix was essentially written by El-Hodiri.

The routine editorial details were El-Hodiri's responsibility with frequent prodding by Otani.

We would like to express our eternal debt to our teachers, especially to J. Chipman, L. Hurwicz and M. Richter. We also have

benefitted a great deal from discussions and collaboration with professors Yuji Kubo, Tony Marino, Toshi Mitsui, Ronald Olsen, James Quirk, Anthony Redwood, Richard Ruppert, Yasuhiko Sakai, Rubin Saposnik, Joe Sicilian, Hirofumi Uzawa, and Fred Van Vleck.

We would like to thank Ms. Anita Williams for her patient expert help in preparing this manuscript, the staff of the Institute for Public Policy and Business Research at the University of Kansas especially Laura Poracky and Amy Waddle for their help in preparing the manuscript, and Alexander Assamein and Frank Hefner for proofreading help. And last, but not least, we would like to express our thanks and appreciation to the economics editorial and book production departments of Springer-Verlag for their patient effient efforts and kind cooperation.

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Part 1 Consumers Demand

The theory of consumers demand is the most complete and most sophisticated part of economic theory and, perhaps because of, or as a result of that, is the most stylized.

We attempt to present in this part a synthesis of what is known by economic theorists about the behavior of consumers. As much as possible we present a general framework so that particular approaches can be studied as special cases.

Our presentation is divided into three chapters. Chapter 1 is devoted to the discussion of the foundations of the theory of consumers behavior; preferences and their relation to utility functions that may be used to represent them. In chapter 2 we introduce demand functions and their general properties. We derive properties of demand functions that are generated by utility maximization and we discuss recoverability of these utility functions from knowledge of demand functions. Finally in chapter 3 we present two additional topics; aggregate demand and demand over time.

Chapter 1 Preferences, Utility and Choice

1. Introduction

A consumer is defined here to be a decision maker who chooses among commodity bundles subject to certain financial and other constraints. The choice is based on a preference ordering of commodity bundles and the financial constraints are summarized by the budget constraint.

A *commodity bundle* is a list of names of commodities with the quantity of each commodity stated next to the name of that commodity. It is convenient, though not inevitable, to fix the list of commodities once and for all so that it is known what the first commodity refers to, what the second commodity refers to, etc., and how many commodities we are talking about, say n. So we index and fix the index. We agree, then, that a commodity bundle is an ordered set of numbers. In symbols we write $x = (x_1, \ldots, x_n)$ where x_i represents the quantity of commodity i, $i = 1, \ldots, n$, and we speak of commodity bundle x. The collection of all commodity bundles is called a *commodity space.* In order to perform some arithmetic operations on bundles, we must endow the commodity space with an algebraic structure. We choose to make the commodity space into a real linear space. Among other things, this implies that the commodities are perfectly divisible and that bundles could be combined to make new bundles. An additional structure, geometric, is needed in order to specify the sense in which two bundles are close to each other. We define being close as being a small distance apart and follow the Euclidean way of measuring distance. Thus our commodity space is now a finite dimensional Euclidean n-space, denoted by R^n. Most of our discussion will be confined to a certain subset X of the commodity space. The set X denotes what we call the *consumption set* for our consumer. The consumption set embodies institutionally and individually determined permissible consumption activities. For instance suppose it is not legal to drink and drive and suppose the consumer would die upon consuming poison in any quantity. Let drinking be good number 1, driving number 2, and let poison be number 3. Let there be one more good, number 4 and allow total freedom in consuming it. Then $x^1 = (1, 1, 2, 3)$ is not in X, written $x^1 \notin X$. But $x^2 = (0, 0, 0, -5) \in X$. In fact if $x_1 x_2 = 0$, $x_3 = 0$ and x_4 is any number, then $x \in X$.

The reader may think of X as the non-negative orthant of our commodity space, i.e., $X = \{x \in R^n | x \geqq 0\}$. Incidently our notation for vector inequalities is as follows. Let z and w be in R^n. We write $x \geqq w$ if $z_i \geqq w_i$ for all i, $i = 1, \ldots, n$. We write $x \geq w$ if $z_i \geqq w_i$ for all i, and $z \neq w$. Finally, $z > w$ if $z_i > w_i$ for all i.

Preferences are defined as an ordering of the bundles in X. When the preferences are represented by a real valued function, the function is called a utility function. Choice maps assign one or more commodity bundles, a subset of X, to each financial situation; prices and income. The choice is based on preferences.

2. Preferences

Our stylized consumer is endowed with a "preference ordering" in the sense of being equipped with a binary relation on the consumption set X, where a binary relation is simply a subset of the Cartesian product $X \times X$. Two bundles x and y in X may be related in the consumer's view in one or more ways. The consumer may strictly prefer x to y. He may be indifferent between x and y. He may weakly prefer x to y, i.e., he may be able to state only that x is at least as good as y. Finally, he may be able to say that x and y are not related at all. We denote the strict preference relation by P, the indifference relation by I, and the weak preference relation by G.

In order to make use of the relations P, I, and G in our analysis, it is necessary to specify certain properties of these relationships. We list now some properties that are usually assumed to hold. It should be noted that one may derive properties of one of these relationships from properties of the other two.

(2.1) *Possible Relational Properties of P:*

P.(i) (Irreflexivity) For every $x \in X$, not xPx.
P.(ii) (Asymmetry) For every $x, y \in X$, if xPy, then not yPx.
P.(iii) (Transitivity) For every $x, y, z \in X$, if xPy and yPz, then xPz.
P.(iv) (Negative transitivity) For every $x, y, z \in X$, if not xPy and not yPz, then not xPz.

(2.2) *Possible Relational Properties of I:*

I.(i) (Reflexivity) For every $x \in X$, xIx.
I.(ii) (Symmetry) For every $x, y \in X$, if xIy, then yIx.
I.(iii) (Transitivity) For every $x, y, z \in X$, if xIy and yIz, then xIz.

(2.3) *Possible Relational Properties of G:*

G.(i) (Reflexivity) For every $x \in X$, xGx.
G.(ii) (Totality or Completeness) For every $x, y \in X$, if $x \neq y$, then either xGy or yGx.
G.(iii) (Transitivity) For every $x, y, z \in X$, if xGy and yGz, then xGz.

Even though these properties appear to be non-restrictive, it is wise to stop and consider their implications in some detail. For instance, consider the transitivity of I and the completeness of G. Some may object to assuming that I is transitive in view of the imprecision of human perception. This point is well-dis-

cussed by Armstrong (1939) and Luce (1956). Even transitivity of P may be questioned particularly when objects in question are composites with multiple characteristics. (See May (1954) and Davidson, McKinsey and Suppes (1955), p. 145.) In the literature, we will find that some authors use P as their primitive concept and others use G as their primitive concept. The main reason of adopting P as a primitive concept is to avoid assuming that I is transitive.

If we begin with G and assume that G is reflexive, total and transitive, then we can define P and I from G as follows: xPy if and only if xGy and not yGx and xIy if and only if xGy and yGx. We can show easily that P and I so defined satisfy:

(2.4) (a) P is irreflexive, asymmetric and transitive on X, and
(b) I is reflexive, symmetric and transitive on X. (See Arrow (1951), p. 14.)

[x]

A binary relation is called an *equivalence relation* if it is reflexive, symmetric and transitive. Thus I defined through G is an equivalence relation. If I is an equivalence relation, then an equivalence class [x] is defined as a subset of X such that $[x] = \{y \in X \mid xIy\}$. We can easily show that the collection of equivalence classes partition X, i.e., $\cup\{\cancel{I(x)} \mid x \in X\} = X$ and for every $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \phi$. An equivalence class of an indifference relation I will be called an indifference class.

Alternatively we can start with P. For example assume that P is asymmetric and negatively transitive. Then we can define an indifference relation I^* by xI^*y if and only if (not xPy) and (not yPx).

Also define G^* by xG^*y if and only if either xPy or xI^*y. Then we can show that:

(2.5) (a) I^* is an equivalence relation and
(b) G^* is reflexive, total and transitive.

Therefore in this case we also end up with the transitivity of an indifference relation. A less restrictive approach is to assume that P is irreflexive and transitive. Note that this implies that P is asymmetric.

Let $\hat{B}$ be a collection of nonempty subsets of X. A set $B \in \hat{B}$ will be called a budget set. Then the notion of "making a choice" may be formulated by a choice map or a choice function which assigns a subset $h(B) \subseteq B$ for each $B \in \hat{B}$.

Starting from a preference over X, we may define a choice function as follows:

$$h(B:G) = \{x \in B \mid (\text{for all } y \in B)\ xGy\}$$

for every $B \in \hat{B}$ or

$$h(B:P) = \{x \in B \mid (\text{for all } y \in B) \text{ not } yPx\}.$$

(2.6) *Theorem:* Suppose that B is a finite set.
(a) IF G is reflexive, total and transitive on X, then $h(B:G) \neq \phi$.
(b) If P is irreflexive and transitive, then $h(B:P) \neq \phi$.

Since (a) is trivial, we only prove (b). Suppose that $h(B:P) = \phi$. Then for every $x \in B$, there exists $y \in B$ such that yPx. Since P is irreflexive, $y \neq x$. So if we fix $x^0 \in B$, we obtain $x^1 \in B$, $x^1 \neq x^0$ and $x^1 P x^0$. Similarly we obtain a chain $x^k P x^{k-1} (k = 1, \ldots, n)$, $x^k \in B$, $(k = 0, 1, \ldots, n)$ and x^k's are all distinct. Since B is finite, there exists $y \in B$ such that for every $x \in B$, yPx. This is a contradiction.

The nonemptiness of a choice is sometimes used as a requirement for a "rational" choice. Of course, some people would not like indecisiveness to be called "irrational." Behaviorally there is no way of distinguishing indecisiveness from a choice of inaction if inaction is an alternative. For further discussion on "rationality" and conditions on nonempty choice, see Davidson, McKinsey and Suppes (1955), Sen (1970), Chapter 1 and 1*, and Patanaik (1971), Chapter 1.

We now state some further properties of our preference relationships. These properties are often discussed in the literature and are useful in deriving properties of consumers' demand functions.

G.(iv) (Weak monotonicity) For any $x, y \in X$, $x \geqq y$ implies xGy.
P.(iv) (Monotonicity) For any $x, y \in X$, $x > y$ implies xPy.
P.(v) (Strong monotonicity) For any $x, y \in X$, $x \geq y$ implies xPy.
P.(vi) (Non-satiation) For any $x \in X$ there exists $y \in X$ such that yPx.
P.(vii) (Local Non-satiation) For any $x \in X$ and any neighborhood $N(x) \subset X$ there is $y \in N(x)$ with yPx.
G.(v) (Weak convexity) For any $x, y \in X$ we have: xGy implies $(\theta x + (1-\theta)y)Gy$ for all $0 \leqq \theta \leqq 1$.
P.(viii) (Convexity) For any x and y in X, xPy implies $(\theta x + 1(1-\theta)y)Py$ for all $0 < \theta < 1$.
G.(vi) (Strong convexity) For any $x, y \in X$ with $x \neq y$, xGy implies $(\theta x + (1-\theta)y)Py$ for all $0 < \theta < 1$.
G.(vii) (Continuity) The sets $C_x = \{y \in X \mid yGx\}$ and $C^x = \{y \in X \mid xGy\}$ are closed in X.

We will give a familiarization (not an interpretation) in terms of graphs with the above assumptions.

In order to illustrate some of the properties stated above, it is helpful to think of the equivalence classes defined by the equivalence relation as a curve in the commodity space and call ~~it~~ an indifference curve*.

For instance weak convexity allows fat indifference curves as in figure 2.1.a below. Convexity rules out fat, but allows flat, indifference curves as in figure 2.1.b. Strong convexity rules out fat and flat indifference curves as figure 2.1.c illustrates. Clearly strong convexity implies convexity which in turn implies weak convexity. The corresponding converses are, also clearly, not true.

Strict monotonicity implies that the direction of preferences in a preference map is to the northwest. Clearly strict monotonicity implies monotonicity as well as local nonsatiation and hence nonsatiation. Once more the converses are not true. For instance if X is the one and only indifference class then we have an example that proves this last assertion. The example of figure 1.a shows that non-

* We will abuse the language by referring to curve, surface or hypersurface as "curve".

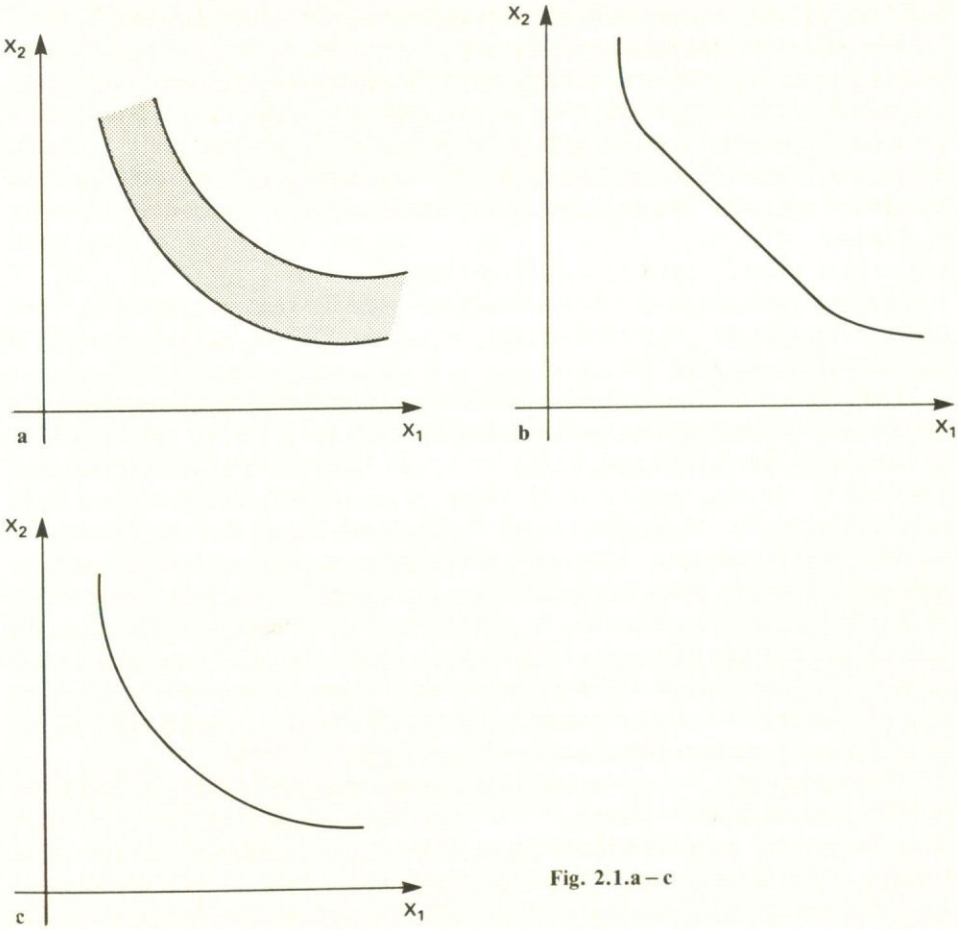

Fig. 2.1.a – c

satiation does not imply local non-satiation. Notice that the same, poor overworked, example shows that monotonicity does not imply local non-satiation. On the other hand local non-satiation does not imply monotonicity since less could be better without violating local nonsatiation. On the, yet, other hand local non-satiation and monotonicity imply strong monotonicity. Finally we note that there are all kinds of combinations of assumptions that imply others. We will not say anything about these except for two. It turns out that strong convexity and monotonicity imply strong monotonicity. However, convexity and strong monotonicity does not imply strong convexity.

3. Preferences and Utility

In this section, we explore the notion of "utility" and its relationship to preferences. Utility is generally believed to be a numerical scale or index of "satisfactoriness" or "desiredness" of a commodity bundle of an economic agent. As such, it would be natural to regard the notion of "satisfactoriness" or "desired-

ness" as a more primitive concept than utility. If the binary relation G (or P respectively) is used to capture the notion of "satisfactoriness" and if we assume that G is reflexive, total and transitive (or P is asymmetric and negatively transitive respectively), then a numerical or real valued function $u: X \to R$ is said to be a *utility function representing G* (or *P* respectively) if for every $x, x' \in X$, $u(x) \geqq u(x')$ if and only if xGx' (or respectively, $u(x) > u(x')$ if and only if xPx'). The above concept of utility is often called "ordinal utility" in economics. Let us first define:

(3.1) *Definition:* Let D be a nonempty subset of R. Then a function $f: D \to R$ is said to be a *monotone increasing transformation* if for every $\alpha, \beta \in D$, $\alpha \geqq \beta$ if and only if $f(\alpha) \geqq f(\beta)$.

Then from the definition of a monotone increasing transformation, it is clear that if $u: X \to R$ is a utility function representing G (or P respectively), then for any monotone increasing transformation f with domain $D = u(X)$, the function $v = f(u): X \to R$ also represents G (or P respectively). Conversely, if both $u: X \to R$ and $v: X \to R$ represent G (or P respectively), then there exists a monotone increasing transformation f with $D = u(X)$ such that $v = f(u)$. Informally, utility functions are called "ordinal" if any monotone increasing transformation of a utility function, say $u: X \to R$, is regarded equivalent to u. To make the concept a little more precise, let $F(X, R)$ be the collection of all real-valued functions on X. Then *ordinal utility* can be defined by an equivalence relation $\sim$ on $F(X, R)$ which is defined by: for every $u, v \in F(X, R)$, $u \sim v$ if and only if there exists a monotone increasing transformation f with $D = u(X)$.

The concept of "cardinal utility" in economics can also be defined analogously if we restrict transformations to monotone increasing affine transformation. That is, cardinal utility is defined by an equivalence relation $\approx$ on $F(X, R)$ as follows: for every $u, v \in F(X, R)$, $u \approx v$ if and only if there exist $\alpha, \beta \in R$, $\alpha > 0$ such that $v = \alpha u + \beta$. One important fact about cardinal utility is that if u and v are equivalent under $\approx$, then the utility difference has a well-defined ordering as follows: if $u \approx v$, then for every $x, x', y, y' \in X$, $u(x) - u(y) \geqq u(x') - u(y')$ if and only if $v(x) - v(y) \geqq v(x') - v(y')$. Therefore we can define a binary relation $\gtrsim$ on $X \times X$ by: for every $(x, y), (x', y') \in X \times X$, $(x, y) \gtrsim (x', y')$ if and only if $u(x) - u(y) \geqq u(x') - u(y')$. Clearly this odering $\gtrsim$ is total, reflexive and transitive and it is independent of the choice of u from an equivalence class under $\approx$.

Our next question is to investigate conditions under which an ordinary utility function representing G (or P) exists and conditions under which a cardinal utility function representing an ordering $\gtrsim$ on $X \times X$ exists. We will mainly confine ourselves to the question of ordinal representability. For the question of cardinal representability, readers may find a good discussion in Fishburn (1970, Chapter 6) and also a closely related topic can be found in Debreu (1960).

In the last section, we considered two alternative approaches with G or P as a primitive concept where G is assumed to be reflexive, total and transitive and P is assumed to be asymmetric and negatively transitive. Since these two apporaches turned out to be almost the same, we will only consider the case of P in this section. As in the previous section, we define xIy if and only if not xPy and not yPx. (We use I instead of I^*.) Let $[x] = \{y \in X \mid xIy\}$ for each $x \in X$, an indif-

ference (or equivalence) class under an equivalence relation I and $X/I = \{[x] = \{[x] \mid x \in X\}$, the set of all indifference classes. We first state a theorem on the ordinal representability of P when the set X/I is at most countable.

(3.2) *Theorem:* If P is asymmetric and negatively transitive and if the set X/I is at most countable, then there exists $u: X \to R$ such that $u(x) > u(y)$ if and only if xPy.

The proof of this theorem is relatively elementary. If the set X/I is finite, we can rank indifference classes according to the ordering P and attach appropriate real numbers. If the set X/I is countably infinite, then a similar procedure can be adopted by using the mathematical induction. A complete proof can be found in Fishburn (1970, pp. 14 – 15).

When the set X/I is uncountable, ordinal representability becomes a little more difficult. Mathematically what we need is a mapping from X with ordering P to R with ordering $>$ so that this mapping preserves orderings, i.e., for every $x, y \in X$, xPy if and only if $u(x) > u(y)$. That is, we want to obtain conditions to establish an order isomorphism from $(X/I, P^*)$ to $(R, >)$ where P^* is an ordering on X/I induced by P, i.e., $[x]P^*[y]$ if and only if xPy. Therefore you may suspect that an additional property similar to some inherent nature of real numbers may be required to obtain an ordinal representability of P.

(3.3) *Definition:* A subset Z of X/I is said to be P^*-order dense if for every $[x], [y] \in X/I \sim Z$ with $[x]P^*[y]$, there exists $[z] \in Z$ such that $[x]P^*[z]$ and $[z]P^*[y]$.

(3.4) *Theorem:* Suppose that the set X/I is uncountable. Then there exists $u: X \to R$ representing P if and only if P on X is asymmetric and negatively transitive and there exists a countable subset of X/I which is P^*-order dense.

Fishburn attributes the above theorem to Cantor in Fishburn (1972, p. 60) and the proof of the theorem can be found in Fishburn (1970, Chapter 3).

A well-known example for which an ordinal (real-valued) utility function does not exist is a case of lexicographic orderings defined on R^2 or R^2_+, the nonnegative quadrant of R^2. A lexicographic ordering P_1 on R^2_+ can be defined as follows: $(x_1, x_2)P_1(x'_1, x'_2)$ if and only if either (a) $x_1 > x'_1$, or (b) $x_1 = x'_1$ and $x_2 > x'_2$. In this case, an indifference class consists of a single point of R^2_+ so we can identify P_1^* by P_1 and X/I, by $X = R^2_+$. First we will argue that there does not exist a countable P_1-dense subset of $X = R^2_+$. Indeed, fix x_1 and consider a subset $A(x_1) = \{(x_1, x_2) \mid x_2 \in R_+\}$. Then we can show that any P_1-dense subset of $A(x_1)$ must be at least countably infinite. A subset $\{(x_1, x_2) \mid x_2 \text{ is rational}\}$ of $A(x_1)$ is such an example. Since x_1 can range over R_+ which is uncountable, any P_1-dense subset of X is uncountable. Therefore, by the above theorem, an ordinal utility function representing the above lexicographic ordering does not exist. A more direct argument to show the nonexistence of an ordinal utility function is as follows. Suppose that an ordinal utility function, say $u_1(x_1, x_2)$, representing P_1 exists. Then for each $x_1 \in R_+$, choose a rational number in the interval $[u_1(x_1, 0), u_1(x_1, 1)]$. Denote this rational number corresponding to x_1 by $f(x_1)$. Since

$x_1 > x_1'$ implies $u_1(x_1, x_2) > u_1(x_1', x_2')$ for any $x_2, x_2' \in R_+$, the mapping f is one-to-one. This establishes a one-to-one mapping from the set R_+ into the set of rational numbers which is impossible. Hence an ordinal utility function representing P_1 defined above does not exist.

We should not conclude from the above discussion that every lexicographic ordering has no ordinal utility representation. The following example will be instructive. Suppose that we take $X = \{(x_1, x_2) | x_1$ is a nonnegative integer and $x_2 \in R_+\}$ and define a lexicographic ordering P_2 on X by: $(x_1, x_2) P_2 (x_1', x_2')$ if and only if either (a) $x_1 > x_1'$, or (b) $x_1 = x_1'$ and $x_2 > x_2'$. Then we can easily check that the following function is an ordinal representation of this lexicographic ordering:

$$u_2(x_1, x_2) = x_1 + \frac{x_2}{1 + x_2}.$$

Various degrees of smoothness of functions such as continuity and differentiability provide us with a powerful set of mathematical techniques. Here we would like to look into the question of the existence of a continuous ordinal utility function representing a preference ordering.

We first give a simple example of a preference ordering P_3 on R_+^2 which has an ordinal utility representation but a utility function representing P_3 can not be continuous with the usual Euclidean topologies on R_+^2 and R. First we define a function u_3: $R_+^2 \to R$ by: for every $(x_1, x_2) \geqq 0$,

$$\begin{aligned} u_3(x_1, x_2) &= x_1 + x_2 \quad \text{if} \quad x_2 \neq 0 \\ &= x_1 + 1 \quad \text{if} \quad x_2 \neq 0 . \end{aligned}$$

Then we define P_3 on R_+^2 by:

$$(x_1, x_2) P_3 (x_1', x_2') \quad \text{if and only if} \quad u_3(x_1, x_2) > u_3(x_1', x_2') .$$

Then clearly u_3 is an ordinal representation of P_3 which is not continuous and also we can show that any monotone increasing transformation of u_3 can not be continuous. Therefore P_3 does not have a continuous ordinal representation.

On the other hand, consider the following function u_4: $R_+ \to R$ and a preference ordering P_4 on R_+ defined by:

$$u_4(x) = \begin{cases} x & \text{for} \quad x \in [0, 1] \\ x + 1 & \text{for} \quad x \in (1, \infty) \end{cases}$$

and

$$x P_4 x' \quad \text{if and only if} \quad u_4(x) > u_4(x') .$$

Then the function u_4 is discontinuous at $x = 1$ and it represents the preference ordering P_4. Let the range of u_4 be denoted by $D = [0, 1] \cup (2, \infty)$. Then define a function f: $D \to R$ by:

$$f(\alpha) = \begin{cases} \alpha & \text{for} \quad \alpha \in [0, 1] \\ \alpha - 1 & \text{for} \quad \alpha \in (2, \infty) . \end{cases}$$

Then the function u_4 is discontinuous at $x = 1$ and it represents the preference ordering P_4. Let the range of u_4 be denoted by $D = [0, 1] \cup (2, \infty)$. Then define a function $f: D \to R$ by:

$$f(\alpha) = \begin{cases} \alpha & \text{for} \quad \alpha \in [0, 1] \\ \alpha - 1 & \text{for} \quad \alpha \in (2, \infty) . \end{cases}$$

Then f is a monotone increasing transformation on D* and $v(x) = f[u_4(x)] = x$ for every $x \in R_+$. Therefore P_4 has a continuous ordinal representation.

Suppose that X has a topology and R has the usual Euclidean topology. Then first observe that a real-valued function $u: X \to R$ is continuous on X if and only if for every $\alpha \in R$, $\{x \in X \mid u(x) > \alpha\}$ and $\{x \in X \mid u(x) < \alpha\}$ are both open subsets of X. Note also that if the function $u: X \to R$ represents a preference ordering P and if for some $\alpha \in R$, there exists $x^0 \in X$ with $u(x^0) = \alpha$, then we have:

$$\{x \in X \mid u(x) > \alpha\} = \{x \in X \mid x P x^0\}$$

and

$$\{x \in X \mid u(x) < \alpha\} = \{x \in X \mid x^0 P x\} .$$

Therefore a necessary condition for existence of a continuous ordinal utility function representing P is that for every $x \in X$, both $\{x' \in X \mid x' P x\}$ and $\{x' \in X \mid x P x'\}$ are open in X. If an ordinal utility function representing P exists, then the above condition also turns out to be a sufficient condition for the existence of a continuous utility representation. This is the well-known theorem due to Debreu (1954 and 1967). Thus we can state:

(3.5) *Theorem:* Suppose that an ordinal utility function representing P exists. Then there exists a continuous ordinal utility function representing P if and only if for every $x \in X$, both $\{x' \in X \mid x' P x\}$ and $\{x' \in X \mid x P x'\}$ are open in X.

For the proof of the above theorem, see Debreu (1964) and Fishburn (1970, Chapter 3).

We conclude this section with a discussion of preferences generated by utility functions. Suppose then that a consumer makes decisions according to an ordinal utility function as defined earlier. Then we may use thus utility function to define the preference relations G, I and P on X. For instance if x and y are two bundles in X, then we can specify G by the rule: $x G y$ is and only if $u(x) \geqq u(y)$. We can either define I and P using G or we can define them directly. The relation I is directly defined by: for any x and y in X we say that $x I y$ if and only if $u(x) = u(y)$. Then the indifference class of x, I_x is given by:

$$I_x = \{y \in X \mid u(y) = u(x)\} .$$

If u is nice, then I_x is a curve in X – an indifference curve. If one does not wish to make reference to x (or any specific bundle) then one defines an indifference curve as a level curve of the utility function is, i.e., as

* Note that this function can not be extended so that it is a monotone transformation on $[0, \infty]$. This is why definition 3.1 is stated in the way adopted here.

$$I_\alpha = \{x \in X \mid u(x) = \alpha\}.$$

The relation P may be defined by: For any x and y in X we say that xPy if and only if $u(x) > u(y)$.

To complete the picture we discuss preference maps. Recall that indifference classes partition X. Indifference curves, not surprisingly, do the same. In fact the collection of indifference curves, as level curves of u, provide a preference map; a geometric rendition of G.

Every bundle lies on one and only one indifference curve. If a bundle does not lie on any indifference curve, then u is not defined on all X and if a bundle lies on two distinct indifference curves, then u is not a function. This is about as much as we can get for free, without further assumptions about u.

If u is monotone, then so is G and a bundle which, according to our vector order, is no more than x can not lie on a lower indifference curve. If u is strictly monotone, then so is P and the indifference curves "stack up." If u is quasi-concave, the G is convex. Quasi-concavity by itself does not yet bring us to the familiar picture of preference maps. Indifference curves may yet be fat and may have flat parts. If u is pseudo concave then G is strongly convex. It is now true that the preference map looks familiar, no fat, no flat. Indifference curves are convex and contain no flat pieces. Finally if u is continuous, then so is G.

Figure 3.1 below depicts a preference map that satisfies pseudo concavity, strict monotonicity, and continuity, with $X = R^2$.

In a preference map that satisfies only the convexity of G, indifference classes could look like those given in figure 2.1.a above.

Several concepts have traditionally been introduced with the help of utility functions, e.g., the concept of marginal utility and, later on, the concept of marginal rate of substitution. Some authors, notably Walras (1926) started with the concept of marginal utility and then defined utility as the integral of marginal utility. This has the advantage of not needing to assume that utility functions are differentiable. In the current literature if *marginal utility* is ever mentioned, it is

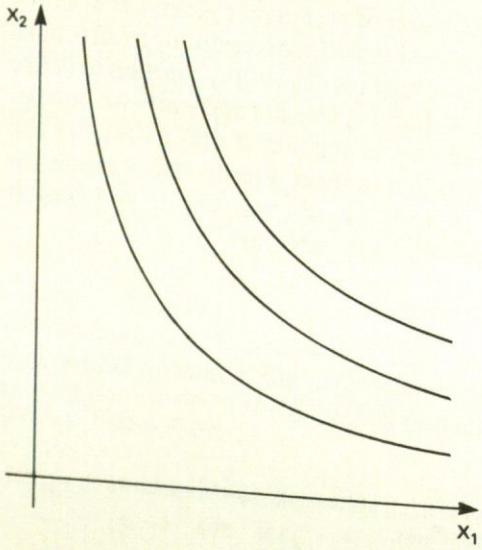

Fig. 3.1

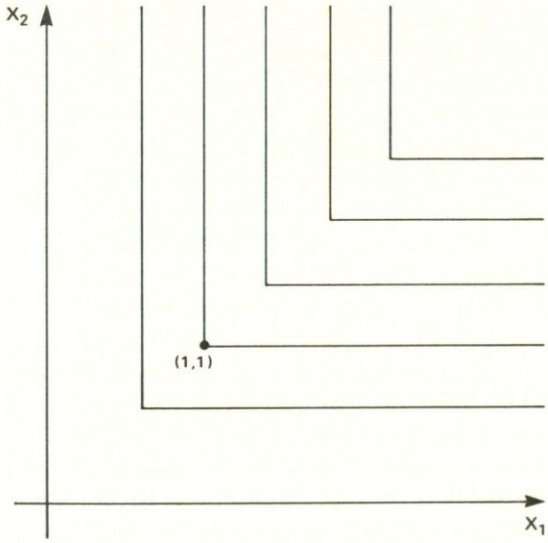

Fig. 3.2

defined as the change in utility due to a change in one commodity while others in the bundle are unchanged. This definition does not a priori require cardinality nor does it require differentiability of utility. For if one accepts the possibility of a set of marginal utilities corresponding to a commodity within a bundle, then neither differentiability nor cardinality is essential. If we insist, however, on having single-valued marginal utility then we need both concepts.

The marginal rate of substitution between two commodities is defined as the change in one commodity due to a change in the other, provided that the consumer stays in the same indifference class. This definition is general enough to allow for fat indifference classes and for indifference curves with kinks in which cases the marginal rate of substitution is a set with more than one element in it. For instance, in the figure 3 below the marginal rate of substitution between good 1 and good 2 at the bundle (1,1) is the set $\{-\infty, 0\}$.

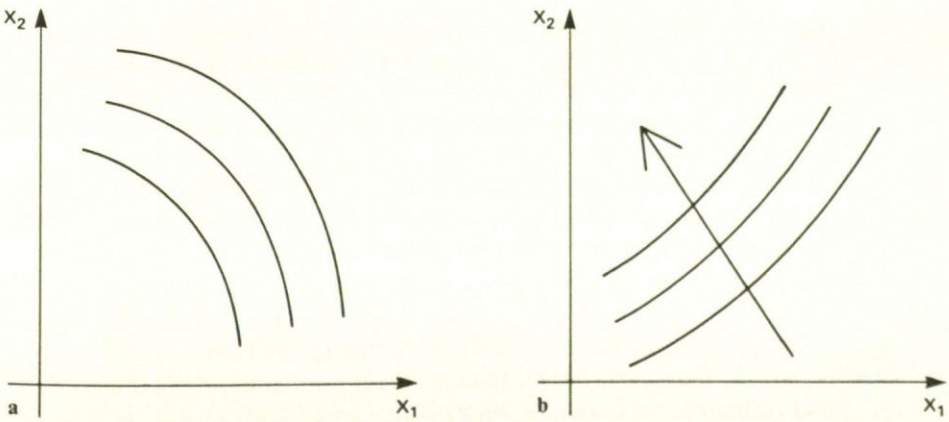

Fig. 3.3.a, b

To make things more concrete, we assume that u is differentiable. This guarantees that the marginal rate of substitution at any bundle is a number. The marginal utility could still be a set in view of ordinality.

The assumption convexity and monotonicity (of G) imply that the marginal rate of substitution (MRS) is negative and that its absolute value decreases as the quantity of the acquired, in place of the sacrificed, good increases.

In figure 3.3.a we violate monotonicity but not convexity and get a positive MRS. In figure 3.3.b we violate convexity and get an increasing MRS.

To make matters slightly more formal, define the marginal rate of substitution between good i, acquired, and good j, forsaken (MRS_{ij}) at bundle x^0 as $\Delta x_j/\Delta x_i$ where $u(x) = u(x^0)$. One may think of $x_j = \xi^j(x_i, x_0)$ where $u(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_{j-1}^0, x_j, x_{j+1}^0, \ldots, x_n^0) = u(x^0)$. If u is differentiable and $u_j^0 \neq 0$, then by the implicit function theorem, we know that ξ^j is a unique function and that it is differentiable. Thus, in the limit, $MRS_{ij} = \partial\xi^j/\partial x_i$.

Monotonicity of u implies that MRS_{ij} is nonpositive since $u_i^0 \geqq 0$, $u_j^0 > 0$ (it is $\neq 0$ by assumption) and since by the implicit function theorem $\partial\xi^j/\partial x_i = -u_i^0/u_j^0$. Pseudo convexity of u implies that the absolute value of $MRS_{ij} = \partial\xi^j/\partial x_i$ is a monotone nonincreasing function in x_i. This follows from the convexity of ξ^j in x_i and from the monotone nonincreasing property of derivatives of differentiable convex functions.

Problems

1. Let
   $X = \{0, 1, 2\}$ and R, S and T be binary relations on X given by:
   $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2), (0, 2)\}$
   $S = \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1)\}$,
   $T = \{(1, 1), (2, 1), (1, 2), (0, 1)\}$.
   (i) Show that R is reflexive, transitive and connected (or total).
   (ii) Show that S is total but neither transitive nor reflexive.
   (iii) Show that T is neither reflexive, nor transitive, nor total.

2. The following binary relation on the set $\{a, b, c, d\}$ is given:
   $R = \{(a, d), (a, c), (a, a), (b, b), (c, d), (c, c), (c, a), (d, d), (d, b)\}$
   (i) Check whether each one of the following properties holds for R or not: reflexivity, transitivity, anti-symmetry, totality or completeness, asymmetry and symmetry.
   (ii) Add the smallest possible pairs to R so that the modified relation becomes transitive. (This is the smallest transitive set containing R and it is sometimes called the transitive closure of R.)

3. Let
   $X = \{a, b, c, d\}$ and consider the following binary relation
   $Q = \{(a, a), (a, b), (c, b), (d, a), (a, b), (a, c)\}$
   (i) Is Q reflexive, total and/or transitive?
   (ii) Find the transitive closure of Q.

4. Let R be a binary relation on a set X. Then
   (a) R is said to be *reflexive* if for every $x \in X$, xRx;
   (b) R is said to be *ireflexive* if, for every $x \in X$, $\sim xRx$;
   (c) R is said to be *symmetric* if, for every $x, y \in X$, xRy implies yRx;
   (d) R is said to be *asymmetric* if, for every $x, y \in X$, xRy implies $\sim yRx$,
   (e) R is said to be *antisymmetric* if, for every $x, y \in X$, xRy and yRx implies $x = y$;
   (f) R is said to be *total* if, for every $x, y \in X$, $x \neq y$ implies that either xRy or yRx;
   (g) R is said to be *transitive* if, for every $x, y, z \in X$, xRy, and yRz implies xRz;
   (h) R is said to the *negatively transitive* if, for every $x, y, z \in X$, $\sim xRy$ and $\sim yRz$ implies $\sim xRz$.

   Prove the following assertions (i) through (v) and show whether or not (vi) is true.
   (i) If R is asymmetric, then R is irreflexive.
   (ii) If R is asymmetric, then R is antisymmetric.
   (iii) If R is irreflexive and transitive, it is asymmetric.
   (iv) If R is reflexive, total and transitive, it is negatively transitive.
   (v) If R is asymmetric and negatively transitive, then R is transitive.
   (vi) If R is symmetric and transitive, then it is reflexive.

5. Let us define (T), (PP), (PI), (IP) and (II) as follows:
   (T) $(\forall x, y, z \in X)$ xGy and $yGz \Rightarrow xGz$
   (PP) $(\forall x, y, z \in X)$ xPy and $yPz \Rightarrow xPz$
   (PI) $(\forall x, y, z \in X)$ xPy and $yIz \Rightarrow xPz$
   (IP) $(\forall x, y, z \in X)$ xIy and $yPz \Rightarrow xPz$
   (II) $(\forall x, y, z \in X)$ xIy and $yIz \Rightarrow xIz$
   (i) Show that if G is reflexive, total, and transitive, then:
   (T) $\Rightarrow$ (PP), (PI), (IP) and (II)
   (ii) Assume that G is reflexive and total. Then show that:
      a. (PP) and (PI) $\Rightarrow$ (T)
      b. (PI) $\Leftrightarrow$ (IP)
      c. (PI) $\Rightarrow$ (II)
      d. (PP) and (II) $\Rightarrow$ (PI)

6. Define a binary relation V on R^2 by:
   $(x_1, x_2)\ V(x'_1, x'_2)$ iff $x_1 \geqq x'_1$ and $x_2 \geqq x'_2$.
   (i) Is V reflexive, total, and/or transitive?
   (ii) A binary relation is said to be negatively transitive if $(\forall x, y, z \in X) \sim xRy$ and $\sim yRz \Rightarrow \sim xRz$.
   Is the binary relation V negatively transitive? (Prove if yes. Give a counterexample if not).

7. Consider the following postulates on a choice function.
   (A_1) $B_1 \subseteq B_2 \Rightarrow B_1 \sim h(B_1) \subseteq B_2 \sim h(B_2)$
   (A_2) $B_1 \subseteq B_2 \Rightarrow B_1 \cap h(B_2) \subseteq h(B_1)$

Suppose that we observe the following three choice functions, h^1, h^2, and h^3 as follows:

(1) $X = \{a, b, c\}$
$h^1[\{a, b\}] = \{b\}$
$h^1[\{a, b, c\}] = \{a\}$

(2) $X = \{a, b, c, d\}$
$h^2[\{a, b, c\}] = \{a, b\}$
$h^2[\{b, c, d\}] = \{c, d\}$
$h^2[\{a, b, c, d\}] = \{a, d\}$

(3) $X = \{a, b, c\}$
$h^3[\{x\}] = \{x\}$ where $x = a$, b or c,
$h^3[\{a, b\}] = \{a\}$
$h^3[\{b, c\}] = \{b\}$
$h^3[\{a, c\}] = \{c\}$

(i) Prove that (A_1) and (A_2) are equivalent.
(ii) Check whether or not each h^i $(i = 1, 2, 3)$, satisfies (A_1) and/or (A_2).
(iii) Can the choice function h^3 be derived from the preference ordering on $X = \{a, b, c\}$ which is reflexive, total, and transitive?

8. Assume that (a) X is a closed and convex subset of R^n, (b) for every $x \in X$, $\{x' \in X \mid x'Gx\}$ and $\{x' \in X \mid xGx'\}$ are closed in R^n.

strict
→ strong

(i) Prove that the strict convexity of G implies the convexity of G.
(ii) Prove that the convexity of G implies the weak convexity of G.
(iii) Prove that if G is strictly convex and weakly monotone, then G is strictly monotone.
(iv) Prove that if G is weakly convex, then for every $x \in X$, $\{x' \in X \mid x'Px\}$ is convex.
(v) Prove that if G is non-saturating and convex, than G is locally non-saturating.

9. Suppose that P is a binary relation on R_+ defined as follows: (xPy iff $u(x) > u(y)$) where $u: R_+ \to R$ is given by $u(x) = x$ for $x \in [0,1]$ $u(x) = 2$ for $x \in [1, 2]$ and $u(x) = x + 1$ for $x \in [2, \infty)$.
Question: Is this preference representable by a continuous (ordinal) utility function?

10. We say that a property is cardinal (or ordinal) iff when the property holds for a utility function u, then the same property holds for every utility function cardinally (resp. ordinally) equivalent to u. Show that concavity is a cardinal property and also show that it is not an ordinal property.

11. Suppose that a function $\phi(x_1, x_2) = f[u(x_1, x_2)]$ and $u(x_1, x_2) = x_1x_2$ where $f: R_+ \to R$. Consider the following two cases for f: (i) $f(x) = x$ for $x \in [0,1)$, $f(x) = 1$ for $x \in [1,2)$ and $f(x) = x - 1$ for $x \in [2, \infty)$; (ii) $f(x) = x$ for $x \in [0, 2)$, $f(x) = -x + 4$ for $x \in [2, 3)$ and $f(x) = x - 2$ for $x \in [3, \infty)$. Check whether or not the following properties hold for ϕ. (a) Satiation; (b) Local nonsatiation; (c) Quasi-concavity.

12. ϕ: $D \to R$, $D \subseteq R$ is said to be strictly monotone increasing if for any real numbers a, $b \in D$, $a > b$ if and only if $\phi(a) \to \phi(b)$. Show that u: $R^n \to R$ is quasi-concave if and only if V: $R^n \to R$ is quasi-concave where $V(x) = \phi(u(x))$ for some strictly monotone increasing ϕ: $D \to R$ with D = Range of u.

13. Let $X \subseteq R^n$ be a convex consumption set. The budget set is defined as $B(p, m) = \{x\colon x \in X \text{ and } p \cdot x \leqq m\}$ for $p \geqq 9$, $m \leqq 0$. Show that for every $(p, m) \leqq 0$, $B(p, m)$ is a convex set.

Chapter 2 Demand

1. Introduction

In this chapter we study the properties of demand functions; those that are implied by utility, or preference, maximization. We present a unified treatment of sensitivity of quantities demanded to changes in prices. We further present an introduction to the partial converse of the above question, i.e. given that demand functions are derived from utility functions, can we recover these utility functions?

In the first section, Sect. 2, we present a basic tool of the analysis; the relation between expenditure minimization and utility maximization. In Sect. 3, we present a general framework for the study of the decomposition of price effects into income effects and substitution effects. We go into a more detailed study of these effects for the case of differentiable demand functions in Sect. 5. In Sect. 4 we study the existence and some of the properties of demand functions derived from utility maximization. In section 6 we present an outline of the problem of recovering utility functions from demand functions derived from them.

2. Utility Maximization and Expenditure Minimization

The relationship between expenditure minimization and utility maximization is one of the most important concepts in the study of consumers' behavior. We present here an exposition of this concept. Our results are more general than the results we actually utilize in subsequent sections. However, the additional generality of the results gives us more flexibility in applying them.

The consumption set of our consumption agent will be a subset of Euclidean n-space, R^n denoted by X. We assume that:

(2.1) (a) X is a nonempty, closed and convex subset of R^n.
(b) X has a lower bound in R^n, i.e., there exists $b \in R^n$ such that for every $x \in X$, $b \leqq x$.

The preference of the consumption agent is assumed to be represented by a real-valued function $u: X \to R$. We define the range of u by:

$$A = \{\alpha \mid u(x) = \alpha \text{ for some } x \in X\}$$

and the set of no-worse-than consumption bundles by: for $\alpha \in R$

$$G(\alpha) = \{x \in X \mid u(x) \geqq \alpha\}.$$

Let R^n_+ and R_+ denote the nonnegative orthant of Euclidean n-space and the set of nonnegative real numbers. The budget set of the consumption agent is defined by: for $p \in R^n_+$, $p \neq 0$ and $m \in R_+$,

$$B(p, m) = X \cap \{x \in R^n \mid p \cdot x \leqq m\}.$$

The relationship between the utility maximization problem and the expenditure minimization problem is summarized as follows.

(2.2) *Theorem:* Assume that the utility function is *locally nonsaturating,* i.e., for every $x \in X$ and for every $\varepsilon > 0$, there exists $x' \in X$ such that $\|x - x'\| < \varepsilon$ and $u(x') > u(x)$. Given $p^0 \geqq 0$, $p^0 \neq 0$ and $m \geqq 0$, if $p^0 \cdot x^0 \leqq m^0$ and $u(x^0) \geqq u(x)$ for every $x \in B(p^0, m^0)$, the $p^0 \cdot x^0 \leqq p^0 \cdot x$ for every $x \in G(\alpha^0)$ where $\alpha^0 = u(x^0)$.

(2.3) *Theorem:* Assume that te utility function is continuous on X. Given $p^0 \geqq 0$, $p^0 \neq 0$ and $\alpha^0 \in A$, if $x^0 \in G(\alpha^0)$ and $p^0 \cdot x^0 \leqq p^0 \cdot x$ for every $x \in G(\alpha^0)$ and also if $m^0 > \inf\{p^0 \cdot x \mid x \in X\}$ where $m^0 = p^0 \cdot x^0$, then $u(x^0) \geqq u(x)$ for every $x \in B(p^0, m^0)$.

Theorem 2.2 states that if the utility function is locally non-saturating then a bundle maximizing the utility function over a budget set is also a bundle that minimizes expenditure over the set of bundles not less preferred than that bundle.

Figure 2.1 shows that local nonsatiation is a nondispensible assumption for this result. In Fig. 2.1, the shaded region represents bundles that are indifferent to one another. Outside the shaded region we have the "usual" preference map. Bundle x^0 maximizes utility over the budget set but it does not minimize expenditure over the no less preferred set, since x^1, in that set yields a lower expenditure.

Theorem (2.3) states that if the *cheaper-point assumption* holds, i.e., there exists a bundle in the consumption set costing strictly less than x^0 at price p^0, then the expenditure minimizing bundle is also utility maximizing. Figure 2.2 gives us an example showing that the cheaper-point assumption is indispensible for this result. This example illustrates what is sometimes called *Arrow's exceptional*

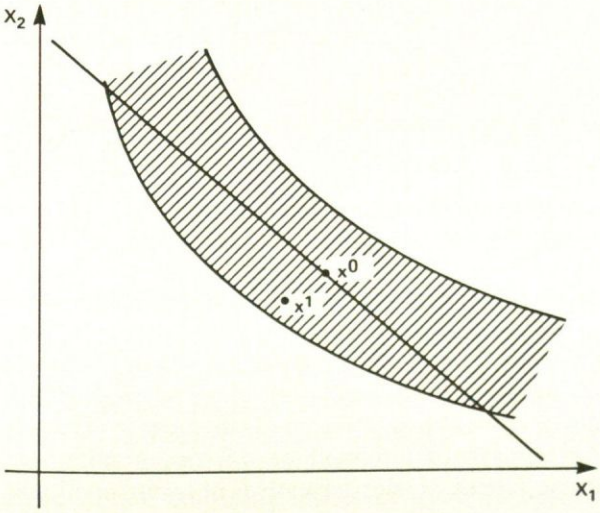

Fig. 2.1

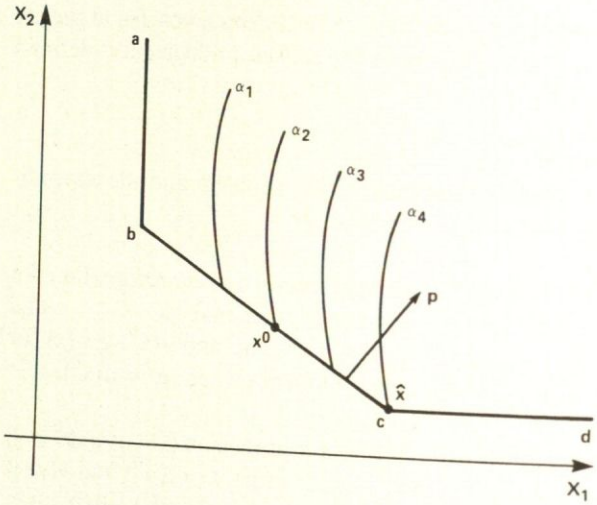

Fig. 2.2

point which was originally introduced in connection with the second classical welfare theorem. In Fig. 2.2, the thick line abcd gives us the lower boundary of the consumption set. Indifference curves indicate utility levels α_i's with $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. When p is the price vector, the budget set is given by the thick line bc. Then x^0 is an expenditure minimizing bundle given $G(\alpha_2)$, but $\hat{x}$ gives a higher utility level.

We can prove (2.2) by way of contradiction. So suppose that there exists a bundle $x^1 \in G(\alpha^0)$ with $p^0 \cdot x^1 < p^0 \cdot x^0$. Then by the continuity of the inner product, we can find an open ball around x^1 with radius $\varepsilon > 0$ so that every bundle in the open ball has its value at p^0 less than our equal to $p^0 \cdot x^0$. Then, by the assumption of local nonsaturation, this open ball must contain a bundle, say $\tilde{x}$, with $u(\tilde{x}) > u(x^1)$ and hence $u(\tilde{x}) > u(x^0)$. But this is a contradiction to the fact that x^0 maximizes $u(x)$ on the budget set $B(p, m)$.

We can also prove (2.3) by way of contradiction. So let us suppose that there exists $x^1 \in B(p^0, m^0)$ which has a higher utility than x^0, i.e., $u(x^1) > u(x^0)$. By the cheaper-point assumption $m^0 > \inf\{p^0 \cdot x \mid x \in X\}$, there must be a bundle x^2 in the consumption set with $p^0 \cdot x^2 < m^0$. We may suppose that $u(x^2) < \alpha^0$, since otherwise we get a contradiction immediately. If we connect x^1 and x^2 by a line segment, then the continuity of u assures that we can find a bundle on that line segment which is indifferent to x^0, but costs strictly less than x^0. This gives us a contradiction.

3. Demand Functions, Substitution Effects and Income Effects

In this section, we define (single-valued) demand functions or ordinary demand functions and provide a basic framework in which the classical analysis of decomposing a price effect into a substitution effect and an income effect is formulated. When we recall the analysis of intermediate microeconomics, the above decomposition should not require us to take derivatives of some functions.

We would like to present here more natural and elementary ways of decomposing a price effect into the above two parts. Also we would like to incorporate both Slutsky's decomposition and Hicks's decomposition into one framework. Most textbooks seem to concentrate heavily on the Hicksian method. But we feel that it is a pity to forget the nice approach by Slutsky which can be tied so well with the revealed preference approach.

In this section, we assume that the consumption set is the nonnegative orthant of Euclidean n-space denoted by R^n_+ and the space of price vectors is the positive orthant of R^n denoted by P^n. Then we call a vector-valued function $h: P^n \times R_+ \to R^n_+$ a *demand function* if it satisfies:

(3.1) (a) h is continuous and positively homogeneous of degree zero in $(p, m) \in P^n \times R_+$, i.e., for every $(p, m) \in P^n \times R_+$ and for every $\lambda \in R$, $\lambda > 0$, $h(\lambda p, \lambda m) = h(p, m)$, and (b) it satis fies the budget constraint as an equality, i.e., for every $(p, m) \in P^n \times R_+$ we have:

$$p \cdot h(p, m) = \sum_{i=1}^{n} p_i h^i(p, m) = m .$$

We can derive a demand function if the consumer in question possesses an appropriate utility function and if he chooses a bundle of goods which maximizes his utility function subject to a competitive budget constraint $B(p, m) = \{x \in R^n_+ \mid p \cdot x \leqq m\}$. Conditions on the utility function to be satisfied are:

(3.2) (a) $u: R^n_+ \to R$ is continuous and strictly quasi-concave on R^n_+; and (b) $u: R^n_+ \to R$ is nonsaturating in $x \in R^n_+$, i.e., for every $x \in R^n_+$, there exists $x' \in R^n_+$ such that $u(x') > u(x)$.

We note that the above two conditions imply that u is locally nonsaturating. When the utility function satisfies the above conditions, it can readily be shown that, for a given $(p, m) \in P^n \times R^n_+$, there exists a unique bundle of goods maximizing $u(x)$ on $B(p, m)$. The existence follows from the strict positivity of p, the continuity of u by applying the Weirstrass maximum and minimum theorem. The uniqueness follows from the strict quasi-concavity of u. Now we may call this bundle $h(p, m)$. To show the positive homogeneity of degree zero of h, it suffices to notice that, for every $(p, m) \in P^n \times R_+$ and for every $\lambda \in R$, $\lambda > 0$, $B(\lambda p, \lambda m) = B(p, m)$. The budget equality $p \cdot h(p, m) = m$ follows simply from the local nonsaturation of u. The only technically difficult part is the proof of the continuity of the demand function. It is known that the possibility of discontinuity of a demand function is also related to Arrow's exceptional point discussed in section 2 of this chapter. For example, suppose that the consumption set is given by $\{(x_1, x_2) \in R^2_+ \mid 0 \leqq x_1 \leqq M\}$ with $M > 1$. Consider a sequence of prices given by $(p_1^v, p_2^v) = (1/n, 1 + (1/n))$ and a sequence of incomes given by $m^v = 1/n$. Then budget lines will rotate as shown in Fig. 3.1. Notice now that we are taking the limit as p_1 goes to zero. Now if the utility function is increasing in x_1, then the optimum bundle at the limit price vector (0, 1) is given by (M, 0). But the point (M, 0) can not be approached by a sequence of bundles in budget sets $B(p^v, m^v)$. Since we are confining ourselves to strictly positive prices, we will be able to

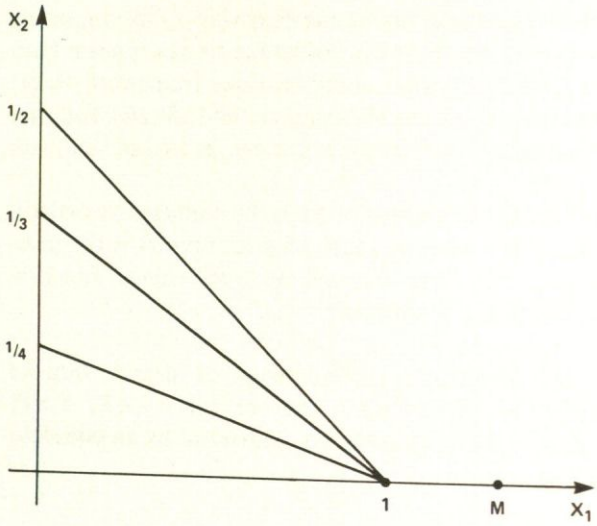

Fig. 3.1

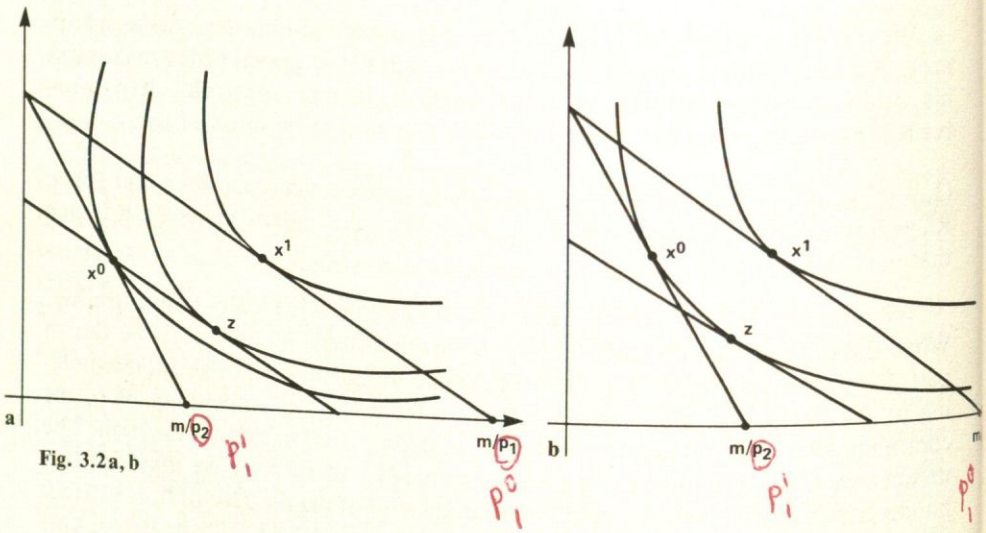

Fig. 3.2a, b

avoid a discontiunity fo this kind. See Debreu (1959) for the proof of the continuity of demand functions.

One of the most fundamental tasks in the theory of consumer's demand is to decompose the effect of a price change into a component called a (pure) substitution effect and a component called an income effect. This decomposition is carried out by choosing an appropriate intermediate bundle. To make it more concrete, let us suppose that $x^0 = h(p^0, m^0)$ and $x^1 = h(p^1, m^0)$ where $p^0/\|p^0\| \neq p^1/\|p^1\|$, i.e., p^0 and p^1 are not proportionate to each other. Let us consider moving from x^0 to x^1 via an intermediate bundle. If we denote that intermediate bundle by z, then this decomposition amounts to the study of: $x^1 - x^0 = (z - x^0) + (x^1 - z)$. According to Slutsky, we choose an intermediate bundle so that it maximizes the utility function satisfying a budget condition

$p \cdot (x^0 - x) \geqq 0$. On the other hand, if we follow the Hicksian approach which seems to be more popular, we choose a bundle which minimizes expenditure $p^1 \cdot x$ subject to a constraint $x \in G(\alpha^0)$ with $\alpha^0 = u(x^0)$. The difference between the two approaches is illustrated in Fig. 3.2 where (a) gives Slutsky's method and (b) gives Hicks's method. When the price of the first good decreases while p_2 and the income remain unchanged. The amount of income needed to let the consumer willingly choose z is usually called "compensated income" and the bundle z "compensated demand." The change from x^0 to z is called a substitution effect and the change from z to x^1 an income effect.

In order to provide a common framework for these two approaches; Slutsky's and Hicks', we will start out with a function μ: $P^n \times A \to R_+$ denoted by $\mu(p, a)$ where A is a certain set of parameters. We call $\mu(p, a)$ a *compensated income* and μ a *compensated income function.* Then we define a *compensated demand function* f: $P^n \times A \to R^n_+$ by $f(p, a) = h(p, \mu(p, a))$. We also assume that the parameter "a" is a function of bundle "x" given by: $a = \tilde{a}(x)$. In order to justify these names, we will impose the following two conditions on the function μ and the function f.

(3.3) If $a^0 = \tilde{a}(x^0)$ and $x^0 = h(p^0, m^0)$, then: (a) $m^0 = \mu(p^0, a^0)$; and (b) for any p^1 with $p^1/\|p^1\| \neq p^0/\|p^0\|$,

$$\mu(p^1, a^0) \leqq p^1 \cdot f(p^0, a^0), \quad \text{and}$$

$$\mu(p^0, a^0) \leqq p^0 \cdot f(p^1, a^0)$$

with at least one strict inequality holding.

The property (a) above requires that, at the initial price p^0 and the initial value of the parameter a; a^0, no income compensation is needed for the consumer to demand x^0. The second property, (b), will give us a result that a change in compensated demand when prices change will lead to what can appropriately be called a substitution effect.

First note that, by the definition of the compensated demand function and by property (a) above we have:

$$f(p^0, a^0) = h(p^0, \mu(p^0, a^0)) = h(p^0, m^0)\,.$$

Also if we let $m^1 = \mu(p^1, a^0)$, then

$$f(p^1, a^0) = h(p^1, \mu(p^1, a^0)) = h(p^1, m^1)\,.$$

Therefore if $x^0 = h(p^0, m^0)$ and $x^1 = h(p^1, m^0)$, then

$$\begin{aligned} x^1 - x^0 &= h(p^1, m^0) - h(p^0, m^0) \\ &= [f(p^1, a^0) - h(p^0, m^0)] + [h(p^1, m^0) - f(p^1, a^0)] \\ &= [f(p^1, a^0) - f(p^0, a^0)] + [h(p^1, m^0) - h(p^1, m^1)]\,. \end{aligned}$$

We shall call the first term $f(p^1, a^0) - f(p^0, a^0)$ in the above expression a *substitution effect* and the second term an *income effect.*

By the property of the budget equality (3.1) (b) and by the definition of the compensated demand function, we have for t = 0, 1,

$$p^t \cdot f(p^t, a^0) = p^t \cdot h(p^t, \mu(p^t, a^0)) = \mu(p^t, a^0) .$$

Therefore the inequalities in (3.3) (b) can be rewritten as:

$$p^1 \cdot [f(p^1, a^0) - f(p^0, a^0)] \leqq 0, \quad \text{and}$$

$$p^0 \cdot [f(p^0, a^0) - f(p^1, a^0)] \leqq 0$$

with at least one strict inequality holding. Thus we get the following basic inequality for the substitution effect: a change in prices and the resulting change in compensated demand are negatively correlated; i.e.

$$[p^1 - p^0] \cdot [f(p^1, a^0) - f(p^0, a^0)] < 0 .$$

When the price of commodity i changes while all other prices remain fixed, i.e., $p_i^0 \neq p_i^1$ and $p_j^0 = p_j^1$ for $j \neq i$, then we have the usual law of demand holding with respect to the ith compensated demand, i.e.,

$$[p_i^1 - p_i^0][f^i(p^1, a^0) - f^i(p^0, a^0)] < 0 .$$

Let us state these two results as a proposition.

(3.4) *Proposition:* If $a^0 = \tilde{a}(x^0)$, $x^0 = h(p^0, m^0)$, and $p^0/\|p^0\| \neq p^1/\|p^1\|$, then we have (a) $h(p^1, m^0) - h(p^0, m^0) = [f(p^1, a^0) - f(p^0, a^0)] + [h(p^1, m^0) - h(p^1, m^1)]$ where $m^1 = \mu(p^1, a^0)$, and

(b) $[p^1 - p^0] \cdot [f(p^1, a^0) - f(p^0, a^0)] < 0$.

We will now show that both Slutsky's approach and Hicks's approach with some additional assumptions yield compensated income functions and compensated demand functions which satisfy (3.3) (a) and (b). We start out with Slutsky's approach. Under Slutsky's method of compensation, the function $\tilde{a}(\cdot)$ is and identity on R_+^n, i.e., $\tilde{a}(x) \equiv x$ and *Slutsky's compensated income function* denoted by M: $P^n \times R_+^n \to R_+$ is simply given by $M(p, x) = p \cdot x$. Then we can define *Slutsky's compensated demand function* denoted by G: $P^n \times R_+^n \to R_+^n$ by $G(p, x) = h(p, M(p, x))$ or $G(p, x) = h(p, p \cdot x)$. We need to add an assumption that if $p^0/\|p^0\| \neq p^1/\|p^1\|$, then $G(p^0, x) \neq G(p^1, x)$. Now let $x^0 = h(p^0, m^0)$ and fix $p^1 \in P^n$ so that $p^0/\|p^0\| \neq p^1/\|p^1\|$. Let $z = G(p^1, x^0) = h(p^1, \ p^1 \cdot x^0)$. Then $p^1 \cdot z = p^1 \cdot x^0$ and $z \neq x^0$. Therefore $u(z) > u(x^0)$. The inequality $u(z) > u(x^0)$ must entail $p^0 \cdot x^0 < p^0 \cdot z$ since x^0 maximizes $u(x)$ on $B(p^0, m^0)$. Thus we get $p^1 \cdot z = p^1 \cdot x^0$ and $p^0 \cdot x^0 < p^0 \cdot z$. (3.3)(b) and (3.3)(a) follow directly from the definition.

Let us now consider the Hicksian approach. Under this approach, the parameter is the level of utility and the function $\tilde{a}(x)$ will be the utility function. The *Hicksian* compensated income function denoted by $m(p, \alpha)$ is defined by the minimum expenditure function as follows:

$$\mu(p, \alpha) = \min\{p \cdot x \,|\, u(x) \geqq \alpha\} .$$

Let the solution of the above expenditure minimization problem be denoted by $g(p, \alpha)$. Then, by Theorem 2.3, $g(p, \alpha)$ also maximizes $u(x)$ on $B(p, m)$ with $m = \mu(p, \alpha)$. Thus we can conclude $g(p, \alpha) = h(p, \mu(p, \alpha))$, namely, $g(p, \alpha)$ is the *Hicksian compensated demand function.* We also add an assumption that if

$p^0/\|p^0\| \neq p^1/\|p^1\|$, then $g(p^0, \alpha) \neq g(p^1, \alpha)$. We now show that (3.3)(a) and (b) are satisfied for the Hicksian case also. If $\alpha^0 = u(x^0)$, and $x^0 = h(p^0, m^0)$, then by Theorem 2.2 x^0 also minimizes $p^0 \cdot x$ on $G(\alpha^0)$ and so $\mu(p^0, \alpha^0) = p^0 \cdot x^0 = p^0 h(p^0, m^0) = m^0$. This proves (3.3)(a). Now to prove (3.3)(b), let $\alpha^0 = u(x^0)$, $x^0 = h(p^0, m^0)$ and $p^1/\|p^1\| \neq p^0/\|p^0\|$. Let $z = g(p^1, \alpha^0)$ and note that $x^0 = h(p^0, m^0) = g(p^0, \alpha^0)$. Then since $x^0 \in G(\alpha^0)$, $z \in G(\alpha^0)$ $x^0 \neq z$, we must have

$$m(p^1, \alpha^0) < p^1 \cdot x^0, \quad \text{and}$$

$$m(p^0, \alpha^0) < p^0 \cdot z$$

which proves (3.2)(b).

It is very interesting to note that the Hicksian approach utilizes the utility function in defining compensated income and demand functions, but the approach by Slutsky does not require the concept of a utility function in defining compensated income and demand functions. In fact in the approach by Slutsky, we can check that the utility function and the utility maximizing behavior are involved only in the proof of the condition (3.3)(b). This is exactly the motivation of Samuelson (1947, for example) in developing the "revealed preference" approach.

If $x^1 = h(p^1, m^1)$, $x^0 \in B(p^1, m^1)$ and $x^1 \neq x^0$, then we say that x^1 is *directely revealed preferred* to x^0, denoted by $x^1 S x^0$. Then Samuelson's *weak axiom of revealed preference* (WARP) can be stated as:

(3.5) (WARP) If the binary relations S defined above is asymmetric, i.e., if $x^1 S x^0$, then it is not the case that $x^0 S x^1$.

The above WARP can be restated more explicitly as follows:

(3.5′) If $x^1 = h(p^1, m^1)$, $p^1 \cdot x^0 \leqq m^1$ and $x^1 \neq x^0$, then $x^0 = h(p^0, m^0)$ for some (p^0, m^0) implies that $p^0 \cdot x^1 > m^0$.

Now following the approach by Slutsky, let us suppose that $x^0 = h(p^0, m^0)$, $z = G(p^1, x^0) = h(p^1, p^1 \cdot x^0)$ and $p^0/\|p^0\| \neq p^1/\|p^1\|$. If we are willing to add a condition that the above entails $z \neq x^0$. Then since $z = h(p^1, m^1)$, $p^1 \cdot x^0 = p^1 \cdot z = m^1$, $z \neq x^0$, and $x^0 = h(p^0, m^0)$, the WARP immediately gives us $p^0 \cdot z > m^0 = p^0 \cdot x^0$. Thus Proposition 3.4 will be valid under the (WARP). The revealed preference approach enables us to replace the concept of utility functions and the utility maximizing behavior by a consistency requirement on the choice function or the demand function of a consumer, namely the (WARP).

4. Differentiable Demand Functions, First-Order and Second-Order Conditions

This section will be more or less a preparatory section for sections that follow. We will introduce conditions on the utility function which generate continuously differentiable demand functions and also compensated demand functions. Also alternative versions of the second-order conditions will be presented. Throughout this section, we shall suppose that $x^0 = h(p^0, m^0) > 0$ for some $(p^0, m^0) > 0$ is

given. Certainly a similar argument can be applied to other bundles chosen at some other $(p, m) > 0$ provided that similar conditions hold there. We shall assume the following first-order and second-order derivative conditions on the utility functions.

(4.1) *Assumption:* (a) $u: R^n_+ \to R$ has continuous first and second derivatives at $x^0 > 0$. (b) $u_i(x^0) \equiv \partial u(x^0)/\partial x_i > 0$ for every $i = 1, 2, \ldots, n$. (c) For every $\eta \in R^n$, $\eta \neq 0$, if $\nabla u(x^0) \cdot \eta = 0$, then $\eta' \nabla^2 u(x^0) \eta < 0$ where $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $\nabla^2 u = [\partial^2 u/\partial x_i \partial x_j]_{i,j=1,\ldots,n}$. We sometimes say that the utility function is *regular* at x^0, if the above conditions (in particular (c)) are satisfied.

First we shall consider the case of the utility maximizing behavior subject to a budget constraint yielding the ordinary demand functions. Since $x^0 \gg 0$ and $p^0 \cdot x^0 = m^0$ under our assumptions, the budget constraint, $m^0 - p^0 x \geqq 0$, is effective. The rank of its Jacobian is one. Thus the first-order necessary conditions* are applicable and there exists $\lambda^0 \in R$ such that:

(4.2) (a) $f^0(\lambda^0, x^0; p^0, m^0) = m^0 - p^0 \cdot x^0 = 0$, and
(b) $f^i(\lambda^0, x^0; p^0, m^0) = u_i(x^0) - \lambda^0 p_i^0 = 0$; $(i = 1, 2, \ldots, n)$.

The second-order sufficient condition for x^0 to be a local maximizer is that $\eta' \nabla^2 u(x^0) \eta < 0$ for every $\eta \in R^n$, $\eta \neq 0$ satisfying $p^0 \cdot \eta = 0$ or equivalently $\nabla u(x^0) \cdot \eta = 0$**. This is clearly satisfied by the regularity condition (4.1)(c) on the utility function. Let $H(x)$ be the bordered Hessian matrix of u at x, i.e., $H(x) = \begin{bmatrix} 0 & , \nabla u(x) \\ [\nabla u(x)]', & \nabla^2 u(x) \end{bmatrix}$. We will show that (4.1)(c) implies the nonsingularity of $H(x^0)$. To do this, let $\eta_0 \in R$, $\eta \in R^n$, and consider:

$$(4.3) \quad H(x^0) \begin{bmatrix} \eta_0 \\ \eta \end{bmatrix} = \begin{bmatrix} [\nabla u(x^0)] \cdot \eta \\ \eta_0 [\nabla u(x^0)]' + \nabla^2 u(x^0) \eta \end{bmatrix} = 0_{n+1}$$

where 0_{n+1} is the $(n+1)$ vector of zeros. Premultiplying $\eta_0 [\nabla u(x^0)]' + \nabla^2 u(x^0) \eta$ by η' and noting $[\nabla u(x^0)] \cdot \eta = 0$ yield:

$$\eta_0 \eta' \cdot [\nabla u(x^0)]' + \eta' \nabla^2 u(x^0) \eta = \eta' \nabla^2 u(x^0) \eta .$$

But the above must be negative unless $\eta = 0$ by (4.1)(c). Consequently $\eta = 0$, and $\eta_0 [\nabla u(x^0)]' + \nabla^2 u(x^0) \eta = \eta_0 [\nabla u(x^0)]' = 0$ gives us $\eta_0 = 0$. Since (4.3) has only the trivial solution, $H(x^0)$ must be nonsingular. Now the Jacobian determinant of the system of Eqs. (4.2) with respect to (λ, x) evaluated at $(\lambda^0, x^0, p^0, m^0)$ becomes as follows:

$$\det \begin{pmatrix} \partial f^0/\partial \lambda, \partial f^0/\partial x_1, \ldots, \partial f^0/\partial x_n \\ \partial f^1/\partial \lambda, \partial f^1/\partial x_1, \ldots, \partial f^1/\partial x_n \\ \vdots \\ \partial f^n/\partial \lambda, \partial f^n/\partial x_1, \ldots, \partial f^n/\partial x_n \end{pmatrix} = \det \begin{bmatrix} 0 & -[p^0]' \\ -p^0, & \nabla^2 u(x^0) \end{bmatrix}$$

* See mathematical appendix, 5.3
** See mathematical appendix, 5.8

$$= \det \begin{bmatrix} 0 & (1/\lambda^0)\,\nabla u(x^0) \\ (1/\lambda^0)\,[\nabla u(x^0)]', & \nabla^2 u(x^0) \end{bmatrix} = (\lambda^0)^{-2} \det H(x^0) \neq 0 .$$

Therefore by the implicit function theorem, there exists a unique continuously differentiable demand function $h(p, m)$ mapping from a neighborhood around (p^0, m^0) into a neighborhood of x^0. When direct demand functions are continuously differentiable around (p^0, m^0), then clearly the Slutsky's compensated demand functions are also continuously differentiable around (p^0, m^0).

Let us now briefly examine the case of the Hicksian compensated demand functions. By a similar reasoning, first-order conditions are applicable, i.e., there exists $\mu^0 \in R$ such that:

(4.4) (a) $\alpha^0 - u(x^0) = 0$, and
(b) $p_i^0 - \mu^0 u_i(x^0) = 0; \quad (i = 1, 2, \ldots, n)$.

The second-order sufficient condition for x^0 to be a local minimizer also turns out to coincide with the regularity condition (4.1)(c). The Jacobian determinant of the system (4.4) with respect to (μ, x) evaluated at $(\mu^0, x^0, p^0, \alpha^0)$ becomes nonzero as follows:

$$\begin{bmatrix} 0 & , -\nabla u(x^0) \\ -\nabla [u(x^0)]', & -\mu^0 \nabla^2 u(x^0) \end{bmatrix} = (-\mu^0)^{n+1} \det H(x^0) \neq 0 .$$

This assures us the continuous differentiability of the Hicksian compensated demand functions $g(p, \alpha)$ in a neighborhood of (p^0, α^0).

We would like to make further digressions on first-order and second-order conditions by converting the above constrained optimization problems into unconstrained problems. We will use the last good or the nth good for the normalization purpose and eliminate x_n using a constraint for each problem. Therefore the following notation will be employed regarding the first $(n-1)$ goods and their normalized prices and the normalized income:

(4.5) $\tilde{x} = (x_1, x_2, \ldots, x_{n-1})'$
$\tilde{p}_i = p_i/p_n$
$\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n-1})'$, and
$\tilde{m} = m/p_n$.

When we solve the budget equality for x_n to get $x_n = (m/p_n) - \sum_{i=1}^{n-1} (p_i/p_n) x_i = \tilde{m} - \sum_{i=1}^{n-1} \tilde{p}_i x_i = \tilde{m} - \tilde{p} \cdot \tilde{x}$, then the utility maximization problem can be rewritten as the one to maximize $V(\tilde{x}; \tilde{p}, \tilde{m})$ with respect to $\tilde{x}$ where $V(\tilde{x}; \tilde{p}, \tilde{m}) = u(\tilde{x}, \tilde{m} - \tilde{p} \cdot \tilde{x})$. Let us denote $V_i = \partial v/\partial x_i$ and $V_{ij} = \partial^2 v/\partial x_i \partial x_j$. Then when the price-income pair is given by $(\tilde{p}^0, \tilde{m}^0)$, the maximizer $\tilde{x}^0$ must satisfy the first-order necessary conditions as follows:

(4.6) $V_i(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0) = u_i(x^0) - u_n(x^0)\tilde{p}_i = 0; \quad (i = 1, 2, \ldots, n-1)$.

The second-order sufficient condition is that the Hessian matrix of V be negative definite at $(\tilde{x}^0, \tilde{p}^0, \tilde{m}^0)$, i.e., for every $\tilde{\eta} = (\eta_1, \eta_2, \ldots, \eta_{n-1})' \in R^{n-1}$, $\tilde{\eta} \neq 0$,

$\tilde{\eta}' \nabla^2 V(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0)\tilde{\eta} = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} V_{ij}(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0)\eta_i\eta_j < 0$. When we partially differentiate (4.6) with respect to x_j, we can obtain that:

$$(4.7)\quad V_{ij}(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0) = u_{ij}(x^0) - \tilde{p}_j^0 u_{in}(x^0) - \tilde{p}_i^0 u_{nj}(x^0) + \tilde{p}_i\tilde{p}_j u_{nn}(x^0)$$

$$= (1, -\tilde{p}_i)\begin{bmatrix} u_{ij}(x^0), & u_{in}(x^0) \\ u_{nj}(x^0), & u_{nn}(x^0) \end{bmatrix}\begin{bmatrix} 1 \\ -\tilde{p}_j \end{bmatrix}.$$

If we define $\eta_n = -\sum_{i=1}^{n-1}\tilde{p}_i^0\eta_i = -\sum_{i=1}^{n-1}(p_i/p_n)\eta_i$, i.e., $p^0 \cdot \eta = \sum_{i=1}^{n} p_i^0\eta_i = 0$ where $\eta' = (\eta_1, \eta_2, \ldots, \eta_n)$, the the computation shows that:

$$\tilde{\eta}' \nabla^2 V(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0)\tilde{\eta} = \eta' \nabla^2 u(x^0)\eta .$$

This confirms for us that the above second-order condition with respect to V is identical to the previous second-order condition, i.e., the regularity condition of u at x^0 given in (4.1)(c). Let us define the ith marginal rate of substitution in terms of the nth good by:

$$(4.8)\quad \pi^i(x) = u_i(x)/u_n(x)\,; \qquad (i = 1, 2, \ldots, n-1)\,.$$

Since, along a budget plane, the nth good is restricted to $x_n = \tilde{m} - \tilde{p} \cdot \tilde{x}$, the ith marginal rate of substitution becomes:

$$(4.9)\quad \gamma^i(\tilde{x}; \tilde{p}, \tilde{m}) = \pi^i(\tilde{x}, \tilde{m} - \tilde{p} \cdot \tilde{x})\,; \qquad (i = 1, 2, \ldots, n-1)\,.$$

Then the first-order conditions (4.6) can be rewritten as:

$$(4.10)\quad \gamma^i(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0) - \tilde{p}_i^0 = 0\,; \qquad (i = 1, 2, \ldots, n-1)\,.$$

The ordinary demand functions $x_1 = h^i(\tilde{p}, 1, \tilde{m}) = h^i(p_1/p_n, \ldots, p_{n-1}/p_n, 1, m/p_n)$; $(i = 1, 2, \ldots, n-1)$ must satisfy the system of Eqs. (4.10) at least in a neighborhood of $(\tilde{p}^0, \tilde{m}^0)$. This fact can be utilized in obtaining some derivative properties of the ordinary demand functions as we do in the next section. For the use in the next section, we will record some derivative properties of π^i and γ^i where a subscript j indicates a partial derivative with respect to x_j:

$$(4.11)\quad \text{(a) } \pi_j^i(x) = [1/u_n(x)]\,[u_{ij}(x) - \pi^i(x)\,u_{nj}(x)]\,;$$

$$(i = 1, 2, \ldots, n-1 \quad \text{and} \quad j = 1, 2, \ldots, n)\,,$$

$$\text{(b) } \gamma_j^i(\tilde{x}; \tilde{p}, \tilde{m}) = \pi_j^i(x) - \pi^j(x)\,\pi_n^i(x)\,|_{x_n = \tilde{m} - \tilde{p} \cdot \tilde{x}}$$

$$= (1/u_n(x))\,V_{ij}(\tilde{x}; \tilde{p}, \tilde{m})\,|_{x_n = \tilde{m} - \tilde{p} \cdot \tilde{x}}\,.$$

We next reformulate the expenditure minimization problem as an unconstrained minimization problem. Since the marginal utility of the nth good is always positive on the positive orthant, we can solve $u(\tilde{x}, x_n) = \alpha$ for x_n to obtain a function $f\colon P^{n-1} \times A \to P$ such that, for every $(\tilde{x}, \alpha) \in P^{n-1} \times A$,

$$(4.12)\quad u(\tilde{x}, f(\tilde{x}, \alpha)) \equiv \alpha$$

holds were A, a subset of R, is the range of the utility function. It is easy to check that $f_i(\tilde{x}, \alpha) = \partial f(\tilde{x}, \alpha)/\partial x_i = -u_i(x)/u_n(x)\,|_{x_n = f(\tilde{x}, \alpha)} = -\pi^i(x)\,|_{x_n = f(\tilde{x}, \alpha)}$. It is also convenient to define the ith marginal rate of substitution along an indifference surface by:

$$(4.13) \quad \beta^i(\tilde{x}, \alpha) \equiv \pi^i(\tilde{x}, f(\tilde{x}, \alpha))\,; \qquad (i = 1, 2, \ldots, n-1)\,.$$

The partial derivative of β^i with respect to x_j becomes:

$$(4.14) \quad \beta^i_j(\tilde{x}, \alpha) = \pi^i_j(x) - \pi^j(x)\,\pi^i_n(x)\,|_{x_n = f(\tilde{x}, \alpha)}\,.$$

The expenditure minimization problem can be restated as the problem of minimizing $E(\tilde{x}; \tilde{p}, \alpha)$ with respect to $\tilde{x}$ where $E(\tilde{x}; \tilde{p}, \alpha) = \tilde{p} \cdot \tilde{x} + f(\tilde{x}, \alpha)$. Given $(\tilde{p}^0, \alpha^0)$, the first-order conditions at $\tilde{x}^0$ become:

$$(4.15) \quad E_i(\tilde{x}^0; \tilde{p}^0, \alpha^0) = \tilde{p}^0_i - \beta^i(\tilde{x}^0, \alpha^0) = 0\,; \qquad (i = 1, 2, \ldots, n-1)\,.$$

Again, at least in a neighborhood of $(\tilde{p}^0, \alpha^0)$, the Hicksian compensated demand functions $x_i = g^i(\tilde{p}, 1, \alpha)$, $(i = 1, 2, \ldots, n-1)$ must satisfy the system of equations given by (4.15). The second-order sufficient condition for $\tilde{x}^0$ to be a local minimizer is the positive definiteness of the Hessian matrix of E with respect to $\tilde{x}$ evaluated at $(\tilde{x}^0, \tilde{p}^0, \alpha^0)$, i.e., for every $\tilde{\eta} = (\eta_1, \eta_2, \ldots, \eta_{n-1}) \in R^{n-1}$, $\tilde{\eta} \neq 0$, $\tilde{\eta}' \nabla^2 E(\tilde{x}^0, \tilde{p}^0, \alpha^0)\,\tilde{\eta} > 0$. We note that

$$E_{ij}(\tilde{x}^0; \tilde{p}^0, \alpha^0) = -\beta^i_j(\tilde{x}^0, \alpha^0), \quad \text{and}$$

$$\begin{aligned}\beta^i_j(\tilde{x}^0, \alpha^0) &= \pi^i_j(x^0) - \pi^j(x^0)\,\pi^i_n(x^0)\\ &= [1/u_n(x^0)]\,\{u_{ij}(x^0) - \pi^i(x^0)\,u_{nj}(x^0) - \pi^j(x^0)\,u_{in}(x^0)\\ &\quad + \pi^i(x^0)\,\pi^j(x^0)\,u_{nn}(x^0)\}\,.\end{aligned}$$

Therefore we can obtain that $\tilde{\eta}' \nabla^2 E(\tilde{x}^0; \tilde{p}^0, \alpha^0)\,\tilde{\eta} = -\eta' \nabla^2 u(x^0)\,\eta$ where $\eta' = (\tilde{\eta}, \eta_n)$ and $\eta_n = -\sum_{i=1}^{n-1} \cancel{\eta}^i(x^0)\,\eta_i$, i.e., $\sum_{i=1}^{n} u_i(x^0)\,\eta_i = \nabla u(x_0) \cdot \eta = 0$. Hence the second order condition once more coincides with the regularity condition (4.1)(c).

At the conclusion of this section we deal with the case where we allow the budget constraint to hold as an inequality and where some of the componets of x can be zeros. Let x^0_i be a local solution to the problem of maximizing $u(x)$ subject to $p \cdot x \leqq m$ and $x \geqq 0$. As we did in the case of equality constraints we first get our characterization by obtaining an explicit expression for solutions to the constraints and then use the method of Lagrange multipliers to get the same result.

If a point x belongs to the budget set, i.e. is as solution of $p \cdot x \leqq m$, $x \geqq 0$, then it could be written as:

$$(4.16) \quad x_i = \frac{\alpha_i}{\alpha_0 + \sum_{i=1}^{n} \alpha_i p_i}\, m, \quad i = 1, \ldots, \cancel{m}\,n,$$

where $\alpha_0 + \sum_{i=1}^{n} \alpha_i p_i > 0$, $\alpha_j \geqq 0$ for all $j = 0, 1, \ldots, n$. In vector form we have

$$(4.17) \quad x = \frac{m\alpha}{(1, p) \cdot (\alpha_0, \alpha)},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $(\alpha_0, \alpha) \cdot (1, p) > 0$, $(\alpha_0, \alpha) \geqq 0$. To verify the above assertion multiply by p in (4.17) and get

$$px = (p\alpha/(\alpha_0 + p\alpha))m \leqq m,$$

since $p\alpha/(\alpha_0 + p\alpha) \leqq 1$. We notice that α_0 is zero in the case of equality constraint and $\alpha_0 > 0$ in case of strict inequality. One is tempted to interpret α_0 as unutilized liquidity especially since it's price is 1 in the expression $(1, p) \cdot (\alpha_0, \alpha)$.

Substituting into the utility function, our problem becomes: maximize $v(\alpha_0, \alpha) = u(\alpha m/[(1, p) \cdot (\alpha_0, \alpha)])$ subject to $\alpha_0 + p \cdot \alpha > 0$, $\alpha \geqq 0$. To obtain the first order characterization we use a theorem about extrema with non-negativity constraints, since $\alpha_0 + p \cdot \alpha > 0$ can be locally neglected. Let (α_0^0, α^0) be a solution to our new problem corresponding to x^0. We have:

$$(4.18) \quad v_j^0 \leqq 0, \quad \alpha_j^0 v_j^0 = 0, \quad j = 0, 1, \ldots, n,$$

where $v_j^0 = \partial v/\partial \alpha_j|_{(\alpha_0, \alpha) = (\alpha_0^0, \alpha^0)}$. Calculating v_j in terms u_i we get

$$(4.19) \quad v_0^0 = -\sum_{i=1}^{n} (m\alpha_i^0/[(1, p) \cdot (\alpha_0^0, \alpha_i^0)]^2) u_i^0 \leqq 0, \ \alpha_0^0 v_0^0 = 0,$$

$$(4.20) \quad v_i^0 = (m/(1, p) \cdot (\alpha_0^0, \alpha_0)) u_i^0 - p_i \sum_{k=1}^{n} (m\alpha_k^0/[(1, p) \cdot (\alpha_0^0, \alpha^0)]^2) u_k^0 \leqq 0,$$

$$\alpha_i^0 v_i^0 = 0, \quad i = 1, \ldots, n.$$

Rewriting (4.19) and (4.20) in terms of x^0 and multiplying by $(1, p)(\alpha_0, \alpha)$ which is positive we get

$$(4.21) \quad \sum_{k=1}^{n} u_k^0 x_k^0 \geqq 0, \quad (m - p \cdot x^0)\left(\sum_{k=1}^{n} u_k^0 x_k^0\right) = 0.$$

$$(4.22) \quad u_i^0 \leqq \left(\frac{p_i}{m}\right) \sum_{k=1}^{n} u_k^0 x_k^0, \quad x_i^0\left(u_i^0 - \frac{p_i}{m} \sum_{i=1}^{n} u_k^0 x_k^0\right) = 0.$$

The equations in (4.21) and (4.22) follow from the fact that $\alpha_0 = 0$ if and only if $m = p \cdot x$ and $x_i = 0$ if and only if $\alpha_i = 0$, $i = 1, \ldots, n$. We can write (4.21) and (4.22) as:

$$(4.23) \quad u_m^0 \geqq 0, \quad u_m^0(m - p \cdot x^0) = 0$$

$$(4.24) \quad u_i^0 \leqq u_m^0 p_i, \quad x_i^0(u_i^0 - u_m^0 p_i) = 0,$$

where $u_m^0 = \frac{1}{m} \sum_{k=1}^{n} u_k^0 x_k^0$.

Thus all "income" is spent if the marginal utility of "income" is positive. If some of the "income" is not spent, then the marginal utility of "income" is zero. It is for the above reason that we think m is alloted expenditure rather than income. If all $x_i^0 > 0$ and $u_i^0 > 0$ for at least one good i with a positive p_i, then we have

$$u_i^0 = u_m^0 p_i$$

$$m = p \cdot x^0 .$$

By way of example let u be homogeneous of degree 1, i.e., an ordinal equivalent of homothetic. Then (4.21) and (4.22) yield:

$$u^0 \geqq 0, \quad (m - p \cdot x^0) u^0 = 0$$

$$u_i^0 \leqq \frac{u^0}{m} p_i, \quad x_i^0 \left(u_i^0 - \frac{u^0}{m} p_i \right) = 0 .$$

This follows immediately from Euler's theorem. In case $u(0) = 0$, $x^0 > 0$ and u is monotone we get: $m = p \cdot x_i^0$ and $u_i^0 = (u^0/m) p_i$.

Of course we could have used a constrained extremum theorem to derive (4.23) and (4.24) which we do now. The point of going the unconstrained way, aside from the fact that constraints are killers of joy, is that the process is demystified and some implicit information is made explicit. It certainly makes more sense to start by talking about u_m than by talking about Lagrange multipliers. We now use the theorems on constrained extrema. We assume that u is differentiable and that at least one commodity, say i_0, is such that $p_{i0} > 0$, $x_{i0}^0 > 0$, where x^0 is, naturally, the bundle that solves our problem.

We then know that there exist constant $(\lambda_0, \lambda_1) \geq 0$ such that:

$$(4.25) \quad \lambda_0 u_i^0 \leqq \lambda_1 p_i, \quad x_i^0 (\lambda p_i - \lambda_0 u_i^0) = 0$$

$$(4.26) \quad p \cdot x^0 \leqq m, \quad \lambda_1 (m - p \cdot x^0) = 0 .$$

We show that $\lambda_0 > 0$. If $\lambda_0 = 0$, then $\lambda_1 p_{i0} = 0$, since $\lambda_0 u_{i0}^0 = \lambda_1 p_{i0}$, and so $\lambda_1 = 0$ but this can't be since $(\lambda_0, \lambda_1) \neq 0$. We now divide (4.25), (4.26) by λ_0 and set $\lambda = \mu_0 / \lambda_1$. Then

$$(4.27) \quad u_i^0 \leqq \lambda p_i, \quad x_i^0 (\lambda p_i - u_i) = 0$$

$$(4.28) \quad p \cdot x^0 \leqq m, \quad \lambda (m - p \cdot x^0) = 0 .$$

Summing the equatiuons in (4.27) over i, we get:

$$(4.29) \quad \sum_{i=1}^{n} u_i^0 x_i^0 = \lambda \sum p_i x_i^0 = \lambda m .$$

Thus λ, here too, is the marginal utility of income; u_m^0. We can now express (4.27) and (4.29) in the form of (4.23) and (4.24). One might note that if u is monotone and if $u_{i0}^0 > 0$ for some good i_0, then $m = p \cdot x^0$, i.e., all income is spent. To see this note that monotonicity implies that $u_{i0}^0 \geqq 0$. If $m > p \cdot x^0$, then $\lambda = 0$. Thus $u_{i0}^0 \leqq 0$. But then $u_{i0}^0 = 0$ which contradicts our assumption that $u_{i0}^0 > 0$.

5. The Slutsky Equations, Substitutions Effects and Income Effects under Differentiability

We have shown in Sect. 3 that a price effect on the ordinary demand functions can be decomposed into a substitution effect and an income effect. In this section, this analysis will be carried out further under the assumption of differentiable demand functions and obtain most of the classical results due to Antonelli (1986), Slutsky (1915) and Hicks-Allen (1934). Throughout this section, we obtain various derivative properties evaluated at particular points denoted by x^0, (p^0, m^0) or α^0 where $x^0 = h(p^0, m^0) = G(p^0, x^0) = g(p^0, \alpha^0)$ and $\alpha^0 = u(x^0)$. Certainly our results are also applicable to other points provided that similar assumptions are satisfied there.

Recall that the Slutsky compensated demand function for the ith good is defined by $G^i(p, x^0) = h^i(p, M(p, x^0))$ where $M(p, x^0) = p \cdot x^0$ and $x^0 = h(p^0, m^0)$. If we partially differentite G^i with respect to p_j and evaluate the result at p^0, then we obtain:

(5.1) $\partial G^i(p^0, x^0)/\partial p_j = \partial h^i(p^0, m^0)/\partial p_j + h^j(p^0, m^0)[\partial h^i(p^0, m^0)/\partial m]$

where $m^0 = p^0 \cdot x^0$ and $x_j^0 = h^i(p^0, m^0)$. The right-hand side of (5.1) is called a Slutsky term denoted by $S_{ij}(p, m)$, i.e.,

(5.2) $S_{ij}(p, m) \equiv \partial h^i(p, m)/\partial p_j + h^j(p, m)[\partial h^i(p, m)/\partial m] \quad (i, j = 1, 2, \ldots, n).$

Thus Slutsky terms are nothing but substitution effects under the Slutsky compensation method and the derivative version of the decomposition of a price effect becomes:

$$\partial h^i(p, m)/\partial p_j = S_{ij}(p, m) - h^j(p, m)[\partial h^i(p, m)/\partial m] \quad (i, j = 1,2, \ldots, n)$$

where of course the first term indicates a substitution effect and the second term an income effect.

For the case of the Slutsky compensated income function, it is trivial that $\partial M(p^0, x^0)/\partial p_j = x_j^0 = h^j(p^0, m^0)$. But for the Hicksian compensated income function, it will not be trivial to establish the following interesting property on the partial derivatives of $m(p, \alpha)$ with respect to p_j.

(5.3) *Lemma:* $\partial m(p^0, \alpha^0)/\partial p_j = g^j(p^0, \alpha^0); \quad (j = 1, 2, \ldots, n)$.

The above lemma is in fact an immediate consequence of the envelope theorem stated and proved in the mathematical appendix, 5.1. But, for pedantic reasons, we will provide the following proof due to Karlin (1959).

Define $F(p) = p^0 \cdot g(p, \alpha^0)$. Then since $x^0 = g(p^0, \alpha^0)$ minimizes $p^0 \cdot x$ on $x \in G(\alpha^0)$, we must have $F(p^0) \leqq F(p)$ for every $p \in P^n$. Therefore the first-order condition at p^0 yields

$$\partial F(p^0)/\partial p_j = \sum_{i=1}^{n} p_i^0[\partial g^i(p^0, \alpha^0)/\partial p_j] = 0 .$$

Noting that $m(p, \alpha) = \sum_{i=1}^{n} p_i g^i(p, \alpha)$, we obtain our desired result:

$$\partial m(p^0, \alpha^0)/\partial p_j = g^j(p^0, \alpha^0) + \sum_{i=1}^{n} p_i^0[\partial g^i(p^0, \alpha^0)/\partial p_j] = g^j(p^0, \alpha^0)$$
$$(j = 1, 2, \ldots, n).$$

Since the Hicksian compensated demand function for the ith good is defined by $g^i(p, \alpha) = h^i(p, m(p, \alpha))$; $(i = 1, 2, \ldots, n)$. Lemma 2 and the partial differentiation of the above with respect of p_j evaluated at (p^0, α^0) yield:

$$(5.4) \quad \partial g^i(p^0, \alpha^0)/\partial p_j = \partial h^i(p^0, m^0)/\partial p_j + h^i(p^0, m^0)[\partial h^i(p^0, m^0)/\partial m] \qquad (i, j = 1, 2, \ldots, n)$$

where we use relations $m(p^0, \alpha^0) = m^0$ and $g^j(p^0, \alpha^0) = h^j(p^0, m^0)$. Therefore we have shown that both the Slutsky substitution effects and the Hicksian substitution effects coincide with the Slutsky terms when these substitution effects are measured by first partial derivatives. We state this in the following proposition.

(5.5) *Proposition:* For every $i, j = 1, 2, \ldots, n$

$$\partial G^i(p^0, x^0)/\partial p_j = \partial g^i(p^0, \alpha^0)/\partial p_j = S_{ij}(p^0, m^0)$$

where $x^0 = h(p^0, m^0)$ and $\alpha^0 = u(x^0)$.

The following properties of the Hicksian compensated income function or the minimum expenditure function is very useful in obtaining further results on the Slutsky terms. It may be noted that the differentiability assumption and derivative conditions of the utility function given by (4.1) are not needed for the following theorem.

(5.6) *Theorem:* (a) $m(p, \alpha)$ is positively homogeneous of degree one in p, i.e., for every $\lambda > 0$, $m(\lambda p, \alpha) = \lambda m(p, \alpha)$; (b) if $p^0/\|p^0\| \neq p^1/\|p^1\|$ entails $g(p^0, \alpha) \neq g(p^1, \alpha)$, then for every $p^0, p^1 \in P^n$ with $p^0/\|p^0\| \neq p^1/\|p^1\|$ and for every $t \in (0, 1)$,

$$m(p^t, \alpha) > t\,m(p^1, \alpha) + (1-t)\,m(p^0, \alpha);$$

and (c) $m(p, \alpha)$ is continuous and concave in $p \in P^n$.

The property (a) is a simple consequence of the fact that, for any $\lambda > 0$, $\min\{(\lambda p) \cdot x \mid u(x) \geqq \alpha\} = \lambda \min\{p \cdot x \mid u(x) \geqq \alpha\}$. To prove (b), choose p^0, p^1 so that $p^0/\|p^0\| \neq p^1/\|p^1\|$ and let $p^t = tp^1 + (1-t)p^0$. Then there exist x^0, x^1 and x^t in R^n_+ so that $m(p^0, \alpha) = p^0 \cdot x^0$, $m(p^1, \alpha) = p^1 \cdot x^1$ and $m(p^t, \alpha) = p^t \cdot x^t$. By the minimization principle, we must have:

$$p^0 \cdot x^0 < p^0 \cdot x^t \quad \text{and} \quad p^1 \cdot x^1 < p^1 \cdot x^t$$

since $p^0/\|p^0\| \neq p^t/\|p^t\|$ and $p^1/\|p^1\| \neq p^t/\|p^t\|$ for $t \in (0, 1)$ and consequently $x^0 \neq x^t$ and $x^1 \neq x^t$. Hence we have:

$$tp^1 \cdot x^1 + (1-t)p^0 \cdot x^0 < tp^0 \cdot x^t + (1-t)p^1 \cdot x^t = p^t \cdot x^t$$

which gives us the property (b). The continuity follows from the fact that a concave function is continuous on its relative interior, see last paragraph in Sect. 4 of mathematical appendix. The concavity of $m(p, \alpha)$ in p is an immediate consequence of (b).

We call the $n \times n$ matrix composed of Slutsky terms defined by (5.2) the Slutsky matrix and denote it by $S(p, m)$, i.e.,

$$(5.7)\quad S(p, m) = \begin{bmatrix} S_{11}(p, m), S_{12}(p, m), \ldots, S_{1n}(p, m) \\ \vdots \\ S_{n1}(p, m), S_{n2}(p, m), \ldots, S_{nn}(p, m) \end{bmatrix}.$$

Lemma 5.3, Proposition 5.5, Theorem 5.6, and the differentiability of compensated demand functions together yield the following proposition on the Slutsky matrix.

(5.8) *Proposition:* (a) The Slutsky matrix is symmetric, i.e., $S_{ij}(p^0, m^0) = S_{ji}(p^0, m^0)$; $(i, j = 1, 2, \ldots, n)$; (b) $S(p^0, m^0)p^0 = 0_n$ where 0_n is an $(n \times 1)$ vector of zeros, and (c) the Slutsky matrix is negative semi-definite, i.e., for every $\eta \in R^n$, $\eta' S(p^0, m^0)\eta \leqq 0$.

Since g^i has continuous partial derivatives with respect to p_j, $(j = 1, 2, \ldots, n)$ at (p^0, α^0), Lemma 5.3 gives us that the Hicksian compensated income function has continuous second partial derivatives with respect to p at (p^0, α^0). Therefore cross partial derivatives of $m(p, \alpha)$ with respect to p at (p^0, α^0) must be symmetric as follows:

$$\partial^2 m(p^0, \alpha^0)/\partial p_i \alpha p_j = \partial^2 m(p^0, \alpha^0)/\partial p_j \partial p_i, \quad \text{i.e.,}$$

$$S_{ij}(p^0, m^0) = \partial g^i(p^0, \alpha^0)/\partial p_j = \partial g^j(p^0, \alpha^0)/\partial p_i = S_{ji}(p^0, m^0)\,.$$

The property (b) follows from Euler's Theorem on homogeneous function applied to $g^i(p, \alpha)$, since $g^i(p, \alpha)$ is homogeneous of degree zero in p. The negative semi-definiteness of $S(p^0, m^0)$ follows from the concavity of $m(p, \alpha)$ in p and the fact the Hessian of a differentiable concave function is negative semi-definite.

The negative semi-definiteness of the Slutsky matrix can be regarded as an infinitesimal version of the property of the substitution effect given by Proposition 3.4(b). In order to see the connection, we will show below how the negative semi-definiteness of $S(p^0, m^0)$ can be derived through a limiting argument from the inequality of Proposition 3.4(b) for the case of Slutsky compensated demand functions. Let us define $p(t) = p^0 + t\eta$. Then the inequality becomes

$$(5.9)\quad [p(t) - p^0] \cdot [G(p(t); x^0) - G(p^0, x^0)]$$
$$= \sum_{h=1}^{n} [p_h(t) - p_h^0][G^h(p(t), x^0) - G^h(p^0, x^0)] < 0\,.$$

For each Slutsky compensated demand function G^h, we can apply the mean value theorem to assert that there exists $p^h(t) \in [p^0, p(t)]$ such that

$$G^h(p(t), x^0) - G^h(p^0, x^0) = \nabla G^h(p^h(t), x^0) \cdot [p(t) - p^0]$$

where $[p^0, p(t)]$ indicates the line segment joining p^0 and $p(t)$. Note that the mean value theorem can not be applied to G, but only to each G^h. Define an $(n \times n)$ matrix S^t by:

$$S^t = [\nabla G^1(p^1(t), x^0), \ldots, \nabla G^n(p^n(t), x^0)]\,.$$

Then we get

$$[p(t) - p^0] \cdot [G(p(t), x^0) - G(p^0, x^0)]$$
$$= [p(t) - p^0]' S^t [p(t) - p^0] = t^2 \eta' S^t \eta < 0\,,$$

i.e., $\eta' S^t \eta < 0$ for every $\eta \in R^n$, $\eta \neq 0$. Using the continuity of ∇G^h in p, we have that $\lim_{t \to 0} \nabla G^h(p^h(t), x^0) = \nabla G^h(p^0, x^0)$, and hence $\lim_{t \to 0} S^t = S(p^0, m^0)$. Therefore we get the result that, for every $\eta \in R^n$, $\eta' S(p^0, m^0) \eta \leqq 0$. Recall that the (WARP) also implies the inequality (5.9). Therefore the (WARP) and the differentiability of the Slutsky compensated demand functions also imply the negative semi-definiteness of the Slutsky matrix. For further discussions related to this aspect, see Kihlstrom, Mas-Colell and Sonnenschein (1976).

The dual to the Hicksian compensated income function is the indirect utility function denoted by $v(p, m)$ which is defined to be the maximum value function for the utility maximization problem subject to a budget constraint, i.e.,

$$(5.10) \quad v(p, m) = \max_{x \in R^n_+} \{u(x) \mid m = p \cdot x \geqq 0\}.$$

By the equivalence between the utility maximization problem and the expenditure minimization problem established in Sect. 2, it should be clear that $\alpha^0 = v(p^0, m^0)$ if and only if $m^0 = m(p^0, \alpha^0)$. If we let the Lagrange multiplier of the utility maximization problem given (p^0, m^0) be denoted by $\lambda^0 = \lambda(p^0, m^0)$, then a straight-forward application of the envelope theorem in 5.11 of the mathematical appendix gives us the following derivative properties of the indirect utility function

$$(5.11) \quad \text{(a)} \quad \partial v(p^0, m^0)/\partial p_i = -\lambda(p^0, m^0) h^i(p^0, m^0), \qquad (i = 1, 2, \ldots, n), \text{ and}$$

$$\text{(b)} \quad \partial v(p^0, m^0)/\partial m = \lambda(p^0, m^0).$$

The relationship $\partial v(p^0, m^0)/\partial p_i = -[\partial v(p^0, m^0)/\partial m] h^i(p^0, m^0)$, $(i = 1, 2, \ldots, n)$, is called *Roy's identity* in the literature.

We would like to obtain further properties of Slutsky terms, in particular, sharpening the results of Proposition 5.8 and also investigate some properties of income effects. Recall that the ordinary demand functions $x_i = h^i(\tilde{p}, 1, \tilde{m})$, $(i = 1, 2, \ldots, n-1)$, must satisfy (4.10) at least in a neighborhood of $(\tilde{p}^0, \tilde{m}^0)$ and the Hicksian compensated demand functions $x_i = g^i(\tilde{p}, 1, \alpha)$, $(i = 1, 2, \ldots, n-1)$, must satisfy (4.15). Let us reproduce them in the following:

$$(5.12) \quad \gamma^i(\tilde{x}; \tilde{p}, \tilde{m}) - \tilde{p}_i = 0, \qquad (i = 1, 2, \ldots, n-1), \text{ and}$$

$$(5.13) \quad \beta^i(\tilde{x}, \alpha) - \tilde{p}_i = 0, \qquad (i = 1, 2, \ldots, n-1).$$

Also recall that $\gamma^i_j(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0) = \pi^i_j(x^0) - \pi^j(x^0)\pi^i_n(x^0) = (1/u_n(x^0)) V_{ij}(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0)$ and $\beta^i_j(\tilde{x}^0, \alpha^0) = \pi^i_j(x^0) - \pi^j(x^0)\pi^i_n(x^0) = -E_{ij}(\tilde{x}^0; \tilde{p}^0, \alpha^0)$. We call the expressions $\pi^i_j(x) - \pi^j(x)\pi^i_n(x)$ Antonelli terms denoted by $A_{ij}(x)$ and the $(n-1) \times (n-1)$ matrix formed by them the Antonelli matrix denoted by $A(x)$, i.e.,

$$A_{ij}(x) = \pi^i_j(x) - \pi^j(x)\pi^i_n(x), \quad (i = 1, 2, \ldots, n-1 \text{ and } j = 1, 2, \ldots, n-1),$$

$$A(x) = \begin{bmatrix} A_{11}(x), \ldots, A_{1,n-1}(x) \\ \vdots \\ A_{n-1,1}(x), \ldots, A_{n-1,n-1}(x) \end{bmatrix}.$$

Clearly $A_{ij}(x) = \gamma^i_j(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0) = \beta^i_j(\tilde{x}^0; \alpha^0) = (1/u_n(x^0)) V_{ij}(\tilde{x}^0; \tilde{p}^0, \tilde{m}^0)$ $= -E_{ij}(\tilde{x}^0; \tilde{p}^0, \alpha^0)$. Therefore we can immediately assert:

(5.14) *Theorem:* The Antonelli matrix $A(x^0)$ is symmetric and negative definite.

Let us denote the $(n-1)\times(n-1)$ northwest principal minor matrix of $S(p, m)$ by $S^n(p, m)$ formed by deleting the nth row and the nth column from $S(p, m)$, i.e.,

$$S^n(p, m) = \begin{bmatrix} S_{11}(p, m), \ldots, S_{1,n-1}(p, m) \\ \vdots \\ S_{n-1,1}(p, m), \ldots, S_{n-1,n-1}(p, m) \end{bmatrix}.$$

Since $g^i_j(p, \alpha)$ is homogeneous of degree -1, we have $g^i_j(\tilde{p}, 1, \alpha) = g^i_j(p_1/p_n, \ldots, p_{n-1}/p_n, 1, \alpha) = p_n g^i_j(p, \alpha)$. Therefore $S(\tilde{p}, 1, \tilde{m}) = p_n S(p, m)$. Now since the Hicksian compensated demand functions $x_i = g^i(\tilde{p}, 1, \alpha)$, where $\alpha = v(\tilde{p}, 1, \tilde{m})$, $(i = 1, 2, \ldots, n-1)$ must satisfay (5.13), $\beta^i_j(\tilde{x}^0; \alpha^0) = A_{ij}(x^0)$ and $g^i_j(\tilde{p}, 1, \tilde{m}) = S_{ij}(\tilde{p}, 1, \tilde{m})$, $(i, j = 1, 2, \ldots, n-1)$, we can apply the inverse function theorem to (5.13) and obtain:

(5.15) *Theorem:* $S^n(\tilde{p}^0, 1, \tilde{m}^0) = p^0_n S^n(p^0, m^0) = [A(x^0)]^{-1}$.

Now since the inverse of a symmetric negative definite matrix is also symmetric and negative definite, we can obtain the following Proposition sharpening Proposition 5.8 as a corollary to Theorem 5.15.

(5.16) *Proposition:* (a) $S^n(p^0, m^0)$ is symmetric and negative definite, and (b)

$$(-1)^r \begin{vmatrix} S_{11}(p^0, m^0), \ldots, S_{1r}(p^0, m^0) \\ \vdots \\ S_{r1}(p^0, m^0), \ldots, S_{rr}(p^0, m^0) \end{vmatrix} > 0, \quad (r = 1, 2, \ldots, n-1).$$

In particular, Proposition 5.16 (b) implies that the own price effect on the ith compensated demand function is strictly negative, i.e., $\partial g^i(p^0, \alpha)/\partial p_i < 0$.

The above analysis was carried out by eliminating the variable x_n and using the nth good to normalize prices. The following mathematical theorem shows us that we could use any variable to eliminate and it provides us with alternative versions of stating the negative definiteness of the Slutsky matrix.

(5.17) *Theorem:* Let S be an $(n\times n)$ real symmetric matrix and S^i be an $(n-1)\times(n-1)$ principal minor matrix of S formed by deleting the ith row and the ith column from S. For a given $p\in R^n$, denote $[p]^\perp = \{\eta\in R^n | \eta\cdot p = 0\}$. Assume that there exists $p\in P^n$ such that (i) $Sp = 0_n$, and (ii) S is negative semi-definite. Then the following are equivalent:

(a) for every $i\in\{1, 2, \ldots, n\}$, S^i is negative definite;
(b) there exists $i\in\{1, 2, \ldots, n\}$ such that S^i is negative definite;
(c) for every $\eta\in R^n$, if $\eta \neq 0_n$ and $\eta/\|\eta\| \neq p/\|p\|$, then $\eta' S\eta < 0$; and
(d) for every $\eta\in R^n$, if $\eta \neq 0_n$ and $\eta\in[p]^\perp$, then $\eta' S\eta < 0$, i.e., S is negative definite on the subspace $[p]^\perp$.

Obviously (a) implies (b) and (c) implies (d). Thus it suffices to prove that (b) implies (c) and (d) implies (a).

Suppose that (b) holds. Because $p\in P^n$ and $p\cdot e^i = p_i > 0$, we have that $p\notin[e^i]^\perp$ where $[e^i]^\perp = \{\eta\in R^n | \eta\cdot e^i = 0\}$. Hence we have that $R^n = [p]\oplus[e^i]^\perp$

where [p] is a linear subspace generated by p and $\oplus$ indicates a direct sum. Choose $\eta \in R^n$ so that $\eta \neq 0$ and $\eta/\|\eta\| \neq p/\|p\|$. Then we can decompose η so that $\eta = \eta^1 + \eta^2$ for some $\eta^1 \in [p]$ and $\eta^2 \in [e^i]^\perp$. Since $\eta/\|\eta\| \neq p/\|p\|$ and $\eta \neq 0_n$, we have that $\eta^1 \neq 0_n$. Define $\mu \in R^{n-1}$ so that $\mu_j = \eta_j^2$ for $j < i$ and $\mu_j = \eta_{j-1}^2$ for $j > i$. Thus, since $S\eta^1 = 0$ and $\eta_i^2 = 0$, $\eta' S \eta = \eta^{2\prime} S \eta^2 = \mu' S^i \mu < 0$. Hence (c) is shown.

Now suppose that (d) holds. Fix any $i \in \{1, 2, \ldots, n\}$ and $\mu \in R^{n-1}$ so that $\mu \neq 0_{n-1}$. Define $\eta \in R^n$ by $\eta_i = 0$, $\eta_j = \mu_j$ for $j < i$ and $\eta_j = \mu_{j+1}$ for $j > i$. Then $\eta \neq 0$ and $\eta/\|\eta\| \neq p/\|p\|$. Since $R^n = [p] \oplus [p]^\perp$, we can write $\eta = \eta^1 + \eta^2$ for some $\eta^1 \in [p]$ and $\eta^2 \in [p]^\perp$. Clearly $\eta^2 \neq 0_n$. Thus $\mu' S^i \mu = \eta' S \eta = \eta^{2\prime} S \eta^2 < 0$ because $S\eta^1 = 0_n$. This proves (a). Since, as shown in Proposition 5.16, the property (b) in Theorem 5.17 holds for the Slutsky matrix. Therefore (a), (c) and (d) are also true for the Slutsky matrix.

So far, we have used the system of equations given by (5.13) to obtain further properties of the Slutsky matrix. We next look at the system of equations given by (5.12) which the ordinary demand functions $x_i = h^i(\tilde{p}, 1, \tilde{m})$ $(i = 1, 2, \ldots, n-1)$ must satisfy. If we apply the implicit function theorem or more particularly the implicit differentiation rule to (5.12), we can obtain:

(5.18)
$$\begin{bmatrix} \partial h^1/\partial p_1, & \ldots, \partial h^1/\partial p_{n-1} \\ \vdots & \vdots \\ \partial h^{n-1}/\partial p_1, & \ldots, \partial h^{n-1}/\partial p_{n-1} \end{bmatrix} = [A(x^0)]^{-1} + [A(x^0)]^{-1} \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix} (\tilde{x}^0)',$$

and

$$\begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^{n-1}/\partial m \end{bmatrix} = -[A(x^0)]^{-1} \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix} \tag{5.19}$$

with $\partial h^i/\partial p_j$ and $\partial h^i/\partial m$ evaluated at $(\tilde{p}^0, 1, \tilde{m})$. Note that in deriving the above we have used relations $\partial \gamma^i/\partial p_j|_{(\tilde{x}^0, \tilde{p}^0, \tilde{m}^0)} = \pi_n^i(x^0)$ obtained by (4.9). When we substitute (5.19) into (5.18), we can again obtain a system of Slutsky equations as follows:

$$\begin{bmatrix} \partial h^1/\partial p_1, & \ldots, \partial h^1/\partial p_{n-1} \\ \vdots & \\ \partial h^{n-1}/\partial p_1, & \ldots, \partial h^{n-1}/\partial p_{n-1} \end{bmatrix} = [A(x^0)]^{-1} - \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^{n-1}/\partial m \end{bmatrix} (\tilde{x}^0)'$$

where $\partial h^i/\partial p_j$ and $\partial h^i/\partial m$ are evaluated at $(\tilde{p}^0, 1, \tilde{m}^0)$.

Hicks and Allen us partial derivatives of compensated demand functions or Slutsky substitution terms in defining substitutes and complements. According to the Hicks-Allen definition, for $i \neq j$, commodity i is said to be a *substitute* to commodity j at (p, m) if and only if $S_{ij}(p, m) > 0$, a *complement* at (p, m) if and only if $S_{ij}(p, m) < 0$ and *neutral* at (p, m) if and only if $S_{ij}(p, m) = 0$. By (5.5) (b), this definition is symmetric in i and j. On the other hand, the definition of substitutes and complements using signs of ordinary demand functions does not possess this symmetry due to income effects.

The Hicks-Allen definition is also invariant under a monotone increasing transformation of a utility function. The easiest way to show this is to note that

(1) x^0 minimizes $p^0 \cdot x$ subject to $u(x) \geqq \alpha^0$ if and only if (2) x^0 minimizes $p^0 \cdot x$ subject to $\phi(u(x)) \geqq \phi(\alpha^0)$ where ϕ is a monotone increasing transformation of real numbers. Let $g^i(p, \alpha)$ be a compensated demand function for i obtained from the problem (1) and $\tilde{g}(p, \phi(\alpha))$ be a compensated demand function for i obtained from the problem (2). Then for every $p > 0$ and α, we have

$$g^i(p, \alpha) \equiv \tilde{g}^i(p, \phi(\alpha)) .$$

Therefore we have

$$\frac{\partial g^i(p, \alpha)}{\partial p_j} = \frac{\partial \tilde{g}^i}{\partial p_j}(p, \phi(\alpha)) .$$

This shows that Slutsky substitution terms are independent of the utility index we use.

From (5.8)(b), we can obtain

$$\sum_{j \neq i} S_{ij}(p, m) p_j = -S_{ii}(p, m) p_i, \qquad (i = 1, 2, \ldots, n) .$$

Since $S_{ii}(p, m) < 0$ in general, we can conclude that for every (p, m) and for every i, there exists a commodity which is a substitute for commodity i. (5.16)(b) places various restrictions on the Slutsky substitution terms but they are not too easy to apply except for $r = 2$. When $r = 2$, we get for $i \neq j$

$$S_{ii}S_{jj} - S_{ij}S_{ji} > 0, \qquad \text{i.e.,}$$

$$S_{ii}S_{jj} > (S_{ij})^2$$

which gives us an upper bound for the absolute magnitude of (S_{ij}).

When we are dealing with a model or consumer's choice with $n = 2$, presumably by some aggregation, then it is clear that $S_{12} > 0$ provided that $S_{11} < 0$. These two commodities are always substitutes under a regularity condition $S_{11} < 0$. Suppose that we have a utility function of the Leontief type, i.e., $u(x_1, x_2) = \min \{x_1/a_1, x_2/a_2\}$. Then we can derive demand functions and compensated demand functions which are differentiable for $(p_1, p_2) > 0$. It is clear that we get $S_{ij}(p, m) = 0$ $(i, j = 1, 2)$ for every $(p_1, p_2) > 0$ and $m \geqq 0$. Therefore if we apply the Hicks-Allen definition, two commodities would be called neutral. It could be noted however that since the Leontief-type functions are not differential, this case does not fit our results quite well.

Edgeworth and Pareto among others use cross partial derivatives of the utility function in defining complements or substitutes. According to this definition, commodity i is a substitute for commodity j at x if $u_{ij}(x) < 0$ and commodity i is a complement to commodity j at x if $u_{ij}(x) > 0$. Following Chipman (1977), we call these the ALEP (*Auspitz-Lieben-Edgeworth-Pareto*) definition. The ALEP definition of complements or substitutes does not possess the independency of a utility index used. For example, consider the following three utility functions for $x_1 \geqq 0$ and $x_2 \geqq 0$ which clearly represent the same preference ordering:

$$u^1(x_1, x_2) = (x_1 + 1)(x_2 + 1)$$

$$u^2(x_1, x_2) = \log u^1(x_1, x_2), \quad \text{and}$$

$$u^3(x_1, x_2) = \{u^2(x_1, x_2)\}^{1/2}$$

noting that $\log(x_i+1) \geqq 0$ for $x_i \geqq 0$. Then we can easily check that for every $x_1 > 0$ and $x_2 > 0$, $\partial^2 u^1/\partial x_1 \partial x_2 = 1$, $\partial^2 u^2/\partial x_1 \partial x_2 = 0$, and $\partial^2 u^3/\partial x_1 \partial x_2 < 0$.

When a preference ordering has a concave utility representation, then Kannai (1977) proposes the following definition of substitutes or complements using the least concave utility function. A utility function $u(x)$ is said to be the *least concave utility function* representing a preference ordering if, for any other concave utility function representing the preference ordering, there exists F mapping from the range of u into R such that $F' > 0$, $F'' < 0$ and $v(x) = F(u(x))$. Since $v(x) = F(u(x))$, we have that:

$$v_{ij} = F'\{u_{ij} + (F''/F')u_i u_j\}.$$

Therefore $v_{ij} > 0$ entails $u_{ij} > 0$ and also $u_{ij} < 0$ implies $v_{ij} < 0$ for any concave representation v. Because of these facts, Kannai defines that commodity i and commodity j are substitutes for each other if $u_{ij} < 0$ and commodity i and commodity j are complements to each other if $u_{ij} > 0$.

Let us now consider income effects. When $\partial h^i(p, m)/\partial m > 0$, then commodity i is said to be a *normal good,* and when $\partial h^i(p, m)/\partial m < 0$, then commodity i is said to be an *inferior good.* In various areas of economics, whether a good is inferior or not turns out to be crucial to obtaining certain results, e.g., in comparative static analyses. We will try to investigate properties of income terms to find out some conditions under which goods become normal or inferior.

Using the budget equality condition, we can immediately obtain the following result.

$$(5.20) \quad \sum_{j=1}^{n} p_j \partial h^j(p, m)/\partial m = 1 \quad \text{or}$$

$$(5.20)' \quad \sum_{j=1}^{n-1} \tilde{p}_j \partial \tilde{h}^j(\tilde{p}, 1, \tilde{m})/\partial m + \partial h^n(\tilde{p}, 1, \tilde{m})/\partial m = 1$$

where $p_n \partial h^j(p, m)/\partial m = \partial h^j(\tilde{p}, 1, \tilde{m})/\partial m, \quad (j = 1, 2, \ldots, n)$.
Thus we can draw an obvious conclusion that all goods can not be inferior. From (5.19), we can get

$$(5.21) \quad \partial h^i(\tilde{p}^0, 1, \tilde{m}^0)/\partial m = -\sum_{j=1}^{n-1} S_{ij}(\tilde{p}^0, 1, \tilde{m}^0)\pi_n^j(x^0) \quad (i = 1, 2, \ldots, n-1).$$

Using (5.20)′ and (5.21), we can derive

$$\begin{aligned}
\partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m &= 1 - \sum_{i=1}^{n-1} \tilde{p}_i^0 \partial h^i(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \\
&= 1 + \sum_{j=1}^{n-1} \left\{ \sum_{i=1}^{n-1} \tilde{p}_i^0 S_{ij}(\tilde{p}^0, 1, \tilde{m}^0) \right\} \pi_n^j(x^0) \\
&= 1 - \sum_{j=1}^{n-1} S_{nj}(\tilde{p}^0, 1, \tilde{m}^0)\pi_n^j(x^0)
\end{aligned}$$

where we have used the fact $\sum_{i=1}^{n-1} \tilde{p}_i^0 S_{ij}(\tilde{p}^0, 1, \tilde{m}^0) + S_{nj}(\tilde{p}^0, 1, \tilde{m}^0) = 0$.

Therefore we summarize what we have obtained in a matrix form as follows:

$$(5.22)\qquad \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^n/\partial m \end{bmatrix} = e_n - \begin{bmatrix} S_{11}, & \ldots, S_{1,n-1} \\ \vdots & \\ S_{n,1}, & \ldots, S_{n,n-1} \end{bmatrix} \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix}$$

where $\partial h^i/\partial m$ and S_{ij} are evaluated at $(\tilde{p}^0, 1, \tilde{m}^0)$. Noting that $\partial h^i(\tilde{p}^0, 1, \tilde{m}^0)/\partial m = p_n^0 \partial h^i(p^0, m^0)/\partial m$ and $S_{ij}(\tilde{p}^0, 1, \tilde{m}^0) = p_n^0 S_{ij}(p^0, m^0)$, we can rewrite (5.22) as follows:

$$(5.22)'\qquad \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^n/\partial m \end{bmatrix} = (1/p_n^0)e_n - \begin{bmatrix} S_{11}, & \ldots, S_{1,n-1} \\ \vdots & \\ S_{n,1}, & \ldots, S_{m,n-1} \end{bmatrix} \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix}$$

with $\partial h^i/\partial m$ and S_{ij} evaluated at (p^0, m^0).

We can get some implications from (5.22). Suppose that we have a case of parallel indifference curves where the utility function can be written in the form $u(x_1, \ldots, x_{n-1}, x_n) = v(x_1, \ldots, x_{n-1}) + a x_n$ with a positive constant "a". Then $\pi^i(x)$ $(i = 1, 2, \ldots, n-1)$ will be independent of x_n resulting in $\pi_n^i(x^0) = 0$ $(i = 1, 2, \ldots, n-1)$. Hence immediately from (5.22), we get

$$\partial h^i(\tilde{p}^0, 1, \tilde{m}^0)/\partial m = 0 \qquad (i = 1, 2, \ldots, n-1)\,, \qquad \text{and}$$

$$\partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m = p_n \partial h^n(p^0, m^0)/\partial m = 1\,.$$

Another immediate implication we can get is the case of two commodities. When $n = 2$, then from (5.22)′, we can obtain:

$$\partial h^1(p^0, m^0)/\partial m = -S_{11}(p^0, m^0)\,\pi_2^1(x^0), \quad \text{and}$$

$$\partial h^2(p^0, m^0)/\partial m = (1/p_2^0) - S_{21}(p^0, m^0)\,\pi_2^1(x^0)\,.$$

If we interchange commodity indices 1 and 2, we also get

$$\partial h^1(p^0, m^0)/\partial m = (1/p_1^0) - S_{12}(p^0, m^0)\,\pi_1^2(x^0), \quad \text{and}$$

$$\partial h^2(p^0, m^0)/\partial m = -S_{22}(p^0, m^0)\,\pi_1^2(x^0)$$

where $\pi^2(x) = u_2(x)/u_1(x)$. Since $S_{11}(p^0, m^0) < 0$ and $S_{22}(p^0, m^0) < 0$, we can conclude that good 1 (respectively good 2) is a normal good if and only if $\pi_2^1(x^0) > 0$ (respectively, $\pi_1^2(x) > 0$). For example if $\pi_2^1(x^0) < 0$, then the marginal rate of substitution $\pi^1(x) = u_1(x)/u_2(x)$ must be decreasing vertically around x^0 as shown in Fig. 5.1. Therefore the income-consumption curve through x^0 corresponding to p_1^0/p_2^0 must cut through as shown by a broken curve through x^0 indicating that good 1 is inferior around x^0.

Using the relationship $\pi_n^i = (1/u_n)\,(u_{in} - \pi^i u_{nn})$ $(i = 1, 2, \ldots, n-1)$, we can rewrite (5.22)′ in terms of the derivatives of the utility function as follows:

$$(5.23)\qquad \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^n/\partial m \end{bmatrix} = (1/p_n^0)e_n - (1/u_n) \begin{bmatrix} S_{11}, & \ldots, S_{1,n-1} \\ \vdots & \\ S_{n,1}, & \ldots, S_{n,n-1} \end{bmatrix} \begin{bmatrix} u_{1n} \\ \vdots \\ u_{n-1,n} \end{bmatrix}$$

$$+ (u_{nn}/u_n^2) \begin{bmatrix} S_{11}, & \ldots, S_{1,n-1} \\ \vdots & \\ S_{n,n}, & \ldots, S_{n,n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}.$$

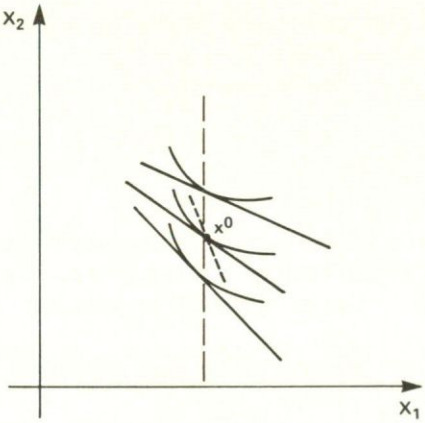

Fig. 5.1

Since $u_i(x^0) = \lambda p_i^0$ and $S(p^0, m^0)p^0 = 0_n$, we have

$$(5.24) \qquad \begin{bmatrix} S_{11}, & \ldots, S_{1,n-1} \\ \vdots & \\ S_{n,1}, & \ldots, S_{n,n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = - \begin{bmatrix} S_{1,n} \\ \vdots \\ S_{n,n} \end{bmatrix} u_n .$$

Therefore, if we substitute (5.24) into (5.23), we obtain the following expression:

$$(5.25) \qquad \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^n/\partial m \end{bmatrix} = (1/p_n^0)e_n - (1/u_n)S(p^0, m^0) \begin{bmatrix} u_{1n} \\ \vdots \\ u_{nn} \end{bmatrix}$$

where $\partial h^i/\partial m$ $(i = 1, 2, \ldots, n)$ are evaluated at (p^0, m^0) and u_n and u_{in} $(i = 1, 2, \ldots, n)$ are evaluated at x^0. When we use good j instead of good n for the normalization, we would get a similar expression as follows:

$$(5.26) \qquad \begin{bmatrix} \partial h^1/\partial m \\ \vdots \\ \partial h^n/\partial m \end{bmatrix} = (1/p_j^0)e_j - (1/u_j)S(p^0, m^0) \begin{bmatrix} u_{1j} \\ \vdots \\ u_{nj} \end{bmatrix} .$$

Now consider good j in (5.26) which will give us $\partial h^j/\partial m = 1/p_j^0 - (1/u_j) \sum_{i=1}^{n} S_{ji}u_{ij}$.

If we use the relation $S_{jj} = - \sum_{i \neq j} S_{ji}(p_i/p_j) = - \sum_{i \neq j} S_{ji}(u_i/u_j)$, then we have

$$\partial h^j/\partial m = (1/p_j^0) - (1/u_j) \sum_{i \neq j} S_{ji}\{u_{ij} - (u_i/u_j)\} .$$

Suppose that either (i) good j is a Hicksian substitute for good i $(i \neq j)$, i.e., $S_{ji} > 0$, and at the same time good j is an ALEP substitute for good i $(i \neq j)$, i.e., $u_{ij} < 0$, or (ii) good j is a Hicksian complement to good i $(i \neq j)$, i.e., $S_{ij} < 0$, and at the same time good j is sufficiently ALEP complementary to good i $(i \neq j)$ in the sense that $u_{ij} > u_i/u_j$. Under this condition, we have that, for every i $(i \neq j)$, $S_{ji}\{u_{ij} - (u_i/u_j)\} < 0$. Therefore $\partial h^j/\partial m > 0$, i.e., good j is a normal good. Essentially what we have shown is that if, under some utility representation, the

Hicksian definition of substitutes or complements agrees with the ALEP definition of substitutes or complements, then goods are likely to be normal.

Similarly when we consider good i ($i \neq n$) in (5.25), we similarly obtain an expression as follows:

$$\partial h^i/\partial m = -(1/u_n) \sum_{j \neq i} S_{ij} \{u_{jn} - (u_j/u_i) u_{in}\}.$$

Hence if $S_{ij} \{u_{jn} - (u_j/u_i)u_{in}\} < 0$ for every i ($i \neq j$), then good i is normal. Note that $S_{ij} \{u_j n - (u_j/u_i) u_{in}\} < 0$ if and only if $S_{ij}(\varepsilon_{jn}(x^0) - \varepsilon_{in}(x^0)) < 0$ where $\varepsilon_{in}(x^0) = \partial(\log u_i(x^0))/\partial(\log x_n)|_{x=x^0}$, the elasticity of the marginal utility of good i with respect to the numeraire good.

Another way to connect income terms and the utility function is to look at the following system of equations from (5.19).

$$(5.27) \quad A(x^0) \begin{bmatrix} \partial h^1(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \\ \vdots \\ \partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \end{bmatrix} = - \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix}.$$

Note that

$$A(x^0) = \begin{bmatrix} \pi_1^1(x^0), & \ldots, \pi_{n-1}^1(x^0) \\ \vdots & \\ \pi_1^{n-1}(x^0), & \ldots, \pi_{n-1}^{n-1}(x^0) \end{bmatrix} - \begin{bmatrix} \pi_n^1(x^0) \\ \vdots \\ \pi_n^{n-1}(x^0) \end{bmatrix} [\pi^1(x^0), \ldots, \pi^{n-1}(x^0)]$$

and $\sum_{i=1}^{n-1} \pi^i(x^0) \partial h^i(\tilde{p}^0, 1, \tilde{m}^0)/\partial m = 1 - \partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m$. If we substitute these into (5.27), we obtain:

$$(5.28) \quad \begin{bmatrix} \pi_1^1(x^0), & \ldots, \pi_n^1(x^0) & \pi_n^1(x^0) \\ \vdots & & \\ \pi_1^{n-1}(x^0), & \ldots, \pi_{n-1}^{n-1}(x^0), & \pi_n^{n-1}(x^0) \end{bmatrix} \begin{bmatrix} \partial h^1(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \\ \vdots \\ \partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \end{bmatrix} = 0_{n-1}.$$

If we substitute $\pi_j^i = (1/u_n)(u_{ij} - \pi^i u_{nj})$ into (5.28), we get:

$$(5.29) \quad (1/u_n(x^0)) \begin{bmatrix} u_{11}(x^0), & \ldots, u_{1n}(x^0) \\ \vdots & \\ u_{n-1,1}(x^0), & \ldots, u_{n-1,n}(x^0) \end{bmatrix} \begin{bmatrix} \partial h^1(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \\ \vdots \\ \partial h^n(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \end{bmatrix}$$

$$= (1/u_n(x^0)) \left\{ \sum_{j=1}^{n} u_{n_j}(x^0) \partial h^j(\tilde{p}^0, 1, \tilde{m}^0)/\partial m \right\} \begin{bmatrix} \pi^1(x^0) \\ \vdots \\ \pi^{n-1}(x^0) \end{bmatrix}.$$

By the homogeneity property, the normalized indirect utility must satisfy

$$(5.30) \quad v(\tilde{p}, 1, \tilde{m}) = u(h^1(\tilde{p}, 1, \tilde{m}), \ldots, h^{n-1}(\tilde{p}, 1, \tilde{m}), \tilde{m} - \sum_{j=1}^{n-1} \tilde{p}_j h^j(\tilde{p}, 1, \tilde{m})).$$

If we partially differentiate (5.30) twice with respect to income, we can show that

$$(5.31)\quad v_{mm}(\tilde{p}^0, 1, \tilde{m}^0) = \sum_{j=1}^{n} u_{nj}(x^0)\,\partial h^j(\tilde{p}^0, 1, \tilde{m}^0)/\partial m\,.$$

Therefore if we supplement (5.31) to the system (5.29), we have the following system of equations

$$(5.32)\quad \begin{bmatrix} u_{11}(x^0), \ldots, u_{1n}(x^0) \\ \vdots \\ u_{n1}(x^0), \ldots, u_{nn}(x^0) \end{bmatrix} \begin{bmatrix} \partial h^1(\tilde{p}^0, 1, \tilde{m}^0)/m \\ \\ \partial h^n(\tilde{p}^0, 1, \tilde{m}0)/\partial m \end{bmatrix} = v_{mm}(\tilde{p}^0, 1, \tilde{m})$$

$$\begin{bmatrix} \pi_1^1(x^0) \\ \vdots \\ \pi^{n-1}(x^0) \end{bmatrix}.$$

Since $v_{mm}(p, m)$ is homogeneous of degree "2" and $\pi^i(x^0) = p_i^0/p_n^0$, (5.32) can be rewritten as

$$(5.32)'\quad [\nabla^2 u(x^0)] \begin{bmatrix} \partial h^1(p^0, m^0)/\partial m \\ \vdots \\ \partial h^n(p^0, m^0)/\partial m \end{bmatrix} = v_{mm}(p^0, m^0)\,p^0\,.$$

The above expression could be obtained more simply from the relationship $v_m(p, m) = (1/p_i)u_i[h(p, m)]$ $(i = 1, 2, \ldots, n)$ and by partially differentiating these with respect to "m" once more as done by Chipman (1977). Another avenue to obtain (5.32)′ is to consider the constrained system of first-order conditions, i.e., $m - p \cdot x = 0$ and $[\nabla u(x)]' - \lambda p = 0$. When we apply the implict differentiation rule and the inversion formula of a partitioned matrix, we can obtain

$$\begin{bmatrix} \partial h^1(p^0, m^0)/\partial m \\ \vdots \\ \partial h^n(p^0, m^0)/\partial m \end{bmatrix} = (1/Q)[\nabla^2 u(x^0)]^{-1} p^0$$

where $Q = (p^0)'[\nabla^2 u(x^0)]^{-1} p^0$ provided that $\nabla^2 u(x^0)$ is nonsingular. The reason for using a longer derivation is simply because of our desire to derive the result from our formulation and our mere curiosity to find its relationship to (5.19) or (5.27). As shown by Chipman (1977), if there exists a concave representation of the preference relation in such a way that $\nabla^2 u(x^0)$ is negative definite and moreover $u_{ij}(x^0) > 0$ $(i, j = 1, 2, \ldots, n)$, i.e., all goods are ALEP complementary to each other, then $-\nabla^2 u(x^0)$ has a nonnegative inverse and $-[\nabla^2 u(x^0)]^{-1} p^0 \gg 0$. (See Nikaido (1968), Chap. II.) Since $v_{mm}(p^0, m^0) = 1/Q < 0$, every good in this case becomes a normal good.

6. Recoverability

If it is *known* that a demand function is derived from utility maximization then, under certain conditions, it is possible to recover that utility function up to a monotone transformation. We emphatically deny that this section is devoted to the study of "sufficient conditions for utility maximization." Indeed, such condi-

tions do not exist. We hasten to declare the great indebtedness of this section to the papers by Hurwicz-Uzawa (1971) and by Hurwicz (1971).

By way of prefacing this treatment, we study an example. Suppose it is known, in a world of 2 goods, that the consumer maximized his utility and obtained the demand function:

$$(6.1)\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x = h(p_1, p_2, m) = \begin{bmatrix} m/2p_1 \\ m/2p_2 \end{bmatrix}.$$

By the first order conditions:

$$(6.2)\quad u_1/u_2 = p_1/p_2,$$

along an indifference curve $u = c$. Differentiating, we get $u_1 dx_1 + u_2 dx_2 = 0$. Thus, by (6.2), we have

$$(6.3)\quad p_2 dx_1 + p_1 dx_2 = 0\,.$$

Thus

$$(6.4)\quad dx_2/dx_1 = -p_1/p_2,$$

along an indifference curve.

By (6.1) we have $p_1/p_2 = x_2/x_1$. Substituting in (6.4) we get

$$dx_2/dx_1 = -x_2 \cdot x_1\,.$$

Separating the variables we have

$$dx_2/x_2 + dx_1/x_1 = 0\,.$$

Integrating, we get $\ln x_2 + \ln x_1 = \text{constant}$, i.e.,

$$(6.5)\quad x_1 x_2 = c$$

as the equation of our indifference curve. Thus our utility function is:

$$(6.6)\quad u(x_1, x_2) = x_1 x_2$$

or any monotone increasing transformation of it. If one were to maximize the utility function in (6.6) subject to the budget constraint, then one would get, in fact, the demand function of (6.1).

In other words if we know that a demand function comes from utility maximization, then we know, up to a monotone increasing transformation, a function that would yield that demand function upon being maximized subject to the budget constraint.

First we discuss Antonelli's approach to recoverability. Suppose now that we are given the function: $\pi(x_1, \ldots, x_{\cancel{m+1}}) = (\pi^1, \ldots, \pi^{\cancel{n}})$ and we *assume* they come from utility maximization. The task is to find a utility function that could have induced these functions. If that were so and U were that, nice and differentiable, utility function then it must satisfy the partial differential equations:

$$(6.7)\quad \frac{\partial u(x)}{\partial x_i} - \pi^i(x)\frac{\partial u(x)}{\partial x_n} = 0\,,$$

where $x = x_1, \ldots, x_{\cancel{m+1}}$.

Following Hurwicz (1971), very closely, we introduce the notation:

$$z = (x_1, \ldots, x_{\cancel{n}\,n-1}) \text{ and } y = x_{\cancel{n-1}\,n}$$

and write (6.7) as

$$(6.8) \quad u_{z_i}(z, y) - \pi^i(z, y)\, u_y = 0\,, \qquad i = 1, \ldots, \cancel{m}\,n.$$

The system (6.7) has a solution $u = u(z, y)$ with $u_y \neq 0$ if and only if

$$(6.9) \quad \frac{\partial \pi^i}{\partial z_j} - \frac{\partial \pi^i}{\partial y}\pi^j = \frac{\partial \pi^j}{\partial z_i} - \frac{\partial \pi^j}{\partial y}\pi^i\,,$$

for all $i, j = 1, \ldots, \cancel{m}\,n$ for all (z, y). As we have seen before, condition (6.9) is nothing else but the symmetry of the Antonelli matrix.

If we have a solution of (6.8) then we have a function that satisfies the first order necessary conditions for a maximum. We don't yet know if it is maximized by these demand functions. To insure maximization we must show that the function we got is quasi-concave. If $u(z, y)$ is a solution of (6.8) then $u_y \neq 0$ and we can solve $u(z, y) = c$ for $y = \varrho(z)$. To show quasi-concavity of u it suffices to show that ϱ is convex. Now $dy = \sum \varrho_i dz_i$. On the other hand

$$u_y d_y + \sum_{i=1}^{n-1} u_{z_i} dz_i = 0\,.$$

So, by (6.6):

$$dy = -\sum_{i=1}^{n-1} (u_{z_i}/u_y)\, dz_i = -\sum_{i=1}^{n} \pi^i dz_i\,.$$

Thus

$$d\varrho = -\sum_{i=1}^{n-1} \pi^i dz_i\,.$$

But π^i are differentiable so $d^2\varrho$ exists and

$$(6.10) \quad d^2\varrho = -\left[\sum_i \sum_j (\pi^i_{z_j} - \pi^i_y \pi^j)\, dz_i dz_j\right].$$

The $d^2\varrho$ is non-negative follows from the negative semi-definiteness of the Antonelli Matrix. This implies that ϱ is convex and hence u is quasi-concave. Thus us is maximized subject to the budget constraints and we have recovered a utility function.

We have presented the outline of a proof of the following proposition:

(6.11) *Proposition:* Suppose the indirect demand functions are continuously differentiable and that the Antonelli matrix is symmetric and negative semi-definite. Suppose furthermore that the indirect demand functions were derived from utility maximization subject to a budget equality constraint. Then there exists a differentiable quasi-concave function which generates these demand functions.

We now discuss recoverability, the Hurwicz-Uzawa way. Let us now go back to our old notation, of an n-dimensional commodity space and not even mention

normalization. Assume that the demand function h(p, m) is derived from utility maximization, is continuously differentiable and that the Slutsky matrix is symmetric and negative semi-definite. Then we can recover a smooth utility function which upon maximization yields the function h.

The procedure parallels that of the last section. First we derive a system of partial differential equations which would have to be satisfied if the demand functions came from utility maximization. Then use the symmetry of the Slutsky matrix to assure the existence of a solution. The solution is the compensated income function. This then is used to define a utility function and show that h(p, m) maximized it subject to the budget constraint.

Since the demand function comes from utility maximization, it must satisfy:

$$(6.12) \quad \frac{\partial m}{\partial p_i} = h^i(p, m) .$$

A necessary and sufficient condition for the unique solvability of (6.11) with initial point m^0, p^0 is that the Slutsky matrix is symmetric which we assumed. Thus these exists a unique function $\mu(p; p^0, m^0)$ satisfying:

$$(6.13) \quad \mu_i = h^i, \mu(p^0; p^0, m^0) = m^0 .$$

Now fix p^0. Define

$$(6.14) \quad u_{p^0}(x) = \mu(p^0; h^{-1}(x)) ,$$

where $h^{-1}(x) = \{(p, m) | x = h(p, m)\}$. It can be easily shown that, even though $h^{-1}(x)$ is not a function, $u_{p^0}(x)$ is a function which is independent of the choice of (p^0, m^0) and which is uniquely maximized by h(p, m). Finally some further notes on recoverability.

Samuelson (1950) starts with direct demand functions, inverts them and then follows Antonelli's path toward recoverability. A further approach toward recoverability starts with an idea of Samuelson; revealed preference. We have shown above that the WARP implies that the Slutsky matrix is negative semi-definite. If, in addition, we can get the symmetry of that matrix then we would have sufficient conditions recoverability. It was hoped that the WARP would yield the symmetry condition, but alas it does not. The strong axiom of revealed preference (SARP) was introduced for that purpose. For the sake of completeness we state the SARP. We say that a bundle x^0 is directly revealed preferred to bundle x^1 if x^0 is chosen by the consumer while x^1 was available, i.e., could have been chosen, we write $x^0 R x^1$. We say that x^0 is indirectly revealed preferred to x^1 if there exists a, possibly empty, finite set of commodity bundles $x^{01}, x^{02}, \ldots, x^{0q}$ such that $x^0 R x^{01}$ and $x^{01} R x^{02}$ and ... and $x^{0q} R x^1$. We denote this by $x^0 R^* x^1$. The SARP may be stated as follows: If $x^0 \neq x^1$ is such that $x^0 R^* x^1$, then it is not true that $x^1 R^* x^0$.

The SARP implies the symmetry of the Slutsky matrix and it also implies the WARP. So if a differentiable demand function satisfies SARP then it is possible to find a utility function which if maximized would yield that demand function. Note that the utility function we found need not represent the consumer's actual preferences since two vastly different preference relations may generate the same demand behavior. To see this last point, consider the demand function:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m/p_x \\ 0 \end{pmatrix}$. This demand function can be generated by the preference relation given by: (x, y) is in different to (x^1, y^1) if $x = x^1$, if $x > x^1$, then (x, y) is preferred to (x^1, y^1). It also can be generated by the preference relation given by: (x, y) is preferred to (x, y^1) if $y > y^1$ and (x, y) is preferred to (x^1, y^1) if $x > x^1$. An analogous example can be constructed for the demand function $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m/2p_x \\ m/2p_y \end{pmatrix}$ with the preference relation represented by $u(x, y) = xy$ and its lexicographic analogue.

It is not necessary to go through the recoverability procedure mentioned above in order to find a plausible preference ordering; one that could have generated the demand function. Indeed, the demand map need not be a function at all in order to obtain recoverability. This much more general and elegant approach was pioneered by Richter (1971) via his Congruence Axiom.

Problems

1. Consider the following functions:
$h^i(p_1, p_2, m) = (\alpha_{i1}p_1 + \alpha_{i2}p_2)^{-1}m + \beta_i p_i$ $(= 1, 2)$ where $p_i \geqq 0$ $(i = 1, 2)$, $m \geqq 0$, and $\alpha_{ij} \geqq 0$ $(i, j = 1, 2)$ and $\beta_i \geqq 0$ $(i = 1, 2)$ are constant parameters. Find appropriate restrictions on parameters α_{ij} $(i, j = 1, 2)$ and β_i $(i = 1, 2)$ so that the above functions satisfy properties of demand functions arising from the classical theory of consumer demand where p_i is the price of good i, m is the income, the consumption set is the nonnegative quadrant and $h^i(p_i p_2, m)$ is the demand function for good i.
(Hint: First check the homogeneity and the budget conditions.)

2. Consider the following functions:
$f(x_1, x_2, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $x_i \geqq 0$ $(i = 1, 2, \ldots, n)$ and α_i's are positive constants, $g(x_1, x_2) = (a_1 x_1^{-\varrho} + a_2 x_2^{-\varrho})^{-1/\varrho}$, $x_i \geqq 0$ $(i = 1, 2)$, a_i's are positive constants with $a_1 + a_2 = 1$ and ϱ is a constant such that $\varrho \neq 0$ and $\varrho > -1$.
i) Show that f is strictly quasi-convace on R^n. (Hint: Show that log f is strictly concave.)
ii) Show that g is strictly qausi-convace.

3. Consider a utility function $u(x_1, x_2)$, $x_1 > 0$, $x_2 > 0$ with $\dfrac{\partial u}{\partial x_i} > 0$. Assume that, for a fixed constant $c \in R$, there exists a function ϕ: $(0, \infty) \to (0, \infty)$ such that $u(x_1, x_2) = c$ iff $x_2 = \phi(x_1)$. Show that the marginal rate of substitution, i.e., $-\dfrac{d\phi(x_1)}{dx_1}$, is strictly decreasing if and only if $\begin{vmatrix} 0, & u_1, & u_2 \\ u_1, & u_{11}, & u_{12} \\ u_2, & u_{21}, & u_{22} \end{vmatrix} > 0$.

4. Consider the following utility functions defined on the positive quadrant.
(a) $u(x_1, x_2) = (ax_1^{-\varrho} + bx_2^{-\varrho})^{-1/\varrho}$, $a > 0$, $b > 0$, $\varrho > -1$,

(b) $u(x_1, x_2) = \text{Min}\left\{\frac{x_1}{a}, \frac{x_2}{b}\right\}, \quad a > 0, b > 0,$

(c) $u(x_1, x_2) = x_1^2 + x_2,$

(d) $u(x_1, x_2) = \min\{3x_1 \neq 6x_2, 4x_1 + 4x_2\}.$

Find demand functions or multivalued demand mappings for each utility function given above.

5. Consider the following utility function $u(x, y) = x + \log y$ where $x \geqq 0, y \geqq 0$. Let p_x, p_y and m be the price of "x", the price of "y" and income.
(a) Find demand functions for "x" and "y".
(b) Find Slutsky terms and check usual properties of Slutsky terms.

6. Consider the following utility function of a competitive consumer

$$u(x, y) = x^{1/2} + y, \; x \geqq 0, \; y \geqq 0$$

where x and y are quantities of two commodities, say commodity "x" and commodity "y". Let p_x, p_y and m be the price of X, the price of Y, and the income of the consumer.
(a) Obtain demand functions for "x" and "y" for the case $x > 0$ and $y > 0$.
(b) Obtain substitution terms and income terms.

7. Consider the following utility function

$$u(x_1, x_2) = x_1 + \alpha(x_2).$$

Assume that $p_i > 0$ $(i = 1, 2)$ and that $m > 0$, and furthermore that they satisfy:

$$\alpha(m/p_2) < p_2/p_1.$$

(a) Find the first order necessary conditions for the maximization problem of the consumer.

(b) Let demand functions be $h^i(p_1, p_2, m)$ $(i = 1, 2)$. Obtain $\frac{\partial h^i}{\partial p_i}$ $(i = 1, 2)$ and $\frac{\partial h^i}{\partial m}$ as explicitly as possible.

(c) Find Slutsky terms using results in (b).

8. Consider the following utility function.

$u(x_1, x_2) = \alpha \log x_1 + \beta \log x_2$, $x_i > 0$ $(i = 1, 2)$ where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$. Find the following: (i) ordinary demand functions, (ii) Slutsky substitution terms and income terms, (iii) indirect utility function, (iv) Hicksian compensated demand function, and (v) Slutsky compensated demand functions, (vi) show that the Slutsky substitution term S_{ij} coincides with the partial derivatives with respect to p_j, the price of good j, of the i-th Hicksian and Slutsky compensated demand functions.

9. Consider the following consumption agent. The consumption set is the non-negative orthant of R^3, 3-dimensional real space. His (weak) preference ordering G is given by:
$(x_1, x_2, x_3) G (y_1, y_2, y_3)$ if and only if $x_1^{\alpha_1} x_2^{\alpha_2} e^{x_3} \geqq y_1^{\alpha_1} y_2^{\alpha_2} e^{y_3}$, where α_1 and α_2 are positive constants. Let $p_i > 0$ be the price of commodity i and $m > 0$ be the income.
(i) Find demand functions of this consumer agent.
(ii) Assume $m/p_3 > \alpha_1 + \alpha_2$ and find Slutsky terms.
(iii) Under the same assumption as (ii), check the negative semidefiniteness of the Slutsky matrix.

10. Consider the following utility function:

$$u(x_1, x_2) = x_1 + \log(x_1 + x_2),\ x_i \geqq 0 \qquad (i = 1, 2)\ .$$

(i) Show that $h^1(p, m) > 0$ iff $p_1 - p_2 < m$ and $h^2(p, m) > 0$ iff

$$\frac{m}{p_1} < \frac{p_1 - p_2}{p_2}\ .$$

(ii) Find income-consumption curves. Is there an inferior good?
(iii) Show that none of goods is a Giffen good.

iff $p_1 + 2p_2 > m$

11. Consider the following utility function:

$$u(x_1, x_2) = (x_1 - 1)(2 - x_2)^{-2},\ x_1 \geqq 1 \text{ and } 0 \leqq x_2 < 2.$$

(i) Draw the indifference map for the above utility function.
(ii) Show that $h^1(p, m) > 1$ and $h^2(p, m) > 0$ iff $p_1 + p_2 < m$.
(iii) Find demand functions for this case.
(iv) Draw a typical income-consumption curve.
(v) Can any good be a Giffen good?

12. Suppose that we have a utility function $u: R^2_+ \to R_+$. Assume that
(i) u is homothetic and u has continuous second partial derivatives
(ii) for every $(x_1, x_2) > 0$, $u_i(x_1, x_2) > 0$ and $\lim_{x_i \to 0} u_i(x_1, x_2) = \infty$ where $u_i = \partial u / \partial x_i$, and
(iii) the usual bordered Hessian of u has a positive determinant. Let $p_i > 0$ be the price of "x_i" and define $z = x_2/x_1$ and $q = p_1/p_2$.
(a) In deriving demand functions from the above utility function, discuss whether it makes any difference if u is assumed to be positively homogeneous of degree one.
(b) Show that the equation $u_1(x_1, x_2)/u_2(x_1, x_2) = q$ can be solved for some function $z = f(q)$ and show analytically that for every $q > 0$, $f'(q) > 0$.
(c) Find demand functions $h^i(p_1, p_2, m)$ for "x_i" $(i = 1, 2)$ as explicitly as possible.
(d) Obtain expressions for Slutsky terms S_{11} and S_{12}.
Hint for (b): If u is positively homogeneous of degree one, then show that

$$\det \begin{pmatrix} 0, & u_1, & u_2 \\ u_1, & u_{11}, & u_{12} \\ u_2, & u_{21}, & u_{22} \end{pmatrix} = (1/x_1x_2)$$

$$\det \begin{pmatrix} 0, & u_1x_1, & u_2x_2 \\ u_1, & u_{11}x_1, & u_{12}x_2 \\ u_2, & u_{21}x_1, & u_{22}x_2 \end{pmatrix} = (u/x_1)(u_2/x_1)(u_2u_{12} - u_1u_{22})$$

13. Suppose that we have a utility function $U(x, y)$ of the form $U(x, y) = u(\alpha(x), \beta(y))$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_r)$, $\alpha: R^n_+ \to R$, $\beta: R^r_+ \to R$, and $u: R^2 \to R$. The functions u, β and β are assumed to have continuous second partial derivatives on some appropriate domains. Let p, q, m be a positive price vector for x, a positive price vector for y, and a positive money income respectively. Let $h^x(p, q, m)$ and $h^y(p, q, m)$ be demand functions for x and y respectively. Assume that $\partial u/\partial\alpha > 0$ and $\partial u/\partial\beta > 0$. Let $\tilde{h}^x(p, m_x)$ be the solution to the following maximization problem, (I_x): Max $\alpha(x)$ subject to $px = m_x$ and let $\tilde{h}^y(q, m_y)$ be the solution to the following maximization problem, (I_y): Max $\beta(y)$ subject to $qy = my$. Define functions M^x and M^y respectively by

$$M^x(p, q, m) = p \cdot h^x(p, q, m) \quad \text{and}$$
$$M^y(p, q, m) = q \cdot h^y(p, q, m).$$

(i) Show that $h^x(p, q, m) = \tilde{h}^x(p, M^x(p, q, m))$ and $h^y(p, q, m) = \tilde{h}^y(q, M^y(p, q, m))$.

(ii) Let $N_x = \{1, 2, \ldots, n\}$ and $N_y = \{1, 2, \ldots, r\}$ be index sets for x and y respectively. Then show that for $i, j \in N_x$ and for $k \in N_y$ we have

(a) $\dfrac{\partial h_i^x}{\partial m} \Big/ \dfrac{\partial h_j^x}{\partial m} = \dfrac{\partial \tilde{h}_i^x}{\partial m_x} \Big/ \dfrac{\partial \tilde{h}_j^x}{\partial m_x}$

(b) $\dfrac{\partial h_i^x}{\partial q_k} \Big/ \dfrac{\partial h_j^x}{\partial q_k} = \dfrac{\partial h_i^x}{\partial m_x} \Big/ \dfrac{\partial h_j^x}{\partial m_x}$

(c) $S_{ik}/S_{jk} = \dfrac{\partial h_i^x}{\partial \cancel{\neq}\, m} \Big/ \dfrac{\partial h_j^x}{\partial m}$

14. Suppose that $\pi^1(x_1, x_2) = \dfrac{\alpha x_2}{\beta x_1}$ and $\pi^2(x_1, x_2) \equiv 1$ where $\alpha > 0$ and $\beta > 0$.

(i) Use $\mu(x_1, x_2) = \dfrac{\beta}{x_2}$ as an integrating factor and find a utility function which generates the above inverse demand function.

(ii) Use $\mu(x_1, x_2) = \beta x_1^{\alpha} x_2^{\beta-1}$ as an integrating factor to find a utility function which generates the above inverse demand function.

(iii) Interpret the above result.

15. Suppose that a consumer has the following lexicographic strict preference ordering P: $(x_1, x_2) P (x_1', x_2')$ if and only if $[f(x_1, x_2) > f(x_1', x_2')]$ or $[f(x_1, x_2) = f(x_1', x_2')$ and $x_1 > x_1']$ where $f(x_1, x_2) = \alpha \log x_1 + \beta \log x_2$ and x_i denotes the nonnegative quantity of commodity i.

(a) Illustrate the set of commodities which are strictly preferred to a given bundle, say $(\hat{x}_1, \hat{x}_2)$.
(b) Is the above preference relation representable by a utility function?
(c) Find demand functions for x_1 and x_2 and also find the Slutsky terms of demand functions.
(d) Are the above demand functions representable by some utility function?

16. Do the following choice functions satisfy WARP and SARP?

$X = \{a, b, c, d\}$ = the set of alternatives

(i) $h[\{a, b, c\}] = \{a, c\}$
$h[\{c, d\}] = \{c\}$
$h[\{a, b, d\}] = \{b\}$
(ii) $h[\{a, c, d\}] = \{d\}$
$h[\{a, b, c\}] = \{c\}$
$h[\{a, d\}] = \{d\}$.

Suppose that $\tilde{x}_2$, $\tilde{x}_2$, are the initial endowments of commodities. Prove that for the case of two commodities with income $m = p_1\tilde{x}_1 + p_2\tilde{x}_2$ the weak axiom of revealed preference implies the strong axiom of revealed preference.

18. Suppose that a consumer has the following utility function:

$$u(x, y) = x + \log y$$

where x denotes leisure and y denotes an aggregated consumption good. Assume that:
(a) $x = \tilde{L} - L$ where L is the labor supply and $\tilde{L}$ is the total hours available for work and leisure.
(b) Let p_x, p_y and W be the price of the consumption good, the cost of leisure and the hourly wage rate respectively.
(c) The consumer obtains the wage income and also non-wage income denoted by π.
(i) Obtain demand functions for the consumption good and leisure for a competitive consumer.
(ii) Find effects of the change of p_x, p_y, W, and π on x and y.

19. Consider the case of a consumer with his utility function given by $u(x, y) = \alpha \log x + \beta \log y$ where x is a consumption good and y is leisure. Suppose that:
(i) $y + L \geqq \tilde{L}$, $y \geqq 0$, $L \geqq 0$ where L is the supply of labor and $\tilde{L}$ is the total hours available.
(ii) Let w, p_x and p_y be the hourly wage rate, the price of the consumption good and the hourly cost of leisure.
(iii) The consumer obtains wage income wL and non-wage income m.
(a) Obtain the optimum combination of x, y and L as functions of p_x, p_y, w and m.
(b) Discuss effects of p_y, w and m on the supply of labor.

20. Consider a farmer who lives close to a town and has two alternatives: (i) to work on his field and/or (ii) to get a job in town.

Assume that: (a) The technology of the farm is characterized by the production function of the form $y = f(L_1)$ where y is the output of, say, wheat and L_1 is labor hours spent on the fields. (b) He works at most L hours ($L_1 + L_2 \leqq L$ where L_2 is hours spent on the job in town). (c) He has a utility function of the form u(c, x) where c is an aggregate of consumption goods (other than wheat) and x is his consumption of wheat. (d) Prices of "c" and wheat are fixed at 1 and p respectively. (e) w denotes the hourly wage rate in town.

Question: (i) Set up the problem of the farmer's decision-making. (ii) Find first order necessary conditions for an optimum and give the interpretation of the conditions. (You can introduce additional appropriate assumptions, for example, indispensability.)

Chapter 3 Topics in Demand Analysis

1. Introduction

In this chapter we present two additional topics in demand theory. The first topic deals with aggregate demand derived from individual demand. It introduces the reader to one of the topics on the current frontiers of economic research even though the problem itself is old. The idea is to explore the possibility of getting better behavior on the part of aggregate demand functions by exploiting the fact that if the number of consumers is very large, then exceptional behavior can be "insignificant" in some well defined sense.

The second topic deals with time and how it affects the analysis of consumers' demand. There is a sense of "aggregation" here except it is across time instances rather than consumers. The way time is modeled in economics is still being discussed. We take here, the most common and simplistic approach, that of linear aggregation. Even under this very simplifying assumption, we find that the range of questions is extensive but the scope of the answers is rather limited.

2. Individual Demand and Aggregate Demand

We have presented various analyses of the properties of an individual consumer's demand function following the classical works by Antonelli (1886), Slutsky (1915), Hicks-Allen (1934) and Samuelson (1947). The major purpose of our interest in the properties of individual demand functions is to obtain some useful information which can be utilized in market equilibrium analyses. Therefore we still need the additional step of aggregating demand functions of various individual consumption agents. Economists often presume or hope that if we aggregate individual demand functions over thousands or millions of consumption agents, then the resulting aggregate demand function would have nicer properties than individual demand functions. Some haphazard behaviors of some consumers may become less conspicuous through the aggregation. Temporal dependencies in consumption where what you ate yesterday influences what you will eat today will become less important through the aggregation. Negative income terms of some consumers may be outweighed by positive income terms of many consumers. Although these presumptions are interesting and they are waiting for some justification, we have not succeeded too well in providing solid theoretical justification for these presumptions. In this section, we try to sort out some of the difficulties and some attempts in trying to establish, in particular, the classical "law of demand" for the aggregate demand function.

A consumption agent will be indexed by t and the set of indices for consumption agents will be denoted by $T = [0, 1]$. The size of consumption agents in T will be measured by the "length" of a subset of T or more precisely by the Legesgue measure on T. Each consumption agent is assumed to thave an individual demand function for commodity i denoted by $h^i(p, m(t), t)$, $(i = 1, 2, \ldots, n)$. If $h^i(p, m(t), t)$ is (Lebesgue-)integrable with respect to t, then the aggregate demand function for good i becomes

$$(2.1) \quad D^i(p, \mu) = \int_T h^i(p, m(t), t)\, d\lambda$$

where μ indicates the distribution of income and λ indicates the Lebesgue measure. If $f(t)$ is the density function of consumption agents on T, then we may consider that $d\lambda = f(t)\,dt$. Of course, the aggregate demand also depends on the distribution of agent's taste, but we do not make the dependency explicit.

(i) The Possibility of the Aggregate Demand Function as a Function of the Aggregate Income

M=

The first question we would like to pose is to find conditions under which the aggregate demand function can be written as $D^i(p, M)$ where $\int_T m(t)\, d\lambda$, $(i = 1, 2, \ldots, n)$.

One sufficient condition well-knwon in the literature is the case of consumers with an identical homothetic preference or utility function. A utility function $u(x)$ is said to be homothetic if $u(x)$ can be written as $u(x) = \phi[f(x)]$ when ϕ is strictly increasing with $\phi(0) = 0$ and f is homogeneous of degree one. Therefore we can start out with a utility function which is homogeneous of degree one, since ϕ is a monotone transformation of a utility index. Then the utility maximization problem with a budget constraint is to find a bundle to maximize $u(x)$ subject to $p \cdot x \leqq m$. Define $a_i = x_i/m$. Then we can convert the original problem into the one to maximize $u(a_1, a_2, \ldots, a_n)$ subject to $p_1 a_1 + p_2 a_2 + \ldots + p_n a_n \leqq 1$. Therefore an optimum bundle "a" can be written as a function of p denoted by $a_i = \alpha_i(p)$. In other words, the ordinary demand functions can be expressed as

$$(2.2) \quad h^i(p, m) = \alpha_i(p)\, m, \qquad (i = 1, 2, \ldots, n).$$

If consumers possess an identical homothetic preference with various income levels, then the individual demand function of agent t becomes $h^i(p, m(t)) = \alpha_i(p)\, m(t)$, $(i = 1, 2, \ldots, n)$. Then the aggregate demand functions can be written as

$$(2.3) \quad \int_T h^i(p, m(t))\, d\lambda = \int_T \alpha_i(p)\, m(t)\, d\lambda = \alpha_i(p) \int_T m((t)\, d\lambda$$
$$= \alpha_i(p) M = h^i(p, M), \qquad (i = 1, 2, \ldots, n).$$

From the form of the aggregate demand functions, it is clear that not only the aggregate demand functions have the form $D^i(p, M)$, but also the aggregate demand $D = (D_1, D_2, \ldots, D_n)$ maximizes $u(D)$ subject to $p \cdot D \leqq M$. In other words, the consumption sector behaves as if the sector maximizes $u[\int_T x(t)\, d\lambda]$ subject to $p \cdot \int_T x(t)\, d\lambda \leqq \int_T m(t)\, d\lambda$. Therefore the theory of demand for an individual consumer is also applicable for the aggregate demand with aggregated Slutsky terms and aggregated income terms.

Another case where the aggregate demand function has the form $D^i(p, M)$ is when the aggregate income M is distributed proportionately, say, $m(t) = \theta(t)M$ with $\theta(t) > 0$ and $\int_T \theta(t)\,d\lambda = 1$. The clearly we have

$$\int_T h^i(p, m(t), t)\,d\lambda = \int_T h^i(p, \theta(t)M, t)\,d\lambda = D^i(p, M)\,.$$

It should be noted that D^i also depends on the distribution of θ as well. Although the aggregate demand functions can be written as $D^i(p, M)$, we can not use the formula, $(\partial D^i/\partial p_j) + D^j(\partial D^i/\partial M)$ to obtain an aggregated Slutsky terms and in fact those terms will not be in general symmetric. Noting that $\partial D^i/\partial M = \int_T \theta(t)(\partial h^i/\partial m)\,dt$, we can see that we have

$$\begin{aligned}
&(\partial D^i/\partial p_j) + D^j(\partial D^i/\partial M) \\
&\quad = \int_T (\partial h^i/\partial p_j)\,d\lambda + [\int_T h^j\,d\lambda]\,[\int_T \theta(\partial h^i/\partial m)\,d\lambda] \\
&\quad = \int_T S_{ij}\,d\lambda + [\int_T h^j\,d\lambda]\,[\int_T \phi(\partial h^i/\partial m)\,d\lambda] - \int_T h^j(\partial h^i/\partial m)\,d\lambda\,.
\end{aligned}$$

There is one special case where the above terms have the nice properties of symmetry and negative semi-definiteness. This happens when each agent possesses a homothetic preference as shown by Chipman (1974). Suppose that the individual demand functions of agent t can be written as

$$h^i(p, m(t), t) = \alpha^i(p, t)m(t), \qquad (i = 1, 2, \ldots, n).$$

Then

$$\begin{aligned}
(2.4)\quad &[\int_T h^j\,d\lambda] + \int_T \theta(\partial h^i/\partial m)\,d\lambda] - [\int_T h^j(\partial h^i/\partial m)\,d\lambda \\
&\quad = M[\int_T \alpha^j(p, t)\theta(t)\,d\lambda]\,[\int_T \alpha^i(p, t)\theta(t)\,d\lambda] - M\int_T \alpha^i(p, t)\alpha^j(p, t)\theta(t)\,d\lambda
\end{aligned}$$

Now define $\bar{\alpha}^i = \int_T \alpha^i \phi\,d\lambda$. Let $b_{ij} = (\alpha^i - \bar{\alpha}^i)(\alpha^j - \bar{\alpha}^j)$, $(i, j = 1, 2, \ldots, n)$ and let $B = [b_{ij}]$. Clearly B is a symmetric matrix. Given

$$\eta = (\eta_1, \eta_2, \ldots, \eta_n)' \in R^n\,,$$

$$\begin{aligned}
\eta' B \eta &= \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha^i - \bar{\alpha}^i)(\alpha^j - \bar{\alpha}^j)\eta_i \eta_j \\
&= \left[\sum_{i=1}^{n} (\alpha^i - \bar{\alpha}^i)\eta_i\right]^2 \geqq 0\,.
\end{aligned}$$

Therefore B is symmetric and positive semi-definite. Note that

$$\begin{aligned}
\bar{b}_{ij} &= \int_T b_{ij}\theta\,d\lambda = \int_T (\alpha^i - \bar{\alpha}^i)(\alpha^j - \bar{\alpha}^j)\theta\,d\lambda \\
&= \int_T \alpha^i \alpha^j \theta\,d\lambda - \bar{\alpha}^i \bar{\alpha}^j\,.
\end{aligned}$$

If we write $\bar{B} = [\bar{b}_{ij}]$, then the matrix $\bar{B}$ is also symmetric and positive semi-definite since $\bar{b}_{ij}$ is an average of b_{ij}. Therefore as shown by Chipman (1974), the matrix consisting of terms of (2.4) becomes symmetric and negative semi-definite. Therefore we can claim the existence of a utility function which has generated the aggregate demand functions $D^i(p, M)$ $(i = 1, 2, \ldots, n)$, although the sum of individual Slutsky terms does not agree with Slutsky terms of the aggregate demand. See also Mas-Colell (1985, 5.7.7) for a similar analysis.

(ii) The Aggregate Demand and the Classical Law of Demand

In this part, we will examine some conditions under which the aggregate income effect becomes nonnegative so that the classical law of demand will be satisfied for the aggregate demand. The exposition of this part depends heavily on the seminal work by Hildenbrand (1983).

Suppose that there exists $T^0 \subseteq T$ such that all consumers in T^0 have an identical taste generating demand functions $h^i(p, m(t))$, $(i = 1, 2, \ldots, n)$ and the income variable is distributed uniformly on $[\underline{m}(p), \bar{m}(p)]$. Then the income effects of own price effect aggregated over T^0 can be written as

$$(2.5)\quad \int_{T^0} h^i(p, m(t))[\partial h^i(p, m(t))/\partial m]\, d\lambda = \int_{\underline{m}(p)}^{\bar{m}(p)} h^i(p, m)[\partial h^i(p, m)/\partial m]\, \partial m .$$

Following Hildenbrand (1983), we can obtain

$$\begin{aligned}(2.6)\quad & \int_{\underline{m}(p)}^{\bar{m}(p)} h^i(p, m)\{\partial h^i(p, m)/\partial m\}\, dm \\ &= (1/2)\int_{\underline{m}(p)}^{\bar{m}(p)} \frac{\partial}{\partial m}\{h^i(p, m)\}^2\, dm \\ &= (1/2)[\{h^i(p, \bar{m}(p))\}^2 - \{h^i(p, \underline{m}(p))\}^2] \\ &= (1/2)[h^i(p, \underline{m}(p)) + h^i(p, \bar{m}(p))][h^i(p, \bar{m}(p)) - h^i(p, \underline{m}(p))] \\ &= \bar{x}_i(p)\, \Delta_m h^i(p)\end{aligned}$$

where $\bar{x}_i(p) = (1/2)[h^i(p, \underline{m}(p)) + h^i(p, \bar{m}(p))]$ and $\Delta_m h^i(p) = h^i(p, \bar{m}(p)) - h^i(p, \underline{m}(p))$. Notice that the above expression $x_i(p)\,\Delta_m h^i(p)$ resembles the usual income effect. Hence if $\Delta_m h^i(p) \geqq 0$ for every p, then the aggregated income effect over T^0 becomes nonnegative.

Now suppose that the set T of consumption agents consists of r intervals T^k $(k = 1, 2, \ldots, r)$ with nonoverlapping interiors, i.e., $T = \sum_{k=1}^{r} T^k$ and $[\text{int } T^k] \cap [\text{int } T^{k'}] = \phi$ for $k \neq k'$. Also assume that, for each k, the taste of consumers is identical on T^k and the income is uniformly distributed over $[\underline{m}_k(p), \bar{m}^k(p)]$. Let $h^i(p, m, k)$, $(i = 1, 2, \ldots, n)$ be the demand functions for agents in T^k on type-k agents. Then if $\Delta_m h^i(p, k) = h^i(p, \bar{m}(p), k) - h^i(p, \underline{m}(p), k) \geqq 0$ for $k = 1, 2, \ldots, r$, then the aggregate demand function $D^i(p, \mu)$ will have the property $\partial D^i(p, \mu)/\partial p_i \leqq 0$.

Let us now consider a case when all consumers in T^0 have identical demand functions $h^i(p, m(t))$, $(i = 1, 2, \ldots, n)$, but the income variable is distributed according to a density function $\varrho(p, m)$. Suppose that $\varrho(p, m) > 0$ for $m \in [\underline{m}(p), \bar{m}(p)]$ and $\varrho(p, m) = 0$ for $m \notin [\underline{m}(p), \bar{m}(p)]$. Let $\bar{\varrho}(p) = \int_0^\infty \varrho(p, m)\, dm$. Define $\eta^i(p, m) = h^i(p, m)\{\partial h^i(p, m)/\partial m\}$ and $\bar{\eta}^i(p) = \int_0^\infty \eta^i(p, m)\, dm$. Then we define the covariance between $\eta^i(p, m)$ and $\varrho(p, m)$ with respect to m denoted by $C(\eta^i, \varrho)(p)$ as follows.

$$(2.7)\quad C(\eta^i, \varrho)(p) = \int_0^\infty \{\eta^i(p, m) - \bar{\eta}^i(p)\}\{\varrho(p, m) - \bar{\varrho}(p)\}\, dm .$$

Then we can assert that

(2.8) *Proposition:* If $\Delta_m h^i(p) = h^i(p, \bar{m}(p)) - h^i(p, \underline{m}(p)) \geqq 0$ for every p, then the aggregate income effect over T^0 of good i is nonnegative if $C(\eta^i, \varrho)(p) \geqq 0$ for every p.

Note first that by (2.6) $\bar{\eta}^i(p) = \bar{x}_i(p)\,\Delta_m h^i(p)$. Then

$$\begin{aligned} C(\eta^i, \varrho)(p) &= \int_0^\infty \{\eta^i(p, m) - \bar{\eta}^i(p)\}\{\varrho(p, m) - \bar{\varrho}(p)\}\,dm \\ &= \int_0^\infty \eta^i(p, m)\,\varrho(p, m)\,dm - \bar{\eta}^i(p)\,\bar{\varrho}(p) \geqq 0\,, \qquad \text{i.e.,} \\ &\int_0^\infty \eta^i(p, m)\,dm\,\varrho(p, m) \geqq \bar{\eta}^i(p)\,\bar{\varrho}(p) \geqq 0\,. \end{aligned}$$

Since the left-hand side of the above is the income effect of good i aggregated over T^0, we have the desired result. One sufficient condition under which $C(\eta^i, \varrho)(p) \geqq 0$ is when $\eta^i(p, m)$ and $\varrho(p, m)$ are similarly ordered with respect to m, i.e., for every $m, m' \in [\underline{m}(p), \bar{m}(p)]$

$$\{\eta^i(p, m) - \eta^i(p, m')\}\{\varrho(p, m) - \varrho(p, m')\} \geqq 0\,.$$

The above inequality implies that

$$\int_0^\infty \eta^i(p, m)\,\varrho(p, m)\,dm \geqq \bar{\eta}^i(p)\,\bar{\varrho}(p)\,, \qquad \text{i.e., } C(\eta^i, \varrho)(p) \geqq 0\,.$$

(See Hardy, Littlewood and Polya (1964), 236 on p. 168.) Interested readers should consult Hildenbrand (1983) for further analysis.

(iii) The Entry Effect of Consumers and the Law of Demand

There is one more price effect which we have neglected in our previous analysis because of our assumption that $x^0 \gg 0$. But intuitively, it appears more appropriate to have a model where, as the price of good i decreases, more consumers will demand the good. We may call this the entry effect of consumers. The first formal model of this variety was first presented by Novshek and Sonnenschein (1979) as far as we are aware of.

Let $h(p, t)$ be a demand function of a particular good for a consumer $t \in T$ with its price p, while the rest of the variables are fixed and they do not appear in $h(p, t)$ explicity. Suppose that $\tilde{p}(t)$ indicates a reservation price for agent t so that $h(p, t) > 0$ if and only if $p < \tilde{p}(t)$. For the sake of concreteness, suppose that $\tilde{p}$ has a shape as shown in Fig. 2.1. Then for a given p^0, we can find $\alpha(p)$ and $\beta(p)$ so that $h(p, t) > 0$ if $t \in (\alpha(p), \beta(p))$.

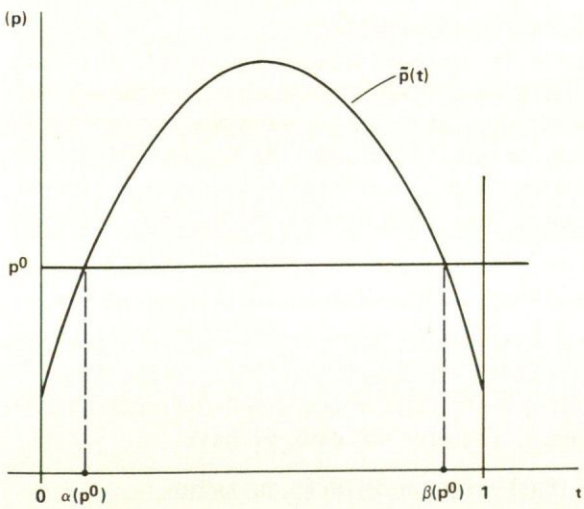

Fig. 2.1

Assume moreover that, at p^0, both $\alpha(p)$ and $\beta(p)$ are differentiable with derivatives $\alpha'(p^0) > 0$ and $\beta'(p^0) < 0$ and also $h(p, t)$ is continuously differentiable with respect to p at p^0 if $t \in (\alpha(p^0), \beta(p^0))$. The aggregate demand function can now be written as

$$D(p) = \int_{\alpha(p)}^{\beta(p)} h(p, t) f(t) dt$$

where $f(t)$ is a density function of the distribution of t on T. If we apply Leibniz's formula for Rieman integrals, then we can obtain

$$\begin{aligned}(2.9)\quad dD(p^0)/dp = &\int_{\alpha(p^0)}^{\beta(p^0)} \{\partial h(p^0, t)/\partial p\} f(t) dt \\ &+ \{h(p^0, \beta(p^0)) f(\beta(p^0)) \beta'(p^0) \\ &- h(p^0, \alpha(p^0)) f(\alpha(p^0)) \alpha'(p^0)\}.\end{aligned}$$

Thus even if the first term of (2.9), the usual price effect, may not be negative, we might get a downward sloping demand curve from the second terms since $\beta'(p^0) > 0$ and $\alpha'(p^0) < 0$. But for this entry effect to be negative, it is necessary that each individual demand function exhibits a discontinuity at its reservation price. If each individual demand function is continuous in price including each reservation price, then, since $h(p, t) = 0$ for $p > \tilde{p}(t)$, we must have $h(\tilde{p}(t), t) = 0$. Because p^0 is a reservation price for consumers $\alpha(p^0)$ and $\beta(p^0)$, we get $h(p^0, \alpha(p^0)) = 0$ and $h(p^0, \beta(p^0)) = 0$. An individual demand function will exhibit a discontinuity at a reservation price if consumers $\alpha(p^0)$ and $\beta(p^0)$ have perfect substitutes at p^0 or if consumers demand discrete quantities. For example, suppose that each consumer consumes only one unit when he demands. Then we have $h(p, t) = 1$ for $t \in (\alpha(p), \beta(p))$ and $h(p, t) = 0$ for $t \notin (\alpha(p), \beta(p))$. Hence

$$D(p) = \int_{\alpha(p)}^{\beta(p)} h(p, t) f(t) dt = \int_{\alpha(p)}^{\beta(p)} f(t) dt .$$

Therefore

$$dD(p^0)/dt = f(\beta(p^0)) \beta'(p^0) - f(\alpha(p^0)) \alpha'(p^0) < 0 .$$

For the case of perfect substitutes, see Novshek and Sonnenschein (1979).

(iv) Income Adjustments and Demand Functions

We have assumed in our analysis that income variables are either fixed or they follow a given distribution. But, in general equilibrium analysis, we usually suppose that incomes are functions of prices. If we do not have observations on incomes, then what we observe may be price vectors and the aggregate quantities $D^i(p) = \int_T h^i(p, m(p, t), t) f(t) dt$ where $m(p, t)$ is an income function of consumer t. Let us define $D^i(p, t) = h^i(p, m(p, t), t)$. Then the own price effect on $D^i(p, t)$ becomes

$$\partial D^i(p, t)/\partial p_i = S_{ii}(p, m, t) + \{(\partial m(p, t)/\partial p_i - h^i(p, m, t)\} \partial h^i(p, m, t)/\partial m$$

where $m = m(p, t)$. Since there is no assurance that $\partial m(p, t)/\partial p_i - h^i(p, m, t)$ is negative, we may not have $\partial D^i(p, t)/\partial p_i < 0$ even if good i is a normal good.

For example, suppose that $m(p, t) = p \cdot \bar{x}(t)$, where $\bar{x}(t)$ is the initial endowment vector of consumption agent t. Then for this case, we have

$$\partial D^i(p, t)/\partial p_i = S_{ii}(p, m, t) + (\bar{x}_i(t) - h^i(p, m, t)) \partial h^i(p, m, t)/\partial m .$$

Thus when good i is normal and agent t is a seller of good i, i.e., $\bar{x}_i(t) - h^i(p, m, t) > 0$, then $(\bar{x}_i(t) - h^i(p, m, t)\partial h^i(p, m, t)/\partial m > 0$. Hence there is nothing peculiar in having $\partial D^i(p, t)/\partial p_i > 0$. This is basically the case of a backward-bending supply curve. Also in the theory of international trade, an offer curve with its import elasticity between zero and one falls in this case. Since even an individual demand is not necessarily downward sloping for a normal good, the aggregation over many consumption agents does not appear to give us much help. Still some correlation arguments can be made as follows. Define

$$
\begin{aligned}
x_i(t) &= h^i(p, m(p, t), t) \\
z_i(t) &= x_i(t) - \omega_i(t) \\
\mu_i(t) &= \partial h^i(p, m(p, t), t)/\partial m \\
S_{ii}(t) &= S_{ii}(p, m(p, t), t) \\
\bar{x}_i &= \int_T x_i(t)\,d\lambda \\
\bar{\omega}_i &= \int_T \omega_i(t)\,d\lambda \\
\bar{z}_i &= \int_T z_i(t)\,d\lambda \\
\bar{\mu}_i &= \int_T \mu_i(t)\,d\lambda \\
\bar{S}_{ii} &= \int_T S_{ii}(t)\,d\lambda\,.
\end{aligned}
$$

Then we can write:

$$
\begin{aligned}
\int_T \partial D^i(p, t)/\partial p_i\,d\lambda &= \int_T S_{ii}(t)\,d\lambda - \int_T z_i(t)\,\mu_i(t)\,d\lambda \\
&= \bar{S}_{ii} - \int_T (z_i(t) - \bar{z}_i)(\mu_i(t) - \bar{\mu}_i)\,d\lambda - \bar{z}_i\bar{\mu}_i \\
&= \bar{S}_{ii} - C(z_i, \mu_i) - \bar{z}_i\bar{\mu}_i
\end{aligned}
$$

where $C(z_i, \mu_i) = \int_T (z_i(t) - \bar{z}_i)(\mu_i(t) - \bar{\mu}_i)\,d\lambda$. Therefore if $z_i(t)$ and $\mu_i(t)$ are positively correlated, then it reinforces the downward slopingness of a demand function. Noting that $z_i(t) = x_i(t) - \omega_i(t)$, we can further decompose $C(z_i, \mu_i)$ as

$$C(z_i, \mu_i) + \bar{z}_i\bar{\mu}_i = C(x_i, \mu_i) + \bar{x}_i\bar{\mu}_i - C(\omega_i, \mu_i) - \bar{\omega}_i\bar{\mu}_i\,.$$

See Mas-Colell (1985, 5.7) for further discussion.

3. Consumers Demand Over Time*

In the previous chapters we assumed that consumers choose among commodity bundles which were elements of a finite dimensional space. In this chapter we provide an introduction to the case where the consumer chooses among time paths such commodity bundles. The theory of demand for such commodity spaces is just beginning to take recognizable shape. Thus we will not deal with the problem in its full generality. We state a general version of the problem and provide a partial discussion of a special case.

* This section is based on El Hodiri (1985)

General Statement of the Problem

Let $[0, T] \subseteq R_+$ be given. We call $[0, T]$ the planning period for the consumer. Let X be the set of functions defined on $[0, T]$ with values in some subset of R^n, say R^n_+. The set X will be our consumption set.

We assume that the consumer has a preference ordering defined on X. Let (m, p) be expected income-price streams which are functions defined on $[0, T]$ with values in R^{n+1}_+. Suppose a budget map B(m, p) is defined so as to assign a subset of X to each income-price stream. We then may define a *demand map* as an operator that assigns to each income-price stream a subset of X which consists of those elements of B(m, p) to which no element of B(m, p) is preferred.

The questions we need to discuss will be similar to those we discussed in our other chapters, e.g. what properties of demand operators can be proved and under what conditions can we recover the preference orderings.

In order to help appreciate the difficulties involved in providing full answers to the above questions, we discuss a very special case.

In the first place we assume that the consumers' preferences are represented by a very special functional; the discounted sum of instantaneous utilities. This assumption embodies a certain amount of additive separability as far as time is concerned. We, furthermore, assume that the budget map is defined in terms of a wealth function. Finally the demand functions that we analyze are lifetime demand functions.

Even for this simple special case, we can't provide full answers to all of our questions. We will only show that cumulative demand functions have "most" of the usual properties of spot demand functions.

Demand Functions, a Special Case

Let the preferences of the consumer be represented by the functional

$$(3.1) \quad U[x] = \int_0^T \phi(x(r))\, e^{-\varrho t} dt$$

where ϕ is the instantaneous utility of consuming bundle x at time t and where $\varrho > 0$ is fixed. If the consumer, with perfect foresight, knows p(t) and m(t) for all $t \in [0, T]$ where (m, p) is the income-price stream, and if we denote by w(t) the consumers' wealth at time t, then

$$(3.2) \quad \dot{w} = \frac{dw}{dt} = m(t) - p(t) \cdot x(t) + r(t)\, w(t)$$

represents the change in wealth when the consumer buys x at price p, where r is the rate of interest. If we know the initial wealth $w(0) = w_0$ and x then we can solve (3.2).

The problem of the consumer is to choose x such that the utility functional is maximized subject to the constraint that $w(T) \geqq 0$, where w(t) is a solution to (3.2) with $w(0) = w_0$. The problem we formulated is a simple problem in optimal control and we use the first order conditions, see mathematical appendix, to explore the nature of solution.

To be able to make use of optimal control theory we must place some restrictions on the paths of bundles and on the utility function ϕ. We shall assume that the paths $x(t)$ are sectionally continuous and that the function ϕ is twice continuously differentiable in all of its arguments.

Let x be a solution to the consumer's problem. Then there exist constants $\lambda_0 \geqq 0$ and $\gamma \geqq 0$ and a function $q\colon [0, T] \to R$ with $(\lambda_0, q, \gamma) \neq 0$ such that

(3.3) $\lambda_0 \phi_i e^{-\varrho t} \leqq q p_i$, with equality if $x_i > 0$, where $\phi_i = \partial\phi/\partial x_i$.

(3.4) $\dot{q} = -rq$, $q(T) = \gamma$, $\gamma w(T) = 0$.

Assume that there always exists a commodity which has a positive price and which is consumed in positive quantity at some time. Then we can show that $\lambda_0 > 0$ and hence may be taken equal to 1. In fact we will assume that $x_i > 0$ at all $t \in [0, T]$ for all i from now on, that all p_i are positive and that m is positive. Then we write (3.3) as

$q = \gamma e^{-R(t)} \Rightarrow q(T) = \gamma e^{-R(T)}$

(3.5) $\phi_i e^{-\varrho t} = q p_i$.

Solving (3.4) our conditions become

(3.6) $\phi_i e^{-\varrho t} = \gamma p_i e^{-R(t)}$, $\gamma w(T) = 0$, where $R(t) = \int_0^t r(\tau)\,d\tau$.

Assuming that there is a commodity i such that the marginal utility of i is positive we see from (3.6) that $w(T) = 0$ since $\gamma > 0$. By (3.6) we have

(3.7) $\phi_x \cdot x e^{-\varrho t} = \gamma (p \cdot x) e^{-R}$.

By (3.7) and (3.2) we have

(3.8) $\dot{w} = rw + m - \dfrac{1}{\gamma} \phi_x \cdot x e^{R - \varrho\tau}$.

Solving (3.8) with $w(T) = 0$ we get

(3.9) $\gamma = \dfrac{\int_0^T \phi_x \cdot x e^{-\varrho t}}{w_0 + \int_0^T m e^{-R}}$

and

(3.10) $w(t) e^{-R} = w_0 + \int_0^t m e^{-R(s)} ds - \dfrac{\int_0^t \phi_x \cdot x e^{-\varrho s}}{\int_0^T \phi_x \cdot x e^{-\varrho t}} (w_0 + \int_0^T m e^{-R})$.

Equation (3.10) provides a plausible microeconomic foundation for a life cycle hypothesis since it gives conditions under which w is negative (borrowing) for t close to zero.

Using (3.9) we write (3.6) as

(3.11) $\phi_i e^{-\varrho t} = \left(\dfrac{\int_0^T \phi_x \cdot x e^{-\varrho t}}{w_0 + \int_0^T m e^{-R}} \right) p_i e^{-R}$.

An Example: In the case where ϕ is log linear, i.e., $\phi(x) = \sum_i \alpha_i \log x_i$, we have $\phi_i = \dfrac{\alpha_i}{x_i}$. Thus $\phi_x \cdot x = \sum_i \alpha_i$ and (3.11) becomes

$$(3.12)\quad \frac{\alpha_i}{x_i}e^{-\varrho t} = \frac{\int_0^T \sum \alpha_i e^{-\varrho t}}{w_0 + \int_0^T m e^{-R}} \cdot p_i e^{-R}.$$

Thus

$$(3.13)\quad x_i = \frac{1}{p_i e^{-R}}\left(\frac{\alpha_i}{\sum \alpha_i}\right)\left(\frac{\varrho e^{-\varrho t}}{1-\varrho^{-\varrho T}}\right)(\overset{w}{\cancel{W}}_0 + \int_0^T m e^{-R} dt).$$

It

~~I~~ can be seen, easily, from (3.13) that the instantaneous own price effect is negative. However, if prices and income change in the same proportion, then x_i would change!

If we multiply x_i by p_i and sum over i, then the total expenditure on consumption at time t, call it c, is given by

$$(3.14)\quad C = [\varrho e^{R-\varrho t}/1 - e^{-\varrho T}](w_0 + \textstyle\sum_0^T m e^{-R} dt).$$

Since the first term of 3.14 is independent of current income, (3.14) provides an instance of the permanent income hypothesis.

Hicks's Generalized Law of Demand

Going back to our original case, we can show that cumulative demand satisfies Hicks's generalized law of demand. To make the exposition simple, we assume that the utility function is concave.

Let (p^1, r^1, m^1, w_0^1) and (p^2, r^2, m^2, w_0^2) be two paths or prices, incomes and interest rates as well as two levels of initial wealth. Let (x^1, w^1) and (x^2, w^2) be the corresponding optimal consumption and wealth paths. Let $(\mathring{x}, \mathring{w})$ be the compensated consumption and wealth paths, i.e., the optimal paths that would be chosen under p^2, r^2 if income, m, is changed so that

$$(3.15)\quad \int_0^T \phi(x) e^{-\varrho t} dt = \int_0^T \phi(x^1) e^{-\varrho t} dt$$

since $\mathring{x}$ satisfies (3.15) and since ϕ is concave we have

$$(3.16)\quad 0 = \int_0^T [\phi(x^1) - \phi(x^0)] e^{-\varrho t}$$
$$\leqq \int_0^T \mathring{\phi}_x \cdot (x^1 - x^0) dt = \int_0^T \mathring{\phi}_x \Delta x e^{-\varrho t} dt, \quad \text{where } \Delta x = x^1 - x^0.$$

But x^0 is optimal, so by (3.6) we have

$$\int_0^T p^2 \cdot \Delta x e^{-R} dt \geqq 0.$$

But $\gamma > 0$, so

$$(3.17)\quad \int_0^T e^{-R^2} p^2 \cdot \Delta x \geqq 0, \quad \text{where } R^2 = \int_0^t e^{-r(s)} ds.$$

Similarly, we can slow

$$(3.18)\quad \int_0^T e^{-R^1} p^1 \cdot (-\Delta x) \geqq 0.$$

Adding (3.17) and (3.18) we get

$$(3.19)\quad \int_0^T \Delta p \cdot \Delta x dt \leqq 0, \quad \text{where } \Delta p = p^1 e^{-R_1} - p^2 e^{-R_2}.$$

Relation (3.18) is a statement of Hicks's generalized law of demand for the present case.

From (3.18) we conclude that the own compensated cumulative price effect is non-positive. The instantaneous own compensated price effect need not, in general, be negative in view of the intertemporal decisions that are explicitly introduced; see Lluch and Morishima (1973).

Problems

1. Suppose that consumers are uniformly distributed over [0, 1] and consumer t has a utility function of a form $u(x, y, t) = (t+1)x + y$, $t \in [0, 1]$. Let p be the price of "x" in terms of the income good "y". All consumers have an identical income $m > 0$ measured in terms of y.

   Derive the aggregate demand function for the good "x".

2. Suppose that consumers are indexed by an income level and the income distribution is uniformly distributed over $[\underline{m}, \bar{m}]$ with a unit density. Let α and β be positive constants satisfying $\alpha < \beta$ and $(\beta/\alpha) < (\bar{m}/\underline{m})$. Assume that the taste of consumers is identical and the individual demand for good x is given as follows: $x = 0$ if $0 \leqq m/p < a$, $x = 1$ if $a \leqq m/p < b$ and $x = 2$ if $b \leqq m/p$.

   Derive the aggregate demand function for good x.

3. Consider the classical situation of consumer's choice. Suppose that a consumer agent has his preference represented by a real-valued function u: $R^n_+ \to R$ denoted typically by $u(x_1, x_2, \ldots, x_n)$. Suppose that x_1 is the consumption of "leisure" measured by hours. Let L denote total hours (fixed) available for "work" and "leisure", i.e., $x_1 + z \leqq L$ where z is "work" hours or the supply of labor. Let p_i $(i = 2, \ldots, n)$ be the price of a unit of "x_i" $(i = 2, \ldots, n)$ and q be the cost of an hour of leisure. The consumer earns two types of income, i.e., the wage income and the nonwage income. Thus we have income $= wz + \pi$ where w is the hourly wage rate and π is the nonwage income.

   (a) Let $p_1 = q + w$. Then *show* that the budget of the consumer can be written as

   $$(*) \quad \sum_{i=1}^{n} p_i x_i \leqq m, \qquad x_i \geqq 0$$

   where $m = wL + \pi$.

   (b) With (*) as the budget inequality, as in the standard case we can obtain the usual demand functions denoted by $x_i = h^i(p_1, \ldots, p_n, m)$ noting $p_1 = q + w$ and $m = wL + \pi$, *indicate how* we can find the demand for leisure and the supply of labor as functions at $(w, w, p_2, \ldots, p_n)$ denoted by:

   $$x_1 = D(w, q, p_2, \ldots, p_n), \quad \text{and}$$

   $$z = S(w, q, p_2, \ldots, p_n).$$

(c) *Express* $\partial D/\partial w$ and $\partial S/\partial w$ in terms of Slutsky terms and $\partial h^i/\partial m$. What is the major difference between the decomposition of $\partial D/\partial w$ and of the "backward bending" supply of labor, i.e., the case with $\partial S/\partial w < 0$.

4. Consider the consumer demand over time problem in R^2. Suppose the utility functional is given by $u(x) = (\int_0^T x_1)(\int_0^T x_2)$. Derive the cumulative demand function.

5. In the consumer's demand over time, in R^2, let the discounting term in the utility functional $u(x) = \int_0^T x_1 x_2 e^{-\varrho(t)}$ be dependent on past consumption, e.g., $\varrho(t) = \int_0^t x_1 + x_2$. Do you still get the "permanent income hypothesis"?

Part II Costs, Production and Price Taking Firms

Introduction

The concept of "technology" is the primitive concept in the theory of production. We devote this part to the study of the economists' understanding of this concept and of its relationship to costs and to decision making by price taking firms.

In Chapter 4 we present the relationship between costs, for a price taking firm, and production. Next, Chapter 5, we study the behavior of a single competitive firm. In Chapter 6 we study competitive markets, i.e. markets where all firms are price takers.

Chapter 4 Costs and Production

1. Introduction

In this chapter we study, in some detail, the concepts of technology set, production function, and cost function. Our presentation is motivated by the desire to provide a rigorous foundation for the theory of costs and production that is usually presented in intermediate microeconomics text books. Therefore, unlike some other advanced microeconomics books, we will not employ the assumption of convex technology sets. Instead, our analysis will be focused on the familiar U-shaped cost curves which are at the center of the traditional theories of perfect and imperfect competition and which are largely the legacy of Alfred Marshall.

In Section 2 we define short-run and long-run technology sets and provide definitions and elementary properties of implicit and, explicit, production functions associated with technology sets. In Section 3 we will consider the short run outlay minimization problem of a firm which is a price taker in the market for inputs and, thus, obtain some basic properties of cost functions and of conditional variable input demand functions. In Section 4 we investigate the duality between production functions and cost functions. Using a calculus approach we show the equivalence of the fact that the marginal cost function is locally increasing and the fact that the underlying production function is locally concave.

2. Technology Sets and Production Functions

Several concepts can serve as primitive concepts in production theory. For instance one could use production processes or one could use the set of input output combinations that are at the firm's disposal. In either case, we are attempting to describe the current state of the art of production which is known and accessible to the firm.

The combination of processes makes up what we will term the technology set of the firm. On the other hand, one may bypass processes and define technology set directly as a subset of the commodity space which is feasible to the firm. Let T denote the technology set, then $T \subseteq R^l_+$, where l is the dimension of the input-output space. We think of an element of T as an l-vector (x, y) of inputs, x and outputs, y. Later on, we will classify inputs into long run and short run and we will classify output into a principal output and byproducts. We take $x \in R^{n+q}_+$ and $y \in R^m_+$.

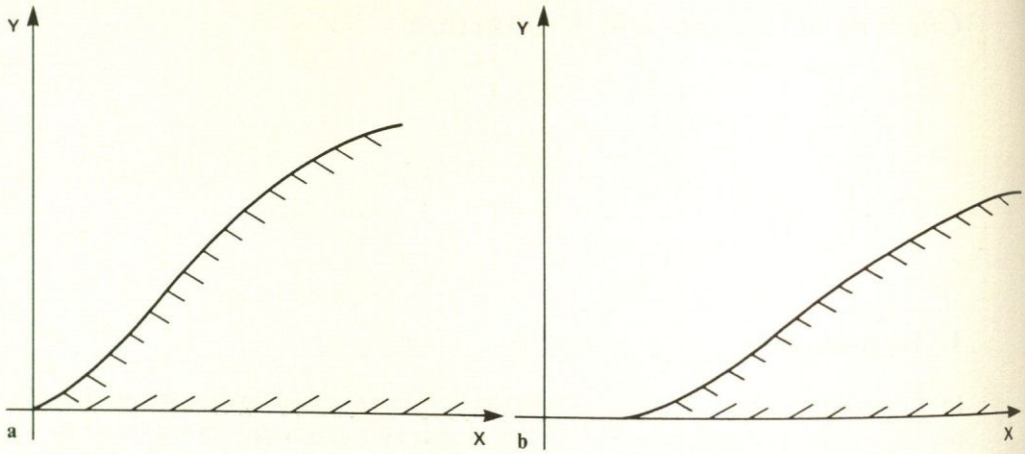

Fig. 2.1.a, b

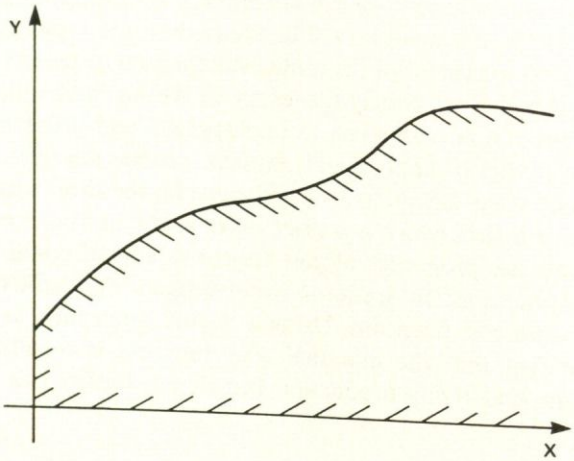

Fig. 2.2

A general description of technology sets may be provided by reviewing some of the standard assumptions about it. These assumptions are not independent from one another and, obviously, are not all invoked at once.

The first assumption asserts the possibility of shut down, i.e., that no inputs are needed if one decides to produce no outputs, i.e.

(2.1) $0 \in T$.

Clearly 2.1 is a long run assumption for it is conceivable that, in the short run, certain inputs will have to be used even though output levels are zeros. Figure 2.1.a gives an example of a set that satisfies 2.1 and Figure 2.1.b gives an example of a set that violates it.

The next assumption asserts the impossibility of producing positive quantities of outputs without positive input level utilization, i.e.

(2.2) $(x, y) \in T$ and $y \geq 0$ imply $x \geq 0$.

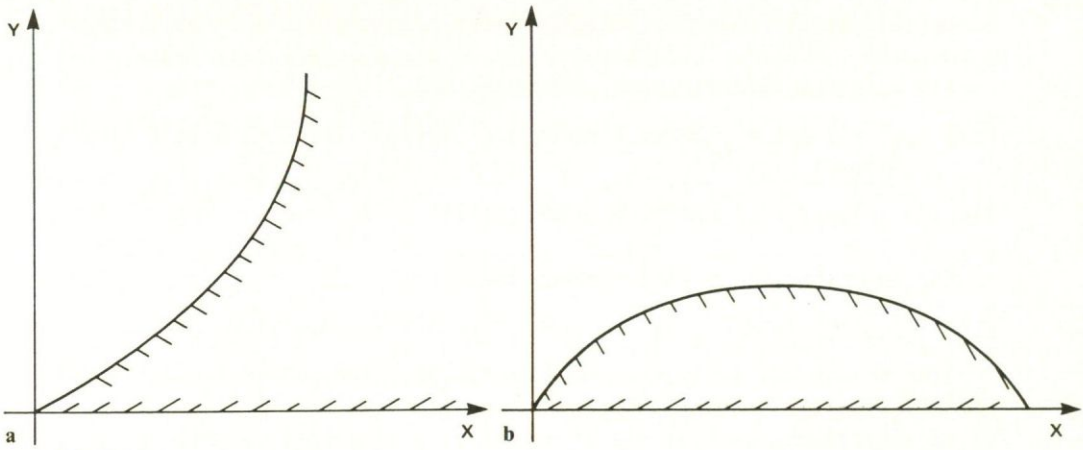

Fig. 2.3.a, b

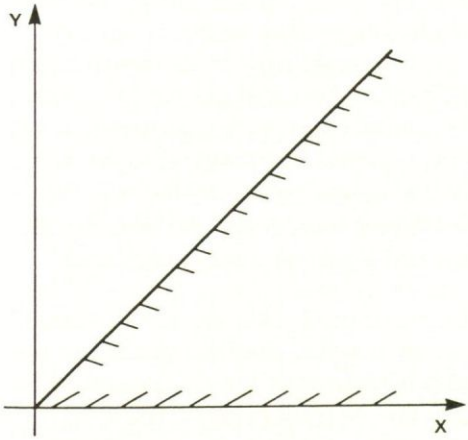

Fig. 2.4

The sets in Figure 2.1 satisfy assumption 2.2 but the set in figure 2.2 does not. The next assumption is that of additivity, i.e.

(2.3) If (x^1, y^1) and (x^2, y^2) are in T, then so is $(x^1 + x^2, y^1 + y^2)$.

The set in Figure 2.3.a satisfies the above assumption, but the one in 2.3.b does not.

The fourth assumption is often referred to as divisibility, i.e., if an input-output vector is possible, then so is any part of it, i.e.,

(2.4) $(x, y) \in T$ implies $\alpha(x, y) \in T$ for any $\alpha \in R$ with $0 < \alpha < 1$.

The set in Figure 2.3.~~a~~b satisfies divisibility while the set in Figure 2.1.a does not. The next assumption is that of constant returns to scale, i.e.

(2.5) If $(x, y) \in T$, then $\theta(x, y) \in T$ for all $\theta > 0$.

None of the sets in Figures 2.1 – 2.3 satisfy this assumption, but the set in Figure 2.4 does.

The next assumption is that of convexity, i.e.,

(2.6) (x^1, y^1) and (x^2, y^2) in T imply $\theta(x^1, y^1) + (1-\theta)(x^2, y^2)$ is in T for all $0 \leqq \theta \leqq 1$.

The sets in Figures 2.4 and 2.3.b satisfy convexity, but the sets in Figure 2.1 do not.

Our final assumption is technical in nature.

(2.7) For every $x \in R_+^{n+q}$, the set $\{y \in R_+^m \mid (x, y) \in T\}$ is closed and bounded.

The relevant sets depicted in all of the figures in this section are closed and bounded. except possibly Figure 2.3 a.

As we mentioned before, the above assumptions are not independent from one another, nor are they necessarily consistent.

For instance, additivity and divisibility imply convexity, but not conversely as seen by Figure 2.3.b which also shows that convexity does not imply constant returns to scale. Convexity does not imply divisibility (one needs 2.1 for that).

Let the first n components of x be variable inputs and let the remaining q components be inputs that vary only in the long run. We write $x = (x^I, x^{II})$ where $x^I \in R_+^n$ and $x^{II} \in R_+^q$. We designate the last component of y, y_m, as the principal product. The remaining outputs, denoted by $y_{)m(}$, may be thought of as by-products. If no inputs are fixed, then we think of T as a *long run technology set*. Otherwise, for each level of fixed inputs x^{II}, we define a *short run technology set* by

$$T(x^{II}) = \{(x^I, y) \mid (x^I, x^{II}, y) \in T\}.$$

Efficient points in T are input-output combinations that are not dominated by other elements of T, i.e., by an element of T which uses less of at least one input – all else being the same – or produces more of at least one output with all else remaining fixed. Formally

(2.2) *Definition.* We say that an element $(x, y) \in T$ is *efficient* if there does not exist another element (x^1, y^1) in T with $(-x^1, y^1) \geq (-x, y)$, in the sense of vector inequality defined ~~in the mathematical appendix~~. on p. 3.

Short run efficiency may be defined by taking $(x) = (x^I)$ in the above, long run, definition.

Note. From now on we assume that $T(x^{II})$ satisfies 2.1, 2.2, 2.7 and

(2.8) (*Assumption*). If $(x, y) \in T$ and $0 \leqq y_m^1 \leqq y_m$, then $(x, y^1) \in T$, where $y^1 = (y_{)m(}, y_m^1)$. Furthermore, if $(x, y) \in T$ and $y \geq 0$, then $y_m > 0$.

The first part of assumption 2.8 asserts the free disposability of the principal product. The second part states that if anything is produced, then the principal product is produced.

We define a production function as a relationship that assigns to each input-byproduct vector, the highest possible value of the principal product. Formally it is a function $F: R_+^{l-1} \to R$ given by $F(x, y_{)m(}) = \max \{y_m \mid (x, y) \in T\}$ if the set

$\{y_m \mid (x, y) \in T\}$ is not empty. Otherwise we take $F = \gamma$ where γ is an arbitrary negative number. We note that F thus defined is indeed a function in view of the compactness of the set used in the first part of the definition. We define an *implicit production function* $f\colon R^{\ell}_{+} \to R$ by

$$(2.9) \qquad f(x, y) = F(x, y_{)m(}) - y_m .$$

We observe that $f(x, y) \geqq 0$ if and only if $(x, y) \in T$. Indeed, suppose $f(x^0, y^0) \geqq 0$ and $(x^0, y^0) \notin T$. Then we have two cases: either the set $\{y_m \mid (x^0, y^0, y_m) \in T\}$ is empty or it is not. If it is empty, then by definition, $f(x^0, y^0) = -)m(- y^0_m < 0$ which is a contradiction. If it is not empty, $(x^0, y^0, z^0) \notin T$ implies that $F(x^0, y^0_{)m(}) - y^0_m < 0$ and so we have a contradiction. If $(x^0, y^0) \in T$ and $f(x^0, y^0) < 0$, then we must have $y^0_m < F(x^0, y^0)$. Buth this contradicts the maximality of y_m. We also note that if (x, y) is efficient, then $f(x, y) = 0$. Furthermore, the function F is continuous. This follows from the maximum theorem. (See mathematical appendix, 7.1.) It should be noted that the representation of the technology set by the function f is not unique. In fact any monotone continuous transformation $\phi(f)$ with $\phi(0) = 0$ is a representation of T in the sense that $T = \{(x, y) \mid f(x, y) \geqq 0\}$.

Define the frontier of T as the set $\{(x, y) \in T \mid f(x, y) = 0\}$. We say that $f(x, y)$ is quasi-concave on the frontier of T is for every $(x^0, y^0) \neq (x^1, y^1)$ with both points on the frontier of T we have $f(x^t, y^t) \geqq 0$, where $(x^t, y^t) = t(x^0, y^0) + (1 - t)(x^1, y^1)$ with $0 \leqq t \leqq 1$. If all the inequalities in that last sentence are strict, then we say that f is strictly quasi-concave on the frontier of T.

If f is quasi-concave, then T is convex. Furthermore T is convex if and only if F is concave. The relationship between the properties of f and those of F is given by

Need to change $t \to \ell$ confusing with transpos. notation

(2.10.) *Theorem:* (a) f is (strictly) quasi-concave in (x, y) along the frontier of T if and only if F is (respectively strictly) concave in $(x, y_{)m(})$. (b) Suppose f and F are twice continuously differentiable in their respective arguments at (x^0, y^0) where $y^0_m = F(x^0, y^0_{)m(})$ and suppose $f^0_{\ell} < 0$ where $f^0_{\ell} = (\partial f/\partial y_m)_{(x^0, y^0)}$. Then for every $0 \neq \tilde{\eta} \in R^{\ell-1}$, we have $\tilde{\eta} F'' \tilde{\eta}' < 0$ if and only if for every $\eta \in R^{\ell}$ with $f'\eta = 0$ we have $\eta f'' \eta' < 0$ where f' and F' denote the gradient vectors of the respective functions evaluated at x^0, y^0 and where f'' and F'' denote the Hessians of these functions evaluated at x^0, y^0. The last assertion is also true when the strong inequalities are replaced by weak ones.

Note that (b) is no more than a local smooth version of (a). To prove (a), suppose first that f is strictly quasi-concave on the frontier of T. Fix (x^0, y^0) and (x^1, y^1) such that $(x^0, y^0) \neq (x^1, y^1)$ so that $y^0_m = F(x^0, y^0_{)m(})$ and $y^1_m = F(x^1, y^1_{)m(})$. Let $(x^t, y^t) = t(x^0, y^0) + (1 - t)(x^1, y^1)$. Then we must have $f(x^t, y^t) > 0$ and hence $F(x^t, y^t_{)m(}) > y^t_m = tF(x^0, y^0_{)m(}) + (1 - t)F(x^1, y^1_{)m(})$. Thus F is strictly concave. The converse and the nonstrict case can be proved in a similar fashion.

To prove (b), note that $f(x^0, y^0) = 0$, $y^0_m = F(x^0, y^0_{)m(})$ and $f^0_{\ell} < 0$. Then, by implicit differentiation we have

$$(T) \qquad F_i = -(f_i/f_{\ell})\,|_{\,y_m = F(x,\, y_{)m(})} ,$$

f and F have arguments (x, y), so f_x, F_x are not appropriate

where subscripts of functions denote partial derivatives, $i = 1, \ldots, l-1$, and where (T) holds in some neighborhood of $(x^0, y^0_{)m(})$. Differentiating (T) one more time we get:

$$\text{(TT)} \quad F_{ij} = -(1/f_l)(1\ F_i)\begin{pmatrix} f_{ij} & f_{il} \\ f_{lj} & f_{ll} \end{pmatrix}\begin{pmatrix} 1 \\ F_j \end{pmatrix}.$$

Thus we can write

$$F'' = -(1/f_l)\begin{bmatrix} & ; & F_1 \\ I_{l-1} & \vdots & \vdots \\ & \vdots & F_{l-1} \end{bmatrix} f'' \begin{bmatrix} & I_{l-1} & \\ \cdots & \cdots & \cdots \\ F_1 & \ldots & F_{l-1} \end{bmatrix}.$$

Note that for $\tilde{\eta} \in R^{l-1}$ we have:

$$\begin{bmatrix} & I_{l-1} & \\ \cdots & \cdots & \cdots \\ F_1 & \ldots & F_{l-1} \end{bmatrix} \tilde{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{l-1} \\ \sum_{j=1}^{l-1} F_j \eta_j \end{pmatrix}.$$

Let $\eta_l = \sum_{j=1}^{l-1} F_j \eta_j = -\sum_{j=1}^{l-1} (f_j/f_l)\eta_j$, i.e., $\sum_{j=1}^{l} f_j \eta_j = 0$; we get:

$$\tilde{\eta} F'' \tilde{\eta}' = -(1/f_l)/\eta f'' \eta'.$$

But $f_l < 0$ and we get our result.

(2.11) Example: Suppose $n = m = 1$ and $q = 0$. Let T be given by

$$T = \{(x, y) \mid 0 \leqq y \leqq x^2(c - x), \quad x \in [0, 3c/4]\},$$

where $c > 0$ is given. The illustration of T is given in Figure 2.5 where the shaded region is the technology set defined above. A production function representing the above technology set can be defined by:

$$\begin{aligned} F(x) &= x^2(c - x) \quad \text{if} \quad x \in [0, 3c/4] \\ &= -1 \quad \text{if} \quad x > 3c/4. \end{aligned}$$

The value of the production function for $x > 3c/4$ is set at -1 which is arbitrarily chosen so that F is defined for every nonnegative real number. We could adopt the convention, used in convex analysis, which defines the value of F to be $-\infty$ when the production is not feasible. The production function is illustrated in Figure 2.6. An implicit production function $f\colon R^2_+ \to R$ can be defined by $f(x, y) = F(x) - y$. Clearly, we have $T = \{(x, y) \in R^2_+ \mid f(x, y) \geqq 0\}$.

(2.12) *Example:* We take $n = 1$, $m = 2$ and $q = 1$ in this example. First we define functions $\phi\colon R_+ \to R_+$, $h\colon R^2_+ a \to R_+$ and $g\colon R^2_+ \to R_+$ by:

$$\phi(v) = c^3 + (v - c)^3,$$
$$h(x^I, x^{II}) = \min(x^I/a, x^{II}/a)$$

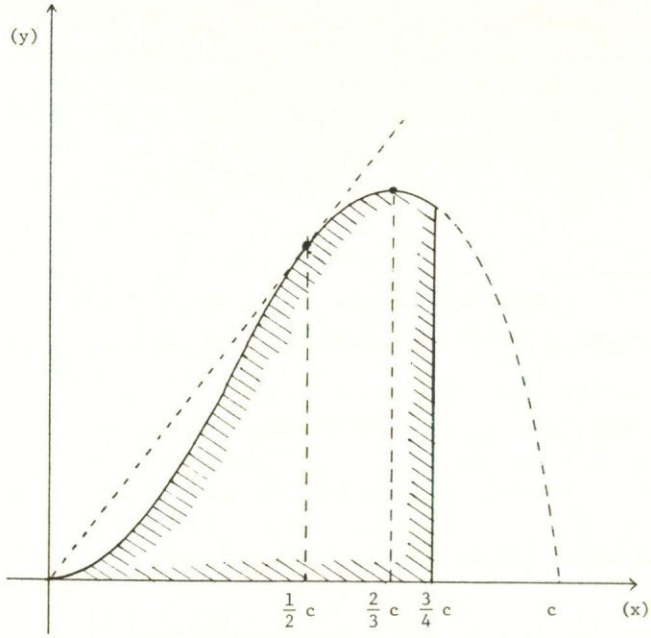

Fig. 2.5

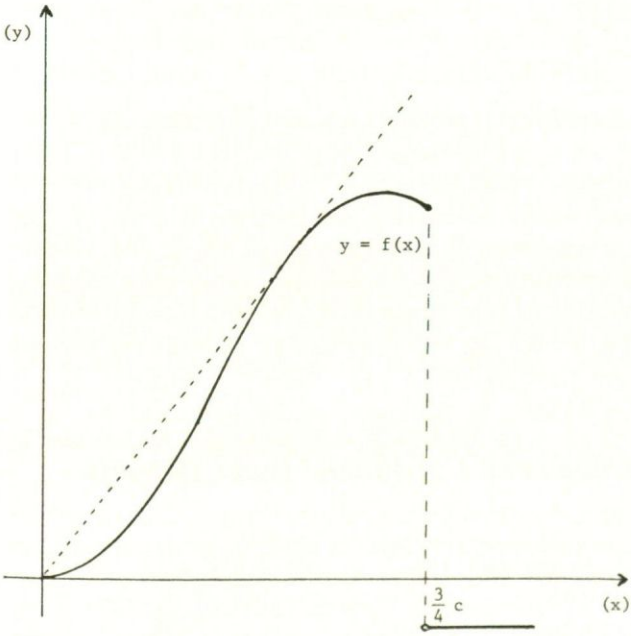

Fig. 2.6

and

$$g(y_1, y_2) = (y_1^2 + y_2^2)^{1/2}.$$

Then we define the technology set by:

$$T = \{(x^{I}, x^{II}, y) \in R_+^4 \mid g(y) \leqq v,\ u = \phi(v),\ h(x^{I}, x^{II}) \geqq u \geqq 0\}.$$

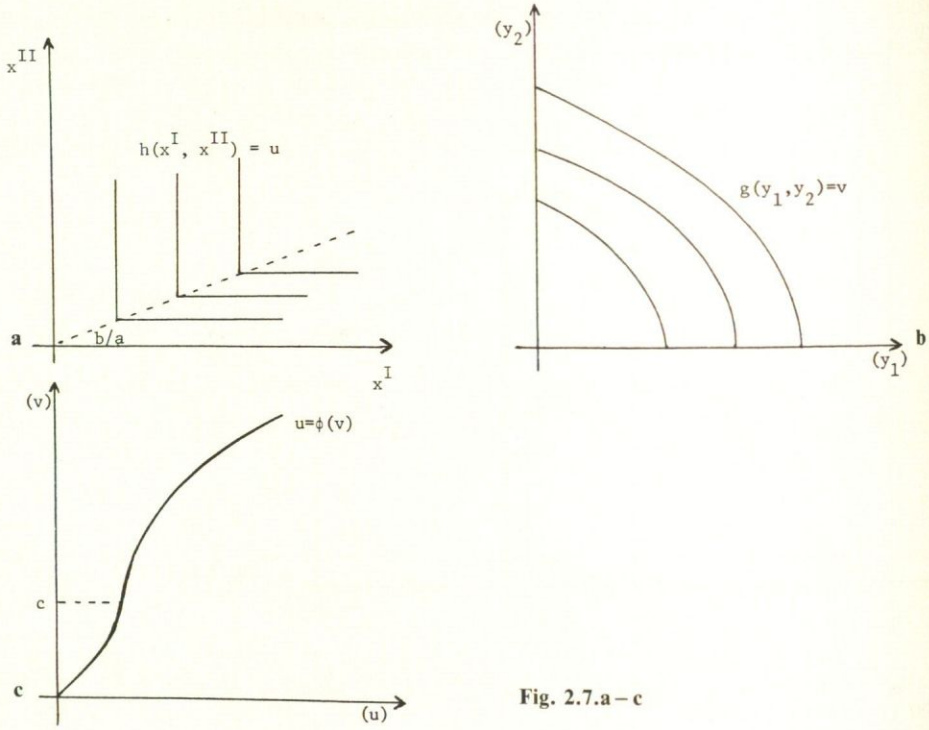

Fig. 2.7.a – c

We may interpret u and v as intermediate products. Under this technology, an input vector (x^I, x^{II}) is transformed into an intermediate product u through h. Then the intermediate product u is transformed into v by ϕ^{-1}. The function ϕ may also be interpreted as a function which determines the returns to scale of this technology since both h and g are positively homogeneous of degree one. Finally the intermediate product v is transformed into a bundle (y_1, y_2) of joint products via the function g. The illustration of level surfaces of functions h and g and the function ϕ is provided in Figure 2.7. In this example, an implicit production function representing T can be given by $f(x^I, x^{II}, y) = h(x^I, x^{II}) - \phi[g(y)]$.

3. Short-run Cost Functions and Conditional Input Demand Functions

In this section we study short-run cost functions and the conditional input demand functions associated with them. The technology set of the firm is assumed to be represented by an implicit production function $f: R^N_+ \to R$, where $N = n + q + m$, as in the preceding section. We will consider the output vector y and the vector of fixed inputs x^{II} among the parameters in the problem of short run outlay minimization. It is convenient to write $v = (x^{II}, y)$ and $s = x^I$. Thus write $f(s; v) = f(x^I; x^{II}, y)$. Define the input-output parameter set V by

May need to change this $(x^I, x^{II}; y)$ to $\phi(x, z; y)$

(3.1) $V = \{v \in R^{q+m}_+ \mid f(x; v) \geq 0 \text{ for some } s \in R^n_+\}$.

Let $w \in P^n$ denote the vector of prices of variable inputs. The short run outlay minimization problem is stated as follows.

Given $w \in P^n$ and $v \in V$ find a variable input vector s which minimizes $w \cdot s$ subject ot $f(s; v) \geqq 0$. We define the *variable cost function* as the minimum variable outlay given (v, w) and provided that $f(s, v) \geqq 0$, i.e. the short run cost function c: $V \times P^n \to R_+$ is given by

$$(3.2) \quad c(v, w) = \min_{\{s \in R^n_+ \mid f(s, v) \geqq 0\}} w \cdot s \ .$$

If the implicit production function is continuous in s, then the variable cost function is well defined and satisfies the convexity and continuity properties stated below.

(3.3) *Proposition:* Let the implicit production function be continuous in s on R^n_+. Then (i) c: $V \times P^n \to R_+$ exists. (ii) c(v, w) is positively homogeneous of degree one in w. (iii) c(v, w) is concave in w for every fixed $v \in V$. (iv) c(v, w) is continuous in w for every fixed $v \in V$.

To prove (i), let $(v, w) \in V\ P^n$ be given and fix $\bar{s}$ such that $f(\bar{s}; v) \geqq 0$. Consider the set D given by

$$D = \{s \in R^n_+ \mid w \cdot s \leqq w \cdot \bar{s}\} \cap \{s \in R^n_+ \mid f(s; v) \geqq 0\} \ .$$

Since $w \in P^n$ and since $w \cdot s$ is continuous in s, the first set in the above intersection is closed and bounded. By continuity of f, the second set is closed. Thus D is a compact subset of R^n_+. Noting that $c(v, w) = \min_{x \in D} w \cdot s$, we conclude that c is well defined.

The proofs of (ii) and (iii) are almost identical with proofs of the homogeneity and the concavity of the minimum expenditure function in Chapter 5, (5.6). The continuity property of (iv) follows from the continuity property of convace functions in convex analysis.

If the implicit production function f is strictly quasi-concave* in s on R^n_+, then it is easy to show that the variable input bundle which minimizes the short-run variable outlay $w \cdot s$ exists uniquely and hence the short-run conditional input demand function minimizing the variable outlay g: $v \times P^n \to R^n_+$ exists and satisfies

$$(3.3) \quad c(v, w) = w \cdot g(v, w) = \sum_{j=1}^{n} w_j g^j(v, w)$$

where $g = (g^1, \ldots, g^j, \ldots, g^n)$. Just as in the case of Hicksian compensated demand functions and the minimum expenditure function in Chapter 2, we can obtain the following properties of the short-run conditional input demand functions and the short-run variable cost function.

V

(3.4) Assume that the implicit production function f is continuous and strictly quasi-concave in s on R^n_+ for each $v \in V$. Then (a) g: $\cancel{v} \times P^n \to R^n_+$ homogeneous of degree zero in $w \in P^n$. (b) Suppose that for every $w^0, w^1 \in P^n, w^0/\|w^0\| \neq w^1/\|w^1\|$

* See mathematical appendix, Section 4.

implies $g(v, w^0) \neq g(v, w^1)$. Then for every $w^0, w^1 \in P^n$ with $w^0/\|w^0\| \neq w^1/\|w^1\|$, we have $(w^1 - w^0) \cdot [g(v, w^1) - g(v, w^0)] < 0$, and for every $t \in (0,1)$, $c(v, w^t) > t\,c(v, w^1) + (1-t)\,c(v, w^0)$ where $w^t = t\,w^1 + (1-t)\,w^0$.

(3.5) *Proposition; Corollary:* Under the assumption of (3.4) (b), if $w^0, w^1 \in P^n$, $w_i^0 \neq w_j^0$ and $w_j^0 = w_j^1$ for every $j \neq i$, then

$$(w_i^1 - w_i^0)\,[g^i(v, w^1) - g^i(v, w^0)] < 0\,.$$

(3.6) Assume that $g^j(v, w) > 0$ for every $j = 1, 2, \ldots, n$ and $g^i(v, w)$ has continuous first partial derivatives with respect to w for every $(v, w) \in V \times P^n$. Then

(a) $\partial c(v, w)/\partial w_j = g^i(v, w) \quad (i = 1, 2, \ldots, n)$,

(b) $\sum_{j=1}^{n} w_j\, \partial g^i(v, w)/\partial w_j = 0 \quad (i = 1, 2, \ldots, n)$,

(c) $\partial g^i(v, w)/\partial w_j = \partial g^j(v, w)/\partial w_i \quad (i, j = 1, 2, \ldots, n)$,

(d) $\left.\dfrac{\partial(g^1, g^2, \ldots, g^n)}{\partial(w_1, w_2, \ldots, w_n)}\right|_{(v,w)}$ is symmetric and negative semi-definite

We say that the implicit production function is *regularly strictly quasi-concave* in $s \in P^n$ for each $v \in V$ with $f(s; v) = 0$ if $v \in V$ and $f(s; v) = 0$ imply that f has continuous second partial derivatives at (s, v) with respect to s and if for every $0 \neq \mu \in R^n$, $f_s \cdot \mu = 0$ implies $\mu' f_{ss} \mu < 0$ where f_s is the gradient vector of f with respect to s and where f_{ss} is the Hessian of f with respect to s. With the assumption of the regular strict quasi-concavity of f in x, we can strengthen the above conclusions of (3.6) (d) and (e)* as follows:

(3.6′) *Proposition:* If the implicit production function f is regularly strictly quasi-concave in $s \in P^n$ for each $v \in V$ and $x \in P^n$ with $f(s, v) = 0$, then for every $(v, w) \in V \times P^n$

(d′) for every $\mu \in R_+^n$, if $\mu/\|\mu\| \neq w/\|w\|$, then $\mu' g_w(v, w)\mu < 0$, and

(e′) $(-1)^r \begin{vmatrix} \partial g^1/\partial w_1, & \ldots, & \partial g^1/\partial w_r \\ \vdots & & \\ \partial g^r/\partial w_1, & \ldots, & \partial g^r/\partial w_r \end{vmatrix} > 0$

$(r = 1, 2, \ldots, n-1)$.

One of the most important properties of short run cost functions relates to the way these functions respond to changes in an output level y_i. For example, we would like to know conditions under which $\partial c/\partial y_i$; the marginal cost with respect to y_i is increasing in y_i. First we will study the global properties of the short run cost function. For this purpose, we introduce the following assumptions about the implicit production function and the set V.

* This is similar to (5.16), Chapter 2

(3.6) *Assumption.* (a) If $f(s, v) > 0$ for some (s, v), then there exists $s' \in R^n_+$ such that $s' \leqq s$ and $f(s', v) = 0$. (b) The set V is convex.

If the implicit production function is monotone increasing in s, i.e., $s' > s$ implies that $f(s', v) > f(s, v)$, then the above assumption (3.6) (a) will be satisfied. The production function derived in Example 1 in Section 2 is not monotone increasing in s for $s \in (2c/3, 3c/4)$, but the above assumption (3.6) (a) is satisfied. (3.6) (b) is no more than a technical convenience. The role of this assumption is to guarantee that if $c(v, w) = w \cdot s$, i.e., s is the outlay minimizing variable input vector for (v, w), then $f(s, v) = 0$. Then we get the following convexity property of the variable cost function with respect to v.

(3.7) Under the assumption (3.6), if the implicit production function f is (strictly) quasi-concave in (s, v) along the production frontier then $c(v, w)$ is (respectively strictly) convex in v.

In order to prove (3.7), choose $v^0, v^1 \in V$, $v^0 \neq v^1$ and let s^0, s^1 be defined by $c(v^0, w) = w \cdot s^0$ and $c(v^1, w) = w \cdot s^1$. By (3.6), we have $f(s^0, v) = 0$ and $f(s^1, v) = 0$. Define $s^t = t s^1 + (1-t) s^0$ and $v^t = t v^1 + (1-t) v^0$ for $t \in (0, 1)$. Since f is strictly quasi-concave along the production frontier, we have $f(s^t, v^t) > 0$. By (3.6), there exists $s' \in R^n_+$ such that $s' < s^t$ and $f(s', v^t) = 0$. Then we obtain that $c(v^t, w) \leqq w \cdot s' < w \cdot s^t = t w \cdot s^1 + (1-t) w \cdot s^0 = t c(v^1, w) + (1-t) c(v^0, w)$.

(3.8) *Corollary:* Under (3.6), if the implicit production function f is (strictly) quasi-concave in (s, v) along the production frontier and if the short-run variable cost function $c(v, w)$ has continuous second partial derivatives with respect to y, then the marginal cost $\partial c(v, w)/\partial y_i$ with respect to y_i is nondecreasing (respectively strictly increasing) in y_i.

This obtains because, if $c(v, w)$ is strictly convex in v, then the Hessian matrix of $c(v, w)$ with respect to v is positive definite for almost all (v, w). In particular, each diagonal element of the Hessian matrix is positive, i.e., $\partial^2 c(v, w)/\partial^2 y_i > 0$, for almost all (v, w). Hence the marginal cost $\partial c(v, w)/\partial y_i$ with respect to y_i must be strictly increasing in y_i. The nonstrict case can be argued similarly.

We have shown earlier in (2.3) (a) that the (strict) quasi-concavity of the implicit production function is equivalent to the (strict) concavity of the associated production function. Therefore (3.7) and (3.8) show that the (strict) concavity of the production function entails the nondecreasing (respectively increasing) marginal cost. But this result is not applicable when the production function is not globally concave. Since the case of U-shaped average or marginal cost curves is important, we would like to obtain results when production functions are not necessarily globally concave. One approach in this direction is to obtain local properties of cost functions in relation to local properties of production functions. To do this, suppose that $(v^0, w^0) > 0$ is given and let $s^0 = g(v^0, w^0) > 0$ and we will impose the following assumption.

(3.9) *Assumption:* (a) The implicit production function is strictly quasi-concave in s. (b) $s^0 > 0$ and f has continuous second partial derivatives with respect to all variables at $(s^0, v^0) = (x^0, y^0)$. (c) $f^0_x > 0$, $i = 1, \ldots, n+q$ and $f^0_y < 0$, $j = 1, \ldots, m$, where f^0_x and f^0_y are gradient vectors of f with respect to x and y respectively evaluated at (x^0, y^0).

We can write the Lagrangian for the minimum outlay problem as follows: $L(\lambda, s; v, w) = w \cdot s - \lambda f(s, v)$, where $\lambda = \lambda(v, w)$ is the Lagrange multiplier associated with the value (v, w) of the parameters.

The following result follows from the envelope theorem in the mathematical appendix, 5 – 11

(3.10) $c_v^0 = -\lambda^0 f_v^0$,

where c_v^0 and f_v^0 are the gradients of c and f with respect to v evaluated at (v^0, w^0). Since $\lambda^0 > 0$, $f_{y_j}^0 < 0$ and $f_{z_k}^0 > 0$, the result (3.10) implies that the marginal cost $c_{y_j} = \partial c/\partial y_j$ with respect to y_j is positive and the variable cost shifts down when z_k is increased around (v^0, w^0). Let $r = m + q$. We say that the implicit production function f is *regularly strict quasi-concave in* (s, v) *at* (s^0, v^0) if, for every $\eta \in R^{n+r}$ with $\eta \neq 0$, $[(f_s^0)', f(f_v^0)']\eta = 0$ implies $\eta' \begin{bmatrix} f_{ss}^0, & f_{sv}^0 \\ f_{vs}^0, & f_{vv}^0 \end{bmatrix} \eta < 0$, and similarly that the short-run variable cost function c is *regularly strictly convex in* v *at* (v^0, w^0) if, for every $\mu \in R^r$ with $\mu \neq 0$, $\mu' c_{vv}^0 \mu > 0$. Then we can obtain the following local version of (3.7).

(3.11) *Proposition:* If the implicit production function f is regularly strictly quasi-concave in (s, v) at (s^0, v^0), then the short-run variable cost function c is regularly strictly convex in v at (v^0, w^0).

Note that, on an open neighborhood around (v^0, w^0), the following equations must hold identically.

$$f[g(v, w), v] = 0,$$

$$\lambda(v, w) f_s[g(v, w), v] = 0, \quad \text{and}$$

$$c_v(v, w) = -\lambda(v, w) f_v[g(v, w), v].$$

If we differentiate the above three systems of equations with respect to v and evaluate the results at (v^0, w^0), then we can get:

(3.12)(a) $f_s^0 g_v^0 + f_v^0 = 0_r$

(3.12)(b) $f_s^0 \lambda_v^0 + \lambda^0 [f_{ss}^0 g_v^0 + f_{sv}^0] = 0_{nr}$

(3.12)(c) $c_{vv}^0 = -f_v^0 \lambda_v^0 - \lambda^0 [f_{vs}^0 g_v^0 + f_{vv}^0]$

where 0_r is and $(r \times 1)$ zero vector and 0_{nr} is an $(n \times r)$ zero matrix. If we premultiply (3.12) (b) by (g_v^0) and note (3.12)(a), then we get:

$$-(f_v^0)(\lambda_v^0) + \lambda^0 [f_{ss}^0 g_v^0 + f_{sv}^0] = 0_{nm},$$

i.e.,

$$(f_v^0)' \lambda_v^0 = \lambda^0 [f_{ss}^0 g_v^0 + f_{sv}^0].$$

If we substitute the above into (3.12)(c) and divide through by λ^0, then we can get:

(3.13) $(1/\lambda^0) c_{vv}^0 = -[(g_v^0)', I_r] \begin{bmatrix} f_{ss}^0, & f_{sv}^0 \\ f_{vs}^0, & f_{vv}^0 \end{bmatrix} \begin{bmatrix} g_v^0 \\ I_r \end{bmatrix}$

* We let $x_k^0 = z_k$ here.

where I_r is and $(r \times r)$ identity matrix. Let $\mu \in \mathbb{R}^r$ with $\mu \neq 0$ and define $\eta \in R^{n+r}$ by $\eta = \begin{bmatrix} g_v^0 \\ I_r \end{bmatrix} \mu$. Then

$$(1/\lambda^0)\, \mu' c_{vv}^0 \mu = -\eta' \begin{bmatrix} f_{ss}^0, & f_{sv}^0 \\ f_{vs}^0, & f_{vv}^0 \end{bmatrix} \eta$$

and, by (3.12)(~~b~~a),

$$[f_s^0, f_v^0]\, \eta = (f_s^0 g_v^0 + f_v^0)\, \mu = 0 .$$

Therefore if f is regularly strict quasi-concave at (x^0, v^0), then we must have $\mu' c_{vv}^0 \mu > 0$ which gives us the desired result.

Note that if the matrix c_{vv}^0 is positive definite, all diagonal elements must be positive. In particular, the marginal cost $c_{y_j}(v^0, w^0)$ with respect to y_j must be increasing at (v^0, w^0).

4. Duality Between Production Functions and Cost Functions

In the previous section, we examined some of the basic properties of the short-run variable cost function derived from an implicit production function. Let $s^0 = g(v^0, w^0) > 0$, $\alpha^0 = c(s^0, w^0)$ and assume that the implicit production function is regularly strict quasi-concave in s for each $v \in V$ and for each s with $f(s, v) = 0$. Then the minimum variable outlay condition requires that

$$(4.1) \quad \frac{f_{s_i}(\cancel{x}s^0, v^0)}{f_{s_n}(\cancel{x}s^0, y^0)} = \frac{w_i^0}{w_n^0}, \quad \text{and} \quad f(s^0, v^0) = 0 .$$

On the other hand, the short-run variable cost function satisfies the inequality of strict concavity for nonproportionate variable input price vectors as stated in (3.4)(b) and, with the regularity of the implicit production function, the Hessian of c satisfies

$$\text{for every } \mu \in R^n, \text{ if } \mu / \|\mu\| \neq w / \|w\|, \text{ then } \mu' c_{ww}(v, w)\, \mu < 0 .$$

Moreover the short-run variable cost function satisfies conditions analogous to (4.1) as follows:

$$(4.2) \quad c_{w_i}(v^0, w^0) = s_i^0, \quad \text{and} \quad c(v^0, w^0/\alpha^0) = 1 .$$

For the case of two variable inputs, we provide illustrations of conditions (4.1) and (4.2) in Figure 4.1 and Figure 4.2 respectively. These two relationships strongly suggest that an implicit production function and the associated variable cost function are no more than alternative parameterizations of the production surfaces which are equivalent to each other. This relationship or correspondence between production functions and cost functions has been termed the duality between these two functions. Although convex analysis has been extensively employed in the literature on the duality between production functions and cost functions, we will obtain similar results through calculus and linear algebra in this section.

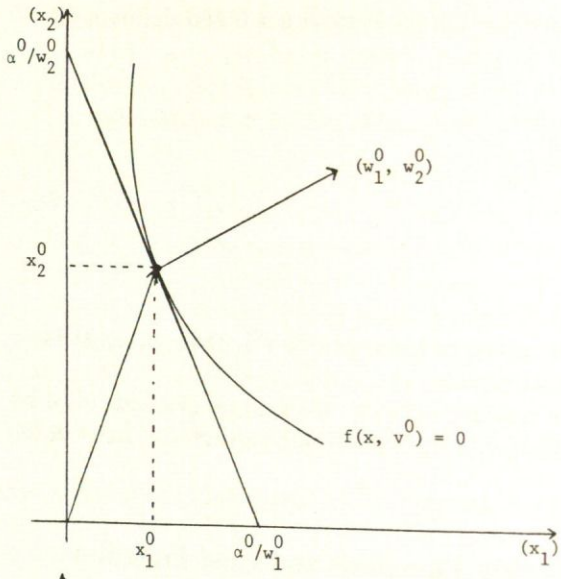

Fig. 4.1

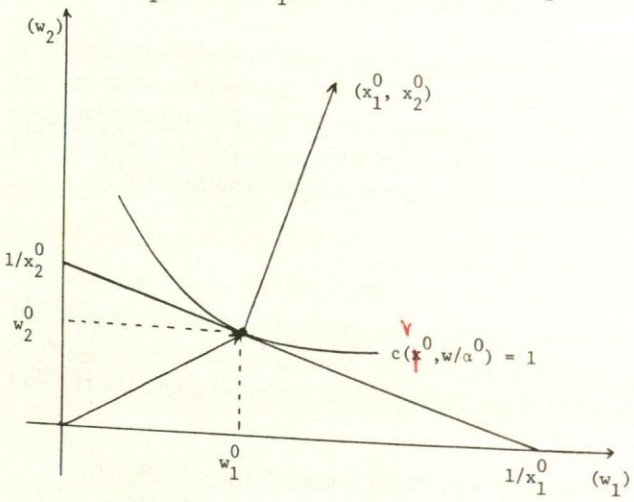

Fig. 4.2

In this section, an implicit production function is assumed to satisfy the following two properties:

(4.3)(a) For every $v \in V$ and for every $s \in P^n$, $f_{s_i}(s, v) > 0$ $(i = 1, 2, \ldots, n)$, and (b) f is regularly strict quasi-concave in $s \in P^n$ with $v \in V$ and $f(s, v) = 0$, i.e., if $s \in P^n$, $v \in V$ and $f(s, v) = 0$, then for every $\mu \in R^n$, $\mu \neq 0$, $f_s(s, v)\mu = 0$ implies that $\mu' f_{ss}(s, v)\mu < 0$.

Let us write $s = (\tilde{s}, s_n)$ and $\tilde{s} = (s_1, \ldots, s_{n-1})$. Then define $s_n = \phi(\tilde{s}, v)$ if and only if $f(\tilde{s}, s_n, v) = 0$. Such a function ϕ exists on an appropriate domain of $(\tilde{s}, v)$ by (4.3)(a).

Define marginal rates of substitution among variable inputs along the production surface by:

$$\pi^i(\tilde{s}, v) = f_{s_i}(\tilde{s}, \phi(\tilde{s}, v), v)/f_{s_n}(\tilde{s}, \phi(\tilde{s}, v), v), \qquad (i = 1, 2, \ldots, n-1).$$

Then, by the implicit differentiation rule, we have that:

(4.4) $\phi_{s_i}(\tilde{s}, v) = \partial\phi(\tilde{s}, v)/\partial s_i = -\pi^i(\tilde{s}, v)\,, \qquad (i = 1, 2, \ldots, n-1).$

Thus first-order conditions of the variable cost minimization imply that:

(4.5) $\phi_{s_i}(\tilde{s}, v) = -w_i/w_n\,, \qquad (i = 1, 2, \ldots, n-1).$

Write $\phi_s = (\phi_{s_1}, \ldots, \phi_{s_{n-1}})$ and $\phi_{ss} = [\phi_{s_i s_j}]_{i,j=1,w,\ldots,n-1}$. Then, by differentiating (4.4) with respect to k_j and arranging terms, we can show that:

$$\phi_{ss} = -(1/f_{s_n})[I_n \vdots (\phi_s)']\, f_{ss} \begin{bmatrix} I_n \\ \cdots \\ \phi_s \end{bmatrix}.$$

Note that, given $\mu \in R^{n-1}$, if we define μ_n by $\mu_n = (\phi_s)'\mu$, the we get $f_s\eta = 0$ where $\eta' = (\mu', \mu_n)$. Therefore, for $\mu \in R^{n-1}$, $\mu'\phi_{ss}\mu = -(1/f_{s_n})\eta' f_{ss}\eta$ with $\eta' = (\mu', \mu_n)$ and $f_s\eta = 0$.

Then conditions (4.3)(a) and (b) on an implicit production function can be equivalently stated by:

(4.6)(a) $s_n = \phi(\tilde{s}, v)$ and $\phi_{s_i}(\tilde{s}, v) = -\pi^i(\tilde{s}, v) < 0, \qquad (i = 1, 2, \ldots, n-1),$

and (b) ϕ is regularly strict convex in $\tilde{s}$, i.e., for every $\mu \in R^{n-1}$, $\mu \neq 0$, $\mu'\phi_{ss}(\tilde{s}, v)\mu > 0$.

Note that if a function satisfying (4.6) is given, then an implicit production function may be defined by $\tilde{f}(\tilde{s}, s_n, v) = s_n - \phi(\tilde{s}, v)$. Then we get $\tilde{f}_{s_n} \equiv 1$.

Suppose that we are given a short-run variable cost function $c(v, w)$ which satisfies the following conditions:

(4.7)(a) $c\colon V \times P^n \to R_+$ is positively homogeneous of degree one in w, i.e., for every $(v, w) \in V \times P^n$ and for every $\lambda \in P$, $c(v, \lambda w) = \lambda c(v, w)$. (b) c has continuous second partial derivatives with respect to w. (c) $c_w(v, w) = (\partial c(v, w)/\partial w_1, \ldots, \partial c(v, w)/\partial w_n) > 0$ for every $(v, w) \in V \times P^n$. (d) For every $(v, w) \in V \times P^n$, $c_{ww}(v, w) = (\partial^2 c(v, w)/\partial w_i w_j)_{i,j=1,2,\ldots,n}$ is symmetric and, for every $\eta \in R^n$, if $\eta \neq 0$ and $\eta/\|\eta\| \neq w/\|w\|$, then $\eta' c_{ww}\eta < 0$.

Note that the above property (d) obtains if the short-run variable cost function is derived from an implicit production function and if the implicit production function is regularly strict quasi-concave, in $s \in P^n$ with $v \in P$ and $f(s, v) = 0$ as shown in (3.6′). Given the short-run variable cost function, we define $g^i\colon V \times P^n \to P$ and $G^i\colon V \times P^{n-1} \to P$ as follows:

$$g^i(v, w) = c_{w_i}(v, w) = \partial c(v, w)/\partial w_i$$

and

$$G^i(v, \omega) = g^i(v, \omega_1, \ldots, \omega_{n-1}, 1)\,.$$

Write $\tilde{G} = (G^1, G^2, \ldots, G^{n-1})'$ and define $W = \{(\tilde{s}, v) \in P^{n-1} \times V \mid \tilde{s} = \tilde{G}(v, \omega)$ for some $v \in V$ and $\omega \in P^{n-1}\}$. The condition (4.7)(d) implies that

$$\tilde{G}_\omega(v, \omega) = \begin{bmatrix} \partial G^1(v, \omega)/\partial\omega_1, \ldots, \partial G^1(v, \omega)/\partial\omega_{n-1} \\ \cdots \\ \partial G^{n-1}(v, \omega)/\partial\omega_1, \ldots, \partial G^{n-1}(v, \omega)/\partial\omega_{n-1} \end{bmatrix}$$

is symmetric and negative definite for every $(v, \omega) \in V \times P^{n-1}$. Therefore $\tilde{G}(v, \cdot)$ is a one-to-one mapping from P^{n-1} onto W. (See Nikaido (1968), Theorem 20.4 and Corollary, pp. 370–371.) Thus there exists a mapping $\pi = (\pi', \ldots, \pi^{n-1})'$: $W \to P^{n-1}$ such that $\omega = \pi(\tilde{s}, v)$ if and only if $\tilde{s} = \tilde{G}(v, \omega)$. Clearly π has continuous first partial derivatives with respect to $s_j (j = 1, 2, \ldots, n-1)$ and $\pi_s(\tilde{s}, v) = (\partial \pi^i(s, v)/\partial s_j)_{i,j=1,2,\ldots,n-1} = [\tilde{G}_\omega(v, \omega)]^{-1}$ by the inverse function theorem.

Now let us define ϕ: $W \to P$ by:

(4.8) $\phi(\tilde{s}, v) = G^n(v, \pi(\tilde{s}, v))$.

Then we will show that the function ϕ satisfies properties given by (4.6). Therefore an implicit production function can be defined by $f(\tilde{s}, s_n, v) = s_n - \phi(\tilde{s}, v)$.

First, we will show that $\phi_{s_i}(\tilde{s}, v)|_{\tilde{s} = G(v,\omega)} = -\omega_i$, i.e. $\phi_{s_i}(\tilde{s}, v) = -\pi^i(\tilde{s}, v) < 0$. Write $\phi_s(\tilde{s}, v) = (\partial\phi(\tilde{s}, v)/\partial s_j)'_{j=1,2,\ldots,n-1}$. Then, by (4.8), we have that:

(4.9) $\phi_s(\tilde{s}, v) = G^n_\omega \pi_s(\tilde{s}, v)$.

Since $c(v, \cdot)$ is positively homogeneous of degree one in w, $g^i(v, \cdot)$ is homogeneous of degree zero in w. Therefore $g^i(v, w_1, \ldots, w_{n-1}, w_n) = G^i(v, w_1/w_n, \ldots, w_{n-1}/w_n, 1)$. From this, it follows that:

$$g^i_{w_n} = -(1/w_n) \sum_{j=1}^{n-1} (w_j/w_n) G^i_{\omega_j} = -(1/w_n) G^i_\omega \omega .$$

Also, since $c_{ww}(v, w) = g_w(v, w)$ is symmetric by (4.7)(d), we have that $g^i_{w_n} = g^n_{w_i} = (1/w_n) G^n_{\omega_i}$. Thus we obtain that $G^n_{\omega_i} = -[G^i_\omega]' \omega$, i.e.,

(4.10) $G^n_\omega = -\omega' G'_\omega$.

Substituting (4.10) into (4.9) and noting the symmetry of $\pi_s = [\tilde{G}_\omega]^{-1}$ yield:

$$\phi_s(\tilde{s}, v) = -\omega' G_w \pi_s(\tilde{s}, v) = -\omega'$$

where $\tilde{s} = G(v, \omega)$ or $\omega = \pi(\tilde{s}, v)$. Therefore we obtain the desired result:

(4.11) $\phi_s(\tilde{s}, v) = -[\pi(\tilde{s}, v)]'$.

An immediate consequence of (4.11) is that:

$$\phi_{ss}(\tilde{s}, v) = -\pi_s(\tilde{s}, v) = [\tilde{G}_\omega(v, \omega)]^{-1}|_{\omega = \pi(\tilde{s}, v)} .$$

Since the matrix $\tilde{G}_\omega(\tilde{s}, \omega)$ is negative definite, we can conclude that $\phi_{ss}(\tilde{s}, v)$ is positive definite for every $(\tilde{s}, v) \in W$. Hence the function ϕ is shown to satisfy (4.6). We can state this result as follows:

(4.12) *Proposition:* If a short-run variable cost function satisfies (4.7), then there exists an implicit production function $\tilde{f}$: $W \times R_+ \to R$ satisfying (4.3) from which the short-run variable cost function is derived through the variable cost minimization.

The problem of obtaining a production function from which a variable cost function is derived may appear quite similar to the integrability problem in consumer theory in which a utility indicator is sought so that a given system of

demand functions can be regarded as if being derived from the maximization of the utility indicator under the constraint of a budget. But it must be noted that the mathematical natures of these two problems are quite different. The relationship between a production function and a variable cost function involves basically the question of whether or not two mappings are inverse to each other. On the other hand, the integrability problem in consumer theory is to find a solution to a system of (partial) differential equations. This may explain why the duality relationship between production functions and cost functions is simpler and easier than the integrability problem in consumer theory.

(4.14) *Example:* Suppose that $n = 2$, $m = 1$, $q = 0$ and a variable cost function is given by $c(y, w) = \alpha^{-\alpha}\beta^{-\beta}w_1^{\alpha}w_2^{\beta}h(y)$ where $h: R_+ \to R_+$ determines returns to scale. Then we have:

$$\partial c(y, w)/\partial w_1 = g^1(y, w) = (\beta/\alpha)^{-\beta}(w_1/w_2)^{-\beta}h(y)$$

and

$$\partial c(y, w)/\partial w_2 = g^2(y, w) = (\beta/\alpha)^{\alpha}(w_1/w_2)^{\alpha}h(y)\,.$$

Hence

$$G^1(y, \omega) = (\beta/\alpha)^{-\beta}\omega^{-\beta}h(y)$$

and

$$G^2(y, \omega) = (\beta/\alpha)^{\alpha}\omega^{\alpha}h(y)\,.$$

$(\beta/\alpha)^{-1}$ $= x_1^{-\alpha/\beta}[h(y)]^{1/\beta}$

By inverting the relationship $x_1 = G^1(y, \omega)$ for the variable ω, we obtain $\omega = \pi(x_1, y) = (\beta/\alpha)x_1^{-1/\beta}[h(y)]^{1/\beta}$. Substituting the above into G^2 yields $x_2 = \phi(x_1, y)$ ~~and~~ $x_1^{-\alpha/\beta}$. Thus an implicit production function can be defined as $\tilde{f}(x_1, x_2, y) = x_2 - x_1^{-\alpha/\beta}[h(y)]^{1/\beta}$ or equivalently as $f(x_1, x_2, y) = x_1^{\alpha}x_2^{\beta} - h(y)$.

The reader may check that the variable cost function we have started from can be derived from the above production function through the variable outlay minimization.

Since one of the most important properties of the variable cost function is the effect on the variable cost of variations of $v = (y, z)$ and, in particular, those of the output vector y, we would like further to investigate the relationship between production functions and variable cost functions with respect to their properties resulting from a change in v.

We have already shown, in (3.11), that if an implicit production function is regularly strict quasi-concave in (s, v) at (s^0, v^0), then the variable cost function is regularly strictly convex in v at (v^0, w^0). We will now show that:

(4.15) *Proposition *:* If an implicit production function is assumed to be regularly strict quasi-concave in s at (s^0, v^0), then the implicit production function is regularly strict quasi-concave both in s and in v at (s^0, v^0) if and only if the associated variable cost function is regularly strict convex in v at (v^0, w^0).

---

* See Marino, Otani and Sicilian (1981) and Quirk and Saposnik (1965) for further details.

When there is a single output, the above result means that, if input isoquants are regularly strictly convex, then an increasing marginal cost at a particular output level and an input price vector is equivalent to the local regular strict concavity of a production function at the corresponding input vector. Needless to say, if the marginal cost is declining at a particular output level and an input price vector, then the production function must exhibit nonconvexity around the corresponding input vector.

If sufficies to prove th "if" part since the "only if" part is nothing but (3.11). First recall (3.13) which is given by:

$$(1/\lambda^0)c^0_{vv} = -[(g^0_v)', I_m]\begin{bmatrix} f^0_{ss}, & f^0_{sv} \\ f^0_{vs}, & f^0_{vv} \end{bmatrix}\begin{bmatrix} g^0_v \\ I_m \end{bmatrix}.$$

We define L, M and N, subspaces of a vector space R^{n+m} by:

$$L = \{\mu \in R^{n+m} \mid [f^0_s, f^0_v]\mu = 0\},$$

$$M = \left\{\mu \in R^{n+m} \mid \text{there exists } \eta \in R^m \text{ such that } \mu = \begin{bmatrix} g^0_v \\ I_m \end{bmatrix}\eta\right\},$$

and

$$N = \{\binom{v}{0} \in R^{n+m} \mid v \in R^n \quad \text{and} \quad f^0_s v = 0.$$

By Assumption (3.9), ranks of $[f_s, f^0_v]$ and f^0_s are both one and the rank of $\begin{bmatrix} g^0_v \\ I_m \end{bmatrix}$ is m. Therefore dim $L = n+m-1$, dim $M = m$ and dim $N = n-1$ where "dim" indicates dimension. Note that N is trivially a subspace of L and also M turns out to be a subspace of L because (3.12)(a) can be rewritten as:

$$[f^0_s, f^0_v]\begin{bmatrix} g^0_v \\ I_m \end{bmatrix} = 0'_m.$$

Since $\eta \in R^m$ and $\eta \neq 0$ implies that $\mu = \begin{bmatrix} g^0_v \\ I_m \end{bmatrix}\eta = \begin{bmatrix} g^0_v\eta \\ \eta \end{bmatrix} \notin N$, therefore L is a direct sum of M and N, i.e., $L = M \oplus N$ where $\oplus$ indicates a direct sum operation. If $\mu \in L$, then μ can be uniquely decomposed into $\mu = \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{bmatrix} g^0_v \\ I_m \end{bmatrix}\eta$ for some $\eta \in R^m$ and for some v with $f^0_s v = 0$. Note that:

$$(v', 0')\begin{bmatrix} f^0_{ss}, & f^0_{sv} \\ f^0_{vs}, & f^0_{vv} \end{bmatrix}\begin{bmatrix} g^0_v \\ I_m \end{bmatrix}\eta$$

$$= v'(f^0_{ss}g^0_v + f^0_{sv})\eta$$

$$= -(1/\lambda^0)\, v'(f^0_s)'\lambda^0_v\eta = 0$$

where we have used (3.12)(b) and the condition $f^0_s v = 0$. Therefore we obtain: for $\mu \in L$, $\mu \neq 0$,

$$\mu' \begin{bmatrix} f^0_{ss}, & f^0_{sv} \\ f^0_{vs}, & f^0_{vv} \end{bmatrix} \mu = \left[\begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} g^0_v \\ I_m \end{pmatrix} \eta \right] \begin{bmatrix} f^0_{ss}, & f^0_{sv} \\ f^0_{vs}, & f^0_{vv} \end{bmatrix} \left[\begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} g^0_v \\ I_m \end{pmatrix} \eta \right]$$

$$= v' (f^0_s)' v + \eta' [(g^0_v)', I_m] \begin{bmatrix} f^0_{ss}, & f^0_{sv} \\ f^0_{vs}, & f^0_{vv} \end{bmatrix} \begin{bmatrix} g^0_v \\ I_m \end{bmatrix} \eta$$

$$= v' f^0_{ss} v - (1/\lambda^0) \eta' c^0_{vv} \eta < 0$$

provided that f is regularly strict quasi-concave in s at (s^0, v^0) and the variable cost function is regularly strict convex in v at (v^0, w^0). This establishes our claim.

The above proof shows very clearly that the regular strict quasi-concavity of an implicit production function or equivalently the regular strict concavity of a production function can be unambiguously decomposed into the regular strict convexity of input isoquants and the regular strict convexity of the variable cost function. In other words, the regular strict convexity along an isoquant and the regular strict concavity of a production function along an output expansion path are equivalent to the regular strict concavity of a production function.

Another important property of a cost function as a function of outputs concerns returns to scale. Suppose that we have a single output production function of a form $y = F(x)$ with fixed inputs subsumed away and suppose that the associated total cost function is written by $c(y, w)$. Then an index $\mu(y, w)$ of returns to scale in terms of the cost function is usually defined by the reciprocal of output elasticity of the total cost, i.e.,

$$\mu(y, w) = \frac{c(y, w)}{y} \Big/ \frac{\partial c(y, w)}{\partial y} .$$

If $\mu(y, w) > 1$ ($= 1$, < 1 respectively), then the returns to scale are said to be increasing (constant, decreasing respectively) at (y, w). The index of returns to scale can be alternatively written in terms of the production function as follows:

$$\mu(y, w) = \frac{k}{F(x)} \frac{\partial F(kx)}{\partial k} \Bigg|_{\substack{k = 1 \\ x = g(y, w)}} = \frac{1}{F(x)} \sum_{i=1}^{n} x_i F_i(x) \Bigg|_{x = g(y, w)} .$$

The equivalence of the definition in terms of the cost function and the one in terms of the production function is easy to see if we note that $\partial c(y, w)/\partial y = \lambda(y, w)$, $\lambda(y, w) F_i(g(y, w)) = w_i$ and

$$c(y, w) = \sum_{i=1}^{n} w_i g^i(y, w) .$$

See Otha (1974) and Hanoch (1975) on the equivalence of these tow definitions. Where there is only one input, this relationship boils down to the well known relationship between average and marginal costs and average and marginal products, i.e.,

$$c(y, w)/y \gtreqless \partial c(y, w)/\partial y \text{ as } f'(x) \gtreqless f(x)/x \text{ with } x = g(y, w).$$

When joint outputs are present with an implicit production function f(x, y) subsuming fixed inputs away, Pazner and Wittig (1977) define and index of returns to scale by

$$\mu(y, w) = \cancel{x}\,c(y, w) / \sum_{j=1}^{m} y_j \, \partial c(y, w)/\partial y_j \, .$$

By noting $\partial c(y, w)/\partial y_j = -\lambda(y, w)\, \partial f(x, y)/\partial y_j \big|_{x = g(y, w)'}$, we can easily show that

$$\mu(y, w) = -\left\{ \sum_{i=1}^{n} x_i \, \partial f(x, y)/\partial x_i \right\} \Big/ \left\{ \sum_{i=1}^{m} y_j \, \partial f(x, y)/\partial y_j \right\} \Bigg|_{x = g(y, w)}$$

Chapter 5 Price Taking Firms

1. Introduction

In this chapter we study the behavior of a price taking profit maximizing firm. Section 1 will be devoted to investigating the consequences of the short run profit maximization. We will obtain the basic properties of the short run profit functions, short run output supply functions and short run derived demand functions for variable inputs. Finally we ill note the duality between production functions and profit functions.

In Section 2, we will first study two general properties of the relationship between an optimization problem and a family of suboptimization problems with some variables fixed. In particular, (i) we will derive an envelope property of the surface generated by an optimum value function which forms an envelope over a family of surfaces generated by optimum value functions parameterized by fixed variables. (ii) We will obtain the LeChatelier-Samuelson principle when objective functions have linear parts. We will discover that hese two general properties can provide us with a rich set of consequences when we apply them to the theory of cost and production with four types of optimization problems, namely, the short-run and the long-run cost minimization problems and the short-run and the long-run profit maximization problems. The last section will be devoted to the case of homothetic production functions and a more particular case of production functions with positive homogeneity of degree one.

2. Short-run Profit Maximization, Short-run Output Supply and Derived Demand for Variable Inputs

In this section, we will investigate consequences of the profit maximizing behavior of a firm in the short-run. In particular, we will obtain some basic properties of a profit function, output supply functions, and (derived) demand functions for variable inputs under the assumption of a competitive firm. The conditional input demand functions for variable inputs obtained in Section 3 of Chapter 4 are derived from the cost minimizing behavior of a competitive firm and, as such, they depend on an output vector, a fixed input vector, and variable input prices. On the other hand, *derived demand functions* for variable inputs are obtained through the profit maximizing behavior of a firm either in competitive markets or in imperfectly competitive product markets with competitive input markets. These demand functions for variable inputs are called "derived"

because variable inputs are considered to be demanded to provide products produced by employing these inputs. In order to distinguish these two types of input demand functions, we will use the term "conditional input demand functions" for the cost minimizing inputs and the term "derived input demand functions" for the profit maximizing inputs. Since the problem of the variable cost minimization and that of the short-run profit maximization are mathematically nearly identical, most results in this section can be deduced from results in the previous two sections. Therefore, for these results, we will not repeat their proofs.

Let $p \in P^m$, $w \in P^n$, and $r \in P^q$ be a vector of output prices, a vector of variable input prices and, a vector of fixed input prices respectively. Let $z = x^{II}$.

(2.1)(a) For each $z \in R^q_+$

$$\Omega(z) = \left\{ (p, w) \in P^{m+n} \,\middle|\, \sup_{(s,y)\in T(z)} (p \cdot y - w \cdot s) < +\infty \right\}, \quad \text{and}$$

(b) for each $(p, w) \in \Omega(z)$, $\pi(p, w, z) = \sup_{(s,y)\in T(z)} (p \cdot y - w \cdot s)$.

We will call $\pi(p, w; z)$ the *short-run profit function*. Since $(0, 0) \in T(z)$ for every $z \in R^q_+$, we have that, for every $(p, w) \in \Omega(z)$, $\pi(p, w, z) \geqq 0$. If the technology set $T(z)$ is not convex, then it is possible that there exists $(s, y) \in T(z)$, $(s, y) \neq 0$ such that $\pi(p, w, z) = p \cdot y - w \cdot s = 0$. Such pairs (s, y) and (p, w) as $\pi(p, w, z) = py - w \cdot s = 0$, $(s, y) \neq 0$ will be respectively called a *shut-down input-output vector* and a *shut-down price vector*. Let us denote the cone-hull of $T(z)$ by Cone $[T(z)]$ which is defined by Cone $[T(z)] = \{\lambda(s, y) \in R^{m+n}_+ \mid \lambda \geqq 0 \text{ and } (s, y) \in T(z)\}$. Then the set of shut-down input-output vectors will be nonzero vectors in the set $T(z) \cap \text{Cone}\,[T(z)]$. When the technology set $T(z)$ is not convex, then the output supply and the derived demand for variable inputs will be multi-valued and exhibit discontinuities at a shut-down price vector. We will define a subset of $\Omega(z)$ eliminating shut-down price vectors from $\Omega(z)$ and also Z, a subset of R^q_+ respectively by:

(2.2) (a) $\Omega^+(z) = \{(p, w) \in \Omega(z) \mid \pi(p, w, z) > 0\}$, and

(b) $Z = \{z \in R^q_+ \mid \Omega^+(z) \neq \phi\}$.

We will assume throughout this section that the output supply function and the derived demand function for variable inputs are defined on $\Omega^+(z)$ for each $z \in Z$.

(2.3) *Assumption:* For every $z \in Z$, there exist functions $y(\cdot, \cdot, z)$: $\Omega^+(z) \to R^m_+$ and $x(\cdot, \cdot, z)$: $\Omega^+(z) \to R^n_+$ such that, for every $(p, w) \in \Omega^+(z)$, $(x(p, w, z), y(p, w, z)) \in T(z)$ and $\pi(p, w, z) = p \cdot y(p, w, z) - w \cdot x(p, w, z)$.

Then we obtain the following basic results.

(2.4)(a) *Proposition:* For every $z \in R^q_+$, $\pi(p, w, z)$ is positively homogeneous of degree one in (p, w) and is a convex function of (p, w) on $\Omega(z)$. (b) $\pi(p, w, z)$ is continuous in (p, w) on the relative interior of $\Omega(z)$. (c) For each z, $y(p, w, z)$ and $x(p, w, z)$ are homogeneous of degree zero in (p, w) on $\Omega^+(z)$. (d) If the implicit production function f is (strictly) quasi-concave in (s, y, z) along the production

surface, then $\pi(p, w, z)$ is (respectively strictly) convex in z. (e) Assume that $(p^0, w^0), (p^1, w^1), (p^t, w^t) \in \Omega^+(z)$ and $(s(p^0, w^0, z), y(p^0, w^0, z)) \neq (s(p^1, w^1, z), y(p^1, w^1, z))$ where $(p^t, w^t) = t(p^1, w^1) + (1-t)(p^0, w^0)$ for some $t \in (0, 1)$. Then we have that

$$\pi(p^t, w^t, z) < t\,\pi(p^1, w^1, z) + (1-t)\,\pi(p^0, w^0, z)\,.$$

The proofs of the linear homogeneity in (a) and the continuity of π in (p, w) in (b) are identical with proofs of analogous properties of the minimum expenditure (and also the short-run variable cost function) in Theorem (5.6) of Chapter 2 (and respectively (3.3) of Chapter 4). Also the proof that π is a convex function in (p, w) on $\Omega(z)$ follows from the corresponding proof that the minimum expenditure function $m(\cdot, \alpha)$ (and the short-run variable cost function $c(v, \cdot)$) is concave in $p \in P$ (and respectively in $w \in P^n$) provided that $\Omega(z)$ is a convex set. To show the convexity of $\Omega(z)$, fix $(p^0, w^0), (p^1, w^1) \in \Omega(z)$. Then, for any $(s, y) \in T(z)$,

$$\begin{aligned} p^t \cdot y - w^t \cdot s &= t(p^1 \cdot y - w^1 \cdot s) + (1-t)(p^0 \cdot y - w^0 \cdot s) \\ &\leqq t\,\pi(p^1, w^1, z) + (1-t)\,\pi(p^0, w^0, z) < +\infty\,. \end{aligned}$$

Therefore $\pi(p^t, w^t, z) = \sup_{(s,y)\in T(z)} (p^t \cdot y - w^t \cdot s) < \infty$.

The proof of (c) is identical with proofs of the homogeneity of the Hicksian compensated demand function and the variable input demand function. For the proof of (d), see the proof of (3.7) of Chapter 4. The property (e) can be proved analogous to (5.6.b) in Chapter 2 and (3.4)(b) of Chapter 4. It may be noted that $\Omega(z)$ is a convex set, but $\Omega^+(z)$ is not generally a convex set. This will be shown in Example (2.10) later in this section.

If the output supply function and the derived demand function have differentiability properties, then we can obtain further results just as in the cases of the Hicksian compensated demand function and the variable input demand function.

(2.5) *Proposition:* Assume that $(p^0, w^0) \in \Omega^+(z^0)$, $s^0 = s(p^0, w^0, z^0)$, $y^0 = y(p^0, w^0, z^0)$, $s^0 > 0$, and $y^0 > 0$ and $s(p, w, z)$ and $y(p, w, z)$ have continuous first partial derivatives with respect to (p, w) at (p^0, w^0, z^0). Then (a) $\pi_p(p^0, w^0, z^0) = y(p^0, w^0, z^0)$ and $\pi_w(p^0, w^0, z^0) = -s(p^0, w^0, z^0)$; (b) $p^0 \cdot y_{p_j}(p^0, w^0, z^0) - w^0 x_{p_j}(p^0, w^0, z^0) = 0$ $(j = 1, 2, \ldots, m)$ and $p^0 \cdot y_{w_j}(p^0, w^0, z^0) - w^0 x_{w_j}(p^0, w^0, z^0)$, $(j = 1, 2, \ldots, n)$; (c) the $(m+n) \times (m+n)$ matrix $\begin{pmatrix} y_p, & y_w \\ -x_p, & -x_w \end{pmatrix}$ evaluated at (p^0, w^0, z^0) is symmetric and positive semi-definite.

(2.6) *Proposition:* In addition to assumptions of (2.5), if the implicit production function is regularly strict quasi-concave in (x, y) at (s^0, y^0, z^0). Then (a): for every $\eta \in R^{m+n}$, if $\eta \neq 0$ and $\eta / \|\eta\| \neq (p^0, w^0) \|(p^0, w^0)\|$, then

$$\eta' \begin{pmatrix} y_p, & y_w \\ -s_p, & -s_w \end{pmatrix}\Bigg|_{(p^0, w, z^0)} \eta > 0\,; \quad \text{and}$$

(b) all principal minors of

$$\left.\begin{pmatrix} y_p, & y_w \\ -\cancel{x}_p, & -\cancel{x}_w \end{pmatrix}\right|_{(p^0, w^0, z^0)} \quad \text{of order } r = 1, 2, \ldots, m+n-1 \text{ are positive.}$$

(2.7) *Proposition:* In addition to assumptions of (5.5), if $z^0 \gg 0$ and $f_z(s^0, y^0, z^0) \gg 0$, then (a) $\pi_z(p^0, w^0, z^0) = \mu^0 f_z(s^0, y^0, z^0) \gg 0$ where μ^0 is the associated Lagrange multiplier for the Lagrangean $p^0 \cdot y - w^0 \cdot s + \mu f(s, y, z^0)$; (b) f is regularly strict quasi-concave in (s, y, z) at (s^0, y^0, z^0) if and only if $\pi(p^0, w^0, z)$ is regularly strict convex in z at z^0.

Proofs of (2.5) and (2.6) are analogous to these of (5.8) and (5.16) respectively in Chapter 2. Result (2.7)(a) follows immediately from the envelope theorem in 5.11 of the mathematical appendix. The proof of (2.7)(b) can be found in proofs of (3.11) and (4.15) of Chapter 4.

The result of (2.5)(c) indicates that various price effects on derived demand for variable inputs and output supply are symmetric. Therefore, unlike consumer's demand functions, we can give definitions of substitutes, complements and others among inputs and outputs in production directly on derived demand functions for variable inputs and output supply functions. We define that: (a) for $j \neq k$, the variable input j is a *substitute* (respectively, a *complement*) to the input k at (p, w, z) in production if $\partial s^j(p, w, z)/\partial w_k > 0$ (respectively, $\partial s^j(p, w, z)/\partial w_k < 0$); (b) for $i \neq h$, the output i is a *substitute* (respectively, a *complement*) to the output h at (p,w, z) in production if $\partial y^i(p, w, z)/\partial p_h < 0$ (respectively, $\partial y^i(p, w, z)/\partial p_h > 0$); (c) the input j is a *normal input in the production of output i* at (p, w, z) if $\partial s^j(p, w, z)/\partial p_i = -\partial y^i(p, w, z)/\partial w_j > 0$; and (d) the input j is an *inferior* (or a *regressive*) *input in the production of output i* at (p, w, z) if $\partial s^j(p, w, z)/\partial p_i = -\partial y^i(p, w, z)/\partial w_j < 0$.

We can obtain the following elementary properties of derived demand for variable inputs and output supply regarding substitutes, complements, normality and inferiority.

(2.8) *Proposition:* Assume that $s = s(p, w, z)$, $y = y(p, w, z)$ and the implicit production function is regularly strict quasi-concave in (s, y) at (s, y, z). Then (a) $m \geqq 2$, if output i is a complement to every other output $h (h \neq i)$ at (p, w), then there exists at least one normal input at (p, w); (b) for $m \geqq 2$, if all inputs are inferior in the production of output i at (p, w), then there exists at least one output $h (h \neq i)$ which is a substitute to the output i at (p, w); (c) if $m = 1$, then at any (p, w), all inputs cannot be inferior in the production of the output; (d) for $n \geqq 2$, if input j is a complement to every other input $k (k \neq j)$ at (p, w), then input j is normal in the production of at least one output at (p, w) and (e) for $n \geqq 2$, if input j is inferior in the production of every output at (p, w), then there exists at least one input $k (k \neq j)$ which is a substitute to input j at (p, w).

All of the above results follow from (2.5)(b) and (2.6)(b). From (2.5)(b), we can obtain:

$$w_j \partial s^j/\partial w_j = \sum_{i=1}^{m} p_i \partial y^i/\partial w_j - \sum_{k \neq j} w_k \partial s^k/\partial w_j \quad \text{and}$$

$$p_i \partial y^i/\partial p_i = -\sum_{h \neq i} p_h \partial y^h/\partial p_i + \sum_{j=1}^{n} w_j \partial s^j/\partial p_i .$$

Then, note that $\partial s^j/\partial w_j < 0$ and $\partial y^i/\partial p_i > 0$ by (2.6)(b).

Consider the case with a single output with a production function $y = F(s)$, $s \in R^n_+$ where fixed inputs are subsumed away. Assume that F has continuous second partial derivatives on P^n and $\partial F(s)/\partial s_i > 0$ for $s \in P^n$. Furthermore it is assumed that (a) for every $s \in P^n$, $F_{ss}(x)$ is negatively definite, and (b) for every $s \in P^n$, for every i, $j \in \{1, 2, \ldots, n\}$ with $i \neq j$, $F_{s_i s_j}(x) > 0$, i.e., all variable inputs are complements to each other in the sense of Wicksell. If $s(p, w)$ is a vector of derived input demand functions, then $s(p, w) > 0$ satisfies the following first-order conditions:

$$p F_s(s, (p, w)) = w .$$

Therefore, by the implicit differentiation rule, we can obtain:

$$p s_w(p, w) = [F_{ss}(s(p, w))]^{-1} .$$

Now the negative definiteness of F_{ss} and the Wicksell complementarity implies that F_{ss}^{-1} is a matrix with all negative elements. (See Nikaido (1968), Theorem 7.4 (ii), p. 107.) Therefore we can conclude that all inputs are complements to each other in the sense we have defined. This result was first pointed out by Rader (1968).

It is possible to adopt the argument developed in Section 4 of the last chapter to establish a dual relationship between production functions and short-run profit functions, except one modification which must be made. One problem which requires us to modify the argument in Section 4 occurs when the implicit production function is not regularly strict quasi-concave in (s, y) for every (s, y) with $f(s, y, z) = 0$. In particular, when the technology set $T(z)$ is not a convex set, then the profit function may not give us any information on portions of the production surface which are nonconvex. For example, if the nonconvexity of the technology set $T(z)$ is confined to a neighborhood of the origin $(x, y) = (0, 0)$, then the short-run profit function will provide us with the information on the technology set corresponding to $\Omega^+(z)$ but it does not necessarily give us appropriate information when $\pi(p, w, z) = 0$. If we examine the proof of (4.12) of Chapter 4, then it is clear that the following local version of (4.12) of Chapter 4 applies in the case of a short-run profit function.

(2.9) *Proposition:* Suppose that we are given a short-run profit function $\pi(p, w, z)$ defined for $z \in Z$ and $(p, w) \in \Omega(z)$ satisfying (a) and suppose that there exists $z^0 \in Z$ and $(p^0, w^0) \in \Omega^+(z^0)$ satisfying (b), (c) and (d) where: (a) $\pi(p, w, z)$ is positively homogeneous of degree one in (p, w) on $\Omega(z)$; (b) π has continuous second partial derivatives with respect to each component of (p, w) at (p^0, w^0, z^0), (c) $\pi_p(p^0, w^0, z^0) > 0$ and $-\pi_w(p^0, w^0, z^0) > 0$, and (d)

$$\begin{pmatrix} \pi_{pp}, & \pi_{pw} \\ \pi_{wp}, & \pi_{ww} \end{pmatrix}\Bigg|_{(p^0, w^0, z^0)}$$

is symmetric and for every $\eta \in R^{n+m}$, if $\eta \neq 0$ and $\eta / \|\eta\| \neq (p^0, w^0) / \|(p^0, w^0\|$, then

$$\eta' \begin{pmatrix} \pi_{pp}, & \pi_{pw} \\ \pi_{wp}, & \pi_{ww} \end{pmatrix}\Bigg|_{(p^0, w^0, z^0)} \eta > 0 .$$

Then there exists an open cone $N(p^0, w^0)$ around (p^0, w^0), a neighborhood $N(z^0)$ around z^0, a neighborhood $N(s^0, y^0)$ around (s^0, y^0) and a function $\tilde{f}$: $N(s^0, y^0) \times N(z^0) \to R$ such that: (a) for each $z \in N(z^0)$, $\max_{(x,y)} \{p \cdot y - w \cdot s \mid \tilde{f}(s, y, z) \geqq 0\}$ has a solution in $N(s^0, y^0)$; (b) for every $(p, w, z) \in N(p^0, w^0) \times N(z^0)$, $\tilde{\pi}(p, w, z) = \pi(p, w, z)$ where $\tilde{\pi}(p, w, z) = \max_{(x,y)} \{p \cdot y - w \cdot s \mid \tilde{f}(s, y, z) \geqq 0\}$; (c) for every $(s, y, z) \in N(x^0, y^0) \times N(z^0)$, $\tilde{f}_s(s, y, z) > 0$ and $-\tilde{f}_y(s, y, z) > 0$, (d) $\tilde{f}$ is regularly strict quasi-concave in (s, y) at every $(s, y) \in N(s^0, y^0)$ for every $z \in N(z^0)$.

It may be noted that, with the presence of noncenvexity in production functions, our calculus approach turns out to be quite useful.

(2.10) *Example:* Let ψ: $R_+ \to R_+$ be defined by:

$$\begin{aligned} \psi(v) &= 0 && \text{for } v \in [0, 1] \\ &= \ln v && \text{for } v \geqq 1 . \end{aligned}$$

Then, for $n = 2$, $m = 1$, $q = 0$, we define a technology set T by

$$T = \{(x_1, x_2, y) \mid y = \psi(v), x_1^\alpha x_2^\beta - v \geqq 0; (x_1, x_2) \in R_+^2\} .$$

Note that the problem to minimize $w_1 x_1 + w_2 x_2$ subject to $x_1^\alpha x_2^\beta - v \geqq 0$ yield the minimum value function $v\phi(w_1, w_2)$ where $\phi(w_1, w_2) = \alpha^{-\alpha} \beta^{-\beta} w_1^\alpha w_2^\beta$ as in Example (4.14) of Chapter 4. Since the shut-down condition is characterized by $\psi'(v) = \psi(v)/v$ as shown in Figure 2.1., the shut-down input-output vectors are characterized by $y = \psi(e) = 1$ and $x_1^\alpha x_2^\beta = e$. Because the profit maximization problem can be written as the one to maximize $p \cdot \ln v - v\phi(w_1, w_2)$ with respect to v, shut-down price vectors are characterized by (p, w_1, w_2) satisfying $e^{-1} = \phi(w_1/p, w_2/p)$. In Figure 2.2, an illustration of the set $\Omega^+ = \{(p, w_1, w_2) \in P^3 \mid \phi(w_1/p, w_2/p) < e^{-1}\}$ is given by the shaded region excluding boundaries.

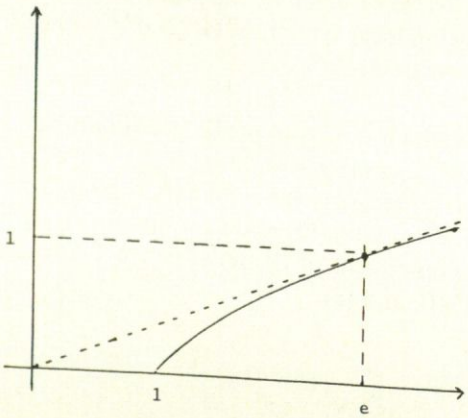

Fig. 2.1

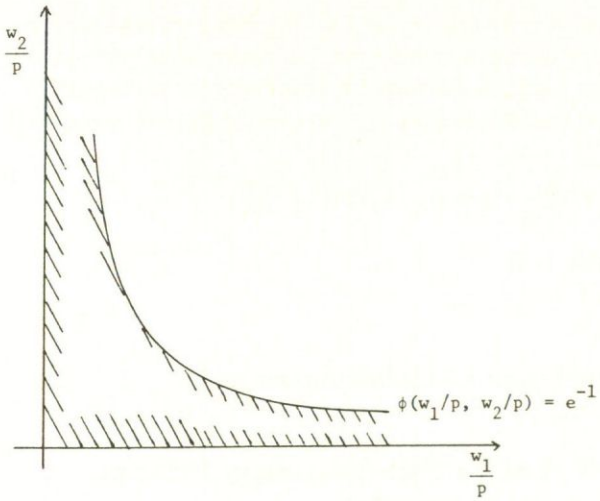

Fig. 2.2

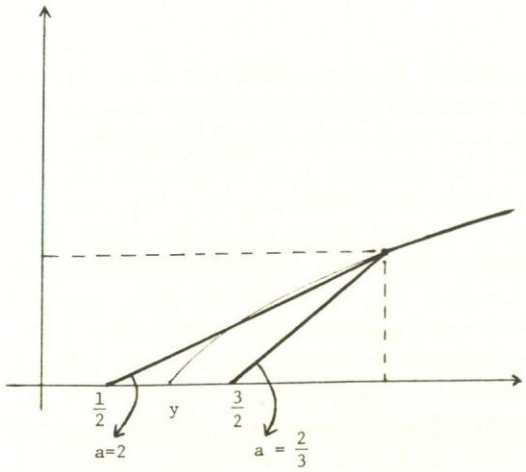

Fig. 2.3

Note that the set Ω^+ in this example is not a convex set. Therefore the profit function becomes $\pi(p, w_1, w_2) = p\{\ln(p/\phi w_1, w_2)) - 1\}$ for $(p, w_1, w_2) \in \Omega = \{(p, w_1, w_2) \in P^3 \mid \phi(w_1/p, w_2/p) \leqq e^{-1}\}$ and $\pi(p, w_1, w_2) = 0$ otherwise.

Now we can start from the above profit function defined on Ω to get back to the technology set. For $(p, w_1, w_2) \in \Omega^+$, we note that:

$$\partial\pi/\partial p \quad = \log(p/\phi(w_1, w_2)) ,$$

$$-\partial\pi/\partial w_1 = (p/\phi(w_1, w_2))\,\partial\phi/\partial w_1,$$

and

$$-\partial\pi/\partial w_2 = (p/\phi(w_1, w_2))\,\partial\phi/\partial w_2 .$$

Therefore we get $p/\phi(w_1, w_2) = e^y$. Now if we eliminate w_1/w_2 from $x_1 = e^y\,\partial\phi/\partial w_1 = e^y\,\alpha^\beta\beta^{-\beta}(w_1/w_2)^{-\beta}$ and $x_2 = e^y\,\partial\phi/\partial w_1 = e^y\,\alpha^{-\alpha}\beta^\alpha(w_1/w_2)^\alpha$, we can obtain $x_1^\alpha x_2^\beta = e^y$. But, since $\phi(w_1/p, w_2/p) < e^{-1}$, this implies that

$e^y = 1/\phi(w_1/p, w_2/p) > e$, i.e., $y > 1$. Hence (x_1, x_2) must be constrained by $x_1^\alpha x_2^\beta > e$. Therefore we can not obtain any information about that part of the technology set corresponding to $y \in (0, 1)$. In fact, the following technology set $\tilde{T}$, for example, yields the identical profit function $\pi(p, w_1, w_2)$ as the one generated by T:

$$\tilde{T} = \{(x_1, x_2, y) \mid y = \tilde{\psi}(v), x_1^\alpha x_2^\beta - v \geqq 0, (x_1, x_2) \in R_+^2\}$$

where

$$\tilde{\psi}(v) = \begin{cases} 0 & \text{for } v \in [0, 1/a] \\ av - 1 & \text{for } v \in [1/a, e] \\ \psi(v) & \text{for } v \geqq e \end{cases}$$

and, of course, $0 < 1/a < e$. See Figure 5.3 for illustrations of $\tilde{T}$.

3. The Envelope Property, LeChatelier-Samuelson Principle and the Theory of Cost and Production

a) Introduction

One of the basic properties of the relationship between a family of short-run total cost functions parameterized by the level of fixed inputs and the long-run total cost function is that the long-run total cost function forms an envelope of a family of short-run cost functions. Consequently, the long-run total average cost function also forms an envelope of a family of short-run total average cost functions and the well-known relationship between the long-run marginal cost curve and short-run marginal cost curves can be obtained. For example, suppose that the long-run marginal cost curve and a short-run marginal cost curve intersect each other at a particular output level, say y^0, and both curves are increasing around the output level y^0. Then the short-run marginal cost curve must be steeper than the long-run marginal cost curve around the output level y^0. See Figure 3.1 where STC, LTC, SMC and LMC indicate respectively a short-run total cost curve, the long-run total cost curve, a short-run marginal cost curve and the long-run marginal cost curve. This fact turns out to be a special case of the LeChatelier-Samuelson principle originally derived by Samuelson (1947, pp. 36–39). In this section, we will show that the envelope property of the long-run cost function and the LeChatelier-Samuelson principle are two rather fundamental properties which are possessed by optimum (maximum or minimum) value functions of two related optimization problems with a specific set of parameters. Then we will provide various applications to the theory of cost and production.

b) The Envelope Property of Maximum Value Functions*

In this subsection, we investigate a relationship between a family of maximization problem parameterized by x^2 denoted by $M(x^2)$ and a related maximization problem denoted by $\tilde{M}$ which are defined respectively by:

* This section is based on Otani (1982).

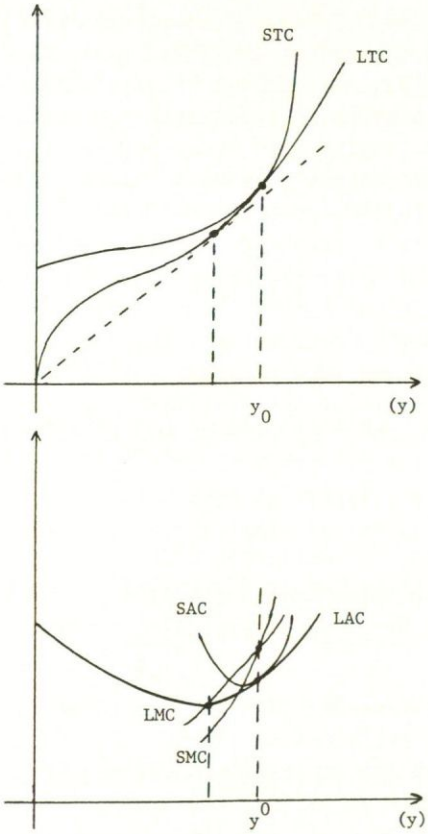

Fig. 3.1

$$(M(x^2)) \quad \underset{x^1 \in R^{n_1}_+}{\text{maximize}}\ H(x^1, x^2, a) \quad \text{subject to} \quad f(x^1, x^2, a) \geqq 0\,,$$

$$(M) \quad \underset{(x^1, x^2) \in R^{n_1+n_2}_+}{\text{maximize}}\ H(x^1, x^2, a) \quad \text{subject to} \quad f(x^1, x^2, a) \geqq 0\,.$$

We then define maximum value functions corresponding to above problems by:

$$F(a, x^2) = \max_{x^1 \in R^{n_1}_+} \{H(x^1, x^2, a) \mid f(x^1, x^2, a) \geqq 0\}$$

and

$$\tilde{F}(a) = \max_{(x^1, x^2) \in R^{n_1+n_2}_+} \{H(x^1, x^2, a) \mid \overset{f}{\cancel{F}}(x^1, x^2, a) \geqq 0\}\,.$$

Let $S(x^2)$ be hypersurfaces generated by the graph of $F(a, x^2)$ parameterized by x^2 and similarly let S be the hypersurface generated by the graph of $\tilde{F}(a)$. Then we shall show that, under a suitable set of assumptions, the surface S is the envelope of the family of surfaces $S(x^2)$ in the sense of an envelope in classical differential geometry. (See, for example, Guggenheimer (1963, p. 43).) In order to familiarize ourselves with the concept of an envelope, we first review the concept of an envelope of a family of curves. Let $I = [0, 1]$ and $F: I^2 \to R^2$ denoted by

$F(t, a)$ be a family of curves parameterized by $a \in I$. Suppose that there exists a C^1-function $\tilde{a}: I \to I$ and define $E_F: I \to R^2$ by $E_F(t) = F(t, \tilde{a}(t))$. Then E_F is said to be an envelope of the family $F(\cdot, a)$ of curves if (a) the tangent of the curve E_F at t coincides with the tangent of the curve $F(t, a_0)$ where $a_0 = \tilde{a}(t)$, and (b) for each t, $\tilde{a}'(t) \neq 0$.

Condition (b) in the definition of an envelope is to guarantee that the envelope $E_F(t)$ and $F(t, a_0)$ have only isolated points in common. Write $F(t, a) = (x_1(t, a), x_2(t\, a))$ and let τ_F and τ_E be tangent vectors to curves $F(t, a_0)$ and $E_F(t)$ respectively with $a_0 = \tilde{a}(t)$. Then τ_F and τ_E are unit vectors proportionate respectively to vectors $(\partial x_1(t, a_0)/\partial t, \partial x_2(t, a_0)/\partial t)$ and $(\partial x_1(t, a_0)/\partial t, \partial x_2(t, a_0)/\partial t) + (\partial x_1(t, a_0)/\partial a, \partial x_2(t, a_0)/\partial a)\tilde{a}'(t)$ when $a_0 = \tilde{a}(t)$. Therefore tangent vectors coincide if $(\partial x_j(t, a_0)/\partial a, \partial x_2(t, a_0)/\partial a)\tilde{a}'(t) = \lambda(t, a_0)(\partial x_1(t, a_0)/\partial t, \partial x_2(t, a_0/\partial t))$ for some proportionality factor $\lambda(t, a_0)$. This condition admits in general only the trivial case which is $\partial x_i(t, a_0)/\partial a = 0$, $(i = 1, 2)$ and $\lambda(t, a_0) = 0$.

The above definition of an envelope of a family of curves can be easily generalized to hypersurfaces of higher dimensions with a space more general than an interval I.

In order to prove the envelope property of maximum value functions, we will impose the following set of assumptions for the above maximization problems.

(3.1) *Assumption:* (i) Functions $H: R_+^{n_1+n_2} \times A \to R$ and $f: R_+^{n_1+n_2} \times A \to R$ are continuously differentiable on an open neighborhood W of $(\bar{x}^1, \bar{x}^2, \bar{a}) \in (R_+^{n_1+n_2} \times A)$ where A is a nonempty subset of a finite dimensional real space; (ii) the problem $(\tilde{M})$ admits a C^1-solution $\tilde{\xi}(a) = (\tilde{\xi}^1(a), \tilde{\xi}^2(a))$ on an open neighborhood $\tilde{U}$ of $\bar{a}$; (iii) the problem $(M(x^2))$ admits a C^1-solution $\xi^1(a, x^2)$ on an open neighborhood U of $(\bar{a}, \bar{x}^2)$ with $\bar{x}^2 = \tilde{\xi}^2(\bar{a})$; (iv) $\{(a, x^2) \mid x^2 = \tilde{\zeta}(a), a \in \tilde{U}\} \subseteq U$, $\{(x^1, x^2, a) \mid x^1 = \xi^1(a, x^2), (a, x^2) \in U\} \subseteq W$, and $\{(x^1, x^2, a) \mid (x^1, x^2) = \tilde{\xi}(a), a \in \tilde{U}\} \subseteq W$; (iv) $\xi^1(a, x^2)$ for each $(a, x^2) \in U$ and $\tilde{\xi}(a)$ for each $a \in \tilde{U}$ satisfy the constraint with equality, i.e., $f(x^1, x^2, a) = 0$ and first-order conditions of corresponding Lagrangean function, i.e., for every $(a, x^2) \in U$, there exists $\lambda(a, x^2) > 0$ such that $H_{x^1}(x^1, x^2, a) + \lambda(a, x^2) f_{x^1}(x^1, x^2, a) = 0'_{n_1}$ with $x^1 = \xi^1(a, x^2)$, and for every $a \in \tilde{U}$, there exists $\tilde{\lambda}(a) > 0$ such that $H_{x^i}(x^1, x^2, a) + \tilde{\lambda}(a) f_{x_i}(x^1, x^2, a) = 0_{n_i}$ ~~$(i = 1, 2)$ with $(x^1, x^2) = \tilde{\zeta}(a)$~~. (v) For each $a \in \tilde{U}$, $\tilde{\xi}(a)$ is a regular maximizer for $(\tilde{M})$, i.e., the second-order sufficient condition is satisfied at $\tilde{\zeta}(a)$.

Conditions (i) through (iii) require that all relevant functions are continuously differentiable and the classical first-order conditions are satisfied locally in each neighborhood of parameters. Condition (v) is a regularity condition to insure that the envelope of a family of surfaces and each surface have only isolated points in common. When we apply The Envelope Theorem (5.11) in the Appendix under (3.1), then we can obtain the following basic envelope property of the maximum value function $\tilde{F}(a)$.

(3.2) *Proposition:* Under (3.1), the surface generated by $\alpha = \tilde{F}(a)$ is the envelope of a family of surfaces generated by $\alpha = F(a, x^2)$ parameterized by x^2 where $a \in \tilde{U}$ and $(a, x^2) \in U^0 = \{(a', x^{2'}) \mid x^{2'} = \tilde{\xi}^2(a'), a' \in \tilde{U}\}$.

To prove the above proposition, first note that, for each $a \in \tilde{U}$, $\tilde{F}(a) = F(a, \tilde{\xi}^2(a))$ and, since $\tilde{\xi}^2(a)$ is a regular maximizer for $(\tilde{M})$, if $(a, x^2) \in U^0$ and $x^2 \neq \tilde{\xi}^2(a)$, then $\tilde{F}(a) > F(a, x^2)$. Therefore the surface $\alpha = \tilde{F}(a)$ and the surface $\alpha = \tilde{F}(a, x^2)$ meet each other only if $x^2 = \tilde{\xi}^2(a)$. Hence it remains only to show that, when two surfaces meet each other, they have the same tangent vector. But this follows from the envelope theorem. Note that $\xi^1(a, \tilde{\xi}^2(a)) = \tilde{\xi}^1(a)$ and $\lambda(a, \tilde{\xi}^2(a)) = \tilde{\lambda}(a)$. Then, by the envelope theorem, we get: for every $a \in \tilde{U}$,

$$\begin{aligned} F_a(a, \tilde{\xi}^2(a)) &= \{H_a(\xi^1(a, x^2), x^2, a) + \lambda(a, x^2) f_a(\xi^1(a, x^2), x^2, a)\}|_{x^2 = \tilde{\xi}^2(a)} \\ &= H_a(\tilde{\xi}(a), a) + \tilde{\lambda}(a) f_a(\tilde{\xi}(a), a) \\ &= \tilde{F}_a(a). \end{aligned}$$

In the theory of cost and production, we can in fact discover four types of envelope relations including the well-known case that the long-run cost surface is an envelope of a family of short-run total cost surfaces parameterized by fixed inputs. Let us define the short-run variable cost function, the long-run total cost function, the short-run profit function and the long-run profit function of an individual firm respectively as follows:

$$c(y, z, w) = \min_x \{w \cdot x \mid f(x, y, z) \geqq 0\},$$

$$\tilde{c}(y, w, r) = \min_{(x, z)} \{w \cdot x + r \cdot z \mid f(x, y, z) \geqq 0\},$$

$$\pi(p, w, z) = \max_{(x, y)} \{p \cdot y - w \cdot x \mid f(x, y, z) \geqq 0\},$$

and

$$\tilde{\pi}(p, w, r) = \max_{(x, y, z)} \{p \cdot y - w \cdot x - r \cdot z \mid f(x, y, z) \geqq 0\}.$$

Then the following results obtain as immediate consequences of Proposition (3.2) although, for the sake of making statements simpler, explicit assumptions corresponding to (3.1) will not be stated.

(3.3) *Proposition:* If conditions corresponding to (3.1) hold and appropriate spaces and subsets are specified, then (i) $\tilde{c}(y, w, r)$ is an envelope of a family of surfaces $c(y, z, w) - r \cdot z$ parameterized by z; (ii) $\tilde{\pi}(p, w, r)$ is an envelope of a family of surfaces $\pi(p, w, z) - r \cdot z$ parameterized by z; (iii) $\pi(p, w, z)$ is an envelope of a family of surfaces $p \cdot y - c(y, z, w)$ parameterized by y; and (iv) $\tilde{\pi}(p, w, r)$ is an envelope of a family of surfaces $p \cdot y - \tilde{c}(y, w, r)$ parameterized by y.

c) The LeChatelier-Samuelson Principle and the Theory of Cost and Production

Both cost minimization and profit maximization problems have linear objective functions and, therefore, price parameters can be expected to exhibit further properties resulting from the linearity of objective functions. It will be shown in

this section that this feature of objective functions leads us to the Le Chatelier-Samuelson principle and provides us with interesting applications to the theory of cost and production.

In this subsection, we introduce more specific maximization problems than in subsection b) with a family of maximization problems parameterized by x^2 denoted again by $M(x^2)$ and a related maximization problem denoted by $\tilde{M}$ as follows:

$$(M(x^2)) \quad \underset{\lambda \in R^{n_1}_+}{\text{maximize}}\ H(x^1, x^2) + a^1 \cdot x^1 \quad \text{subject to} \quad f(x^1, x^2) \geqq 0\,, \quad \text{and}$$

$$(\tilde{M}) \quad \underset{(x^1, x^2) \in R^{n_1+n_2}_+}{\text{maximize}}\ H(x^1, x^2) + a^1 \cdot x^1 + a^2 \cdot x^2 \quad \text{subject to} \quad f(x^1, x^2) \geqq 0\,.$$

When two maximization problems have the above structure, then we will say that $(M(x^2))$ is a subproblem of $(\tilde{M})$ symbolically denoted by $(M(x^2)) < (\tilde{M})$. We will impose the following assumption.

(3.4) *Assumption:* There exists $(\bar{a}^1, \bar{a}^2) \in R^{n_1+n_2}$ such that (i) the problem $(\tilde{M})$ has a C^1-solution $\tilde{\xi} = (\tilde{\xi}^1, \tilde{\xi}^2)$ on an open neighborhood $\tilde{U}$ of $(\bar{a}^1, \bar{a}^2)$; (ii) the subproblem $(M(x^2))$ has a C^1-solution ξ^1 on an open neighborhood U of $(\bar{a}^1, \bar{x}^2)$ where $\bar{x}^2 = \tilde{\xi}^2(\bar{a}^1, \bar{a}^2)$; and (iii) $\{(a^1, x^2) \mid x^2 = \tilde{\xi}^2(a^1, a^2) \text{ and } (a^1, a^2) \in \tilde{U}\} \subseteq U$.

Under Assumption (3.4), for every $(a^1, a^2) \in \tilde{U}$, $\tilde{\xi}^i(a^1, a^2)$ is continuously differentiable in (a^1, a^2) and if $x^2 = \tilde{\xi}^2(a^1, a^2)$, then $(a^1, x^2) \in U$ and $\xi^1(a^1, x^2)$ is also continuously differentiable in (a^1, x^2). Therefore we can apply the chain rule to $\tilde{\xi}^1_i(a^1, a^2) = \xi^1_i(a^1, \tilde{\xi}^2(a^1, a^2))$ to obtain that, for every $(a^1, a^2) \in \tilde{U}$,

$$(3.5) \quad \partial\tilde{\xi}^1_i/\partial a^1_j = \partial\xi^1_i/\partial a^1_j + \sum_{k=1}^{n_2} (\partial\xi^1_i/\partial x^2_k)(\partial\tilde{\xi}^2_k/\partial a^1_j)\,.$$

By the symmetry result of Lemma (5.13) in the Appendix, we have $\partial\tilde{\xi}^2_k/\partial a^j_i = \partial\tilde{\xi}^1_j/\partial a^2_k$ and, apply the chain rule again to $\tilde{\xi}^1_j(a^1, a^2) = \xi^1_j(a^1, \tilde{\xi}^2(a^1, a^2))$ with respect to a^2_k to obtain that:

$$(3.6) \quad \partial\tilde{\xi}^1_j/\partial a^2_k = \sum_{h=1}^{n_2} (\partial\xi^1_j/\partial x^2_h)(\partial\tilde{\xi}^2_h/\partial a^2_k)\,.$$

Substituting (3.6) into (3.5) yields:

$$(3.7) \quad \partial\tilde{\xi}^1_i/\partial a^1_j = \partial\xi^1_i/\partial a^1_j + (\partial\xi^1_i/\partial x^2)(\partial\tilde{\xi}^2/\partial a^2)(\partial\xi^1_j/\partial x^2)'.$$

If we take $i = j$ in (3.7) and note that the matrix $\partial\tilde{\xi}^2/\partial a^2$ is positive semi-definite by Lemma (5.13) in the Appendix. The we can obtain the Le Chatelier-Samuelson principle as stated in (i) and (ii) of the following proposition. When $i \neq j$, we can state (iii) of the following proposition immediately from (3.7).

(3.8) *Proposition:* (i) For every $(a^1, a^2) \in \tilde{U}$

$$0 \leqq \partial\xi^1_i(a^1, x^2)/\partial a^1_i\,|_{x^2 = \tilde{\xi}^2(a^1, a^2)} \leqq \partial\tilde{\xi}^1_i(a^1, a^2)/\partial a^1_i\,; \qquad (i = 1, 2, \ldots, n_1)\,;$$

(ii) the second inequality on the right in the above expression of (i) becomes strict if the matrix $\partial\tilde{\xi}^2/\partial a^2$ is nonsingular and $\partial\xi^1_i/\partial x^2_k \neq 0$ for at least one k; (iii) for $i \neq j$ $(i, j = 1, 2, \ldots, n_1)$, we have that for every $(a^1, a^2) \in \tilde{U}$,

$$\partial\tilde{\xi}_i^1(a^1, a^2) \gtreqless \partial\xi_i^1(a^1, x^2)/\partial a_j^1 \,|_{x^2 = \tilde{\xi}^2(a^1, a^2)}$$

as

$$(\partial\xi_i^1/\partial x^2)'(\partial\tilde{\xi}^2/\partial a^2)(\partial\xi_j^1/\partial x^2)\,|_{x^2 = \tilde{\xi}^2(a^1, a^2)} \gtreqless 0\,.$$

If assumptions of Lemma (5.14) (ii) in the Appendix are met, then the result (i) of Lemma (3.8) can be strengthened as follows by applying the Lemma in the Appendix.

(3.9) *Proposition:* If $H(x) \equiv 0$ and f is regularly strict quasi-concave in (x^1, x^2) at $(\bar{x}^1, \bar{x}^2)$ with $\bar{x}^i = \tilde{\xi}^i(\bar{a}^1, \bar{a}^2)$, then an appropriate $\tilde{U}$ can be chosen so that for every $(a^1, a^2) \in \tilde{U}$

$$0 < \partial\xi_i^1 a^1, x^2)/\partial a_i^1 \,|_{x^2 = \tilde{\xi}^2(a^1, a^2)} < \partial\tilde{\xi}_i^1(a^1, a^2)/\partial a_i^1\,; \qquad (i = 1, 2, \ldots, n_1).$$

We can apply Propositions (3.8) and (3.9) to the theory of cost and production. As in the previous section b, we will consider the short-run variable cost minimization (SC), the long-run total cost minimization (LC), the short-run profit maximization (SP) and the long-run profit maximization (LP) defined as follows:

$$\text{(SC)} \quad \underset{x \in R_+^n}{\text{maximize}}\,(-w)\cdot x \quad \text{subject to} \quad f(x, y, z) \geqq 0\,,$$

$$\text{(LC)} \quad \underset{(x, z) \in R_+^{n+q}}{\text{maximize}}\,(-w)\cdot x + (-r)\cdot z \quad \text{subject to} \quad f(x, y, z) \geqq 0\,,$$

$$\text{(SP)} \quad \underset{(x, y) \in R_+^{n+m}}{\text{maximize}}\; p\cdot y + (-w)\cdot x \quad \text{subject to} \quad f(x, y, z) \geqq 0\,, \quad \text{and}$$

$$\text{(LP)} \quad \underset{(x, y, z) \in R_+^{n+m+q}}{\text{maximize}}\; p\cdot y + (-w)\cdot x + (-r)\cdot z \quad \text{subject to} \quad f(x, y, z) \geqq 0\,.$$

Using the notation on subprograms, we can readily notice two chains of subprogram relations which are (SC) < (LC) < (LP) and (SC) < (SP) < (LP). Since the chain (SC) < (SP) < (LP) yields more interesting results, we will focus our applications to this case. Problems (SC), (SP) and (LP) respectively generate $g(y, z, w)$ the short-run variable input demand function, $x(p, w, z)$ and $y(p, w, z)$ the short-run derived variable input demand and the short-run output supply functions, and $\tilde{x}(p, w, r)$, $\tilde{y}(p, w, r)$ and $\tilde{z}(p, w, r)$ the long-run derived variable input demand, the long-run output supply and the long-run derived fixed input demand functions. In Table (3.1) we will list notations used in applications to (SC) < (SP) and (SP) < (LP) and their relationship to notations used in Propositions (3.8) and (3.9) for easier reference.

For the following applications, we will omit quantifiers such as "for every $(p, w) \in \tilde{U}_I$", but results are supposed to apply for appropriate neighborhoods. Also since, in our applications, the nonlinear part $H(x)$ is zero, we will assume that the implicit production function is regularly strict quasi-concave in (x, y, z) at a particular input $(\bar{x}, \bar{y}, \bar{z})$. Of course, this will imply that when prices change in an appropriate neighborhood, then the resulting inputs and/or outputs will stay in a neighborhood of $(\bar{x}, \bar{y}, \bar{z})$ and the implicit production function will be regularly strict quasi-concave at such an input-output vector (x, y, z). We may

Table 3.1

	(SC) < (SP)	(SP) < (LP)
a^1	$-w$	$(p, -w)$
a^2	p	$-r$
x^1	x	(x, y)
x^2	y	z
$\xi^1(a^1, x^2)$	$g(y, z, w)$	$(x(p, w, z), y(p, w, z))$
$\tilde{\xi}^1(a^1, a^2)$	$x(p, w, z)$	$(\tilde{x}(p, w, r), y(p, w, r))$
$\tilde{\xi}^2(a^1, a^2)$	$y(p, w, z)$	$\tilde{z}(p, w, r)$
$(a^1, x^1) \in U$	$(w, y) \in U_I$	$(p, w, z) \in U_{II}$
$(a^1, a^2) \in \tilde{U}$	$(p, w) \in \tilde{U}_I$	$(p, w, r) \in \tilde{U}_{II}$

call this assumption the regularity assumption. Now let us consider the case (SC) < (SP). Note that the short-run derived variable input demand function $x^i(p, w, z)$ is related to the short-run input demand function $g^i(y, z, w)$ by $x^i(p, w, z) = g^i(y(p, w, z), z, w)$. Therefore by the chain rule, we have

$$(3.10) \quad \partial x^i/\partial w_i = \partial g^i/\partial w_i + (\partial g^i/\partial y)(\partial y/\partial w_i) .$$

The first term on the right-hand side of (3.10) is the pure substitution effect along an isoquant and then second term on the right-hand side is the effect through the output expansion. Since $\partial x^i/\partial w_i < \partial g^i/\partial w_i < 0$ by the Le-Chatelier-Samuelson principle under the regularity assumption, the output expansion effect $(\partial g^i/\partial y)'(\partial y/\partial w_i)$ must be negative. Therefore, unlike consumer's demand functions, the derived input demand functions have both the substitution effect and the output expansion effect negative. This result was first obtained by Sakai (1973 and 1974). Now consider the case of a single output and assume that expansion paths are strictly increasing so that $\partial g^i/\partial y > 0$ for every $i = 1, 2, \ldots, n$. Noting that $\partial y/\partial p > 0$ under the regularity assumption and using (3.7) for $i \neq j$ $(i, j = 1, 2, \ldots, n)$, we obtain that:

$$\partial x^i/\partial w_j = \partial g^i/\partial w_j + (\partial g^i/\partial y)(\partial y/\partial p)(\partial g^j/\partial y) < \partial g^i/\partial w_j .$$

Thus we can conclude that derived input demand functions tend to exhibit more complementarity than cost minimizing input demand functions.

We next consider (SP) < (LP). One immediate consequence of the Le-Chatelier-Samuelson principle is that:

$$0 < \partial y^i(p, w, z)/\partial p_i|_{z = \tilde{z}(p, w, r)} < \partial \tilde{y}^i(p, w, r)/\partial p_i ,$$

i.e., the long-run supply elasticity of an individual firm is larger than the short-run supply elasticity of an individual firm. A similar result for derived demand for variable inputs becomes:

$$\partial \tilde{x}^i(p, w, r)/\partial w_i < \partial x^i(p, w, z)/\partial w_i|_{z = \tilde{z}(p, w, r)} < 0 .$$

Now consider the case of a single fixed input. First, assume that the fixed input is a normal input to the production of every output in the long-run, i.e., $-\partial \tilde{y}^i/\partial r = \partial \tilde{z}/\partial p_i > 0$. Then since $\tilde{y}^i(p, w, r) = y^i(p, w, \tilde{z}(p, w, r))$, we have $\partial \tilde{y}^i/\partial r = (\partial y^i/\partial z)(\partial \tilde{z}/\partial r) > 0$. Because $\partial \tilde{z}/\partial r < 0$ under the regularity assumption,

we obtain that $\partial y^i/\partial z|_{z=\tilde{z}(p,w,r)} > 0$. Hence $(\partial y^i/\partial z)(\partial \tilde{z}/\partial r)(\partial y^j/\partial z)|_{z=\tilde{z}(p,w,r)} < 0$ which, by Proposition (3.8) (iii), yields that:

$$\partial y^i(p,w,z)/\partial p_j|_{z=\tilde{z}(p,w,r)} < \partial \tilde{y}^i(p,w,r)/\partial p_j .$$

Therefore we can conclude that output supplies of an individual firm tend to become more complementary to each other in the long-run.

Continuing the case of a single fixed input, now assume that the fixed input is complementary to every variable input in the long-run, i.e., $\partial \tilde{x}^j/\partial r = \partial \tilde{z}/\partial w_j < 0$. Then, using $\tilde{x}^j(p,w,r) = x^j(p,w,\tilde{z}(p,w,r))$, we get $\partial \tilde{x}^j/\partial r = (\partial x_j/\partial z)(\partial \tilde{z}/\partial r)|_{z=\tilde{z}(p,w,r)} < 0$. Since $\partial \tilde{z}/\partial r < 0$, we must have $\partial x^j/\partial z|_{z=\tilde{z}(p,w,r)} > 0$ for every $j = 1, 2, \ldots, n$. Therefore $(\partial x^i/\partial z)(\partial \tilde{z}/\partial r)(\partial x^j/\partial z)|_{z=\tilde{z}(p,w,r)} < 0$ and, again by Proposition (3.8) (iii), we obtain that:

$$\partial \tilde{x}^i/\partial w_j < \partial x^i/\partial w_j|_{z=\tilde{z}(p,w,r)} .$$

This means that the demand for variable inputs tends to become complementary to each other in the long-run strenghtening Rader's contention in this paper (1968) discussed in Section 2 of this chapter. If the fixed input is both a normal input to the production of every output and a complement to every variable input in the long-run, then similarly we must have $(\partial y^i/\partial z)(\partial \tilde{z}/\partial r) \cdot (\partial x^j/\partial z)|_{z=\tilde{z}(p,w,r)} < 0$. Thus Proposition (3.8) (iii) implies that:

$$\begin{aligned}\partial \tilde{y}^i/\partial w_j &= -\partial \tilde{x}^j/\partial p_i < \partial y^i/\partial w_j|_{z=\tilde{z}(p,w,r)} \\ &= -\partial x^j/\partial p_i|_{z=\tilde{z}(p,w,r)} .\end{aligned}$$

This shows that every variable input tends to become more normal in the production of any output in the long-run.

4. The Case of Homothetic Production Functions

Throughout this section, we will consider only the case of a single output and without fixed inputs and we will examine various implications when production functions are homothetic. Shephard (1953, p. 41) introduced the concept of "homothetic" production functions which are defined as follows:

(4.1) *Definition:* A function $F: U \to R$, where $U \subseteq R^n$, $0 \in U$ and U is a cone, is said to be *homothetic* if there exists $\phi: D \to R$ and $H: U \to R$ such that (i) $D = H(U)$ and for every $x \in U$, $F(x) = \varphi[H(x)]$, (ii) ϕ is continuous and strictly increasing on D with $\phi(0) = 0$, and (iii) H is positively homogeneous of degree one on U.

We will further assume the following conditions on the homothetic production function.

(4.2) (i) $U = R^n_+$, (ii) ϕ and H are continuously differentiable on interiors of each domain, and (iii) $H_j(x) = \partial H(x)/\partial x_j > 0$ on the interior of R^n_+.

Define $D' = \phi(D)$ and let $\psi: D' \to D$ be the inverse of ϕ, i.e., $y = \psi(v)$ if and only if $v = \psi(y)$. Since H is positively homogeneous of degree one on R^n_+, we

have that $g(y, w)$ minimizes $w \cdot x$ subject to $F(x) - y = \phi(H(x)) - y \geqq 0$ if and only if $g(y, w)$ minimizes $w \cdot x$ subject to $H(x) - \psi(y) \geqq 0$ if and only if $g(y, w)$ minimizes $w \cdot (x/\psi(y))$ subject to $H(x/\psi(y)) - 1 \geqq 0$. Therefore, it suffices to investigate the problem to minimize $w \cdot a$ subject to $H(a) - 1 \geqq 0$. Let the Lagrangean function of this minimization problem be:

$$\underset{a \in R^n_+}{\text{minimize}}$$

$$L(a, \lambda: w) = w \cdot z + \lambda(1 - H(a)) .$$

Then the first-order conditions become:

(4.3.a) $w_j - \lambda H_j(a) \geqq 0 ; \quad (j = 1, 2, \ldots, n),$

and

(4.3.b) $a_j(w_j - \lambda H_j(a)) = 0 ; \quad (j = 1, 2, \ldots, n).$

If the function H is strictly quasi-concave, the above minimization problem is well-defined and possessed a unique solution, say $a(w) = (a^j(w))_{j=1,\ldots,n}$ for every $w > 0$. Let $\lambda(w)$ be the associated Lagrange multiplier. Summing (4.3.b) over j yields:

$$(4.4) \quad \sum_{j=1}^{n} w_j a^j(w) = \lambda(w) \sum a^j(w) H_j(a(w)) = \lambda(w)$$

where we have used the fact that $H(a) = \sum_{j=1}^{n} a_j H_j(a)$ by Euler's theorem on homogeneous functions. (See (3.2) of the mathematical appendix for the statement of Euler's theorem.) Since $a^j(w) = g^j(y, w)/\psi(y)$; $(j = 1, 2, \ldots, n)$, we obtain that:

$$g^j(y, w) = \psi(y) a^j(w) ; \quad (j = 1, 2, \ldots, n) \quad \text{and}$$

$$C(y, w) = \psi(y) c(w)$$

where $c(w) = \sum_{j=1}^{n} w_j a^j(w)$. Therefore input demand functions and the cost function can be factored out into multiplicative forms involving y and w separately. If $g^n(y, w) > 0$, then $g^j(y, w)/g^n(y, w) = a^j(w)/a^n(w)$ independent of y. Therefore for each $w > 0$, the expansion path will be a straight line through the origin.

The value of the function H denoted by ν can be interpreted as an aggregate of n inputs or as an intermediate input which is produced by $\nu = H(x)$ and used in producing y through $y = \phi(\nu)$. Therefore we can unambiguously define returns to scale independent of input combinations by:

$$\alpha(\nu) = \frac{d \log \phi(\nu)}{d \log \nu} = \nu \phi'(\nu)/\phi(\nu)$$

which is equivalent to the reciprocal of the output elasticity of the cost function

$$\beta(y) = 1/(d \log C(y, w)/d \log Y)$$
$$= \psi(y)/(y \psi'(y)) .$$

These two elasticities are equivalent because $\alpha(\nu) = \beta[\phi(\nu)]$ or $\beta(y) = \alpha[\psi(y)]$.

(4.5) *Example:* Suppose that ϕ satisfies that (i) ϕ: $R_+ \to R_+$; (ii) for every $v \in (0, \infty)$, $\phi'(v) > 0$; (iii) there exists $\bar{v} \in (0, \infty)$ such that $\phi''(v) > 0$ for $v \in (0, \bar{v})$ and $\psi''(v) < 0$ for $v \in (\bar{v}, \infty)$ and (iv) $\lim_{v \to \infty} \phi'(v) = 0$. Note that ϕ is strictly convex on $(0, \bar{v})$. Therefore, if $v \in (0, \bar{v})$, we have $\phi(v) < \phi'(v)v$ or $\phi(v) - \phi'(v)v < 0$. Also, since ϕ is strictly concave on $(\bar{v}, \infty)$, we have that for every v^0, $v^1 \in (\bar{v}, \infty)$ $\phi(v^1) - \phi(v^0) < \phi'(v^0)(v^1 - v^0)$ or $\phi(v^1) - \phi'(v^0)v^1 < \phi(v^0) - \phi'(v^0)v^0$. If we let v^0 become large enough, then we can have $\phi(v^1) - \phi'(v^0)v^1 > 0$ because $\lim_{v \to \infty} \phi'(v) = 0$. Hence there exists $v \in (\bar{v}, \infty)$ such that $\phi(v) - \phi'(v)v > 0$. Thus there must exist a unique v^* such that $\phi(v^*) - \phi'(v^*)v^* = 0$, i.e., $\phi'(v^*) = \phi(v^*)/v^*$. Therefore we obtain that $\alpha(v) > 1$ for $v \in (0, v^*)$, $\alpha(v^*) = 1$ and $0 < \alpha(v) < 1$ for $v \in (v^*, \infty)$. The average cost in this case achieves its minimum at $y^* = \phi(v^*)$ because $C(y, w)/y = (\psi(y)/y)c(w) = (v/\phi(v))c(w)$ and $\phi(v)/v$ achieves its maximum when $\phi(v)/v = \phi'(v)$, i.e., when $v = v^*$. See Figure 4.1 for the illustration. We can also incorporate a production function with three stages, i.e., ϕ: $[0, c] \to R$ and there exist $\bar{v}$ and $\hat{v}$ such that $0 < \bar{v} < \hat{v} < c$, $\phi'(v) > 0$ and $\phi''(v) > 0$ for $v \in (0, \bar{v})$, $\phi'(v) > 0$ and $\psi''(v) < 0$ for $v \in (\bar{v}, \hat{v})$ and $\phi'(v) < 0$ for $v \in (\hat{v}, c)$. For example, consider $\phi(v) = v^2(c - v)$ fore $v \in [0, c]$. The we get that $\bar{v} = (1/3)c$, $v^* = (1/2)c$, $\hat{v} = (2/3)c$ and returns to scale become $\alpha(v) = 3 - c/(c - v)$. See Figure 4.2.

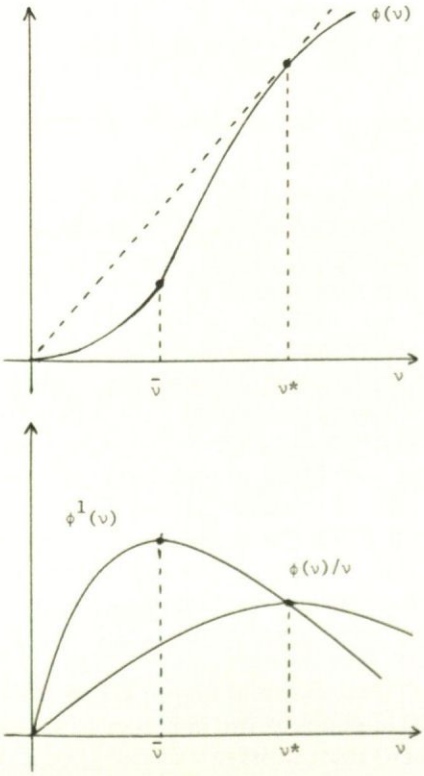

Fig. 4.1

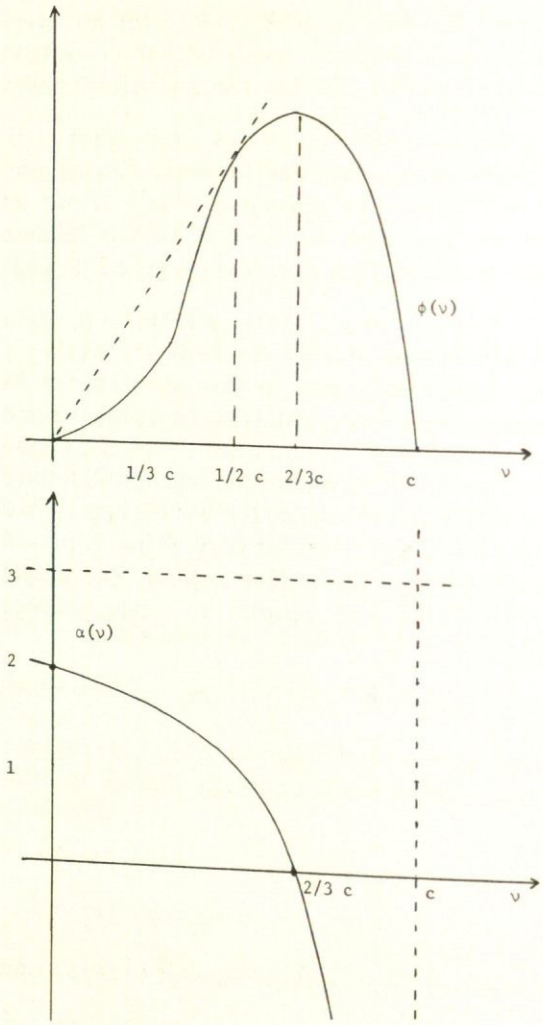

Fig. 4.2

A special class for homothetic functions is functions which are positively homogeneous of degree one. When a production function is positively homogeneous of degree one, then the production is said to obey constant returns to scale and the technology set underlying the production function is a cone. Since $\psi(y) = y$ under constant returns to scale, the cost function and input demand functions become $C(y, w) = y\,c(w)$ and $g^j(y, w) = y\,a^j(w)$, $(j = 1, 2, \ldots, n)$. Therefore the unit cost function $c(w)$ is nothing but the average cost (also the marginal cost) function. Because the profit can be written as $\pi(y, p, w) = y(p - c(w))$, the output supply behavior of a firm can be summarized by:

(4.6) $p - c(w) \leqq 0$, and $y(p - c(w)) = 0$.

Therefore we get the horizontal supply schedule as shown in Figure 4.3 by thick lines. By (4.4), the Lagrange multiplier $\lambda(w)$ is equal to the unit cost function $c(w)$. Therefore variable input coefficients $a(w)$ must satisfy:

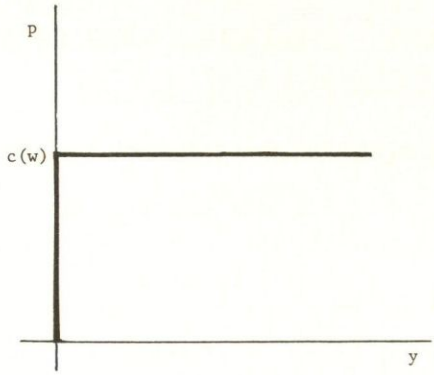

Fig. 4.3

$$pF_j(a(w)) \leqq c(w)F_j(a(w)) \leqq w_j \qquad (j = 1, 2, \ldots, n)$$

where the first inequality on the left becomes equality when $y > 0$ and the second on the right becomes equality when $a^j(w) > 0$.

The profit condition (4.6) implies that $py = yc(w) = \sum_{j=1}^{n} w_j y a^j(w) = \sum_{j=1}^{n} w_j g^j(y, w)$ for every (p, w) compatible with the profit conditions, i.e., the total revenue is equal to the total payment to factors of production. This is sometimes called the product exhaustion property and often explained as a consequence of Euler's theorem in the mathematical appendix.

If we use the necessity of Euler's Theorem and (4.4), we obtain

$$y = \sum_{j=1}^{n} x_j F_j(x)|_{x = g(y,w)} = \frac{1}{\lambda(w)} \sum_{j=1}^{n} w_j g^j(y, w), \quad \text{i.e.,} \quad yc(w) = \sum_{j=1}^{n} w_j g^j(y, w)$$

since $\lambda(w) = c(w)$. Then by the profit condition, $py = yc(w)$, the exhaustion property obtains for every (p, w) compatible with the profit condition.

On the other hand, the condition of a long-run competitive market equilibrium with free entry and identical technology among firms requires that the pure or economic profit be eliminated, i.e., $p^0 = C(y^0, w^0)$ where y^0 is the long-run equilibrium output of an individual firm and (p^0, w^0) is a pair of the output price and input prices prevailing at the long-run equilibrium. When the production function is homothetic with a convex-concave function as in example (4.5), then y^0 must satisfay the condition $y^0 = \phi(v^0)$ and $\psi(v^0)/v^0 = \phi'(v^0)$ which is the condition of the minimum average cost. Therefore, letting $x^0 = g(y^0, w^0)$, we have that:

$$\begin{aligned} F(x^0) &= \phi[H(x^0)] = \phi'(v^0)H(x^0) \\ &= \phi'(v^0) \sum_{j=1}^{n} x_j^0 H_j(x^0) \\ &= \sum_{j=1}^{n} x_j^0 \phi'(v^0) H_j(x^0) = \sum_{j=1}^{n} x_j^0 F_j(x^0). \end{aligned}$$

But the condition $F(x^0) = \sum_{j=1}^{n} x_j^0 F_j(x^0)$ does not necessarily satisfy the sufficiency part of Euler's theorem because Euler's theorem requires that $F(x) = \sum_{j=1}^{n} x_j F_j(x)$ *for every* $x \in U$. Therefore the production function does not have to be positively homogeneous of degree one.

Chapter 6 Industry Supply and Industry Equilibrium in a Competitive Market

1. Introduction

In this chapter, we will show how short-run and long-run supply maps for an industry can be derived in a more rigorous fashion than a standard approach in intermediate level textbooks so that short-run and long-run supply maps can be used in analyzing various cases of market equilibrium.

In deriving both short-run and long-run supply maps, we will assume the existence of infinitely many firms or, in particular, a continuum of potential firms so that discontinuities of individual supply maps arising from the presence of shutdown points can be smoothed out through the aggregation over firms.

In the long-run situation, we will explicitly introduce entrepreneurial factors which are assumed to be indispensable and indivisible in production. We will argue that the presence of an indispensable and indivisible factor called an entrepreneur turns out to be crucial in distinguishing a long-run supply map of an individual firm an a long-run supply map for an industry.

When entrepreneurs are not homogeneous in quality, then we will show that a long-run supply map for an industry exhibits an increasing cost and a long-run market equilibrium will generate positive Ricardian rents to intramarginal entrepreneurs. On the other hand, when an infinite supply of homogeneous entrepreneurs is available to an industry, the industry will exhibit a constant cost with zero pure profit to each active firm. Furthermore, we will show that a constant cost industry supply map can be regarded as if it is derived from an aggregate production function with constant returns to scale through profit maximization.

2. Short-run Industry Supply and Short-run Industry Equilibrium

In the short-run situation of an industry, it is supposed that there exists a predetermined set of active firms denoted by $[0, T]$ when the size of firms is measured by the Lebesgue measure (or the measurement of lengths) so that an individual firm $t \in [0, T]$ is an infinitesimal. Let $av(y, t)$ and $m(y, t)$ be the short-run average variable cost function and the short-run marginal cost function of firm $t \in [0, T]$ which will be assumed to satisfy: (i) $av(y, t)$ achieves its minimum at $y_0(t) \geqq 0$ with the minimum value $p_0(t) = \min \{av(y, t) \mid y \geqq 0\} > 0$; (ii) $m(y, t)$ is defined on $(0, \bar{y}(t))$ where $\bar{y}(t) \in (0, +\infty]$ and it is continuous and strictly in-

creasing in y on $(y_0(t), \bar{y}(t))$. Let us define $\bar{p}(t) = \lim_{y \to \bar{y}(t)} m(y, t)$ which could be $+\infty$. Then the short-run supply map denoted by $y(p, t)$ of firm $t \in [0, T]$ can be defined by:

$$y(p, t) = \begin{cases} 0 & \text{if } p \in [0, p_0(t)) \\ \{0, y_0(t)\} & \text{if } p = p_0(t) \\ y \text{ such that } p = m(y\,t) & \text{if } p \in (p_0(t), \bar{p}(t)), \quad \text{and} \\ \bar{y}(t) & \text{if } p \geqq \bar{p}(t). \end{cases}$$

We provide an example of an individual supply map in Figure 6.1.

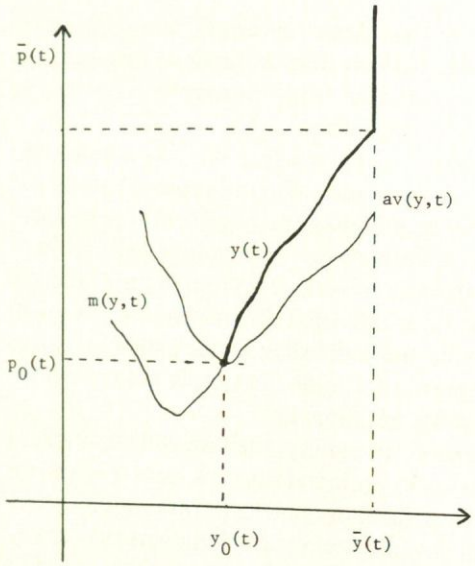

Fig. 6.1

(6.1) *Remark:* We will not be concerned with technicalities on the measurability of functions and mappings. But we note that if $m(y, t)$ and $av(y, t)$ are Borel measurable (or more particularly if they are continuous with respect to t), then $y_0(t)$, $p_0(t)$, $\bar{y}(t)$, $\bar{p}(t)$ and $y(p, t)$ can be shown to be Borel measurable in t for a given p.

Our next step is to obtain the aggregate supply map denoted by $y(p)$ of this industry. Let us define $p_0 = \inf_{t \in T} p_0(t)$. Then for $0 \leqq p < p_0$, $y(p) = 0$. Now suppose that $p > p_0$ and $\lambda[T_0(p)] = 0$ where $T_0(p) = \{t \mid p_0(t) = p\}$ and λ denotes the Lebesgue measure. Then the aggregate supply of the industry at p is defined by:

$$y(p) = \int_{T(p)} y(p, t)\, d\lambda$$

where $T(p) = \{t \mid p_0(t) < p\}$.

Now suppose that $\lambda[T_0(p)] > 0$. In this case, there exists infinitely many firms with a positive measure like a nondegenerate interval which have discontinuities due to shutdown possibilities at p since, for $t \in T_0(p)$, $y(p, t) = \{0, y_0(t)\}$. In order

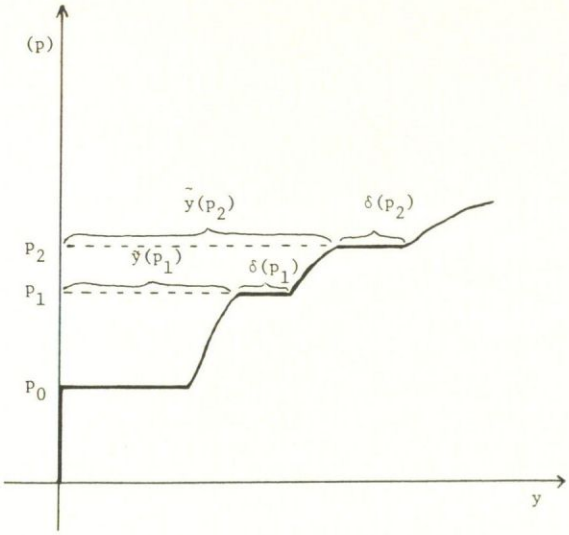

Fig. 6.2

to integrate $y(p, t)$ over $T_0(p)$, we have to borrow the idea of the integral of multivalued mappings, see e.g. Hildenbrand (1974) for more results. This is defined as follows:

$\int_{T_0(p)} y(p, t)\, d\lambda =$ the set of all $\int_{T_0(p)} f(t)\, d\lambda$ such that $f(t)$ is integrable over $T_0(p)$ and $f(t) \in y(p, t)$ for almost every $t \in T_0(p)$.

Noting that the supremum value of $\int_{T_0(p)} f(t)$ for the particular mapping $y(p, t)$ is $\int_{T_0(p)} y_0(t)\, d\lambda = b(p)$ and the smallest value of $\int_{T_0(p)} f(t)\, d\lambda$ is zero, we can convince ourselves that $\sum_{T_0(p)} y(p, t)\, d\lambda = [0, b(p)]$ (or $[0, \infty)$ if $b(p) = +\infty$). Denote this convex interval by $\delta(p) = [0, b(p)]$. Then the aggregate supply of this industry at p can be written as:

$$y(p) = \int_{T(p) \cup T_0(p)} y(p, t)\, d\lambda = \int_{T(p)} y(p, t)\, d\lambda + \delta(p)$$
$$= \tilde{y}(p) \neq \delta(p)$$

where $\tilde{y}(p) = \int_{T(p)} y(p, t)\, d\lambda$ and

$$\tilde{y}(p) + \delta(p) = [\tilde{y}(p), \tilde{y}(p) + b(p)].$$

Therefore, we can conclude that the aggregate supply map of this industry is a continuous nondecreasing map possibly with flat convex segments (in the sense of the textbook type supply schedule with a quantity variable on the horizontal axis and with a price variable on the vertical axis). See Figure 6.2 for an illustration of the short-run supply map of an industry.

(6.2) *Example:* Suppose that $p_0(t) = p_0 + t$ for some $p_0 > 0$, $y_0(t) = y_0$ for some $y_0 > 0$ and an individual supply map $y(p, w)$ is given by:

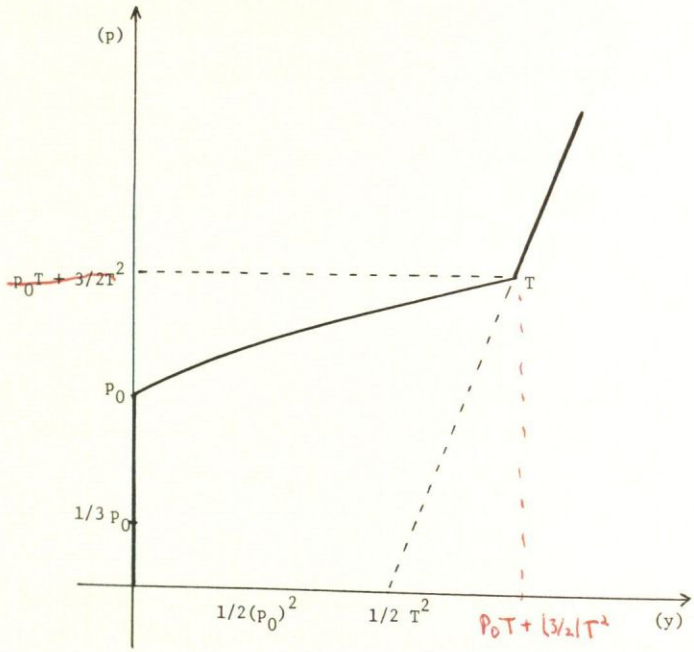

Fig. 6.3

$$
\begin{aligned}
y(p, t) &= 0 && \text{for} \quad p \in [0, p_0(t)) \\
&= \{0, y_0\} && \text{for} \quad p = p_0(t) \\
&= p + t && \text{for} \quad p > p_0(t).
\end{aligned}
$$

Then for $p \in [p_0, p_0 + T]$, $T_0(p) = \{t \mid p_0(t) = p\} = \{p - p_0\}$ which is a singleton and $T(p) = \{t \mid p_0(t) < p\} = [0, p - p_0)$. Therefore the aggregate supply map for the industry becomes:

for $p \leqq p_0$, $y(p) = 0$, for $p \in (p_0, p_0 + T)$,

$$
\begin{aligned}
y(p) &= \int_{T(p)} y(p, t)\, dt = \int_0^{p - p_0} (p + t)\, dt \\
&= (1/2)(p - p_0)(3p - p_0)
\end{aligned}
$$

and for $p \geqq p_0 + T$

$$
y(p) = \int_0^T (p + t)\, dt = pT + (1/2)T^2.
$$

We provide an illustration of the supply map by a thick curve in Figure 6.3. Since firms shut down continuously, the supply curve has no flat segment in this example.

(6.3) *Example:* Let $T = 2$, $y_0(t) = y_0 > 0$ for every $t \in [0, 2]$, $p_0(t) = p_0 > 0$ for $t \in [0, 1]$ and $p_0(t) = 2p_0$ for $t \in (1, 2]$. Individual supply maps for $p > p_0(t)$ will be assumed to be:

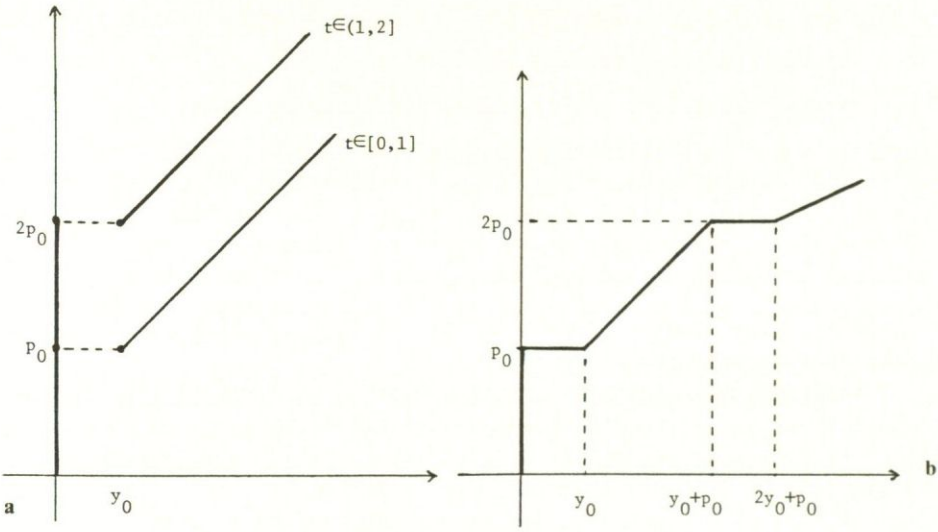

Fig. 6.4.a, b

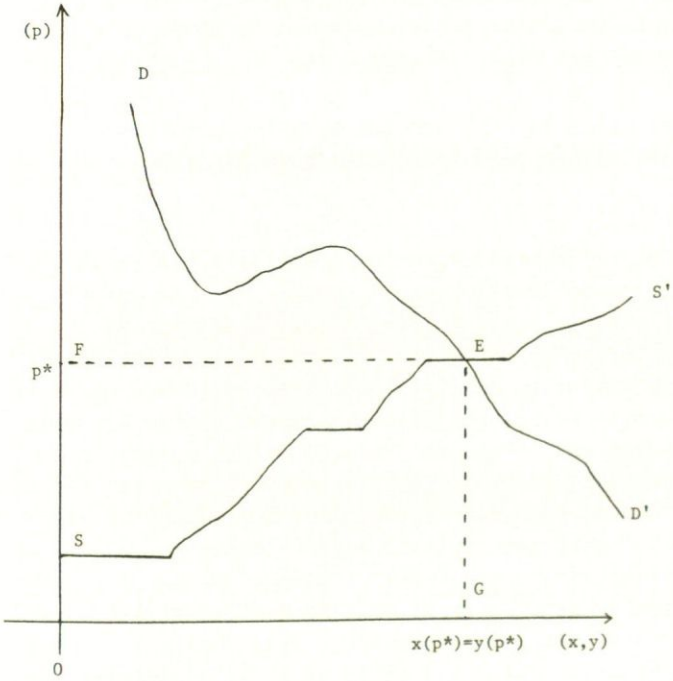

Fig. 6.5

$$y(p, t) = p - p_0 + y_0 \quad \text{for } t \in [0, 1]$$
$$= p - 2p_0 + y_0 \quad \text{for } t \in [1, 2].$$

Then it is easy to check that the aggregate supply map becomes:

$$y(0) = \begin{cases} 0 & \text{for } p \in [0, p_0) \\ [0, y_0] & \text{for } p = p_0 \\ p - p_0 + y_0 & \text{for } p \in (p_0, 2p_0) \\ y_0 + p_0 + [0, y_0] & \text{for } p = 2p_0 \\ 2p - 3p_0 + 2y_0 & \text{for } p > 2p_0. \end{cases}$$

Individual supply maps are shown in Figure 6.4(a) and thick lines in Figure 6.4(b) show the aggregate supply map.

If we let $x(p)$ be an aggregate market demand for the output produced by the industry concerned, then a competitive market equilibrium price in the short-run, say p^*, is defined to be the market price which equates the aggregate market demand and the short-run aggregate industry supply, i.e., $x(p^*) - y(p^*) = 0$ as shown in Figure 6.5 where DD′ is the graph of $x(p)$ and SS′ is the graph of $y(p)$. The area between the demand curve DE and EF is the Marshallian consumers' surplus, the area ESOG under the supply curve is the total sum of marginal costs over firms producing $y(p^*)$ which amounts to the total sum of variable costs over firms and finally the area EFSE between the price line and the supply curve is the total sum of rents and quasi-rents over producing firms.

3. Long-run Industry Supply and Long-run Equilibrium of an Industry

In this section, we will investigate how a long-run supply of an industry is derived and we discuss some basic properties of a long-run equilibrium of an industry. A long-run situation of an industry is characterized by the following conditions, i.e., (a) the industry faces "many" potential entrants to the industry, (b) the entry to the industry and the exit from the industry is free in the sense that the entry cost and the exit cost are both absent, and (c) every potential entrant has an access to an identical long-run technology set. Throughout this section, we will identify the long-run technology set by the long-run total cost function. A firm will be characterized by the long-run total cost function and an *indivisible and indispensable input* called an entrepreneurial input or an entrepreneur. We denote an entrepreneurial input by e_t where t is an index of names of entrepreneurs with $t \in T = [0, \infty)$, e_t is assumed to lie in $I = [a, b]$ for some real numbers $a < b$ and the set I is an index set for entrepreneurial quality types. The size of entrepreneurs will be measured by the Lebesgue measure on T. An entrepreneur will be assumed to be the sole owner and the sole manager of the firm. Later we will discuss the validity of this assumption. The long-run total cost function will be denoted by $\tilde{C}(y, e_t)$ where y is the level of output and all variables other than (y, e_t) are subsumed away. Since the total cost function does not directly depend on t, it is identical among firms, but the quality of entrepreneurial inputs may be different among entrepreneurs. The ability of entrepreneurs can be different as

entrepreneurs but when they choose to earn income as workers, we will assume that they earn the same wage rate. In particular, we will assume that $\tilde{C}(0, e_t) = \bar{w} > 0$ for every $t \in T$ and $\bar{w}$ is the wage rate which entrepreneurs can earn as workers. Therefore, in the long-run, all inputs other than an entrepreneurial input are assumed to be variable. Also note that the cost of an entrepreneur above $\bar{w}$ is not included in the total cost function mainly because of the clarity in our exposition.

For the sake of simplicity, we will assume that e_t, considered as a mapping from T the set of entrepreneur's names into I the set of entrepreneur's quality types, induces a continuous density function $f(e)$ on I with its support I, i.e., I = the closure of $\{e \mid f(e) > 0\}$.

The long-run average total cost function denoted by $\tilde{a}(y, e)$ and the long-run marginal cost function denoted by $m(y, e)$ will be defined respectively by: for $y > 0$, $\tilde{a}c(y, e) = \tilde{C}(y, e)/y$ and $\tilde{m}(y, e) = \partial\tilde{C}(y, e)/\partial y$. The long-run average cost function will be assumed to have a U-shape and the long-run marginal cost function is assumed to be strictly increasing above the output which attains the minimum total average cost. More specifically, we will assume that:

(6.4)(a) $\tilde{a}c(y, e)$ and $\tilde{m}(y, e)$ are assumed to be continuous in $y > 0$, (b) for each $e \in I$, there exists $\tilde{y}_0(e) > 0$ such that $\tilde{a}c(\tilde{y}_0(e), e) < \tilde{a}c(y, e)$ for every $y > 0$ with $y \neq \tilde{y}_0(e)$, (c) $\tilde{m}(y, e)$ is strictly increasing in y for $y > \tilde{y}_0(e)$, and (d) $\tilde{y}_0(e)$ is uniformly bounded below by a positive number, i.e., there exists $\varepsilon > 0$ such that for every $e \in I$, $\tilde{y}_0(e) \geqq \varepsilon$.

Let us denote the minimum average cost of $e \in I$ be $\tilde{a}c_0(e) = \tilde{a}c(\tilde{y}_0(e), e)$ and we will introduce the following simplifying condition on $\tilde{a}c_0(e)$.

(6.5)(a) $\tilde{a}c_0(e)$ is a continuous and strictly increasing function on I, $0 < \underline{p} = \lim_{e \to a} \tilde{a}c_0(e) < \bar{p} = \lim_{e \to b} \tilde{a}c_0(e) < +\infty$. See Figure 6.6 for the illustration of $\tilde{a}c_0(e)$. Then the long-run supply map denoted by $\tilde{y}(p, e)$ of each firm-type or an entrepreneur-type e can be defined by: $\tilde{y}(p, e) = 0$ for $p \in [0, \tilde{a}c_0(e))$, $\tilde{y}(p, e) = \{0, \tilde{y}_0(e)\}$

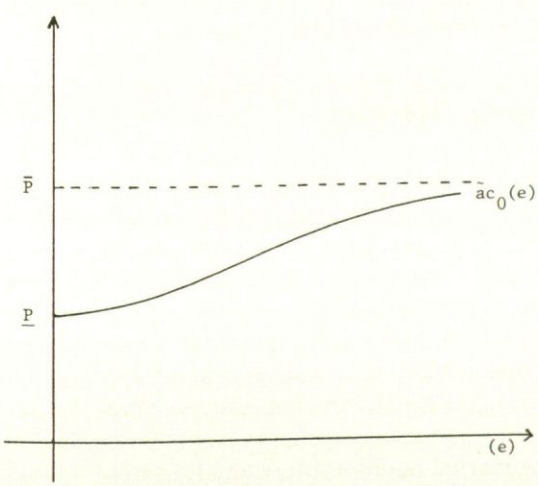

Fig. 6.6

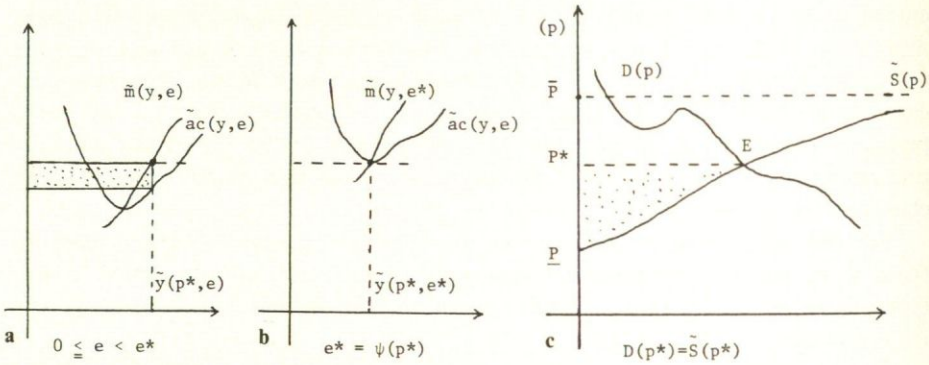

Fig. 6.6.a – c

for $p = \tilde{a}c_0(e)$ and for $p > \tilde{a}c_0(e)$, $\tilde{y}_0(p, e) = y$ such that $\tilde{m}(y, e) = p$ if such a y exists and $\tilde{y}(p, e) = \infty$ if otherwise.

Define the set of active firm-types at p by $I(p) = \{e \in I \mid \tilde{a}c_0(e) \leqq p\}$. Then the long-run aggregate supply map for the industry can be defined as:

$$\tilde{S}(p) = \int_{I(p)} \tilde{y}(p, e) f(e) de .$$

Note that $\lim_{p \to \underline{p}} \tilde{S}(p) = 0$ and $\lim_{p \to \bar{p}} \tilde{S}(p) = \infty$. The latter holds because $\lim_{p \to \bar{p}} \int_{T(p)} \tilde{y}(p, e) f(e) de \geqq \lim_{t \to \infty} \int_0^t \cancel{e}\, ds = \infty$. For the sake of easier exposition, we will assume that for every $p \in (\underline{p}, \bar{p})$, $0 < \tilde{S}(p) < +\infty$. Then for each $p \in (\underline{p}, \bar{p})$, there exists an entrepreneur e with $\tilde{a}c_0(e) = p$. Such a firm-type or an entrepreneur-type will be called a *marginal firm-type* or a *marginal entrepreneur-type* at p. Let us denote an inverse map to $\tilde{a}c_0(e) \neq p$ by $e = \psi(p)$. Then $\psi(p)$ is the marginal entrepreneur-type at p. Suppose that the marginal cost function is differentiable for $y > \tilde{y}_0(e)$ and also $\psi(p)$ is differentiable on $(\underline{p}, \bar{p})$. Then for every $p \in (\underline{p}, \bar{p})$ we have that:

$$\tilde{S}'(p) = \int_a^{\psi(p)} \tilde{y}'(p, e) f(e) de + \tilde{y}(p, \psi(p)) \psi'(p) f(\psi(p))$$

$$= \int_a^{\psi(p)} \tilde{y}'(p, e) f(e) de + \tilde{y}_0(\psi(p)) \psi'(p) f(\psi(p)) .$$

Therefore the long-run marginal cost will be increasing because the long-run marginal cost is increasing and also because of the entry of the marginal entrepreneur-type with inferior quality.

Let D(p) be the aggregate demand for the product produced by the industry concerned. Then the long-run market equilibrium price is given by $p^* \in (\underline{p}, \bar{p})$ such that $D(p^*) - \tilde{S}D(p^*) = 0$. Then $e^* = \psi(p^*)$ is the marginal entrepreneur-type earning $\bar{w}$ and zero Ricardian rent. Under our assumptions, all active entrepreneur-types $e \in [a, e^*)$ will earn positive Ricardian rent and potential but inactive entrepreneur-types $e \in (e^*, +\infty)$ will become workers earning $\bar{w}$. See Figure 6.6 for an illustration of a long-run market equilibrium where (c) shows a long-

run equilibrium, (b) a situation of the marginal entrepreneur-type and (a) a situation of an active entrepeneur-type $e \in [a, e^*)$ earning positive Ricardian rent of the dotted area. Also note that the shaded region in (c) between p^*E and the long-run supply curve for the industry gives us the total Ricardian rent for the industry accruing to all active entrepreneurs. Let us denote the Ricardian rent of entrepreneur-type e by $\tilde{\pi}(y, p, e) = py - \tilde{C}(y, e)$. Then the long-run average cost inclusive of the Richardian rent becomes $\hat{a}c(y, p, e) = \tilde{a}c(y, e) + \tilde{\pi}(y, p, e)/y$ for $y > 0$. Since $p \equiv \hat{a}c(y, p, e)$ identically by definition and for $p > \tilde{a}c_0(e)$ the individual supply map follows $p = \tilde{m}(y, e)$, for $p > \tilde{a}c_0(e)$ we have that $\tilde{m}(y, e) = \hat{a}c(y, p, e)$. Therefore, in this sense, it is sometimes stated that the long-run supply curve for the industry is the horizontal sum of individual long-run marginal cost curves and it is also the horizontal sum of individual long-run average cost curves *inclusive of rent,* i.e., a long-run average cost curve for the industry inclusive of rent. (See Viner (1931) on "Ricardian" increasing costs.) But it should be noted that since the Ricardian rent is a result of a long-run market equilibrium, individual long-run average costs inclusive of rent become indeterminate unless we know a long-run equilibrium price. (See Friedman (1962, p. 117) on this point.)

Suppose that each potential firm adopts its capacity at each efficient scale. Therefore if we denote a short-run supply map by $y(p, e)$ and a long-run supply map by $\tilde{y}(p, e)$, then we must have $y(\tilde{a}c_0(e), e) = \tilde{y}(\tilde{a}c_0(e), e)$ and for $p > \tilde{a}c_0(e)$, $y(\tilde{a}c_0(e), e) < \tilde{y}(\tilde{a}c_0(e), e)$ for the regular case. Consider the following simultaneous adjustment process of the market price p and the entrepreneur-type $\hat{e}$ where entrepreneur-types $e \leqq \hat{e}$ are assumed to be active and entrepreneur types $e > \hat{e}$ inactive:

$$\frac{dp}{d\tau} = D(p) - S(p, \hat{e})$$

$$\frac{d\hat{e}}{d\tau} = p - \tilde{a}c_0(\hat{e})$$

where $S(p, \hat{e}) = \int_a^{\hat{e}} y(p, e) f(e) de$ and τ indicates the adjustment time.

Then the Jacobian matrix of the above system is given by:

$$A = \begin{bmatrix} D'(p) - S'(p, \hat{e}), & -y(p, \hat{e}) f(\hat{e}) \\ 1 & , -\tilde{a}c_0'(\hat{e}) \end{bmatrix}$$

where $D'(p) = dD(p)/dp$ and $S'(p, \hat{e}) = \partial S(p, \hat{e})/\partial p$.

Let (p^*, e^*) be a long-run equilibrium. Then if $D'(p^*) - S'(p^*, e^*) < 0$, then $\operatorname{tr} A = D'(p^*) - S'(p^*, e^*) - \tilde{a}_0'(e^*) < 0$ and $\det A = -(D'(p^*) - S'(p^*, e^*)) \times \tilde{a}c_0'(e^*) + \tilde{y}_0(e^*) f(e^*) > 0$. Therefore the long-run equilibrium will be either a stable node or a stable focus. We will illustrate the case of a stable focus on Figure 6.7.

The long-run supply map we have derived is increasing in the market price because of the assumption that, as the market price increases, entrepreneur-types with inferior quality enter the market. A case like this is often called and *increasing cost industry*. Another interesting case is the case of a *constant cost* industry.

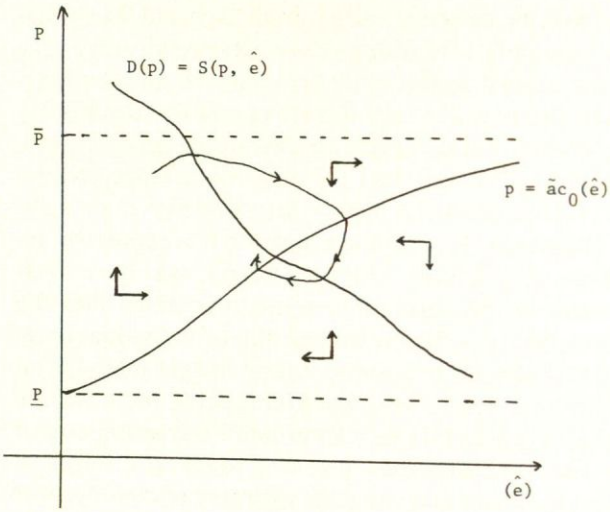

Fig. 6.7

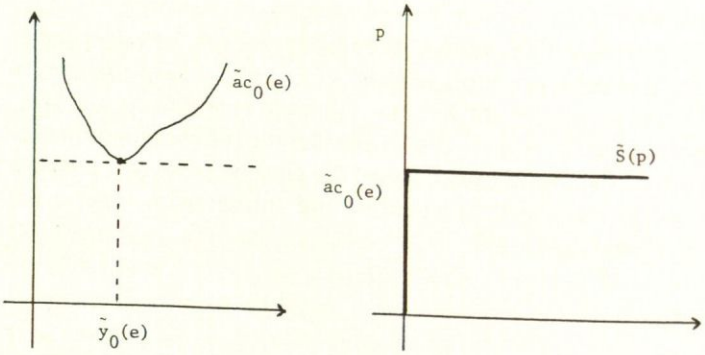

Fig. 6.8

A constant cost industry occurs when entrepreneurs are homogeneous or have identical quality. Therefore when prices of all inputs other than an entrepreneur input are given, then $\tilde{a}c_0(e)$ becomes constant on I. Under this condition, the aggregate supply map $\tilde{S}(p)$ can be characterized by $\tilde{S}(p) = \{0\}$ if $0 \leqq p < \tilde{a}c_0(e)$ and if $p = \tilde{a}c_0(e)$, then $\tilde{S}(p) = [0, +\infty)$. See Figure 6.8 for the illustration of an aggregate supply map and the cost condition of a representative firm. At a market equilibrium, each entrepreneur earns his opportunity cost $\bar{w}$ and the Ricardian rent to entrepreneur will disappear by the homogeneity of entrepreneurial quality. Therefore when the supply of entrepreneurs with homogeneous quality for the industry is available, then this industry can be characterized by an aggregate technology set with constant returns to scale even though an individual technology set exhibits variable returns to scale. This result can be shown for a general case as follows.

Let $\tilde{c}(y, w)$ be a long-run total cost function where y is a scalar of an output produced and w is an $(n \times 1)$ vector of prices of all inputs other than an entre-

preneurial input. Let the long-run average cost denoted by $A(y, w)$ be defined by $A(y, w) = \tilde{c}(y, w)/y$ for $y > 0$. We will assume that $\tilde{c}(y, w)$ and $A(y, w)$ satisfy the following conditions.

(6.6)(a) $\tilde{c}(y, w)$ is continuous, concave and positively homogeneous of degree one in $w \in P^n$ and if $w^0, w^1 \in P^n$ and $w^0/\|w^0\| \neq w^1/\|w^1\|$, then $\tilde{c}(y, w^t) > t\tilde{c}(y, w^1) + (1-t)\tilde{c}(y, w^0)$ for every $t \in (0, 1)$ where $w^t = tw^1 + (1-t)w^0$. (b) For every $w \in P^n$, there exists a unique $y_0(w) > 0$ such that $A(y_0(w), w) < A(y, w)$ for every $y > 0$ with $y \neq y(w)$.

Let us define the minimum value function of the long-run average cost by $\phi(w) = A(y_0(w), w) = \min_{y>0} A(y, w)$. Then we can prove the following Lemma.

(6.7) *Lemma:* ϕ: $P^n \to R$ is continuous and convex in w and if $w^0, w^1 \in P^n$ and $w^0/\|w^0\| \neq w^1/\|w^1\|$, then $\phi(w^t) > t\phi(w^1) + (1-t)\phi(w^0)$ for every $t \in (0, 1)$ where $w^t = tw^1 + (1-t)w^0$.

To prove this Lemma, let us write $y^0 = y_0(w^0)$, $y^1 = y_0(w^1)$ and $y^t = y_0(w^t)$ for $t \in (0, 1)$. Then $\phi(w^0) = A(y^0, w^0) \leqq A(y^t, w^0)$ and $A(y^1, w^1) \leqq A(y^t, w^1)$. Since the properties of $\tilde{c}(y, w)$ with respect to w are carried by $A(y, w)$, A is concave in w and if $w^0/\|w^0\| \neq w^1/\|w^1\|$, then $A(y, w^t) > tA(y, w^1) + (1-t)A(y, w^0)$. Therefore $A(y^t, w^t) \geqq tA(y^t, w^1) + (1-t)A(y^t, w^0)$ and the strict inequality holds when $w^0/\|w^0\| \neq w^1/\|w^1\|$. Hence we obtain $\phi(w^t) = A(y^t, w^t) \geqq tA(y^1, w^1) + (1-t)A(y^0, w^0) = t\phi(w^1) + (1-t)\phi(w^0)$ proving the concavity of ϕ and moreover the strict inequality of the above holds when $w^0/\|w^0\| \neq w^1/\|w^1\|$.

Let $g^j(y, w)$ be the input demand function for input j ($j = 1, 2, \ldots, n$) and let $a^j(y, w)$ be the input coefficient function for input j ($j = 1, 2, \ldots, n$) defined by $a^j(y, w) = g^j(y, w)/y$ for $y > 0$. Then $\tilde{c}(y, w) = \sum_{j=1}^{n} w_j g^j(y, w)$ and $A(y, w) = \sum_{j=1}^{n} w_j a^j(y, w)$. We then investigate derivative properties of $\phi(w)$ by adding the following conditions on the long-run total cost function $\tilde{c}(y, w)$, the long-run average cost function $A(y, w)$ and $y_0(w)$.

(6.8)(a) $\tilde{c}(y, w)$ is twice continuously differentiable in $(y, w) \in P^{n+1}$. (b) $[\tilde{c}_w(y, w)]' = g(y, w)$ for every $(y, w) \in P^{n+1}$. (c) For every $(y, w) \in P^{n+1}$, for every $\eta \in R^n$, if $\eta \neq 0$ and $\eta/\|\eta\| \neq w/\|w\|$, then $\eta'\tilde{c}_{ww}\eta < 0$. (d) For every $w \in P^n$, $y_0(w)$ is continuously differentiable and $A_{yy}(y_0(w), w) = \partial^2 A(y_0(w), w)/\partial y^2 > 0$.

Then we will establish that the minimum value function $\phi(w)$ possesses similar properties as those of $\tilde{c}$ in (6.8). In particular, we will show that:

(6.9)(a) For every $w \in P^n$, $\phi_w(w)$ where $\tilde{a}(w) \neq \tilde{a}(w) = a(y_0(w), w)$. (b) For every $w \in P^n$, if $\eta \in R^n$, $\eta \neq 0$ and $\eta/\|\eta\| \neq w/\|w\|$, then $\eta'\phi_{ww}(w)\eta < 0$.

The first property (a) follows from the evelope theorem, i.e., $[\phi_w(w)]' = [A_w(y_0(w), w)]' = g(y_0(w)\, w)/y = a(y_0(w), w)$. Since $\phi_w(w) = A_w(y_0(w), w)$ for every $w \in P^n$, we get $\phi_{ww}(w) = A_{wy}(y_0(w), w) \cdot y_{0w}(w) + A_{ww}(y_0(w), w)$. Also because the first-order condition $A_y(y_0(w), w) = 0$ holds for every $w \in P^n$, we

$y_0(w)$

$= \tilde{a}(w)$ where $\tilde{a}(w) = a(y_0(w), w)$

have $A_{yy}\, y_{0w}(w) + A_{yw} = 0$. Thus $\phi_{ww} = -A_{yy}(y_{0w})(y_{0w})' + A_{ww} = -A_{yy}(y_{0w})(y_{0w})' + \tilde{c}_{ww}/y$ which proves the assertion (b).

Thus we have established that the function $\phi(w)$ possesses all properties of a unit cost function derived from a production function which is positively homogeneous of degree one and which satisfies (4.1) of Chapter 4. Therefore we can apply the duality relationship between production functions and cost functions to assert that there exists a production function which is positively homogeneous of degree one which satisfies (4.1) of Chapter 4 and which generates the unit cost function $\phi(w)$.

When the original long-run production function is homothetic, then the long-run total cost function can be written as $\tilde{c}(y, w) = \psi(y)\, c(w)$ as shown in Section 4 of Chapter 5. Define y_0 by $\psi(y_0)/y_0 < \psi(y)/y$ for every $y > 0$. If ψ is differentiable, the y_0 can be defined by $\psi'(y_0) = \psi(y_0)/y_0$. Then for this case, $\phi(w) = (\psi(y_0)/y_0)\, c(w)$ and $y_0(w) = y_0$. Therefore the minimum value of the long-run average cost is achieved at y_0 independent of w. When the original long-run production function is not homothetic, then $y_0(w)$ will not be constant and thus $\tilde{a}(w)$ will lie on different isoquants of the original production functions as w varies. But our result shows that it is possible to define an aggregate production function which is concave and possesses constant returns to scale and moreover we can regard $\phi(w)$ to be the long-run unit (or average) cost function derived from the unit cost minimization subject to this aggregate production function.

We have several remarks regarding our approach in formulating the long-run supply map for an industry.

(6.10) *Remark:* We have assumed that an entrepreneurial input is indispensable to a firm and moreover indivisible in the sense that a firm requires an input of a single entrepreneur for its operation and a firm with more than one entrepreneur is less efficient than a firm with a single entrepreneur. Unless there exists an indispensable and indivisible unit of an input, the distinction between the long-run supply map of one firm and the long-run supply map for an industry gets blurred. Imagine a case where a firm can operate equally efficiently with the replication of many homogeneous entrepreneurial units. Then the long-run supply curve of a firm with multiple plants each with an entrepreneur or a manager will become horizontal just as in the case of the long-run supply curve for an industry with constant cost. Therefore we need justify the existence of an efficient scale for a firm distinct from the way in which the long-run supply for an industry gets expanded through the entry of new firms. Thus an entrepreneurial input can play an important role in providing a clear distinction between a firm's supply map and an industry's supply map.

(6.11) *Remark:* We have considered the case of a classical firm where an entrepreneur is both the owner and the manager of the firm. The concept of an owner of a firm should be interpreted as the owner who holds the residual claim of the firm. When uncertainties are present, a firm has to take another function, namely a function of risk bearing in addition to a function of management. Fama (1980) argues that the function of risk bearing gets separated from the managerial function through security ownership generating the well-known

separation of security ownership and management of the firm. This also suggests that the assumption of a classical firm may be justifiable in an economy without uncertainties. With no uncertainties, there is no reason to separate ownership and management. In fact, it can be argued that the separation of ownership and management may be an inefficient form of organization in the absence of uncertainties because managers may lose incentive to efficiently manage the firm.

(6.12) *Remark:* Cases of an increasing cost industry with a rising long-run supply schedule and a decreasing cost industry with a declining long-run supply schedule are often attributed to, what Viner (1937) calls, pecuniary external diseconomies or pecuniary external economies respectively. For example, the case of pecuniary external diseconomies is said to arise when prices of primary factors or materials rise as the industry's expansion of its output increases purchases of its inputs. One problem of this concept on pecuniary externalities is that it tries to incorporate a part of general equilibrium interdependencies into a partial equilibrium analysis. Thus it tends to make the meaning of "cetris paribus" conditions rather ambiguous. Also unless the technology set itself depends on prices, pecuniary externalities can not be conceived as a source of market failures. Therefore we believe that cases of increasing or decreasing cost industries should be explained either by the presence of Ricardian differential rents on a very specific factor as an entrepreneurial input or by the presence of Marshallian externalities explained in the next remark which are external to individual firms, but internal to the industry.

(6.13) *Remark:* The Ricardian differential rent due to differences in entrepreneurial quality can explain the case of an increasing cost industry. But we need to introduce Marshallian external economies in order to explain a decreasing cost industry for a competitive market. Needless to say, Marshallian external diseconomies can be used in explaining an increasing cost industry, but we will discuss mostly Marshallian external economies in this remark. Marshallian external economies must be economies external to individual firms so that the presence of economies is compatible with the assumption of a competitive industry. On the other hand, economies must be reaped in this industry so that we can justify the case of decreasing cost. Therefore economies must be internal to the industry. Examples of this kind of economies are not so easy to give, but it is usually attributed to either organizational improvement of input markets or organizational and informational improvement among entrepreneurs as the industry expands. To make the analysis simpler, suppose that entrepreneurs are all homogeneous, the long-run production function is homothetic when the total industry's output Y is fixed and has a scale effect as Y expands. In particular, suppose that a long-run individual technology is given by a production function of the form $\psi(y, Y) - f(x) = 0$ where f is homogeneous of degree one, ψ is assumed to have properties similar to the inverse function $y = \phi(v)$ of Example 4.5 of Chapter 5 for a fixed Y and $\partial\psi(y, Y)/\partial Y > 0$. Then the above production function generates a long-run total cost function of the form $C(y, w, Y) = \psi(y, Y)\,c(w)$ where w is a price vector for input x. Let us define $\tilde{\psi}(Y) = \min_{y>0} \psi(y, Y)/y$ assuming that $\tilde{\psi}(Y)$ is well-defined as in Example 4.5 of Chap-

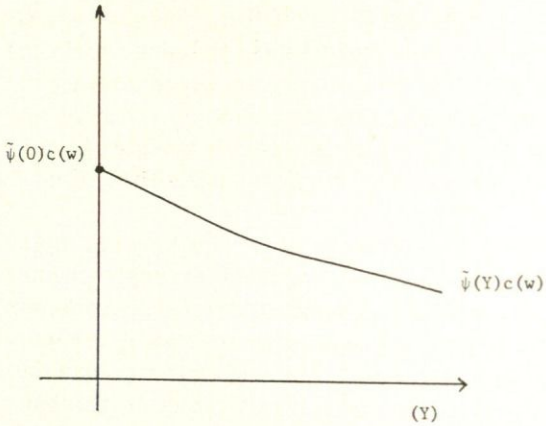

Fig. 6.9

ter 5. Note that, by the envelope theorem, $\tilde{\psi}'(Y) = (1/y)\,\partial\psi(y, Y)/\partial Y$ where y should be evaluated at the minimizer of $\psi(y, Y)$/for $y > 0$. Then $\min_{y>0} C(y, w, Y)/y = \tilde{\psi}(y)\,c(w)$ becomes the minimum value of the long-run average cost. Therefore equation $p = \tilde{\psi}(Y)\,c(w)$ defines the inverse supply map for the industry which is decreasing as Y expands as shown in Figure 6.9. When $\partial\psi(y, Y)/\partial Y > 0$, then we will have a case of Marshallian external diseconomies generating an increasing supply map.

Problems for Part II

1. The following functions are defined on the nonnegative quadrant of R^2. For each function obtain demand functions for inputs and the cost function if it is a reasonable production function.

   (a) $F(x_1, x_2) = x_1^a x_2^b \quad (a > 0, b > 0)$,

   (b) $F(x_1, x_2) = e^{x_1} + e^{x_2}$,

   (c) $F(x_1, x_2) = x_1 + 2x_2$,

   (d) $F(x_1, x_2) = x_1^2 + x_2^2$,

   (e) $F(x_1, x_2) = g\left[\mathrm{Min}\left(\frac{x_1}{a}, \frac{x_2}{b}\right)\right]$ or

   $h(y) = \mathrm{Min}\left(\frac{x_1}{a}, \frac{x_2}{b}\right), \quad (a > 0, b > 0)$,

   (f) $F(x_1, x_2) = g[(ax_1^{-\varrho} + bx_2^{-\varrho})^{-1/\varrho}]$ or

   $h(y) = (zx_1^{-\varrho} + bx_2^{-\varrho})^{-1/\varrho}, \quad (\varrho > -1)$

   (g) $F(x_1, x_2) = g\left[a x_2 \log\left(\frac{x_1}{x_2}\right) + b x_2\right]$ or

   $h(y) = a x_2 \log\left(\frac{x_1}{x_2}\right) + b x_2, \quad (a > 0, b > 0)$.

In (e), (f) and (g), the function $g(z)$ is twice continuously differentiable and $g'(z) > 0$ for every $z > 0$. Furthermore, the function h is the inverse function of g.

2. Find cost functions of the following production functions and show that cost functions are self-dual, i.e., cost functions have the same form as production functions.

   (a) $f(x_1, x_2) = x_1^{\alpha} x_2^{\alpha}$, $x_1 > 0$, $x_2 > 0$, $\alpha > 0$, $\beta > 0$.

   (b) $f(x_1, x_2) = (a x_1^{-\varrho} + b x_2^{-\varrho})^{-1/\varrho}$, $x_1 > 0$, $x_2 > 0$, $a > 0$, $b > 0$ and $\varrho > -1$ $(\varrho \neq 0)$.

3. Consider the following production function of a firm:

$$
\begin{array}{lll}
x = 0 & \text{for} & 0 \leqq L \leqq 5 \\
x = 2L - 10 & \text{for} & 5 \leqq L \leqq 10 \\
x = (1/2)L + 5 & \text{for} & 10 \leqq L \leqq 20 \\
x = 15 & \text{for} & 20 \leqq L
\end{array}
$$

   where x is output and L is labor input which is the only input required. Let p be the price of a unit of output and w be the wage rate of a unit of labor.

   Question: Obtain the supply schedule for x and the demand schedule for L when the firm is competitive (or a price taker).

4. Obtain the derived input demand schedule and the output supply schedule for the production function $F: R_+ \to R_+$ defined as follows:

   $f(x) = x$ for $x \in [0, 4/3)$, $f(x) = 4(x-1)$ for $x \in [4/3, 2)$, $f(x) = x + 2$ for $x \in [2, 4)$ and $f(x) = (1/2)x + 4$ for $x \in [4, \infty)$.

5. Consider a production function with two inputs and one output denoted by $y = f(x_1, x_2)$. Assume that the Hessian matrix of f is negative definite for $(x_1, x_2) > 0$. Let $x^i(p, w_1, w_2)$ be the derived demand function for input x_i.

   (a) Find explicit expressions for $\partial x^i / \partial w_j$ and show that $\partial x^i / \partial w_i < 0$ and $\partial x^2 / \partial w_1 = \partial x^1 / \partial w_2 \gtreqless 0$ as $f_{12} = \partial^2 f / \partial x_1 x_2 \lesseqgtr 0$.

   (b) Show that if $f_{12} > 0$, then both inputs are not inferior (or regressive).

6. Consider the following production function of a competitive producer: $x = f(z)$, $z = L^{\alpha} K^{\beta}$ where x is an output, L is an input of labor, K is an input of capital, α and β are constants with $\alpha + \beta = 1$ and $f(z)$ satisfies that $f'(z) > 0$, $f''(z) < 0$ and $f(0) = 0$. (z is just a definitional notation).

   Questions: (a) From the cost minimization principle, obtain the demand for L, the demand for K and the cost function as functions of w (wage rate), r (rental rate) and x.

(b) Assume further that $\alpha = \beta$ and $f(z) = z^{1/2}$. Find the supply function of x and derive demand functions for L and K from the profit maximization principle.

(c) Using the result in (b), show than an increase in w will decrease both the demand for L and the demand for K. Give some economic interpretations of this effect.

7. Find derived demand functions for inputs and the output supply function for the following production function:

$$y = f(x_1, x_2) \text{ where } f(x_1, x_2) = h(z) \text{ and } z = F(x_1, x_2),$$
$$h(z) = z^2(A - z),\ F(x_1, x_2) = x_1^{\alpha} x_2^{\beta},\ A > 0,\ \alpha > 0,\ \beta > 0,$$
$$\alpha + \beta = 1. \qquad (x_1 \geqq 0, x_2 \geqq 0, z \geqq 0, y \geqq 0.)$$

Hint: Consider (z) as an intermediate input. First find the unit cost function for (z) noting $\alpha + \beta = 1$. Then consider the profit maximization with the production function $y = h(z)$.

You have to find appropriate imputed prices of "z". Let p_y, w_1, w_2 be prices of "y", "x_1", "x_2" respectively.

8. Suppose that we have an implicit production function of the form:

$$F(x_1, x_2, y_1, y_2) = f(x_1, x_2) - g(y_1, y_2) \quad \text{where} \quad x_i \geqq 0 \quad (i = 1, 2),$$

$y_i \geqq 0$, $(i = 1, 2)$, x_i is the quantity of input i and y_i is the quantity of output i. Assume that:

(i) f is continuous, strictly concave for $(x_1, x_2) \geqq 0$ and homothetic in (x_1, x_2).

(ii) g is continuous, strictly quasi-concave for $(y_1, y_2) \geqq 0$ and positively homogeneous of degree one in (y_1, y_2).

(iii) f and g are increasing in arguments, i.e.,

$$(x_1', x_2') > (x_1, x_2) \Rightarrow f(x_1', x_2') > f(x_1, x_2) \quad \text{and}$$
$$(y_1', y_2') > (y_1, y_2) \Rightarrow g(y_1', y_2') > g(y_1, y_2).$$

Let p_i be the price of a unit of "y_i" and w_i be the price of a unit of "x_i". ($p_i > 0$ and $w_i > 0$).

Consider the following maximization problem:

$$\max_{(x,y) \in R_+^4} p_1 y_1 + p_2 y_2 - w_1 x_1 - w_2 x_2 \quad \text{subject to} \quad F(x, y) \geqq 0.$$

Show that the above problem reduces to the following maximization problem. $\max_{(z,v) \in R_+^2} pz - wv$ subject to $h(z) - z \geqq 0$ where

$$h: R_+ \to R_+,\ p = \tilde{p}(p_1, p_2),\ w = \tilde{w}(w_1\ w_2)$$
$$\tilde{p}: R_{++}^2 \to R_{++} \quad \text{and} \quad \tilde{w}: R_{++}^2 \to R_{++}.$$

(You may consider that z, v, p and w are *aggregate* variables.)

9. Consider a production function $y = f(x_1, x_2)$, $x_i \geqq 0$. $(i = 1, 2)$. Assume that f has continuous partial derivatives for $x_i > 0$ and $f_i(x_i, x_2) = \frac{\partial f}{\partial x_i}(x_1, x_2) > 0$ for $x_i > 0$. Let $\tilde{x}_i$: $R^2_{++} \to R$ be defined by:

$$f(\tilde{x}_1(k, y), \tilde{x}_2(k, y)) - y = 0$$
$$k\tilde{x}_1(k, y) - \tilde{x}_2(k, y) = 0.$$

The marginal rates of substitution as a function of (x_1, x_2) and also as a function of (k, y) are defined respectively by:

$$\tilde{\phi}(x_1, x_2) = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

and

$$\phi(k, y) = \tilde{\phi}(\tilde{x}_1(k, y), \tilde{x}_2(k, y)).$$

Then we define the elasticity of substitution as a function of (k, y) given by

$$\sigma(k, y) = \frac{\phi(k, y)}{k} \frac{1}{\frac{\partial \phi(k, y)}{\partial k}}.$$

(i) Show that

$$\sigma(k, y) = \frac{x_1^2}{x_1 f_1 + x_2 f_2} \frac{\Delta}{f_1 f_2} \quad \text{where}$$

$$\Delta = \det \begin{pmatrix} 0, & f_1, & f_2 \\ f_1, & f_{11}, & f_{12} \\ f_2, & f_{21}, & f_{22} \end{pmatrix}.$$

(ii) If is positively homogeneous of degree one, then σ depends only on k so that we can write $\sigma(k)$.

10. Suppose that $f(x_1, x_2) = (a_1 x_1^{-\varrho} + a_2 x_2^{-\varrho})^{-1/\varrho}$ where $a_i > 0$

$$a_1 + a_2 = 1, \ -1 < \varrho \quad \text{and} \quad \varrho \neq 0.$$

(i) Show that $\sigma(k) = \frac{1}{1+\varrho}$.

(ii) Show that

$$\lim_{\sigma \to 0} f(x_1, x_2) = \min\{x_1, x_2\},$$

and

$$\lim_{\sigma \to 1} f(x_1, x_2) = x_1^{a_1} x_2^{a_2}.$$

11. Suppose that $f(x_1, x_2) = g\left[\left(\frac{a_1 x_1^{-\varrho} + b_1}{a_2 x_2^{-\varrho} + b_2}\right)^c\right]$ where $c \neq 0$, g is a differentiable function from $R_{++} \to R_+$ with $g'(z) > 0$ for $z > 0$. Show that $\sigma(k, y) = \frac{1}{1+\varrho}$.

12. Consider the following situation involving two firms, firm A and firm B.

(a) Firm B produces an (intermediate) output "x" using a labor input "L_x". Firm B is a price-taker in both markets.
(b) Firm A produces an output "y" using an input "x", and a labor input "L_y". Firm A is a price taker in the market for "y" and in the labor market. But firm A is the sole buyer for "x" and acts as a pure monopsonist.
(c) Let p_x, p_y, w be prices of "x", "y" and labor.
(d) Firm B has a production function $x = \phi(L_x)$ with $\phi(0) = 0$, $\phi'(L_x) > 0$, $\phi''(L_x) < 0$ for $L_x > 0$.
(e) Firm A has a production function $y = f(x, L_y)$ which satisfies

$$\frac{\partial f}{\partial x} > 0, \quad \frac{\partial f}{\partial L_y} > 0, \quad \frac{\partial^2 f}{\partial x^2} < 0, \quad \frac{\partial^2 f}{\partial L_y^2} < 0, \quad \frac{\partial^2 f}{\partial x \partial L_y} > 0,$$

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial L_y^2} - \left(\frac{\partial^2 f}{\partial x \partial L_y}\right)^2 > 0$$

and $f(0, 0) = 0$.

Questions: (i) Examine profit maximizing behaviors of firm A and firm B.
(ii) Analyze effects of a change in parameters w and P_y on firm A, i.e., effects on the supply of "Y" demands for "x" and "L_y".

Hint: (i) Let the inverse of $\phi(L_x)$ be denoted by $L_x = g(x)$. Show $g'(x) > 0$, $g''(x) > 0$.
(ii) Obtain an inverse supply function of x by B as $P_x = s(x, w)$.
(iii) You will get a nice result if $d/d_x(wg'(x) + wxg''(x)) \geqq 0$.
Interprete the meaning of this condition.

13. Consider the following labor allocation problem of a consumption agent.

(a) The preference of this consumer is represented by a utility function $u(x_1, x_2, x_3) = \phi(x_1, x_2) + x_3$, $(x_1, x_2, x_3) \in R^3_+$.
(b) The technology of the domestic production of good 1 and good 2 is represented by a production function $v = f(y_1, y_2)$, $f\colon R^2_+ \to R_+$ where y_i is the domestic output of good i and v is the domestic labor input.
(c) The total labor supply available is fixed at $L > 0$ and the constraint is given by: $x_3 + v + z \leqq L$ where z is the supply of labor for an outside job and x_3 denotes the demand for leisure.
(d) The Hessian of f is assumed to be positive definite for every $(y_1, y_2) > 0$.

Our major task here is to analyze the effect of the wage rate w for the outside job on the labor supply for the outside job. Denote the labor supply function for the outside job by $z(p_1, p_2, w)$ where p_i is the price of good i.
Denote the usual demand functions for x_i by $h^i(p_1, p_2, w, m)$, $(i = 1, 2, 3)$ which are, of course, the outcome of maximizing $u(x_1, x_2, x_3)$ subject to $p_1x + p_2x_2 + wx_3 \leqq m$. Define the (imputed) profit function for the domestic production by:

$$\pi(p_1, p_2, w) = \max\{p_1y_1 + p_2y_2 - wv \mid f(y_1, y_2) = v\}$$

and $y_i(p_1, p_2, w)$ and $v(p_1, p_2, w)$ be the domestic output supply function for good i and the domestic labor demand function respectively as the outcome of the above imputed profit maximization. Define the gross income function by:

$$M(p_1, p_2, w) = wL + \pi(p_1, p_2, w).$$

Assume that, at the optimum, we have $(x_1, x_2, x_3) > 0$, $(y_1, y_2) > 0$, $(v, z) > 0$. Work out the following questions.

(i) Define $x^3(p_1, p_2, w) = h^3(p_1, p_2, w, M(p_1, p_2, w))$ and show that $z(p_1, p_2, w) = -x^3(p_1, p_2, w) - v(p_1, p_2, w)$.

(ii) Show that $\partial\pi(p_1, p_2, w)/\partial w = -v(p_1, p_2, w)$.

(iii) Show that $\partial v(p_1, p_2, w)/\partial w < 0$.

Hint: Try to derive the following equation:

$$\partial v/\partial w = -\frac{1}{w}(f_1, f_2)(H_f)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$ where H_f is the Hessian matrix of f and $f_i = \partial f/\partial y_i$.

(iv) Show that:

$\partial z(p_1, p_2, w)/\partial w = -(S_{33} + \partial v/\partial w) - z\partial h^3/\partial M$ and try to provide an intuitive interpretation of this result.

Part III Market Structure

Introduction

In this part we investigate the implications of various models of market structure on equilibrium market outcomes. We confine our attention to those models of imperfect competition which are fairly standard and which are cast in a partial equilibrium framework. We have attempted to be more careful and more explicit, than is usual in textbooks at this level, in stating definitions and assumptions and in the standard of rigor which is imposed on the analysis.

We start with the case of monopoly in Chapter 7. Then we present the cases of duopoly and oligopoly in Chapter 8. In Chapter 9 we discuss monopolistic competition. Finally, some variations on the traditional themes of modelling firms are presented in Chapter 10.

Chapter 7 Monopoly

1. Introduction

The simplest case of an imperfectly competitive market is that of a monopolist, i.e. of a single seller and of buyers who are price takers who do not have a close substitute to that seller's product. The sources of monopoly power could be institutional, historical, technological or they could be based on the nature of the demand for the product. We will simply take that power to be a given in our analysis.

In the Section 2 of this chapter we present an analysis of a monopolist who perceives a demand function for his product. In Section 3 we present some efficiency considerations of monopoly. The case of a discriminating monopolist is discussed in Section 4 for single markets and in Section 5 for several separate markets.

2. Monopoly with Perceived Demand Functions

Consider a firm that produces a single product and sells it in a market where it has monopoly power. Let $p = D(q)$ denote an inverse demand function for monopolist's product and let $p = \tilde{D}(q, \theta)$ denote a family of perceived demand functions perceived by the monopolist with a parameter θ which is a real number in $[\underline{\theta}, \bar{\theta}]$. For the sake of simplicity, we assume that perceived demand functions are linear in q so that they can be written as:

(2.1) $p = b(\theta) - a(\theta)\, q$

where $b(\theta)$ and $a(\theta)$ are assumed to be positive real numbers for every θ and to be continuous in $\theta \in [\underline{\theta}, \bar{\theta}]$. Let $C(q)$ denote the firm's total cost function. Since there is no entry considered in this section, it is immaterial whether $C(q)$ is the long-run cost or the short-run cost. Then the profit function of the firm can be written as:

(2.2) $\pi(q, \theta) = q\,\tilde{D}(q, \theta) - C(q)$.

If θ is considered fixed, then the first-order necessary condition for $q^* > 0$ to be a local maximizer is the usual equality of the marginal revenue and the marginal cost at q^*, i.e.,

(2.3) $mr(q^*, \theta) = C'(q^*)$

where

$$mr(q^*, \theta) = \tilde{D}(q^*, \theta) + q^* \frac{\partial \tilde{D}}{\partial q}(q^*, \theta)$$

$$= \tilde{D}(q^*, \theta)(1 - 1/\eta(q^*, \theta))$$

and

$$\eta(q^*, \theta) = -(q^*/\tilde{D}(q^*, \theta)) \frac{\partial \tilde{D}}{\partial q}(q^*, \theta).$$

Suppose now that the monopolist obtains some information about the parameter θ by observing a price and a quantity demanded at that price. This is an estimation process which may be expressed as a function $\theta = \tilde{\theta}(p, q)$. We will impose the following *consistency on the estimation* * *process* of θ

(2.4) For every (p, q), $p = D(q) = \tilde{D}(q, \tilde{\theta}(p, q))$.

Note that firm's decision making over the quantity produced in (2.3) presupposes a fixed parameter which may not be consistent with p^* and q^* with $p^* = D(q^*)$. Therefore it may be reasonable to impose an equilibrium condition requiring a consistency of an estimated parameter with a price and a quantity which are observed along the demand curve.

(2.5) *Definition:* We call (p^*, q^*, θ^*) a *consistent monopoly equilibrium* if $p^* = D(q^*)$, $\theta^* = \tilde{\theta}(p^*, q^*)$, $\pi(q^*, \theta^*) \geqq 0$, $q^* > 0$ and q^* locally maximizes $\pi(q, \theta^*)$.

The definition of a consistent monopoly equilibrium above presumes a certain form of "bounded rationality" in which the firm decides the quantity produced without recognizing that parameters estimated do depend on (p, q). We may imagine that the firm consists of two divisions, one division decides on the quantity produced with a given parameter provided by the other division estimating the parameter. But the production division has no information on the estimation procedure the other division is using. Since only local information is used in the estimation of a parameter, we feel it more reasonable to require only local maximizing behavior of the firm as in our definition.

If perceived demand functions are linear as in (2.1) and the estimation process is consistent in the sense of (2.4), then we have

$$D(q) = b(\theta) - a(\theta)q, \quad \text{or}$$

$$b(\theta) = D(q) + a(\theta)q.$$

Therefore there is only one degree of freedom in the family of perceived demand functions. Note that $mr(q, \theta) = b(\theta) - 2a(\theta)q = D(q) - a(\theta)q$.

Now we shall consider a very specific example.

(2.6) *Example:* Suppose that the true inverse demand function is linear with a form $D(q) = \beta - \alpha q$, $\beta > 0$ and $\alpha > 0$ and the total cost function is also linear with a form $C(0) = 0$ and $C(q)$ ~~and~~ $cq + F$ for $q > 0$ where $c > 0$ is a constant

* This follows the approach of Negishi (1961).

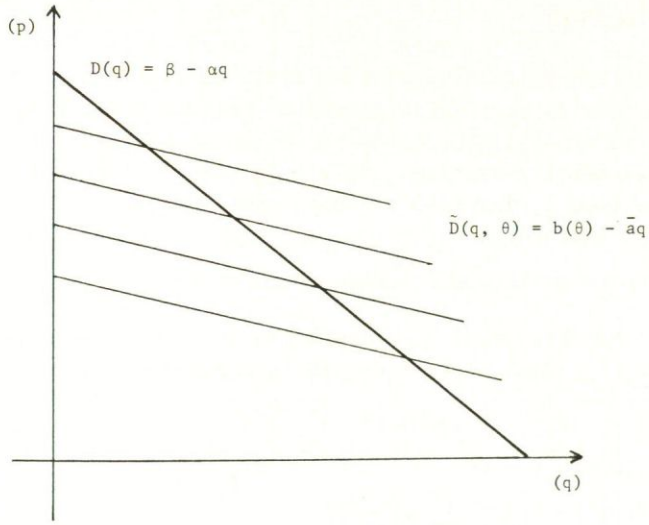

Fig. 2.1

marginal cost and $F > 0$. If the firm knows the true demand function, then the usual monopoly output satisfying the first order condition is given by $q_M = (\beta - c/2\alpha)$ and the profit at q_M is given by $[(\beta - c)^2/4\alpha] - F$. Therefore the profit at q_M is nonnegative if and only if $4\alpha F \leqq (\beta - c)^2$.

Suppose now that the firm faces a family of perceived demand functions $\tilde{D}(q, \theta) = b(\theta) - a(\theta)q$ such that $a(\theta) = \bar{a} > 0$ for every $\theta \in [\underline{\theta}, \bar{\theta}]$. See Figure 2.1 for an illustration of this family of perceived demand functions for the case $\bar{a} < \alpha$. Then, with the consistency condition imposed, $mr(q, \theta) = D(q) - a(\theta)q = \beta - (\alpha + \bar{a})q$. Therefore the first-order necessary condition for the profit maximization entails that $q^* = (\beta - c)/(\alpha + \bar{a})$ and $q_M < q^*$ if and only if $\bar{a} < \alpha$.

One the other hand, suppose that $a(\tilde{\theta}(p, q)) = a_0 + (a_1/q)$ where $a_0 > 0$ and $a_1 > 0$. Then the first-order necessary condition for the profit maximization requires that $q^{**} = (\beta - c - a_1)/(\alpha + a_0)$. If $a_0 < \alpha$, then $q_M < q^{**}$.

It should be noted that, in both families of perceived demand functions, there is no guarantee that the profit at q^* or at q^{**} is nonnegative. For example, if $4\alpha F = (\beta - c)^2$, then the profit will be nonnegative only when $q = 0$ or $q = q_M$. Therefore $\bar{a} < \beta$ for the first case or $a_0 < \alpha$ for the second case will imply the nonexistence of a consistent monopoly equilibrium. Therefore further conditions must be introduced to guarantee the existence of a consistent monopoly equilibrium.

First of all, there must be an interval $[\underline{q}, \bar{q}]$ with $0 > \underline{q} < \bar{q}$ where the total average cost curve and the marginal cost curve must lie below the true demand curve or more precisely for every $q \in [\underline{q}, \bar{q}]$, $ac(q) \leqq D(q)$ and $C'(q) \leqq D(q)$ where $ac(q) = C(q)/q$ for $q > 0$.

Define $\tilde{a}(q) = a(\tilde{\theta}(D(q), q))$ and $\tilde{mr}(q) = mr(q, \tilde{\theta}(D(q), q))$. Since the first-order condition for the profit maximization under the consistency condition becomes $\tilde{mr}(\tilde{q}) - C'(q) = D(q) - \tilde{a}(q)q - C'(q) = 0$, an obvious sufficient condition for the existence of a consistent monopoly equilibrium will be that there exist $q_1, q_2 \in [\underline{q}, \bar{q}]$ such that $D(q_1) - C'(q_1) \leqq \tilde{a}(q_1)q_1$ and $D(q_2) - C'(q_2) \geqq \tilde{a}(q_2)$.

3. Monopoly and Efficiency

Suppose that the monopolist has complete information about the inverse demand function $p = D(q)$ and suppose further that all consumers have the same utility function which is of the form $U(q_h, y_h) = y_h + u(q_h)$. If each consumer maximizes $U(q_h, y_h)$ subject to a budget constraint $pq_h + y_h = \bar{y}_h$ where $\bar{y}_h$ is an initial endowment of the income good y, then the first-order necessary condition for an interior solution implies that $u'(q_h) = p$ or $q_h = (u')^{-1}(p)$. Therefore $\bar{q} = \frac{1}{n}\sum_h q_h = (u')^{-1}(p)$ or $u'(\bar{q}) = p$. Thus the inverse demand function is exactly the marginal utility of the consumption of $\bar{q}$. If consumers are buying the total quantity $q^* = n\bar{q}^*$ at price p^*, then consumers' surplus measures the net benefit to consumers as follows:

$$\begin{aligned}\int_0^{q^*} D(q)\,dq - p^*q^* &= [n\,u(\bar{q})]\,|_0^{q^*} - p^*q^* \\ &= n(u(\bar{q}^*) - u(0)) + \sum_h (y_h^* - \bar{y}_h) \\ &= \sum_h \{U(\bar{q}^*, y_h^*) - U(0, \bar{y}_h)\}\end{aligned}$$

where y_h^* is assumed to satisfy consumer h's budget constraint $p^*q_h^* + y_h^* = \bar{y}_h$.

At the monopolist equilibrium, we have that

$$mr(q^*) = D(q^*) + q^*D'(q^*) = C'(q^*)\,.$$

If $u''(\bar{q}) = nD'(q) < 0$, then $D(q^*) > C'(q^*)$. Since the welfare of consumers is maximized when $u'(\bar{q}) = D(q)C'(q)$ or at q_c in Figure 3.1, the monopoly equilibrium is not efficient and the efficiency loss or the deadweight loss can be measured by the area bounded by a, b and c in Figure 3.1.

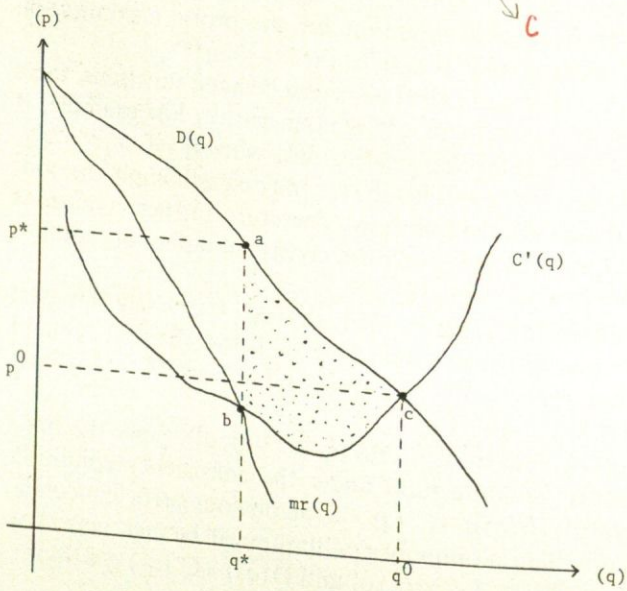

Fig. 3.1

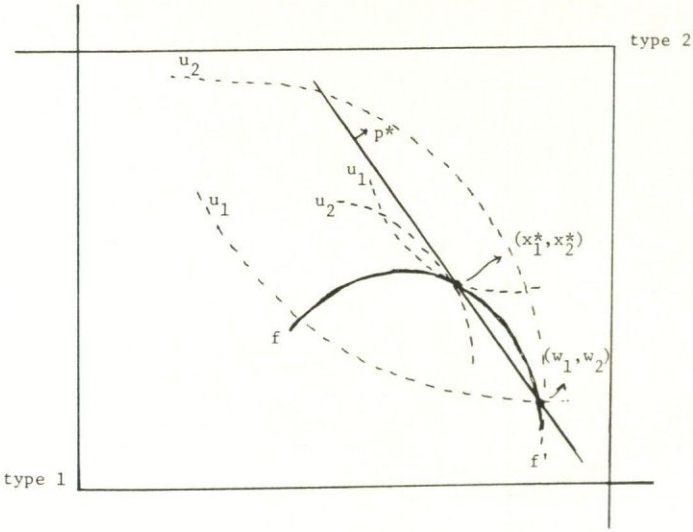

Fig. 3.2

When the concept of consumer's surplus is not appropriate, the welfare loss of a monopoly equilibrium should be interpreted as Pareto inefficiency. We will provide two illustrations of Pareto inefficiency of monopoly equilibria below. First, we will consider a pure exchange economy with two goods and two types of consumers. Type 1 consumer is a monopolist with a unit mass and type 2 consumers are represented by points in a unit interval [0, 1]. The weight of type 2 consumers is measured by a length. So all type 2 consumers together have a unit size. We assume that type 2 consumers are characterized by an identical utility function $u_2(x_2(t))$ and an identical initial endowment $w_2(t)$ where $x_2(t)$, $w_2(t) \in \mathbb{R}^2_+$. Type 1 consumer has a utility function $u_1(x_1)$, $x_1 \in \mathbb{R}^2_+$ and an initial endowment $w_1 \in \mathbb{R}^2_+$. Utility functions are assumed to be continuous and strictly quasi-concave. Suppose that, in this economy, only market exchanges with a single price are allowed, no coalition formations are allowed and moreover type 1 consumer possesses complete information on the total competitive offer curve of type 2 consumers. Then a monopoly equilibrium will occur at (x_1^*, x_2^*, p^*) as shown in Figure 3.2 where $x_2^*(t) = x_2^*$ for every $t \in [0, 1]$. Clearly the allocation (x_1^*, x_2^*) is not Pareto efficient.

The next model has a simple production sector. The production sector produces a consumption good y employing an input of labor L and the technology is represented by a production function $y = f(L)$. We will assume that $f(0) = 0$, $f'(L) > 0$ and $f''(L) < 0$ for every $L > 0$. There is one type 1 consumer with a unit mass and a continuum of type 2 consumers indexed by $t \in [0, 1]$. Type 1 consumer is the sole owner of the firm with a utility function $u_1(x_1)$, $x_1 \in \mathbb{R}_+$ and a budget constraint $px_1 = \pi$ where p is the price of a consumption good and π is the profit of the firm. Type 2 consumers have an identical differentiable and strictly quasi-concave utility funciton $u_2(x_2(t), z(t))$, $(x_2(t), z(t)) \in \mathbb{V}^2_+$ where $x_2(t)$ is the quantity of a consumption good and $z(t)$ is the consumption of leisure. The budget

R

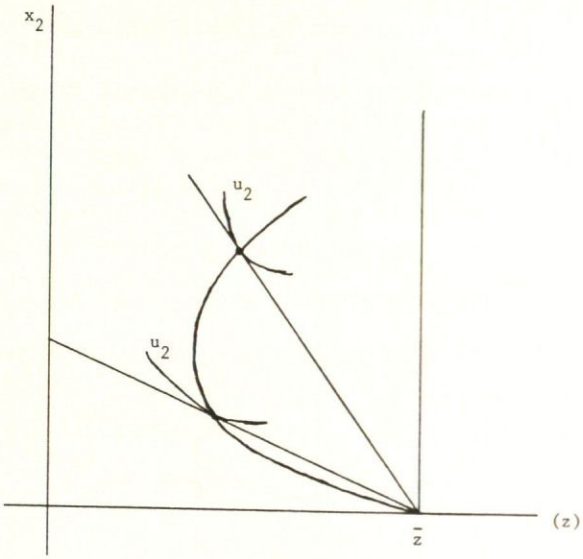

Fig. 3.3

constraint of a type 2 consumer is given by $p x_2(t) + w z(t) = w \bar{z}$ where $\bar{z}$ is the initial endowment of labor hours and w is the hourly wage rate. Thus $\bar{z} - z(t)$ is considered to be the supply of labor by consumer t of type 2. Since type 2 consumers are identical with strictly quais-concave utility functions, we can treat them as if we have one price taking consumer with a unit mass. An offer curve of type 2 consumers is illustrated in Figure 3.3. We assume that the firm is a monopolist with complete information on the offer curve of type 2 consumers. Let $x_2(w/p)$ and $z(w/p)$ be competitive demand functions for a consumption good and leisure by type 2 consumers respectively. Define the supply function of labor by $L(w/p) = \bar{z} - z(w/p)$. Then the profit of the monopolist measured in terms of the consumption good can be written as:

$$\pi(w/p) = f[L(w/p)] - (w/p) L(w/p) .$$

The first-order condition for the profit maximization yields:

$$(3.1) \quad \pi'(w/p) = \{f'[L(w/p)] - (w/p)\} L'(w/p) - L(w/p) = 0$$

which can be rewritten as:

$$(3.2) \quad \frac{w}{p} = f'(\cancel{l}\,L) - \frac{L(w/p)}{L'(w/p)} = f'(L) - \frac{w}{p} \frac{1}{\eta(w/p)}$$

where

$$\eta(w/p) = (w/p) \frac{L'(w/p)}{L(w/p)} .$$

Note that the profit is not maximized over the part of the offer curve which bends backward since the same labor hours can be supplied at the lower real wage rate.

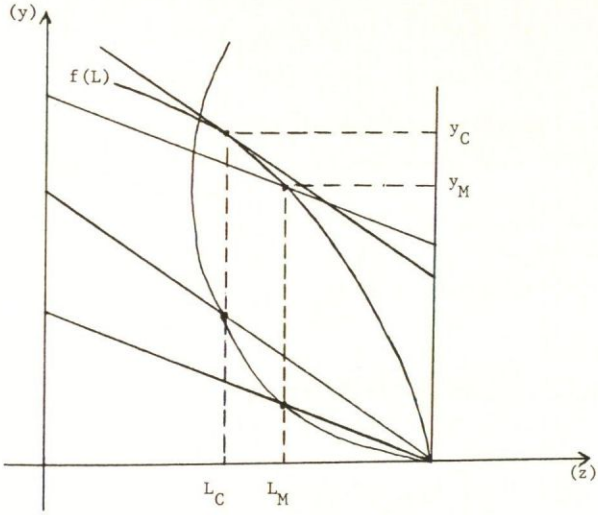

Fig. 3.4

Also, if $L'(w/p) = 0$, then $\pi'(w/p) = -L(w/p) < 0$. Therefore when the profit is maximized, then $L'(w/p) > 0$ which in turn implies that

$$(3.3) \quad \frac{w}{p} < f'(L) .$$

Therefore the monopoly equilibrium must occur so that $L_M < L_C$ and $y_M < y_C$ where M and C indicate a monopoly equilibrium and a competitive equilibrium respectively. We illustrate this result in Figure 3.4. Since $\dfrac{\partial u_2/\partial z}{\partial u_2/\partial x_2(t)} = w/p < f'(L)$ at the monopoly equilibrium, we can conclude that the monopoly equilibrium is not Pareto efficient.

We have shown the inefficiency of the standard monopoly equilibrium using three different models. But, then, this immediately implies that it is possible for the monopolist to offer a (possibly non-market) contract which is Pareto-superior to the monopoly equilibrium. Consider the first partial equilibrium model of the monopoly. Let (p_M, q_M) be the monopolist price-quantity pair and let (p_C, q_C) be an outcome at the intersection of the demand curve and the marginal cost curve which we call a competitive outcome although the profitability condition may be violated. When the marginal cost is rising sufficiently near q_C, then the monopolist at the monopoly equilibrium (p_M, q_M) can offer an additional quantity $q_S = q_C - q_M$ at price p_C as shown in Figure 3.5 where p_C is assumed to be higher than $C'(q_M)$, the marginal cost at q_M. Clearly this benefits both the owners of the monopolist and consumers of the product. In Figure 3.5, the dotted region is the additional benefit to consumers and the shaded area is the additional profit accruing to the monopolist firm. If the marginal cost is decreasing at q_C, then the monopolist can not profitably offer an additional contract at p_C. But it could offer q_S at price $C'(q_M)$ as shown in Figure 3.6. It is possible to approximate the competitive output level q_C by offering

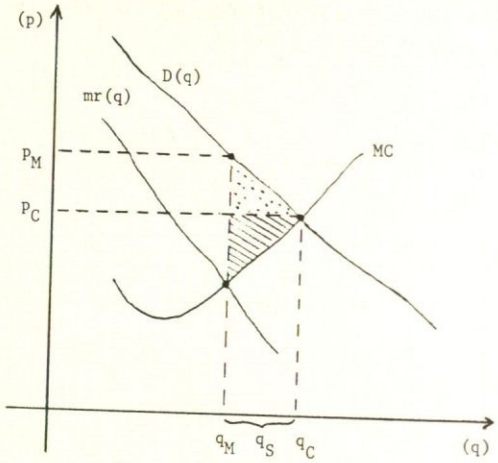

Fig. 3.5

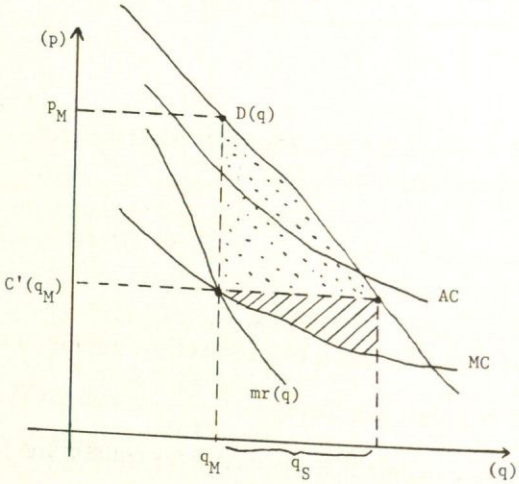

Fig. 3.6

several contracts successively. Coupons and quantity limited sales may be considered as examples of some contracts offered by the monopolist which also benefit consumers.

4. Discriminating Monopoly in a Single Market

As we have discussed in the previous section, the inefficiency of the standard monopoly equilibrium suggests that a more sophisticated monopolist will try to offer additional non-market contracts or try to employ multiple-pricing schemes to improve its profit. In this section, we will consider two cases of a multiple-pricing or discriminating monopoly facing a single market. Of course it requires some schemes to successfully employ multiple prices generally incurring a further cost of discrimination. But, in this section, we will disregard the consideration of the cost of discrimination so that we can find simpler and more transparent anal-

ysis and its implications. Throughout this section, we will assume that the monopolist faces a linear inverse demand function of the form:

$$p = b - aQ$$

where $b > 0$, $a < 0$ and Q ist the total output sold. This demand function may be interpreted as a perceived demand function without requiring the consistency, which we required in Section 2.

Suppose that the monopolist firm plans to employ N discriminating prices, say $p_1, p_2, \ldots, p_N$ where $p_1 > p_2 > \ldots > p_N$ and let q_n be the quantity sold at price p_n. Define $Q_n = q_1 + q_2 + \ldots + q_n$. Then the total revenue function and the profit function can be written as:

$$R(q_1, q_2, \ldots, q_N) = \sum_{n=1}^{N} (b - aQ_n) q_n$$

and

$$\pi(q_1, q_2, \ldots, q_N) = R(q_1, q_2, \ldots, q_N) - C(Q_N).$$

Noting that $(\partial R/\partial q_n) = b - aQ_n - a(q_n + q_{n+1} + \ldots + q_N)$, we get first-order condition for the profit maximization for the interior solution as:

$$(4.1) \quad b - aq_n - aQ_n = C'(Q_N).$$

Then clearly, we have $q_1 = q_2 = \ldots = q_N = Q_N/N$. We rewrite (4.1) as:

$$(4.1)' \quad mr_N(q_N) = C'(Q_N)$$

where

$$mr_N(Q_N) = b - \left(1 + \frac{1}{N}\right) aQ_N.$$

Note that $\lim_{N \to \infty} mr_N(Q) = D(q) = b - aQ$. Now if the marginal cost curve cuts through the marginal revenue curves mr_N as shown in Figure 4.1 or more specifically if the set $\{Q \mid mr_N(Q) \geqq C'(Q)\}$ is a convex interval for each N, then the solution to (4.1)′ increases monotonically to the competitive output level Q_C as N increases, i.e.,

$$Q_N < Q_{N+1} < Q_C$$

and

$$\lim_{N \to \infty} Q_N = Q_C$$

where Q_C is a solution to $D(Q) = C'(Q)$. In particular, if $C'(Q) = c$, where c is a positive constant, then we have that $Q_N = \frac{N}{N+1} \frac{b-c}{a}$, $q_N = \frac{1}{N+1} \frac{b-c}{a}$ and $Q_C = \frac{b-c}{a}$. Thus $Q_C - Q_N = \frac{1}{N+1} \frac{b-c}{a}$ and the rate of convergence is $1/(N+1)$. The limiting case of this discriminating pricing is called the *perfectly discriminating monopolist* when the total output is Q_C but each q is infini-

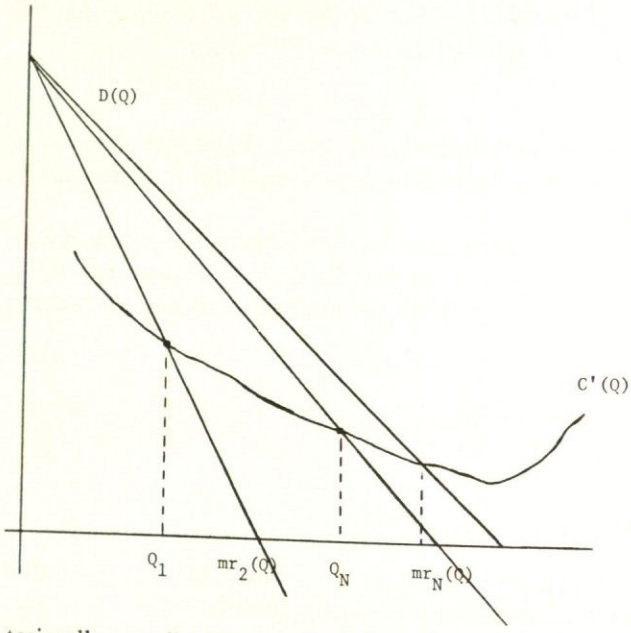

Fig. 4.1

tesimally small. The total consumers' surplus with N prices can be calculated as $(1/2)Na(q_N)^2 = (1/2)a(Q_N^2/N)$ and also the total profit with N discriminating pricing becomes $\int_0^{Q_N} (b - aQ)\,dQ - (1/2)a(Q_N^2/N) - C(Q_N) = bQ_N - (1/2)a\left(1 + \frac{1}{N}\right)Q_N^2 - C(Q_N)$. Therefore as N increases, the total consumers' surplus monotonically decreases toward zero and is absorbed into the monopolist's profit. Therefore increasing N does not constitute a Pareto-superior movement although the limiting outcome of the perfectly discriminating monopoly is efficient.

We will next present an alternative scheme of a discriminating monopolist in a single market. One interesting feature of this scheme is that as the number N of discriminating prices increases, the outcomes become Pareto improving in the sense that both consumers' surplus and monopolist's profit will increase as N increases. For the sake of computational simplicity, we will assume that the total cost function has the form $C(Q) = cQ + A$ where $c > 0$ is the constant marginal cost and $A > 0$ is the fixed cost. We start out from the standard monopoly outcome (p_1, q_1) when $q_1 = (1/2)\,(b - c)/a$ and $p_1 = b - aq_1$. Given q_1, the monopolist will realize that it can still increase its profit by selling q_2 at price $p_2 = b - a(q_1 + q_2)$. The extra revenue will be denoted by $R(Q_1, q_2) = [b - a(Q_1 + q_2)]\,q_2$ where $Q_1 = q_1$. The total profit then becomes $p_1 q_1 + R(Q_1, q_2) - C(Q_1 + q_2)$. The monopolist can maximize the above profit with respect to q_2. This second stage of the monopolist's production and pricing decision is illustrated in Figure 4.2. Suppose that the monopolist successively proceeds in this manner up to the $(N-1)$st stage and performs the N-th stage optimization. Let $Q_n = q_1 + q_2 + \ldots + q_n$ with $Q_0 = 0$. Then, given $q_1, q_2, \ldots, q_{N-1}$, the monopolist's total profit of producing an extra quantity q_N becomes:

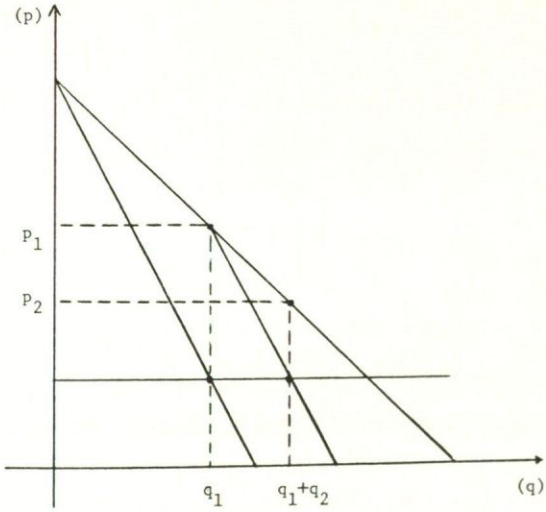

Fig. 4.2

$$\pi(q_1, q_2, \ldots, q_{N-1}, q_N) = \sum_{n=1}^{N} R(Q_{n-1}, q_n) - C(Q_{N-1} + q_N)$$

where $R(Q_{n-1}, q_n) = [b - a(Q_{n-1} + q_n)]\, q_n$.

Then the first-order conditions for the profit maximization with respect to q_N becomes

$$(b - a\,Q_{N-1}) - 2\,a\,q_N = c$$

i.e.,

$$(4.2) \quad q_N = (b-c)/(2a) - (1/2)\,Q_{N-1}\,.$$

Since a similar condition must hold for $N-1$, we have

$$q_{N-1} = (b-c)/(2\,a) - (1/2)\,Q_{N-2}\,.$$

Subtracting $(1/2)\,q_{N-1}$ from both sides yields

$$\begin{aligned}(1/2)\,q_{N-1} &= (b-c)/(2a) - (1/2)(Q_{N-2} + q_{N-1}) \\ &= (b-c)/(2a) - (1/2)\,Q_{N-1}\,.\end{aligned}$$

Therefore we get $q_N = (1/2)\,q_{N-1}$. Since $q_1 = (b-c)/(2a)$ = the standard monopoly output, we obtain:

$$q_n = \left(\frac{1}{2^n}\right)\frac{b-c}{a}, \qquad (n = 1, 2, \ldots, N)$$

and

$$Q_N = \sum_{n=1}^{N} q_n = \left(1 - \frac{1}{2^N}\right)\frac{b-c}{a}\,.$$

Define $\pi_n = (p_n - c)\,q_n = (1/4^n)\,\dfrac{(b-c)^2}{a}$. Then the total profit becomes:

$$\pi_N = \sum_{n=1}^{N} \pi_n - A = (1/3)\left(1 - \frac{1}{4^N}\right)\frac{(b-c)^2}{a} - A\,.$$

Also the total consumers' surplus CS_N can be computed as:

$$CS_N = \sum_{n=1}^{N} (1/2)(p_{n-1} - p_n)q_n \quad \text{where} \quad p_0 = b.$$

Then we get

$$CS_N = (1/\overset{6}{\cancel{7}})\left(1 - \frac{1}{\underset{4}{\cancel{8}}^N}\right)\frac{(b-c)^2}{a}\,.$$

Therefore π_N and CS_N are both monotonically increasing in N and $\lim_{N\to\infty} \pi_n = (1/3)\dfrac{(b-c)^2}{a} - A$ and $\lim_{N\to\infty} CS_N = (1/\overset{6}{\cancel{7}})\dfrac{(b-c)^2}{a}$. Note that the standard monopoly profit is $\pi_1 = (1/4)\dfrac{(b-c)^2}{a} - A$. Therefore, even if $\pi_1 < A$, for sufficiently large N, π_N can be positive provided that $(1/3)\dfrac{(b-c)^2}{a} > A$. Therefore under this successive monopoly pricing scheme, the outcomes improve both consumers' surplus and monopolist's profit as N increases. Also note that Q_N monotonically increases to the competitive output level $Q_C = (b-c)/a$ and the speed of the convergence is very rapid with the rate of convergence $(Q_C - Q_N)/Q_C = 1/2^N$.

5. Discriminating Monopoly in Separate Markets

Consider a single seller who faces n markets in which it is too costly for consumers in one market to participate in other markets. Assume that, in each market, the monopolist faces an inverse demand function $D_i(q_i)$ with $D_i'(q_i) < 0$, $(i = 1, 2, \ldots, n)$ and let the total cost function be denoted by $C(\overset{Q}{\cancel{q}})$ where $Q = \sum_{i=1}^{n} q_i$. We assume that $C'(Q) > 0$. The total profit function of the monopolist is then given by:

$$\pi(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} R_i(q_i) - C(Q)$$

where $R_i(q_i) = q_i D_i(q_i)$.

The first-order conditions for the profit maximization become:

$$(5.1) \quad \frac{\partial \pi}{\partial q_i} = R_i'(q_i) - C'(Q) = 0\,, \qquad (i = 1, 2, \ldots, n)$$

where $R_i'(q_i) = D_i(q_i) + q_i D_i'(q_i)$ provided that positive solutions are admitted. Let $q_i^* > 0$, $(i = 1, 2, \ldots, n)$ be the solution to (5.1) and let p_i^* be defined by $p_i^* = D_i(q_i^*)$. From (5.1), we get

$$p_i^* + q_i^* D_i'(q_i^*) = p_j^* + q_j^* D_j'(q_j^*)$$

or

$$p_i^*\left(1-\frac{1}{\eta_i(q_i^*)}\right)=p_j^*\left(1-\frac{1}{\eta_j(q_j^*)}\right)$$

where $\eta_i(q_i) = -\frac{1}{q_i}\frac{D_i(q_i)}{D_i'(q_i)}$. This can be rewritten as:

$$(5.2)\quad \frac{p_i^*}{p_j^*}=\frac{1-(1/\eta_j(q_j^*))}{1-(1/\eta_i(q_i^*))}.$$

Noting that $R_i'(q_i^*)>0$ implies that $1-\frac{1}{\eta_i(q_i^*)}>0$, we can conclude that $p_i^*>p_j^*$ if and only if $\eta_i(q_i^*)<\eta_j(q_j^*)$.

We now introduce a further assumption about inverse demand functions: that inverse demand functions are linear in q_i, i.e.,

$$(5.2)\quad D_i(q_i)=b_i-a_iq_i,\qquad (i=1,2,\ldots,n)$$

where $b_i>0$ and $a_i \overset{>}{\cancel{<}} 0$. Then (5.1) evaluated at an optimum solution q_i^* and $Q^*=\sum_{i=1}^{n} q_i^*$ can be written as:

$$b_i-2a_iq_i^*=C'(Q^*)$$

or

$$\frac{b_i}{a_i}-2q_i^*=\frac{1}{a_i}C'(Q^*).$$

Summing over i yields

$$(5.3)\quad \sum_{i=1}^{n}(b_i/a_i)-2Q^*=C'(Q^*)\sum_{i=1}^{n}(1/a_i).$$

Now suppose that the firm sets the same price in every market. Then the quantity the firm can sell in market i when the price is p will be $q_i=(b_i/a_i)-(1/a_i)p$. Summing over i and solving for p yields the total inverse demand function for the firm when the uniform pricing across all markets is applied

$$(5.4)\quad p=\left[\sum_{i=1}^{n}(1/a_i)\right]^{-1}\left[\sum_{i=1}^{n}(b_i/a_i)-Q\right].$$

Then the first-order condition for the profit maximization subject to the uniform pricing condition becomes:

$$(5.5)\quad \left[\sum_{i=1}^{n}(1/a_i)\right]^{-1}\left[\sum_{i=1}^{n}(b_i/a_i)-2Q\right]=C'(Q)$$

or

$$\sum_{i=1}^{n}(b_i/a_i)-2Q=C'(Q)\sum_{i=1}^{n}(1/a_i)$$

which is identical with (5.3). Therefore when the inverse demand function is linear, then the total quantity produced under the discriminating pricing is the same as the one under the uniform pricing. Let q_i^0 be the quantity sold in market i at the uniform pricing. Then (5.5) can be rewritten as:

$$\left[\sum_{i=1}^{n}(1/a_i)^{-1}\right]\left[\sum_{i=1}^{n}(1/a_i)R_i'(q_i^0)\right]=C'(Q^*)$$

when $R_i'(q_i^0)=b_i-2a_iq_i^0$. Thus, under the uniform pricing, a weighted sum of the marginal revenues is set equal to the marginal cost $C'(Q^*)$. When the monopolist switches from the uniform pricing at the discriminating pricing, it will expand the quantity sold and decrease the price in market i if and only if $R_i'(q_i^0)>C'(Q^*)$. Since $R_i'(q_i^0)=p_i^0\left(1-\frac{1}{\eta_i(q_i^0)}\right)$ the condition for the market expansion and the price reduction can also be expressed by $\frac{1}{p_i^0}(p_i^0-C'(Q^*))>1/\eta_i(q_i^0)$. Since the total quantity produced does not change when the monopolist switches from the uniform pricing to the discriminating pricing, the welfare effect on consumers is purely distributional in which consumers in an expanded market get better off, but those in a contracted market get worse off.

The assumption of the linearity of the inverse demand function is crucial in obtaining the result that the total quantity produced under the discriminating pricing coincides with the one under the uniform pricing. The major reason why this is so is because, with the linear inverse demand function, the horizontal sum of marginal revenue curves $R_i'(q_i)$ coincides with the marginal revenue of the horizontal sum of demand curves $D_i(q_i)$ provided that $q_i>0$ for every $i=1,2\ldots,n$. Let us be more precise. Let $q_i=F_i(p)$ if and only if $p_i=D_i(q_i)$ and define $F(p)=\sum_{i=1}^{n}F_i(p)$. Then the total inverse demand function $p=D(Q)$ under the uniform pricing is the inverse function to $Q=F(p)$. Let us also define $q_i=M_i(\mu)$ if and only if $R_i'(q_i)=\mu$ and $Q=M(\mu)$ if and only if $R'(Q)=\mu$ where $R'(Q)=D(Q)+QD'(Q)$. When $D_i(q_i)$ is linear in q_i, then we can easily show that $\sum_{i=1}^{n}M_i(\mu)=M(\mu)$ when $M_i(\mu)>0$ for every $i=1,2,\ldots,n$.

Chapter 8 Duopoly and Oligopoly

1. Introduction

In this chapter, we will study simple interactions of firms that are few in number and that recognize the influence of each other's actions on the outcome of their own actions.

We start, in Section 2, with the Cournot-duopoly-oligopoly model. In Section 3 we present Bertrand-Edgeworth duopoly and in Section 4 we present symmetric Cournot equilibrium with free entry and small efficient scale. In Section 5 we discuss perceived demand functions, conjectural variations and consistent conjectural variations.

2. Cournot Duopoly and Oligopoly

a) Introduction

In this model, formulated and studied by Cournot (1878), each firm has information about the output levels of the other firms and believes that these output levels remain the same. Let Q_i be the sum of output levels of firms other than firm i that are currently observed by firm i. Then firm i believes that the portion of the demand curve beyond Q_i will be its to supply. We denote by $D(Q)$ the inverse demand function with

$$Q = \sum_{i=1}^{N} q_i$$

and we assume that the products of all firms are perfect substitutes for one another. Then firm i believes that the demand left for it to satisfy when

$$Q_i = \sum_{j \neq i} q_j$$

is given will be

$$\tilde{D}(q_i : Q_i) = D(q_i + Q_i) .$$

We call this demand function a *residual* (or *contingent) demand function* for firm i given Q_i. (See Figure 2.1 for an illustration). When firm i chooses to supply the quantity q_i, then the market price, p, will be determined by a demand curve so that $p = D(q_i + Q_i)$. Therefore the price is determined by the market demand

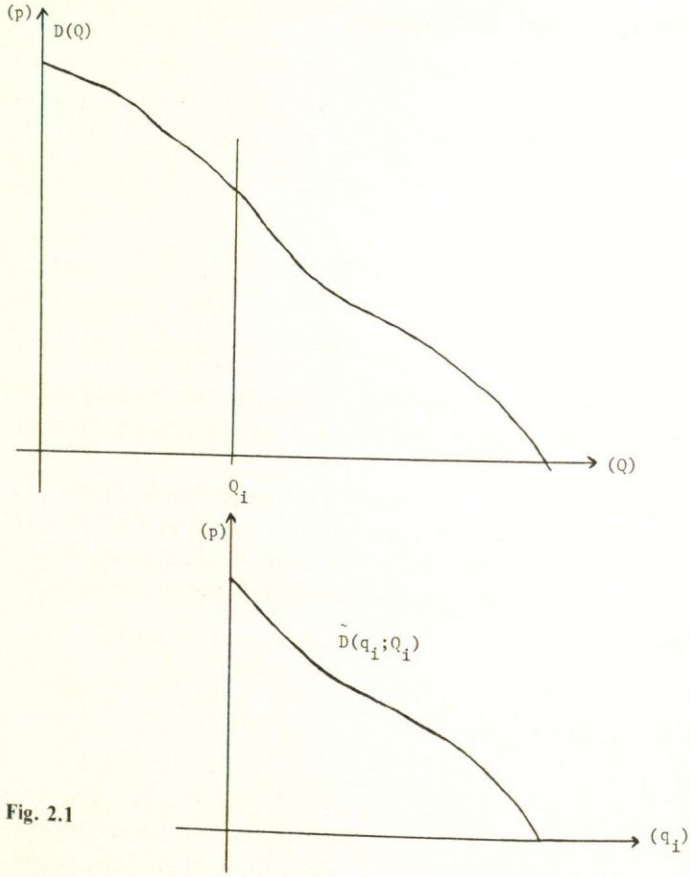

Fig. 2.1

and is not an active strategic variable for each firm. We may interpret the role of price in Cournot's theory as the following assumption about the way firms perceive each other's pricing policies: Each firm, say firm i, perceives that if it increases its price then such an increase will be exactly and immediately followed by all other firms.

We note that residual demand functions are none other than the perceived demand functions of Chapter 7.

b) Static Cournot Duopoly and Oligopoly

We will throughout assume that the inverse demand function is linear in the total quantity and the marginal cost is constant. This is of course for the sake of its computational simplicity. Let $D(Q) = b - aQ$ be the inverse demand function and let c be the constant marginal cost where $a > 0$, $0 < c < b$. Then the profit function of firm i can be written as:

$$\pi(q_i, Q) = [b - a(q_i + Q_i)]\, q_i - c q_i = (b - c)\, q_i - a Q_i q_i - a q_i^2$$

where the fixed cost is assumed to be zero. Then the first-order condition for the profit maximization becomes:

$$\frac{\partial \pi}{\partial q_i} = (b-c) - aQ_i - 2aq_i = 0$$

provided that it admits a positive solution q_i. Therefore the reaction function or the best response function, denoted by $\phi_i(Q_i)$ of firm i becomes

$$(2.1) \quad \phi_i(\overset{Q}{\cancel{q_i}}) = \max\left\{0, -\frac{1}{2}Q_i + \frac{1}{2}\frac{b-c}{a}\right\}.$$

Suppose that $q_j = -1/2\,Q_j + 1/2\,(b-c)/a$ for every $j \neq i$. Then we can rewrite this as $q_j = -Q_i - q_i + (b-c)/a$. Summing these over $j \neq i$ yields:

$$(2.2) \quad Q_i = -\frac{N-1}{N}q_i + \frac{N-1}{N}\frac{b-c}{a}.$$

The system (2.2) and (2.3) given by

$$(2.3) \quad q_i = -1/2\,Q_i + 1/2\frac{b-c}{a}$$

has a solution

$$q_i^* = \frac{1}{N+1}\frac{b-c}{a}$$

$$Q_i^* = \frac{N-1}{N+1}\frac{b-c}{a}.$$

This gives us a Cournot-Nash equilibrium which is symmetric and unique. The total quantity is given by

$$Q^* = \frac{N}{N+1}\frac{b-c}{a}.$$

The total quantity produced at the competitive equilibrium defined by $D(Q_C) = c$ is given by:

$$Q_C = \frac{b-c}{a}.$$

The total profit of all firms joined together is

$$\sum_{i=1}^{N} \pi(q_i, Q_i) = \sum_{i=1}^{N} \{[b - aQ]\,q_i - cq_i\} = (b-c)\,Q - aQ^2.$$

Therefore the total output at the cartel monopoly is given by

$$Q_M = \frac{1}{2}\frac{b-c}{a}.$$

When the symmetry condition is imposed, then individual firm's quantities produced at the competitive equilibrium and at the cartel monopoly are given respectively by:

$$q_c = \frac{1}{N}\,\frac{b-c}{a}$$

$$q_M = \frac{1}{2N}\,\frac{b-c}{a}.$$

Figure 2.2 illustrates these solutions where the thick line indicates the reaction curve of firm i, the broken line indicates the graph of (2.2) and CE, CNE and CM are respectively symmetric solutions under the competitive equilibrium, the Cournot-Nash equilibrium and the cartel monopoly. Clearly if N goes to infinity, the Cournot-Nash equilibrium coincides with the competitive equilibrium. Cournot calls this as the case of "unlimited competition."

We now confine ourselves to the duopoly case where $N = 2$ and examine consequences in more detail. The profit function of firm i for the duopoly case becomes:

$$\pi(q_i, q_j) = [(b-c) - a(q_i + q_j)]\,q_i, \qquad i \neq j.$$

Note that $\pi(q_i, q_j) = 0$ if and only if either $q_i = 0$ or $q_1 + q_2 = (b-c)/a$. Therefore isoprofit curves of firm i are hyperbolas with asymptotes $q_i = 0$ and $q_1 + q_2 = (b-c)/a$. Then we can superimpose isoprofit curves of firm 1 and firm 2 in the illustration of Figure 2.2 and illustrate various outcomes for the case of two firms in Figure 2.3. In Figure 2.3, CNE indicates the duopoly or Cournot-Nash equilibrium and CE and CM are symmetric outcomes under the competitive and cartel monopoly cases respectively. M_i indicates firm i's monopoly output. Finally S_i indicates a Stackelberg equilibrium where firm $j\,(j \neq i)$ has the

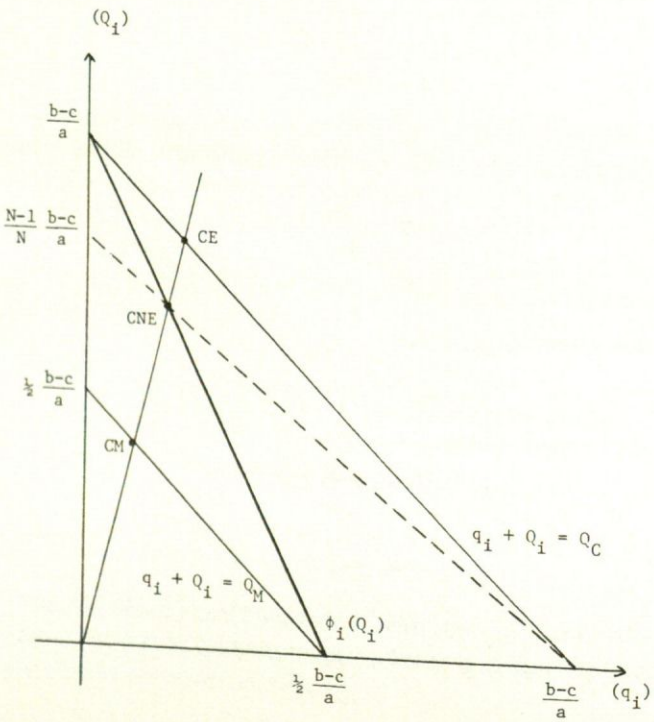

Fig. 2.2

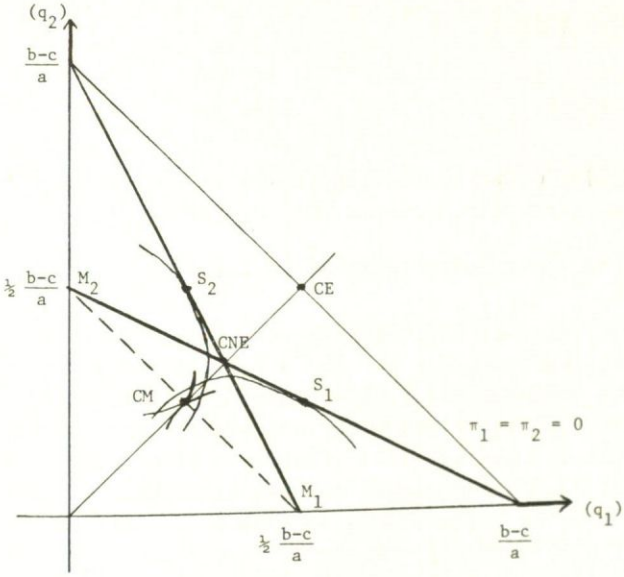

Fig. 2.3

behavior of the Cournot best response, but firm i has a complete information on firm j's reaction function and it chooses the profit maximizing output level along firm j's reaction curve.

Suppose that, in order to arrive at the duopoly equilibrium, each firm adjusts its quantity produced continuously as follows starting from an initial condition $(q_1(0), q_2(0)) \geqq 0$.

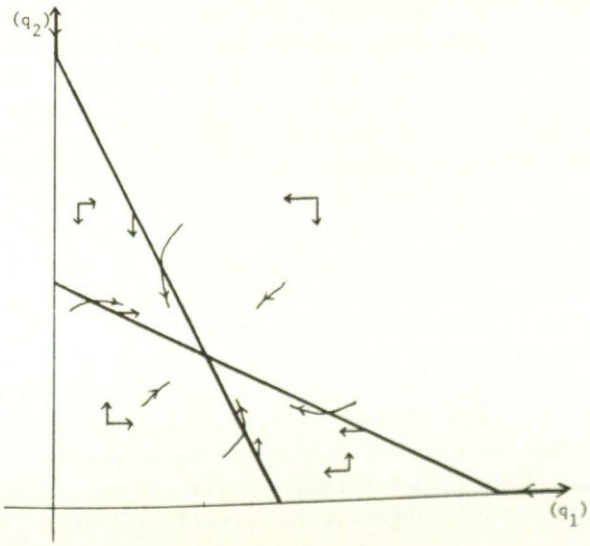

Fig. 2.4

$$\frac{dq_1(t)}{dt} = k_1[\phi_1(q_2(t)) - q_1(t)]$$

(2.4)

$$\frac{dq_2(t)}{dt} = k_2[\phi_2(q_1(t)) - q_2(t)]$$

where $\phi_i(q_j) = \max\{0, -(1/2)q_j + (1/2)(b-c)/a\}$ $(i \neq j)$ and $k_i > 0$ $(i = 1, 2)$. The Jacobian of the above linear system at the Cournot-Nash equilibrium is given by $\begin{pmatrix} -k_1 & , & -1/2\,k_1 \\ -1/2\,k_2, & & -k_2 \end{pmatrix}$. The characteristic polynomial associated with this Jacobian becomes $P(\lambda) = \lambda^2 + (k_1 + k_2)\lambda + (3/4)k_1k_2$. The discriminant D of this polynomial is $D = (k_1+k_2)^2 - 3k_1k_2 = (k_1-k_2)^2 + k_1k_2 > 0$ since $k_i > 0$ $(i = 1, 2)$. Hence $P(\lambda) = 0$ has real roots and it is easy to check that they are both negative. The phase diagram of (2.4) is shown in Figure 2.4 and we can see that the Cournot-Nash equilibrium is a globally stable solution of (2.4). Figure 2.4 also shows the instability of the cartel monopoly outcome although it is not a very convincing argument since the system (2.4) presupposes a noncooperative behavior of firms.

c) Cournot Duopoly with a Fixed Entry Cost

Suppose that there are two firms in a market where firm 1 is an active incumbent firm, but firm 2 is a potential entrant to the market. We assume that the total cost function of firm 1 is given by $C_1(q_1) = cq_1$ with $c > 0$, but the total cost function of firm 2 if firm 2 enters the market is given by $C_2(q_2) = cq_2 + K$ with $K > 0$. The rest of our model will be the same as in the previous sections. K represents a fixed cost of entering the market for firm 2 and, of course, if firm 2 decides not to enter the market, firm 2 will incur no cost. Firm 2's decision on whether or not to enter the market depends critically on its perception of firm 1's behavior. If, under a particular perception of firm 1's behavior, firm 2 believes

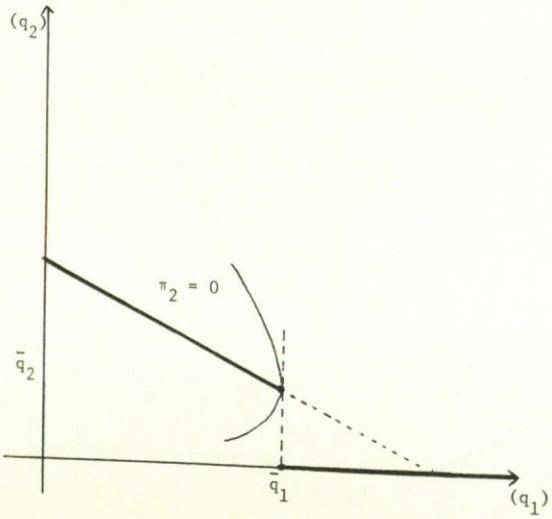

Fig. 2.5

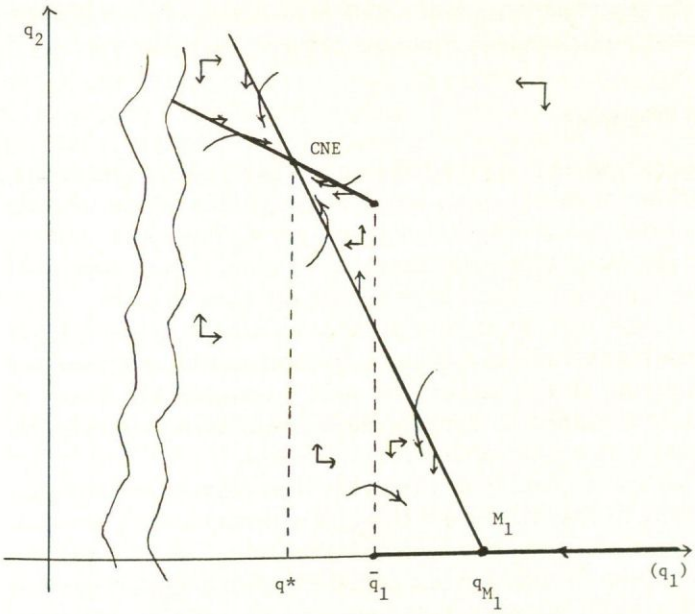

Fig. 2.6

that it can earn a profit at least as large as the break even level, then it will decide to enter the market.

Firm 2's best response function or reaction function constrained by the break even condition is shown in Figure 2.5 where a discontinuity occurs at $q_1 = \bar{q}_1$. $\bar{q}_1$ and $\bar{q}_2$ of Figure 2.5 satisfy conditions $AC_2(\bar{q}_2) = b - a(\bar{q}_1 + \bar{q}_2)$ and $AC_2'(\bar{q}_2) = -a$ where $AC_2(q_2) = C_2(q_2)/q_2$. A simple computation will yield $\bar{q}_1 = (b-c)/a - 2(K/a)^{1/2}$ and $\bar{q}_2 = (K/a)^{1/2}$. Notice that this is the best response which firm 2 computes before entering the market. Once firm 2 enters the market, it has to bear the cost K and, since the average variable cost is equal to the constant marginal cost, the shutdown level becomes $q_2 = 0$. Thus, once firm 2 enters the market, the discontinuity disappears and we regain the same reaction function as we had in section b.

Suppose that $q^* < \bar{q}_1 < q_{M_1}$ as shown in Figure 2.6 where $q^* = \frac{1}{3}(b-c)/a$ and $q_{M_1} = \frac{1}{2}(b-c)/a$. Given the constrained reaction function of firm 2, Figure 2.6 shows that there are two Cournot-Nash equilibria before the entry by firm 2. One equilibrium is the same as the previous symmetric Cournot-Nash equilibrium (CNE) and the other is the monopoly equilibrium (M_1) of firm 1. Notice first that, once firm 2 decides to enter the market, then M_1 becomes no more a Cournot-Nash equilibrium since the reaction function of firm 2 after its entry decision does not have the discontinuity. Also suppose that firm 1 is setting its output at q_{M_1} before firm 2's entry and suppose that firms follow the previous adjustment process (2.4) in reaching a Cournot-Nash equilibrium. Then firm 2 may realize that, if the initial condition $q_2(0)$ of firm 2 is sufficiently large, firm 1 will modify its quantity which firm 1 wants to produce and eventually the symmetric Cournot-Nash equilibrium may be reached as shown in the phase diagram of Figure 2.6. In game theory, an equilibrium CNE is called a (subgame) perfect

equilibrium and, on the other hand, q_{M_1} is an example of an equilibrium which is not perfect since it ceases to be an equilibrium once firm 2 enters the market.

d) Repeated Cournot Oligopoly

So far in previous sections, we only considered an oligopoly game in one period. Without the consideration of entries, there was a unique Cournot-Nash equilibrium as an outcome of our noncooperative oligopoly game. Therefore, sustaining an outcome like the cartel monopoly outcome requires some cooperative efforts. This situation is almost the same as prisoner's dilemma in game theory. Luce and Raiffa (1957) show that, if the same game is repeated indefinitely many times, then the cooperative outcome in prisoner's dilemma can be sustained as a noncooperative equilibrium. In this section, we shall investigate the nature of outcomes which can be sustained as noncooperative equilibria in a repeated Cournot oligopoly game with a finite or an infinite horizon.

Let $S_i = [0, \bar{q}]$ be the set of quantity strategies for firm i in one period Cournot oligopoly game where we may choose $\bar{q} = (b-c)/a$ without loss of generality. Let $S = \prod_{i=1}^{N} S_i$ and a generic element of S be denoted by $s = (q_1, q_2, \ldots, q_N)$. We may write the profit function for firm i in one period Cournot oligopoly game as $\pi_i(s) = \pi_i(q_i, Q_i)$ with a slight abuse of notations were $Q_i = \sum^{(j \neq i)} q_j$. Suppose that the same one period Cournot oligopoly game is to be played repeatedly T times consecutively. T will be called the horizon of the game and it may be a finite integer or infinite. Let $t \in \{1, 2, \ldots, T\}$ be a time period, $S_i(t) = S_i$ and $S(t) = \prod_{i=1}^{N} S_i(t)$. Then a strategy of firm i can be defined as a sequence of functions $f_i \equiv (f_i(t))_{t=1}^{T}$ such that

$$f_i(1) \in S_i(1) = S_i$$

and, for $t = 2, \ldots, T$,

$$f_i(t) : S(1) \times \ldots \times S(t-1) \to S_i(t) .$$

Let us denote $f = (f_i)_{i=1}^{N}$ and the outcome at t under f be denoted by $\sigma(f)(t)$, i.e.,

$$\sigma(f)(1) = f(1) = (f_1(1), f_2(1), \ldots, f_N(1))$$

and

$$\sigma(f)(t) = (f_i(t)(\sigma(f)(1), \ldots, \sigma(f)(t-1))_{i=1}^{N} .$$

When the horizon T is finite, then the payoff function for the repeated game with horizon T which we use will be the average payoff over time periods defined by

$$\pi_i^T(f) = \frac{1}{T} \sum_{t=1}^{T} \pi_i(\sigma(f)(t)) .$$

When the horizon is infinite, then we will use $\pi_i^{\infty}(f) = \lim_{T \to \infty} \pi_i^T(f)$ if the limit exists. If the limit does not exist, we may consider either the limit inferior or the limit superior.

Suppose that the horizon T is finite. Consider the last period T with a given history up to $T-1$. Then, since this reduces to one period Cournot oligopoly

game, the previous Cournot-Nash equilibrium outcome must result in period T. Then, by using backward induction, the only Cournot-Nash equilibrium outcome with a finite horizon is that, in each period, firms produce the one period Cournot-Nash equilibrium output. Therefore, unless we introduce an infinite horizon or modify our equilibrium concept with a finite horizon, we can not expect to generate other outcomes as noncooperative equilibria. First we will consider the case of an infinite horizon.

(i) An Infinite Horizon Case

Let us define $U = \{(u_1, u_2, \ldots, u_N) \mid$ there exists $s \in S$ such that $u_i = \pi_i(s)$, $(i = 1, 2, \ldots, N)\}$ and $u_i^* = \pi_i(s^*)$, $s^* = (q_i^*)_{i=1}^N$ where s^* is the one period Cournot-Nash equilibrium, i.e., $q_i^* = \dfrac{1}{N+1}\,\dfrac{b-c}{a}$. We also denote the best response function of one period Cournot oligopoly game by $q_i = \phi_i(\cancel{q_i}\ Q)$ where $Q_i = \sum_{j \neq i} q_j$ and then define $\tilde{\pi}_i(Q_i) = \pi_i(\phi_i(Q_i), Q_i)$.

Choose an arbitrary $\bar{u} \in U$ so that $u^* \leqq \bar{u}$ and let $\bar{s} = (\bar{q}_i)_{i=1}^N \in S$ satisfy $\bar{u} = \pi(\bar{s})$. Then we will show that $\bar{s}$ can be sustained as a stationary noncooperative equilibrium in a repeated Cournot game with an infinite horizon. In order to show this, we define a strategy $\bar{f} = (\bar{f}_i)$ as follows:

(2.5) $\bar{f}_i(1) = \bar{q}_i$

and, for $t > 1$, $\bar{f}_i(t)$

$\bar{f}_i(s(1), s(2), \ldots, s(t-1)) = \bar{q}_i$ if $s(1) = s(2) = \ldots = s(t-1) = \bar{s}$,

and $\bar{f}_i(t)$

$\bar{f}_i(s(1), s(2), \ldots, s(t-1)) = q_i^*$ if otherwise.

Obviously $\pi_i^\infty(\bar{f}) = \bar{u}_i$. Suppose that $s(1) = s(2) = \ldots = s(\tau-1) = \bar{s}$ and firm i tries to defect from $\bar{q}_i$ in period τ by choosing the best reply output level $\phi_i(Q_i)$ whenever $s(1) = \ldots = s(\tau-1) = \bar{s}$. In other words, f_i^τ be any strategy which satisfies that

(2.6) $f_i^\tau(t) = \bar{f}_i(t)$ for $t = 1, 2, \ldots, \tau-1$

and $f_i^\tau(\tau)$

$f_i^\tau(s(1), \ldots, s(\tau-1)) = \phi_i(\bar{Q}_i)$ if $s(1) = \ldots = s(\tau-1) = \bar{s}$.

Let $u_i' = \tilde{\pi}_i(\bar{Q}_i)$ where $\bar{Q}_i = \sum_{j \neq i} \bar{q}_j$ and $\bar{f}/f_i^\tau = (\bar{f}_1, \ldots, \bar{f}_{i-1}, f_i^\tau, \bar{f}_{i+1}, \ldots, \bar{f}_N)$. Then for $t > \tau$,

(2.7) $\pi_i^T(\bar{f}/f_i^\tau) = \dfrac{\tau-1}{T}\bar{u}_i + \dfrac{1}{T}u_i' + \dfrac{T-\tau}{T}u_i^*$.

Therefore, $\lim_{T\to\infty} \pi_i^T(\bar{f}/f_i^\tau) = u_i^* \leqq \bar{u}_i$. Since this is true for any τ, we can assert that the strategy N-tuple $\bar{f} = (\bar{f}_i)_{i=1}^N$ is a Cournot-Nash equilibrium in a repeated game with an infinite horizon and it yields a stationary outcome $\bar{s} = (\bar{q}_i)_{i=1}^N$.

Thus we have shown that any feasible payoff vector which is Pareto superior to one period Cournot-Nash equilibrium payoff can be sustained as a stationary outcome of a noncooperative equilibrium in a repeated Cournot game with an in-

finite horizon. In particular, the cartel monopoly outcome can be sustained as a noncooperative equilibrium in an infinitely repeated game.

Let us see what will happen to our results if we use a discounted payoff function instead of the average payoff function. Suppose that, for the infinitely repeated Corunot game, the payoff function of firm i is given by $\pi_i^\delta(f) = \sum_{t=1}^{\infty} \delta^{t-1} \pi_i(\sigma(f)(t))$ for some $\delta \in (0, 1)$. Since S_i is bounded, this sum is well-defined. We consider the same strategies $\bar{f} = (\bar{f}_i)_{i=1}^N$ as before. Let f_i^τ be any other strategy for firm i such $f_i^\tau(t) = \bar{f}_i(t)$ for $t = 1, 2, \ldots, \tau - 1$, and $f_i^\tau(s(1), \ldots, s(\tau-1)) = \phi_i(\bar{Q}_i)$ if $s(1) = \ldots = s(\tau-1) = \bar{s}$. Then $\pi_i^\delta(\bar{f}/f_i^\tau) - \pi_i^\delta(\bar{f}) = \delta^\tau(u_i' - \bar{u}_i) + \sum_{t=\tau+1}^{\infty} \delta^t(u_i^* - \bar{u}_i)$ where u_i', u_i^* and $\bar{u}_i$ are as defined previously. Therefore $\pi_i^\delta(\bar{f}/f_i^\tau) - \pi_i^\delta(\bar{f}) \leqq 0$ if and only if

$$\left(u_i' + \frac{\delta}{1-\delta} u_i^*\right) - \left(\bar{u}_i + \frac{\delta}{1-\delta} \bar{u}_i\right) = (u_i' - \bar{u}_i) - \frac{\delta}{1-\delta}(\bar{u}_i - u_i^*) \leqq 0, \quad \text{i.e.,}$$

the defection of firm i from playing $\bar{f}_i$ will not benefit firm i if and only if the benefit of the defection in period τ does not exceed the discounted loss after τ. The above condition is also equivalent to

$$(2.8) \quad (u_i' - u_i^*) - \frac{1}{1-\delta}(\bar{u}_i - u_i^*) \leqq 0 .$$

If this condition is satisfied, then the stationary payoff $(\bar{u}_i)_{i=1}^N$ can be sustained as a noncooperative equilibrium with strategies $(\bar{f}_i)_{i=1}^N$. Note that if $u_i^* < \bar{u}_i$, then we can always find δ sufficiently close to one so that the condition (2.8) is satisfied.

(ii) Finitely Repeated Oligopoly Game and ε-Equilibria

When the horizon is finite, then in order to yield outcomes other than the stationary one period Cournot-Nash outcome the concept of an equilibrium must be modified. In this section, we will explore the concept of an ε-equilibrium due to Radner (1980) and its application to a finitely repeated oligopoly game. As the reader can see, this section owes very heavily to the paper by Radner (1980). We say that a strategy N-tuple $\bar{f} = (\bar{f}_i)_{i=1}^N$ constitutes an ε-equilibrium of a repeated game with horizon T if, for every i and for every strategy f_i,

$$\pi_i^T(\bar{f}) \geqq \pi_i^T(\bar{f}/f_i) - \varepsilon$$

where ε is a positive constant.

As in the previous section, choose $\bar{u} \in U$ and $\bar{s} = (\bar{q}_i)_{i=1}^n \in S$ so that $\bar{u} = \pi(\bar{s})$ and $u^* \leqq \bar{u}$. Let $\bar{f} = (\bar{f}_i)_{i=1}^N$ be the same strategy N-tuple defined by (2.5) and let f_i^τ be an alternative strategy of firm i defecting at τ with $0 < \tau \leqq T$. Then, as in (2.7),

$$\pi_i^T(\bar{f}/f_i^\tau) - \pi_i^T(\bar{f}) = \frac{1}{T}(u_i' - \bar{u}_i) + \frac{T-\tau}{T}(u_i^* - \bar{u}_i) .$$

Since $u_i^* \leqq \bar{u}_i \leqq u_i'$,

$$\pi_i^T(\bar{f}/f_i^\tau) - \pi_i^T(\bar{f}) \leqq \pi_i^T(\bar{f}/f_i^T) - \pi_i^T(\bar{f}) = \frac{1}{T}(u_i' - \bar{u}_i) .$$

Therefore, $\bar{f} = (\bar{f}_i)_{i=1}^n$ is an ε-equilibrium if

$$(2.9)\quad \frac{1}{T}(u_i' - \bar{u}_i) \leqq \varepsilon .$$

It should be noted that, given $\bar{f} = (\bar{f}_i)_{i=1}^{n}$ and $\varepsilon > 0$, the strategy N-tuple $\bar{f} = (\bar{f}_i)_{i=1}^{N}$ can be sustained as an ε-equilibrium if T is sufficiently large or more precisely if $T \geqq T_\varepsilon$ where T_ε is the smallest integer larger than $(1/\varepsilon)(u_i' - \bar{u}_i)$.

Let us consider a slightly more general strategy which Radner calls a trigger strategy of order k with $(\bar{q}_i, q_i^D)$ where $1 < k \leqq T$, $\bar{q}_i$ indicates an output level when firm i does not defect and q_i^D is a defecting output level of firm i. As before, we fix $\bar{u} \in U$ and $\bar{s} = (\bar{q}_i)_{i=1}^{N} \in S$ so that $\bar{u} = \pi(\bar{s})$ and $u^* \leqq \bar{u}$. A trigger strategy of order k with $(\bar{q}_i, q_i^D)$ denoted by $\bar{f}_i^k$ will be defined by:

$$(2.10)\quad \bar{f}_i^k(1) = \bar{q}_i ,$$

for $1 < t \leqq k$,

$$\bar{f}_i^k(t)(s(1), \ldots, s(t-1)) = \begin{cases} \bar{q}_i & \text{if} \quad s(1) = \ldots = s(t-1) = \bar{s} \\ q_i^* & \text{if} \quad \text{otherwise,} \end{cases}$$

for $t = k+1$,

$$\bar{f}_i^k(k+1)(s(1), \ldots, s(k)) = \begin{cases} q_i^D & \text{if} \quad s(1) = \ldots = s(k) = \bar{s} \\ q_i^* & \text{if} \quad \text{otherwise,} \end{cases}$$

and, for $k+1 < t \leqq T$

$$f_i^k(t)(s(1), \ldots, s(t-1)) = q_i^* .$$

Now suppose that all firms other than firm i are using trigger strategies of order k with $(\bar{q}_j, q_j^D)$ denoted by $\bar{f}_j^k$, $j \neq i$. We assume that $Q_i^D = \sum_{j \neq i} q_j^D$ is sufficiently large so that $Q_i^D \geqq Q_i^* = \sum_{j \neq i} q_j^*$.

If firm i also uses a trigger strategy of order k with $(\bar{q}_i, q_i^D)$ denoted by $\bar{f}_i^k$, then firm i's payoff will be

$$\pi_i^T(\bar{f}^k) = \frac{k}{T}\bar{u}_i + \frac{1}{T}\pi_i(q_i^D, Q_i^D) + \frac{T-k-1}{T}u_i^* .$$

On the other hand, let f_i^τ be an alternative strategy for firm i trying to defect from the strategy $\bar{f}_i^k$ at period τ. Since other firms will be producing at q_j^* for $\tau \geqq k+2$, firm i will not be able to improve its payoff more than when it plays $\bar{f}_i^k$ if $\tau \geqq k+2$. Therefore we may take $\tau \leqq k+1$. Suppose that $\tau \leqq k$ and $f_i^\tau(\tau)(s(1), \ldots, s(\tau-1)) = \phi_i(\bar{Q}_i)$ when $s(1) = \ldots = s(\tau-1) = \bar{s}$. Then

$$\pi_i^T(\bar{f}/f_i^\tau) = \frac{\tau-1}{T}\bar{u}_i + \frac{1}{T}u_i' + \frac{T-\tau}{T}u_i^*$$

$$= \frac{\tau}{T}(\bar{u}_i - u_i^*) - \frac{1}{T}\bar{u}_i + \frac{1}{T}u_i' + u_i^* .$$

Since $\bar{u}_i - u_i^* \geqq 0$, $\pi_i^T(\bar{f}/f_i)$ is nondecreasing in τ and, for $\tau \leqq k$, we have

$$\pi_i^T(\bar{f}^k/f_i^\tau) \leqq \pi_i^T(\bar{f}^k/f_i^k) .$$

Now suppose that $\tau = k+1$ and $f_i^{k+1}(k+1)(s(1), \ldots, s(\tau-1)) = \phi_i(Q_i^D)$ if $s(1) = \ldots = s(k) = \bar{s}$. It is clear that $\phi_i(Q_i^D)$ is the best response of firm i in period $k+1$ when $s(1) = \ldots = s(k) = \bar{s}$. Then

$$\pi_i^T(\bar{f}^k/f_i^{k+1}) = \frac{k}{T}\bar{u}_i + \frac{1}{T}\tilde{\pi}_i(Q_i^D) + \frac{T-k-1}{T}u_i^* .$$

Thus we obtain

$$\begin{aligned}
&\pi_i^T(\bar{f}^k/f_i^k) - \pi_i^T(\bar{f}^k/f_i^{k+1}) \\
&= -\frac{1}{T}\bar{u}_i + \frac{1}{T}u_i' - \frac{1}{T}\tilde{\pi}_i(Q_i^D) + \frac{1}{T}u_i^* \\
&= \frac{1}{T}(u_i' - \bar{u}_i) + \frac{1}{T}(u_i^* - \tilde{\pi}_i(Q_i^D)) \geqq 0
\end{aligned}$$

because $u_i' \geqq \bar{u}_i$ and $Q_i^D \geqq Q_i^*$ implies $u_i^* \geqq \tilde{\pi}_i(Q_i^D)$. Therefore, we can conclude that if all other firms are using strategies $\bar{f}_j^k$ $(j \neq i)$, then the best alternative to firm i is to try to defect in period k. Hence the strategy n-tuple $\bar{f}^k = (\bar{f}_i^k)_{i=1}^N$ is an ε-equilibrium if, for every i,

$$\text{(2.11)} \quad \pi_i^T(\bar{f}^k/f_i^k) - \pi_i^T(\bar{f}^k) = \frac{1}{T}(u_i' - \pi_i(q_i^D, Q_i^D)) - \frac{1}{T}(\bar{u}_i - u_i^*) \leqq \varepsilon .$$

Note if $q_i^D = q_i^*$ for every $i = 1, 2, \ldots, n$, then the condition for $\bar{f}^k$ to be an ε-equilibrium becomes

$$\pi_i^T(\bar{f}^k/f_i^k) - \pi_i^T(\bar{f}^k) = \frac{1}{T}(u_i' - \bar{u}_i) \leqq \varepsilon$$

which is identical with the previous condition (3.9).

3. Bertrand-Edgeworth Duopoly

a) Bertrand Duopoly Model

The duopoly and oligopoly models of Cournot in previous sections are formulated under an assumption that products produced by competitors are perfect substitutes as assumed by Cournot (1838, Chapter VII). J. Bertrand (1883) has criticized Cournot's model where quantities of products produced are used as strategic variables of firms in a market with identical products. According to Bertrand, any firm which cuts its price to where it is below competitors' will be able to capture the whole market provided that the marginal cost of production is constant and there is no capacity constraint in production. Thus Bertrand considers prices set by firms as strategic variables and postulates a contingent demand function of each firm for given prices of other firms which is quite distinct from Cournot's contingent demand function. The difference between Cournot's model and Bertrand's model leis not just in their choices of strategic variables, but more critically in their postulates on properties of contingent

demand function of each firm for given prices of other firms which is quite distinct from Cournot's contingent demand function. The difference between Cournot's model and Bertrand's model lies not just in their choices of strategic variables, but more critically in their postulates on properties of contingent demand functions. We will only consider the duopoly case. The extension to the case of more than two firms will be fairly straightforward.

Let $D(p) = \text{Max}\ \{\beta - \alpha p, 0\}$ for $p \geqq 0$ be the market demand function for a particular product with $\alpha > 0$ and $\beta > 0$. The marginal costs of production are assumed to be constant and identical between firms and we denote the constant unit cost by $c > 0$. Define a function δ: $\mathbb{R} \to \mathbb{R}$ by:

$$(3.1) \quad \delta(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

We will assume that, when the two firms set the same price, i.e., $p_i = p_j$ $(i \neq j)$, then the share of the market for each firm is one half of the total quantity demanded. Then Bertrand's contingent demand function of firm i given firm j's price p_j $(j \neq i)$ is defined by:

$$D(p_i, p_j) = \delta(p_j - p_i)\, D(p_i) + (1 - \delta(p_i - p_j) - \delta(p_j - p_i))(1/2)\, D(p_i).$$

Note that if $p_i < p_j$, then $D(p_i, p_j) = D(p_i)$, if $p_i = p_j$, then $D(p_i, p_j) = (1/2)\, D(p_i)$ and if $p_i > p_j$, $D(p_i, p_j) = 0$. We illustrate Bertrand's contingent demand curve $D(p_i, \bar{p}_j)$ in Figure 3.1 for a given $\bar{p}_j$ $(j \neq i)$. A unique feature of Bertrand's contingent demand curve is that it is discontinuous when $p_i = p_j$. Without loss of generality, we can confine prices to lie in the closed interval $[c, \beta/\alpha]$. We define real valued functions π_L and $\pi_=$ defined on $[c, \beta/\alpha]$ by

$$\pi_L(p) = (p - c)\, D(p), \quad \text{and}$$

$$\pi_=(p) = (1/2)(p - c)\, D(p).$$

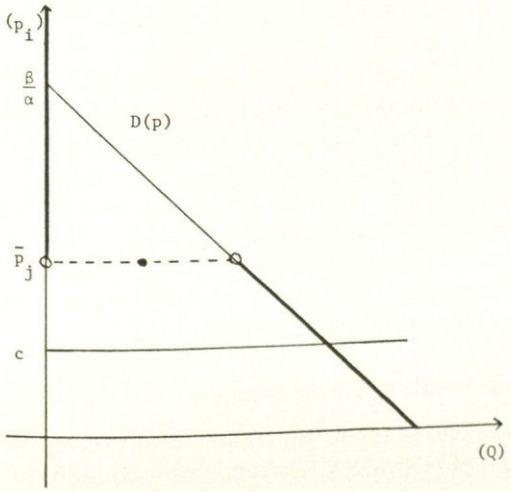

Fig. 3.1

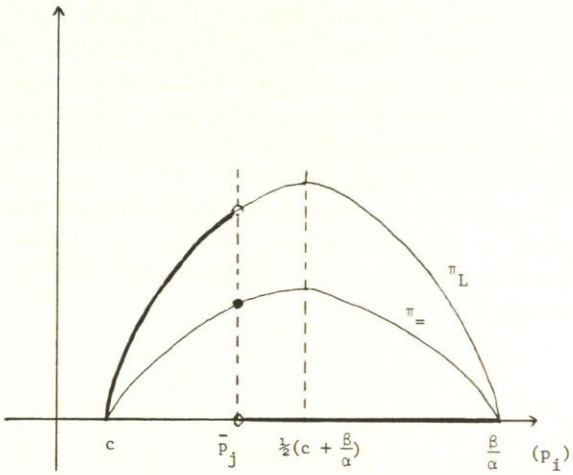

Fig. 3.2

Then the profit function of firm i can be written as follows:

$$\begin{aligned}\pi_i(p_i, p_j) &= (p_i - c)\, D(p_i, p_j)\\ &= \delta(p_j - p_i)\, \pi_L(p_i)\\ &\quad + (1 - \delta(p_i - p_j) - \delta(p_j - p_i))\, \pi_=(p_i)\,.\end{aligned}$$

In Figure 3.2, a discontinuous thick curve illustrates the profit function for a given $\bar{p}_j$. Let p_M be the price which a monopolist will charge, i.e., $p_m = (1/2)\left(c + \frac{\beta}{\alpha}\right)$. Suppose that $p_j \in (p_M, \beta/\alpha]$. Then the best reply of firm i ($i \neq j$) is obviously $p_i = p_M$. Now if $p_j \in (c, p_m]$, then the profit function is strictly increasing in $p_i \in (c, p_j)$ along $\pi_L(p_i)$ but it discontinuously drops to $\pi_=(p_i)$ if $p_i = p_j$ and to zero if $p_i > p_j$. Therefore firm i will try to undercut p_j ($j \neq i$) but, since π_L is increasing in $p_i \in (c, p_j)$, the best response price of firm i does not exist if $p_j \in (c, p_M)$. Finally if $p_j = c$, then any price in the interval $[c, \beta/\alpha]$ yields zero profit and, hence, the best response is any price in $[c, \beta/\alpha]$. On Figure 3.3, we illustrate best response maps or reaction curves of two firms when the solid lines and the broken lines are respectively firm 1's and firm best response maps. The dotted line signifies unsuccessful undercutting effort of firms. Clearly the unique Bertrand-Nash equilibrium price pair is given by $(\hat{p}_1, \hat{p}_2) = (c, c)$ and the total quantity produced is $\hat{Q} = \beta - \alpha c$. In Cournot's model, the competitive outcome will occur in the case of "unlimited competition" or in the case when the number of competing firms become infinite. But, under Bertrand's formulation of oligopoly in a market with identical products or perfect substitutes, the competitive outcome will result whenever there are more than two firms competing each other driving prices down to the constant cost of production.

b) Bertrand-Edgeworth Duopoly Model with Capacity Limits

Edgeworth (1897) introduces capacity constraints to Bertrand's duopoly model and he gives an example that there is no pure strategy Bertrand-Nash equilibrium

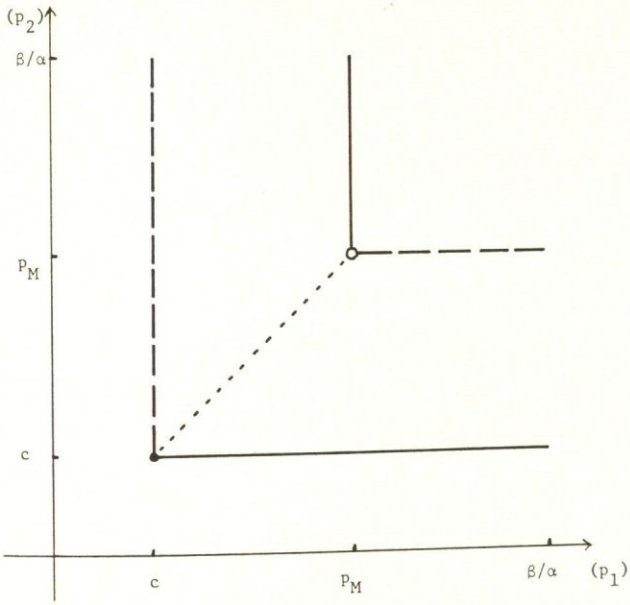

Fig. 3.3

and prices fluctuate indeterminately within an interval. We will consider the case of an identical capacity limit, say $K > 0$, for two firms and show that the above so-called Edgeworth cycle appears if and only if $K \in (q^*, Q_c)$ where $Q_c = \beta - \alpha c$ is the competitive output level and $q^* = \frac{1}{3} Q_c$ is a firm's output level at the Cournot-Nash duopoly equilibrium. Our analysis in this section owes heavily to the work by Levitan and Shubik (1972).

If $K \geqq Q_c$, then the capacity limit is not effective and the previous Bertrand's equilibrium $(\hat{p}_1, \hat{p}_2) = (c, c)$ will result. Therefore we may assume that $0 < K < Q_c$. Then Bertrand's contingent demand function with a capacity constraint can be written as:

$$(3.2) \quad D(p_i, p_j) = \delta(p_j - p_i) \min\{D(p_i), K\} + \delta(p_i - p_j) \max\{0, \min\{D(p_i) - K, K\}\}$$
$$+ (1 - \delta(p_i - p_j) - \delta(p_j - p_i)) \min\{(1/2) D(p_i), K\}$$

for $j \neq i$. Discontinuous thick curves in Figure 3.3 (a) and (b) illustrate the above contingent demand curve when $\bar{p}_j$ satisfies $0 < D(\bar{p}_j) < K$ and $K < D(\bar{p}_j) < 2K$ respectively. The profit function of firm i becomes

$$\pi(p_i, p_j) = (p_i - c) D(p_i, p_j)$$

for $j \neq i$.

Let $p_K = (1/\alpha)(\beta - K)$, $p_m = (1/2)[(\alpha/\beta) + c]$ which is the monopoly price without a capacity limit, and $\bar{p} = \max\{p_K, p_M\}$. If the firm j sets its price so high that $p_j > \bar{p}$, then the best response of firm i $(i \neq j)$ is clearly $p_i = \bar{p}$. Let $p_{2K} = (1/\alpha)(\beta - 2K)$ and $\underline{p} = \max\{c, p_{2K}\}$. If $p_i \in [\underline{p}, p_K]$, then $K \leqq D(p_i) \leqq 2K$ or $0 \leqq D(p_i) - K \leqq K$. Therefore for $i \neq j$, if $p_j < p_i \leqq p_k$, then $D(p_i, p_j) = D(p_i) - K$, if $p_{2K} \leqq p_i < p_j$, then $D(p_i, p_j) = K$, and, if $p_i = p_j$, then $D(p_i, p_j)$

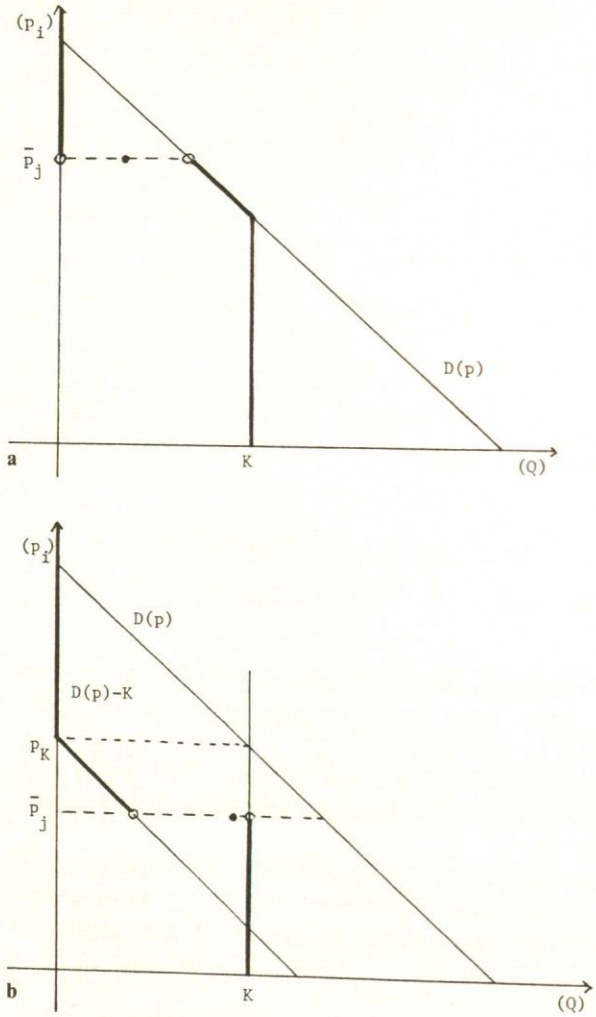

Fig. 3.3.a, b

$= (1/2)\, D(p_i)$. Define real-valued functions π_H, π_L and $\pi_=$ defined on $[c, \beta/\alpha]$ by:

$$\begin{aligned}
\pi_H(p_i) &= (p_i - c)(D(p_i) - K) \\
&= \alpha(p_i - c)(p_K - p_i)\,, \\
\pi_L(p_i) &= (p_i - c)\,K\,, \quad \text{and} \\
\pi_=(p_i) &= (1/2)(p_i - c)\,D(p_i)\,.
\end{aligned}$$

Then if $p_i, p_j \in [\underline{p}, p_K]$ for $i \neq j$,

$$\begin{aligned}
\pi(p_i, p_j) = {} & \delta(p_i - p_j)\,\pi_H(p_i) + \delta(p_j - p_i)\,\pi_L(p_i) \\
& + (1 - \delta(p_i - p_j) - \delta(p_j - p_i))\,\pi_=(p_i)\,.
\end{aligned}$$

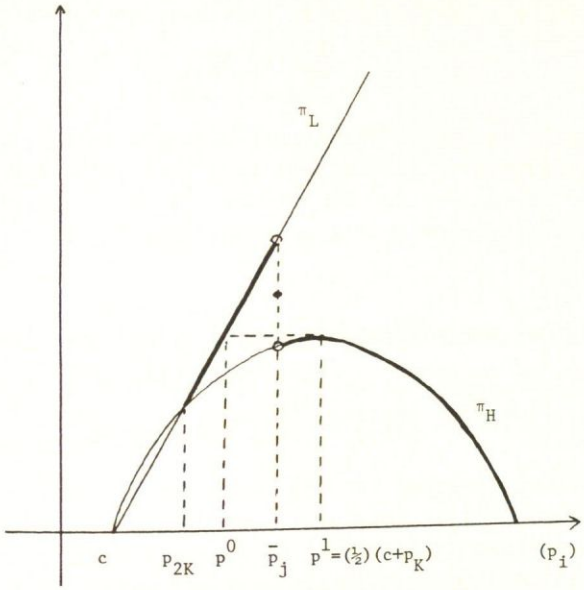

Fig. 3.4

For a given $\bar{p}_j \in [\underline{p}, p_K]$, we illustrate the profit function by a thick curve for $p_i \in [\underline{p}, p_K]$ in Figure 3.4 where $\pi_=(p_i)$ curve is not shown except one point $\pi_=(\bar{p}_j)$. It may be noted that the intersection of π_H and π_L occurs at p_{2K} and p_{2K} is assumed to be larger than c in Figure 3.4. Also, in Figure 3.4, it is assumed that $\pi_H(p^1) < \pi_L(p^1)$ where $p^1 = (1/2)(c+p_K)$. Note that $\pi_H(p_i)|_{p_i=(1/2)(c+p_K)} = (1/4)\alpha(p_K-c)^2$ and $\pi_L(p_i)|_{p_i=(1/2)(c+p_K)} = (1/2)(p_K-c)K$. Therefore $\pi_H(p^1) < \pi_L(p^1)$ where $p^1 = (1/2)(c+p_K)$ if and only if $K > (1/3)(\beta - \alpha c)$. Now if the condition $K > (1/3)(\beta - \alpha c)$ holds, then there exists $p^0 \in [\underline{p}, p_K]$ such that

$$\pi_L(p^0) = \pi_H(p^1), \quad \text{i.e.,} \quad (p^0 - c)K = (1/4)\alpha(p_K - c)^2.$$

Since $\pi_H(p)$ attains its maximum value when $p = p^1$, we have $p_{2K} < p^0$. Now we are ready to describe the best response of firm i when $p_j \in [c, \bar{p}]$ $(j \neq i)$. First suppose that $p_j \in (p^0, \bar{p}]$. Then firm i will try to undercut its price below p_j and $\pi(p_i, p_j)$ achieves its supremum value when p_i converges to p_j from below, but the value of $\pi(p_i, p_j)$ falls discontinuously at $p_i = p_j$. Therefore the best response of firm i does not exist when $p_j \in (p^0, \bar{p}]$. On the other hand, if $p_j \in [c, p^0]$, the values of $\pi_L(p_i)$ for $p_i < p_j$ are below the value of $\pi_H(p_i)$ at $p_i = p^1$. Therefore, the best response of firm i is not to undercut its price below p_j but is to raise its price to $p_i = p^1$. Thus for $p_j \in (p^0, p^1]$, firm i tries to undercut its price below p_j, but if $p_j \in [c, p^0]$, then firm i will raise its price to p^1. Therefore we can conclude that, if $K \in (q^*, Q_c)$, then a pure strategy Bertrand-Nash equilibrium does not exist and prices tend to indeterminately fluctuate in the interval $[p^0, p^1]$ generating the so-called Edgeworth cycle. In Figure 3.5, thick lines and broken lines indicate the best replies of firm 1 and firm 2 respectively, a dotted line indicates undercutting of prices and arrows indicate the Edgeworth cycle.

When an equilibrium in pure strategies does not exist, then it is natural to ask whether or not an equilibrium in mixed strategies can be found. A mixed strategy

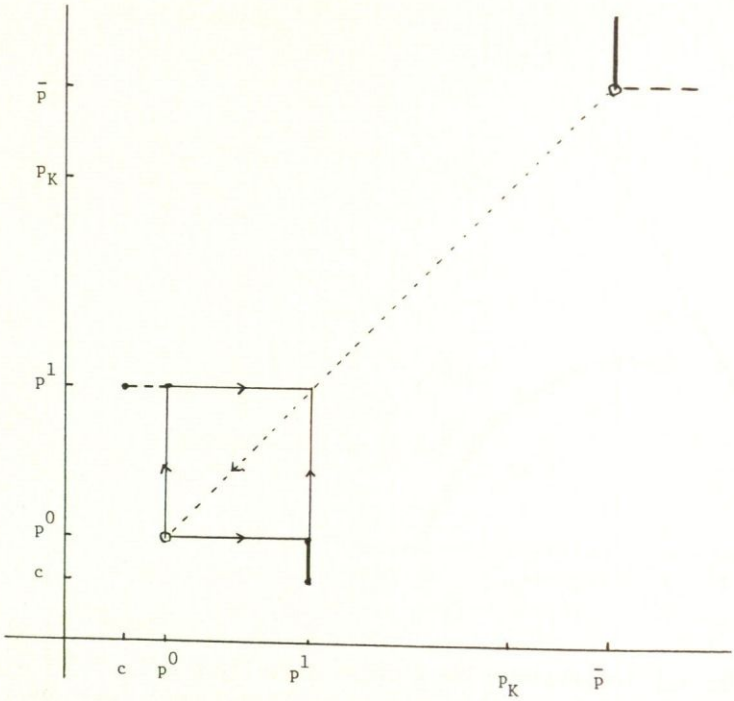

Fig. 3.5

in our case is a probability measure in the space of prices $[c, \beta/\alpha)$. Let μ_i be a mixed strategy or a probability measure on $[c, \beta/\alpha]$ and define a probability distribution function F_i by

$$F_i(p) = \mu_i[\{p' \mid p' \leqq p\}].$$

We will assume that the support of μ_i, or the smallest closed subset of $[c, \beta/\alpha]$ on which the probability is one, is an interval and each probability distribution function is a continuous function. We will show that mixed strategies μ_i^* or F_i^* satisfying the above properties in fact exist.

Without loss of generality, we may assume that $F_i(p) = 1$ if $p > p^1$ and $F_i(0) = 0$ if $p < p^0$. This is because, if the subset $[c, \beta/\alpha] \sim [p^0, p^1]$ has a positive probability assigned, then each firm can profitably shift the probability weight from $[c, \beta/\alpha] \sim [p^0, p^1]$ into $[p^0, p^1]$. Define $\tilde{\pi}_i(p, F_j)$ to be the expected profit of firm i when F_j $(j \neq i)$ is the mixed strategy used by firm j and when firm i uses a pure strategy $p \in [p^0, p^1]$. Then we can write:

$$\tilde{\pi}_j(p, F_j) = F_j(p)\,\pi_H(p) + (1 - F_j(p))\,\pi_L(p)$$

for $j \neq i$ and for $p \in [p^0, p^1]$. Let $[p', p''] \subseteq [p^0, p^1]$ be the support of μ_j. Then $\tilde{\pi}_i(p'', F_j) = \pi_H(p'')$ because $F_j(p'') = 1$. Since π_H is strictly increasing in $p \in [p^0, p^1]$, we must have $p'' = p^1$. On the other hand, $\tilde{\pi}_i(p', F_j) = \pi_L(p')$. Since $\tilde{\pi}_i(\cdot, F_j)$ must be constant on $[p', p^1]$ and $\pi_L(p') = \pi_H(p^1)$, we can conclude that $p' = p^0$. Define $A = \pi_H(p^1) = (1/4)\,\alpha(p_K - c)^2$. Then an equilibrium mixed strategy F_j^* must satisfy the following condition so that $\tilde{\pi}_i(\cdot, F_j^*)$ becomes constant on $[p^0, p^1]$:

$$F_j^*(p)\,\pi_H(p) + (1 - F_j^*(p))\,\pi_L(p) = A, \qquad \text{i.e.,}$$

$$F_j^*(p) = \frac{\pi_L(p) - A}{\pi_L(p) - \pi_H(p)}.$$

Note that if $p \in (p^0, p^1)$, then $0 < \pi_L(p) - A < \pi_L(p) - \pi_H(p)$. Therefore $F_j(p^0) = 0$, $F_j^*(p^1) = 1$ and $F_j^*(p) \in (0, 1)$ for every $p \in [p^0, p^1]$. Therefore we can conclude that the pair (F_1^*, F_2^*) is a symmetric mixed strategy equilibrium in the Bertrand-Edgeworth duopoly with capacity limits.

4. Symmetric Cournot Equilibrium with Free Entry and Small Efficient Scale*

a) Introduction

In previous sections on Cournot oligopoly models, we have made assumptions that the inverse demand function is linear in the quantity demanded and that the marginal cost is constant. These two assumptions are sufficient to guarantee that the profit function is concave in the quantity of each firm and we obtain continuous (in fact linear) Cournot reaction functions except in the case with fixed entry costs. Roberts and Sonnenschein (1977) showed, using an example, that theoretically, it would be very difficult to justify the concavity of profit functions with respect to the quantity produced. Without the concavity of profit functions, Cournot reaction functions may exhibit discontinuities and, hence, the existence of a Cournot equilibrium may not be assured. On the other hand, most price theory books provide a story that, when the number of potential firms is large relative to demand, then each firm perceives a flat demand curve and, with free entry, the long-run competitive market equilibrium of a firm will be achieved at the quantity which minimizes the average cost of production eliminating the economic profit. Let us define an efficient scale as that quantity which minimizes average cost. Following the intuition behind this story, Novshek (1980) proves that a symmetric Cournot equilibrium with free entry exists provided that an efficient scale of firms is sufficiently small relative to demand and moreover the total output at the symmetric Cournot equilibrium is close to the perfectly competitive output level.

In this section, we will present a simplified version of Novshek's result of proof. The basic assumption of our approach is that the market demand curve is downward sloping when the price is at the minimum average cost and that the average cost function achieves a regular minimum at an efficient scale. Then, at the long-run competitive market equilibrium, the profit function will attain its unique regular global maximum at the efficient scale. At this point the profit is zero. We then employ the implicit function theorem and continuity arguments to obtain Novshek's result. The result of this section not only provides the proof of the existence of a Cournot-Nash equilibrium when firms are sufficiently small

* This section is based on Otani, T. and Sicilian, J.: "A Simple proof of Novshek's result on Cournot Equilibrium with Free Entry" (January 1985, unpublished).

relative to demand and there are many firms with free entry and exit, but also it provides a rationale to a long-run equilibrium in a perfectly competitive market.

b) The Model, Equilibrium Concept and Assumptions

All firms in this model are assumed to have the same technology producing a homogeneous single output y which is characterized by a total cost function C: $[0, \infty] \to [0, \infty)$. The average total cost function ac: $(0, \infty) \to (0, \infty)$ is defined by $ac(y) = C(y)/y$, and it will be assumed that there will be a unique positive level of output y^* at which the total average cost is minimized. The level of output y^* minimizing the total average cost will be called the efficient scale for a firm with a total cost function C. Market demand will be assumed to be characterized by an inverse demand function F: $(0, \infty) \to (0, \infty)$.

In this section, we will restrict our attention to symmetric equilibria where output levels y_i are identical among firms. Let a profit function $\tilde{\pi}$: $[0, \infty) \times (0, \infty) \to (-\infty, \infty)$ be defined by: $\tilde{\pi}(y, z) = F(z+y)y - C(y)$ where z is interpreted as the total output level produced by all firms other than the firm concerned. Given (C, F), a symmetric market equilibrium on a symmetric Cournot equilibrium with free entry is an integer n, a number of firms and an output levely satisfying the following three conditions.

(4.1) *Definition:* $(\hat{n}, \hat{y})$ is a *symmetric market equilibrium* for (C, F) if (i) $\hat{n}$ is a positive integer; (ii) $\tilde{\pi}(\hat{y}, (\hat{n}-1)\hat{y}) \geqq \tilde{\pi}(y, (\hat{n}-1)\hat{y})$ for every $y \geqq 0$, and (iii) $\tilde{\pi}(y, \hat{n}\hat{y}) < 0$ for every $y > 0$.

Novshek (1980) was concerned with markets for which the efficient scale y^* is small relative to market quantity demanded at the price $p^* = ac(y^*)$. In order to characterize a size of an efficient scale relative to market demand, we introduce Novshek's parameterizations of market demand and an efficient scale. We define an inverse demand function F_β for each $\beta > 0$ by $F_\beta(q) = F(q/\beta)$, and a total cost function C_α for each $\alpha > 0$ by $C_\alpha(y) = \alpha C(y/\alpha)$. If we let $ac_\alpha(y) = ac(y/\alpha)$, then ac_α has a unique minimum at αy^* and $F(q^*) = ac(y^*)$ implies that $F_\beta(\beta q^*) = ac_\alpha(\alpha y^*)$. See Figures 4.1 (a) and 4.1 (b) for illustrations of $F_\beta(q)$ and $ac_\alpha(y)$ when $\beta > 1$ and $0 < \alpha < 1$. We say that an efficient scale is small relative to market demand for the market (C_α, F_β) if α/β is close to zero. In the next section, we shall prove under the following set of assumptions, that there exists $\bar{k} > 0$ such that for every $\alpha > 0$, $\beta > 0$ with $\alpha/\beta \leqq \bar{k}$, the market (C_α, F_β) has a symmetric market equilibrium.

(4.2) *Assumption:* (C.1) $C(0) = 0$; (C.2) if $y > 0$ and $y \neq y^*$, then $ac(y) > ac(y^*)$; (C.3) $\liminf_{y \to 0} ac(y) > ac(y^*)$ and $\liminf_{y \to \infty} ac(y) > ac(y^*)$; (C.4) C is twice continuously differentiable on $(0, \infty)$; (C.5) $C''(y^*) > 0$; (F.1) F is twice continuously differentiable on $(0, \infty)$; (F.2) there is $q^* > 0$ with $F(q^*) = ac(y^*)$ satisfying (i) for every $q > q^*$, $F(q) < F(q^*)$ and (ii) $F'(q^*) < 0$.

Assumptions (C.2) and (C.3) will be satisfied if the total average cost function is U-shaped as in textbooks. (F.2) is a little weaker than the assumption of downward sloping demand. The sign of ac' may change several times, but ac must

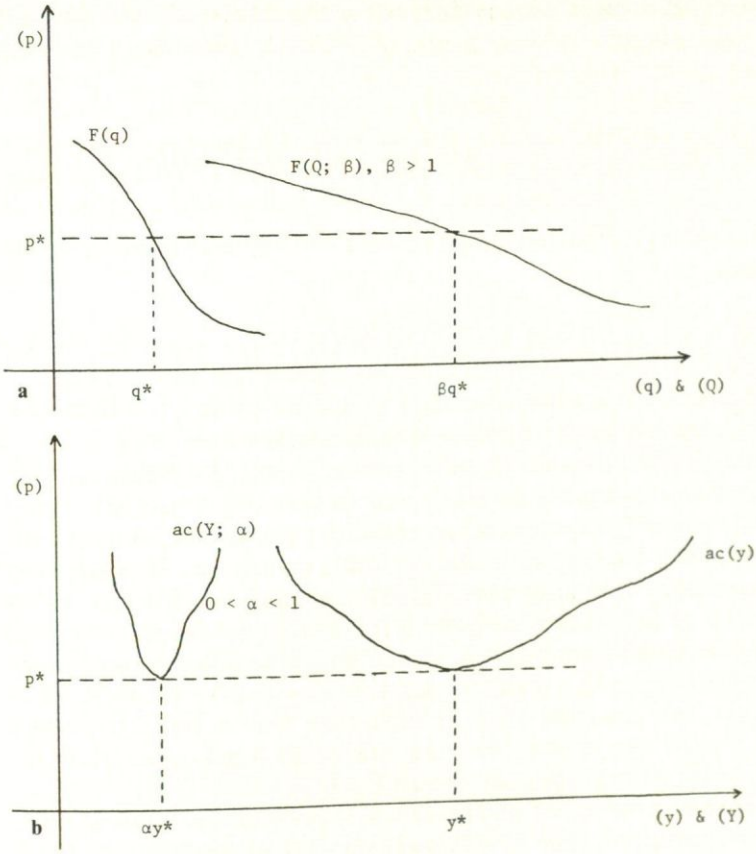

Fig. 4.1.a, b

have a unique regular global minimum at y^*. On the other hand, the sign of F' may change several times, there may be several quantities q for which $F(q) = ac(y^*)$ and q^* in (F.2) will be the largest q with $F(q) = ac(y^*)$.

c) The Existence of a Symmetric Cournot Equilibrium with Free Entry

In this subsection, we shall prove the following theorem.

(4.3) *Theorem:* Under the set of assumptions of (4.2), there exists $\bar{k} > 0$ such that, for every $\alpha > 0$, and $\beta > 0$, if $\alpha/\beta \leqq \bar{k}$, then the market (C_α, F_β) has a symmetric market equilibrium.

Our proof of the above theorem will make use of an artificial equilibrium concept. Define π: $[0, \infty) \times (0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ by $\pi(y, z, k) = F(z + ky)y - C(y)$. Strictly speaking the domain of π ought to be restricted so that $z + ky > 0$. But since in our proof below we will consider (y, k) in a neighborhood of $(q^*, 0)$, we will not be concerned about the looseness in our definition.

(4.4) *Definition:* $(\hat{n}, \hat{y})$ is called a k-equilibrium for the market (C, F) if (i) $\hat{n}$ is a positive integer; (ii) $\pi(\hat{y}, (\hat{n}-1)k\hat{y}, k) \geqq \pi(y, (\hat{n}-1)k\hat{y}, k)$ for every $y \geqq 0$; and (iii) $\pi(y, \hat{n}k\hat{y}, k) < 0$ for every $y > 0$.

Since

$$\alpha\pi(y, z, \alpha/\beta) = F\left(\frac{\beta z + \alpha y}{\beta}\right)\alpha y - \alpha C\left(\frac{\alpha y}{\alpha}\right) = F_\beta(\beta z + \alpha y)\alpha y - C_\alpha(\alpha y)$$

and also since $\hat{z} = (\hat{n}-1)\dfrac{\alpha}{\beta}\hat{y}$ if and only if $\beta\hat{z} = (\hat{n}-1)\alpha\hat{y}$, the following lemma is easy to establish.

(4.5) *Lemma:* If $\alpha > 0$, $\beta > 0$ and $k = \alpha/\beta$, then $(\hat{n}, \hat{y})$ is a k-equilibrium for (C, F) if and only if $(\hat{n}, \alpha\hat{y})$ is a symmetric market equilibrium for (C_α, F_β).

Because of Lemma 4.5, it suffices to show that there exstis $\bar{k} > 0$ such that, for every $k \in (0, \bar{k}]$, the market (C, F) has a k-equilibrium.

To begin with, we shall provide an outline of the proof. We define a supply map $H(z, k)$ of a firm maximizing its profit and we note that $y^* \in H(q^*, 0)$ and moreover y^* satisfies the first-order and second-order conditions. Then, by the implicit function theorem, we can find a local regular profit maximizer $y = \psi(z, k)$ in a neighborhood of $(q^*, 0)$. We will show that $\psi(z, k) \in H(z, k)$ for some neighborhood of $(q^*, 0)$ provided that $k \geqq 0$ and $\pi(\psi(z, k), z, k) \geqq 0$. We next define $\tilde{z}(k)$ in a neighborhood of zero so that $\psi(z, k)$ satisfies the zero profit condition, i.e., $\pi(\psi(z, k), z, k) = 0$ and we define $\tilde{y}(k) = \psi(\tilde{z}(k), k)$. By modifying $\tilde{z}(k)$ and $\tilde{y}(k)$, we can find $(\hat{n}, \hat{y}, \hat{z})$ such that $\hat{n}$ is a positive integer, $\hat{y} = \psi(\hat{z}, k)$ and $\hat{z} = (\hat{n}-1)k\hat{y}$. Then we show that $(\hat{n}, \hat{y})$ is a k-equilibrium for (C, F) for $k \in (0, \bar{k}]$ for an appropriately chosen $\bar{k} > 0$.

Now we proceed with our proof of Theorem 4.3. For $k \in (-\infty, \infty)$ and $z > 0$, let us define an output supply map of a firm maximizing its profit by:

$$H(z, k) = \{y \geqq 0 \mid \text{for every } y' \geqq 0,\ \pi(y, z, k) \geqq \pi(y', z, k)\}.$$

By (C.1), $\pi(0, q^*, 0) = \pi(y^*, q^*, 0)$ and, by (C.2), if $y > 0$ and $y \neq y^*$, then $\pi(y, q^*, 0) = y(p^* - ac(y)) = y(ac(y^*) - ac(y)) < 0$. Therefore $y^* \in H(q^*, 0)$. Moreover, by (C.4) and (C.5), $\partial\pi/\partial y(y^*, q^*, 0) = 0$ and $\partial^2\pi/\partial y^2(y^*, q^*, 0) = -C''(y^*) < 0$. Therefore y^* is a regular global maximizer of $\pi(y, q^*, 0)$.

(4.6) *Lemma:* There exist an open rectangle $U_1 \subseteq R^2$ with $(q^*, 0) \in U_1$, an open interval $V \subseteq R$ with $y^* \in V$ with $0 \notin \bar{V}$ (the closure of V in R) and a unique function $\psi: U_1 \to V$ such that, for every $(z, k) \in U_1$, $\partial\pi/\partial y(\psi(z, k), z, k) = 0$. Moreover U_1 and V can be chosen so that, for every $(z, k) \in U_1$, $\partial^2\pi/\partial y^2(\psi(z, k), z, k) < 0$, $\partial\psi/\partial z(z, k) < 0$ and $F'(z) < 0$.

Since $\partial\pi/\partial y(y^*, q^*, 0) = 0$ and $\partial^2\pi/\partial y^2(y^*, q^*, 0) < 0$, the first part of Lemma 4.6 on the existence of a unique function $\psi: U_1 \to V$ can be obtained by applying the implicit function theorem. Noting that

$$\frac{\partial\psi}{\partial z}(q^*, 0) = -\left.\frac{\partial^2\pi/\partial y\,\partial z}{\partial^2\pi/\partial y^2}\right|_{(y^*, q^*, 0)} = F'(q^*)/C''(y^*) < 0 \quad \text{and} \quad F'(q^*) < 0,$$

not italic

it should be clear that U_1 and V can be chosen so that $\partial^2\pi/\partial y^2(\psi(z, k), z, k) < 0$, $\partial\psi/\partial z(z, k) < 0$ and $F'(z) < 0$ for every $(z, k) \in U_1$ by the continuity of these functions.

We note that, since $\pi(y, q^*, k) = F(q^*, ky)y - C(y)$ and $q^* > 0$, the function $\psi(q, k)$ is mathematically defined in a neighborhood of $k = 0$ in R. But, of course, our interest will be on the nonnegative side of $k = 0$. The above lemma shows that $y = \psi(z, k)$ for $(z, k) \in U_1$ is a regular local maximizer of $\pi(\cdot, z, k)$ and the next lemma asserts, that, with further conditions, $\psi(z, k)$ is in fact a global maximizer.

(4.7) *Lemma:* There exists an open rectangle $U_2 \subseteq U_1$ with $(q^*, 0) \in U_2$ such that, if $(z, k) \in U_2$, $k \geqq 0$ and $\pi(\psi(z, k), z, k) \geqq 0$, then $\psi(z, k) \in H(z, k)$.

Because of (C.2) and $\liminf_{y \to 0} ac(y) < ac(y^*)$, there exists $\varepsilon > 0$ such that if $y > 0$ and $y \notin V$, then $ac(y) \geqq ac(y^*) + \varepsilon$. On the other hand, by (F.2), there exist $\varepsilon' \in (0, \varepsilon]$ and $\hat{q} \in (0, q^*)$ such that if $z \geqq \hat{q}$, then $F(z) < F(q^*) + \varepsilon'$. Choose an open rectangle $U_2 \subseteq U_1$ so that $(q^*, 0) \in U_2$ and $\hat{q} \leqq z$ for every $(z, k) \in U_2$. Suppose that $(z, k) \in U_2$, $k \geqq 0$ and $\pi(\psi(z, k), z, k) \geqq 0$. If $y \notin V$, then since $\hat{q} \leqq z \leqq z + ky$, we have that $F(z + ky) < F(q^*) + \varepsilon' \leqq ac(y^*) + \varepsilon \leqq ac(y)$. Hence $\pi(y, z, k) < 0 \leqq \pi(\psi(z, k), z, k)$. Since $\psi(z, k)$ is a local maximizer on V, we can assert that $\psi(z, k) \in H(z, k)$.

Now let us define $f: U_2 \to R$ by $f(z, k) = \pi(\psi(z, k), z, k)$. Then

$$f(q^*, 0) = \pi(y^*, q^*, 0) = 0 \text{ and}$$

not italic

$$\frac{\partial f}{\partial z}(q^*, 0) = \left(\frac{\partial \pi}{\partial y}\frac{\partial \psi}{\partial z} + \frac{\partial \pi}{\partial z}\right)\Bigg|_{(y^*, q^*, 0)} = \frac{\partial \pi}{\partial z}(y^*, q^*, 0) = F'(q^*)y^* < 0.$$

Therefore, again by the implicit function thereom, there exists a unique function $\tilde{z}$ from a neighbordhood of $k = 0$ into a neighborhood of q^* such that $f(\tilde{z}(k), k) = 0$ for every k in this neighborhood of $k = 0$ and $\tilde{z}(0) = q^*$. Since our interest is on the nonnegative side of $k = 0$, we can restrict $k \in [0, k_1]$ and $\tilde{z}$: $[0, k_1] \to W$ for some $k_1 > 0$ and for some open interval W with $q^* \in W$ and $0 \notin \bar{W}$. Using the continuity argument, the above k_1 can be chosen so that for every $k \in [0, k_1]$, $(\tilde{z}(k), k) \in U_1$ and $\tilde{z}(k) - k\tilde{y}(k) \in U_2$ where $\tilde{y}(k) = \psi(\tilde{z}(k), k)$.

See Figure 4.2 for an illustration of $(\tilde{y}(k), \tilde{z}(k))$. We state these results as the following lemma.

(4.8) *Lemma:* There exist $k_1 > 0$, an open interval W with $q^* \in W$ and $0 \notin \bar{W}$ and a unique function $\tilde{z}$: $[0, k_1] \to W$ such that $\tilde{z}(0) = q^*$, and for every $k \in [0, k_1]$, $f(\tilde{z}(k), k) = 0$, $(\tilde{z}(k), k) \in U_1$ and $\tilde{z}(k) - k\tilde{y}(k) \in U_2$.

Since V, a neighborhood of y^*, and W, a neighborhood of q^*, do not contain 0 in their closures, there exists $k_2 > 0$ such that, for every $y \in V$ and for every $z \in W$, $z \geqq k_2 y$. Let us define $\bar{k} = \min\{k_2, k_3\}$. We now show that for each $k \in (0, \bar{k}]$ the market (C, F) has a k-equilibrium.

Fix $k \in (0, \bar{k}]$. Then there exist an integer $\hat{n}$ and $r \in R$ such that $0 \leqq r < k\tilde{y}(k)$ and $\tilde{z}(k) = (\hat{n} - 1)k\tilde{y}(k) + r$. Since $z \geqq k_2 y$ for every $y \in V$ and $z \in W$ $\tilde{z}(k) = (\hat{n} - 1)k\tilde{y}(k) + r \geqq k_2\tilde{y}(k)$. Hence $(\hat{n} - 1)k_2\tilde{y}(k) + k_2\tilde{y}(k) > k_2\tilde{y}(k)$ which yields $\hat{n} - 1 > 0$.

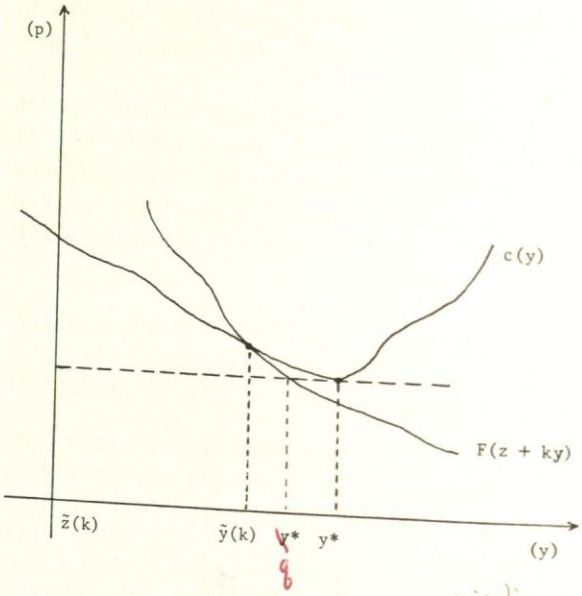

Fig. 4.2

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We now have $\tilde{z}(k) \geqq (\hat{n}-1)\,k\,\psi(\tilde{z}(k), k)$ and, since $\partial\psi/\partial z(z, k) < 0$ for $(z, k) \in U_1$, $\tilde{z}(k) - k\,\tilde{y}(k) < \tilde{z}(k) - r = (\hat{n}-1)\,k\,\psi(\tilde{z}(k), k) < (\tilde{n}-1)\,k\,\psi(\tilde{z}(k) - k\,\tilde{y}(k), k)$. Therefore there exists $\hat{z} \in (\tilde{z}(k) - k\,\tilde{y}(k), \tilde{z}(k)]$ such that $\hat{z} = (\hat{n}-1)\,k\,\psi(\hat{z}, k)$. (See Figure 4.3 for an illustration of this result.) Define $\hat{y} = \psi(\hat{z}, k)$ so that $\hat{z} = (\hat{n}-1)\,k\,\hat{y}$. We will show that $(\hat{n}\ \hat{y})$ is a k-equilibrium.

Since $\hat{z} \leqq \tilde{z}(k)$ and $F'(z) < 0$ for $z \in [\hat{z}, \tilde{z}(k)[$, we have $0 = \pi(\tilde{y}(k), \tilde{z}(k), k) \leqq \pi(\tilde{y}(k), \hat{z}, k) \leqq \pi(\hat{y}, \hat{z}, k)$. Therefore, by Lemma 4.7, $\hat{y} \in H(\hat{z}, k)$ and (i) and (ii) of the definition of a k-equilibrium are satisfied. In order to show the condition (iii) of a k-equilibrium, note that $\tilde{y}(k) = \psi(\tilde{z}(k), k) \leqq \psi(\hat{z}, k) = \hat{y}$ since $\partial\psi/\partial z(z, k) < 0$ for $z \in [\hat{z}, \tilde{z}(k)]$. Thus $\hat{z} + k\,\hat{y} \geqq \hat{z} + k\,\tilde{y}(k) > \tilde{z}(k)$. Let $y > 0$ and first consider a case with $\hat{z} + k\,\hat{y} + k\,y \leqq q^*$. Since $F'(z) < 0$ for $z \in [\tilde{z}(k) + k\,y, \hat{z} + k\,\hat{y} + k\,y]$, $\pi(y, \hat{z} + k\,\hat{y}, k) < \pi(y, \tilde{z}(k), k) \leqq 0$. On the other hand, if $\hat{z} + k\,\hat{y} + k\,y > q^*$, then $F(\hat{z} + k\,\hat{y} + k\,y) < F(q^*) = a\,c(y^*)$ and $a\,c(y^*) \leqq a\,c(y)$. Hence $\pi(y, \hat{z} + k\,\hat{y}, k) = y\,\{F(\hat{z} + k\,\hat{y} + k\,y) - a\,c(y)\} < 0$. Therefore we have shown that the condition (iii) of a k-equilibrium holds at $(\hat{n}, \hat{y})$. This concludes the proof of Theorem 4.3.

Needless to say, $(\hat{y}, \hat{z})$ with $\hat{z} = (\hat{n}-1)\,k\,\hat{y}$ is in a neighborhood of (y^*, q^*) with $\hat{y} < y^*$ and $\hat{z} < q^*$ and, as k approaches zero from the positive side, $(\hat{y}, \hat{z})$ will converge to (y^*, q^*). Therefore, if k is sufficiently small, a symmetric Cournot equilibrium will be very close to a long-run equilibrium of a perfectly competitive market.

Figure 4.2 shows earlier that the output level $\tilde{y}(k)$ corresponds to a situation akin to a textbook-type equilibrium under monopolistic competition with free entry and $\hat{y}$ corresponds to a similar equilibrium output where the number of firms is required to assume integers. The basic difference between our analysis and the case of monopolistic competition is that, in our case, an individual firm faces the demand curve as formulated by Cournot and it is not conceived as the demand curve for one product with many other close substitutes in a market with product differentiation.

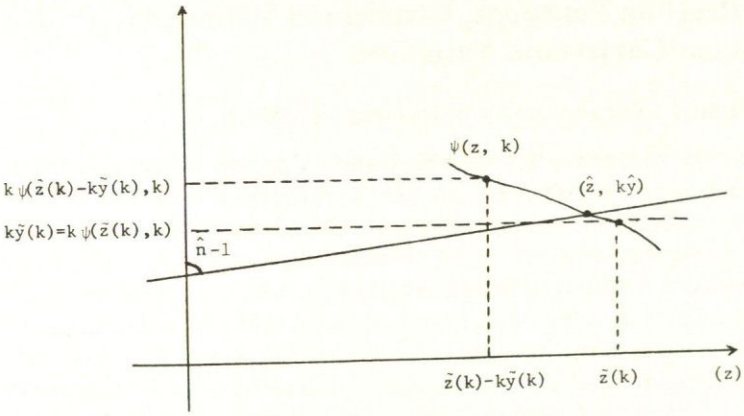

Fig. 4.3

Therefore it will be interesting to find out whether we can obtain a conclusion often claimed for an equilibrium with monopolistic competition with free entry that there will be many firms, each producing below the efficient scale. First it is clear that, at a symmetric Cournot-Nash equilibrium with free entry, we have $\hat{y} < y^*$, i.e., an equilibrium output level of each firm is below the efficient scale y^*. Therefore it remains to find out some properties on the equilibrium number of firms. Here we will compare the number of firms at a symmetric Cournot-Nash equilibrium with free entry and an optimum number of firms which is defined by the number of firms needed to produce q^* with each firm producing at the efficient scale y^*.

From Figure 4.2, it can be seen easily that $\tilde{z}(k) + k y^* > q^*$, i.e., $\tilde{z}(k) > q^* - k y^*$. We will first consider a case where numbers of firms can assume positive real numbers neglecting the integer-valuedness of numbers of firms. Define $\tilde{n}(k)$ and $\tilde{n}^*(k)$ respectively, by $\tilde{z}(k) + k\tilde{y}(k) = \tilde{n}(k) k \tilde{y}(k)$ and $q^* = \tilde{n}^*(k) k y^*$. Then we have

$$\begin{aligned}\tilde{z}(k) &= [\tilde{n}(k) - 1] k \tilde{y}(k) > q^* - k y^* \\ &= [\tilde{n}^*(k) - 1] k y^* > [\tilde{n}^*(k) - 1] k \tilde{y}(k)\end{aligned}$$

Therefore $\tilde{n}(k) - 1 > \tilde{n}^*(k) - 1$, i.e., $\tilde{n}(k) > \tilde{n}^*(k)$. Thus we can conclude that the equilibrium numbers of firms at a Cournot-Nash equilibrium with free entry is always larger than the optimum number of firms provided that we can neglect the integer-valuedness of numbers of firms.

We next show that a similar, but a little weaker, conclusion follows if numbers of firms are confined to integer-values. So we define integers $\hat{n} > 1$ and $\hat{n}^* > 1$ by $\tilde{z}(k) + k\tilde{y}(k) = \hat{n} k \tilde{y}(k) + r$ for some $r \in [0, k\tilde{y}(k))$ and $q^* = \hat{n}^* k y^* + r^*$ for some $r^* \in [0, k y^*)$. Using $\tilde{z}(k) > q^* - k y^*$ again yield $\tilde{z}(k) = [\hat{n} - 1] k \tilde{y}(k) + r > \hat{n}^* k y^* + r^* - k y^*$. Since $k\tilde{y}(k) > r$ and $r^* \geqq 0$, we obtain $\hat{n} k \tilde{y}(k) > [\hat{n}^* - 1] k y^*$. Because $y^* > \tilde{y}(k)$, we get that $\hat{n} > \hat{n}^* - 1$. Therefore we can conclude that the equilibrium number of firms at a Cournot-Nash equilibrium free entry is always no smaller than the optimum number of firms.

with

5. Perceived Reaction Functions, Conjectural Variations and Consistent Conjectural Variations

a) Perceived Reaction Functions and Conjectural Variations

We have seen that the distinction between Bertrand's duopoly model and Cournot's duopoly model rests crucially on each firm's perception of the behavior of other firms which results in a particular type of perceived demand functions. We call each firm's perception of other firms' behavior a *perceived reaction function*. We assume that a perceived reaction function of firm i can be written as a function $\psi^i(q_i; \theta_i)$ where q_i is the output level of firm i and θ_i is a parameter of the function and the value of the function is the quantity produced by other firms reacting to q_i. The derivative of ψ^i with respect to q_i is termed a *conjectural variation* by R. Frisch. Very often conjectural variations are introduced as the primitive concept, but we prefer to consider the perceived reaction function to be the more basic concept and regard conjectural variations to be derived from perceived reaction functions. We believe that this approach makes the concept clearer and our analysis more transparent.

As before, firm i's profit function in an industry consisting of competing identical firms can be written as $\tilde{\pi}^i(q_i, Q_i) = q_i P(q_i + Q_i) - C(q_i)$. When firm i has a perceived reaction function $\psi^i(q_i; \theta_i)$, then the profit function becomes $\pi^i(q_i; \theta_i) = \tilde{\pi}^i(q_i, \psi^i(q_i; \theta_i))$. The parameter θ_i will be assumed to depend on $(q_1, q_2, \ldots, q_n)$ which is currently observed. Let us give several examples of perceived reaction functions. If firms have competitive perceived reaction functions, then we may write $\psi^i(q_i; \theta_i) = \theta_i - q_i$ where $\theta_i = Q$. Then the competitive conjectural variation becomes -1. In the case of Cournot's oligopoly, we may take $\psi^i(q_i; \theta_i) = \theta_i$ with $\theta_i = Q_i$. Thus Cournot's conjectural variation becomes zero. Now consider the case of a symmetric cartel monopoly where $q_i = Q/n$ and $Q_i = (n-1)q_i = \frac{n-1}{n}Q$. This case coincides with the outcome of the case when perceived reaction functions are given by $\psi^i(q_i; \theta_i) = (n-1)q_i$. Suppose that the inverse demand function and the cost function are both linear in quantities and are given respectively by: $P(Q) = b - aQ$ and $C(q_i) = cq_i$. Then the Cournot-Nash reaction function of firm i becomes $q_i = \psi^i_{CN}(Q_i) = (1/2)\frac{b-c}{a} - (1/2)Q_i$. Assume that firm 1 is a Stackelberg leader and the rest of the firms are Stackelberg followers. Then summing over the Cournot-Nash reaction functions for $i \neq 1$ and noting that $Q_i = Q_1 + q_1 - q_i$, we obtain

$$(5.1) \quad Q_1 = \sum_{i \neq 1} \psi^i_{CN}(Q_i) = (1/2)\left[(n-1)\frac{b-c}{a} - (n-2)Q_1 - (n-1)q_1\right].$$

Note that $-(1/2)(n-1)$ is the direct effect on the output of the rest of firms responding to a unit increase of firm 1's output. Assume that firm 1, the Stackelberg leader, also computes the mutual interactions among the rest of firms. Then we can solve (5.1) for Q_1 to get

$$(5.2)\quad Q_1 = \frac{n-1}{n}\,\frac{b-c}{a} - \frac{n-1}{n}\,q_1 = \psi^1(q_1)$$

which is the perceived reaction function for the rest of firms as perceived by the Stackelberg leader. Therefore, the conjectural variation for the Stackelberg leader becomes $-(n-1)/n$. Note that if the Cournot-Nash reaction function is not linear, then finding a perceived reaction function for the Stackelberg leader and its conjectural variation becomes much more complex if the Stackelberg leader wants to take interactions among followers into account. One method will be first to solve $q_i = \psi^i_{CN}(Q_1 + q_1 - q_i)$ for q_i, say $q_i = F^i(Q_1 + q_1)$ and then solve $Q_1 = \sum_{i \neq 1} F^i(Q_1 + q_1)$ for Q_1.

Sometimes, it is claimed that, when the conjectural variations of duopolists are constant and zero, then this case reduces to a Bertrand duopoly. But we feel that this is a little misleading because the case of a Bertrand duopoly crucially depends on more global characteristics of perceived demand functions with a discontinuity with respect to a price variable.

b. Perceived Reaction Functions and Consistent Conjectural Variations

In this section, we will introduce two kinds of consistency of perceived reaction functions, namely (i) the consistency of parameters θ_i, and (ii) the consistency of conjectural variations. In order to facilitate our discussion, we will, throughout this section, consider the case of only two firms with perceived reaction functions $q_j = \psi^i(q_i; \theta_i)$, $(i, j = 1, 2$ and $i \neq j)$. Parameters are assumed to be determined by the observation of quantities currently produced. This can be expressed by $\theta_i = \tilde{\theta}_i(q_1, q_2)$. Given a parameter θ_i, firm i chooses its output q_i so that its profit $\pi^i(q_i, \theta_i) = \tilde{\pi}^i(q_i, \psi^i(q_i; \theta_i))$ is maximized. This approach is similar to the one we have employed in the section on "Monopoly with Perceived Demand Functions" in Chapter 7. We may suppose that each firm is composed of a division to estimate a parameter θ_i and a production division. Decision makings of two divisions are decentralized. The production division is not aware of the function $\theta_i = \tilde{\theta}^i(q_1, q_2)$ and it simply takes θ_i as a parameter fed by the estimation division. Following Negishi (1961), we introduce the following consistency condition on parameter estimation functions:

(5.3) For every $(q_1, q_2) \geqq 0$,

$$\psi^i(q_i, \tilde{\theta}^i(q_1, q_2)) = q_j, \qquad (i, j = 1, 2 \text{ and } i \neq j)\,.$$

This means that if (q_1^0, q_2^0) is the output pair currently observed, then the parameter $\theta_1^0 = \tilde{\theta}^1(q_1^0, q_2^0)$, for example, must satisfy the condition that the graph of the perceived reaction function of firm 2 as perceived by firm 1 passes through (q_1^0, q_2^0), i.e., $q_2^0 = \psi^1(q_1^0, \theta_1^0)$. not italic

Assume that, for each θ_i given, there exists a unique output q_i maximizing firm i's profit $\pi^i(q_i, \theta_i)$. We denote this relation by $q_i = \phi^i(\theta_i)$. Then a pair (q_1^0, q_2^0) is said to be an *equilibrium* if $q_i^0 = \phi^i(\theta_i^0)$ and $\theta_i^0 = \tilde{\theta}^i(q_1^0, q_2^0)$, $(i = 1, 2)$. Note that this equilibrium concept is so general that equilibria under a competitive market, a Cournot duopoly case, a Stackelberg leader-follower case and

a Cartel monopoly can be obtained as special cases of the above equilibrium concept by appropriately defining perceived reaction functions. Since the above concept of equilibrium allows may outcomes depending on perceived reaction functions, we may try to restrict perceived reaction functions by imposing further conditions. One natural condition to employ for this purpose is to impose a consistency between actual reaction functions and perceived reaction functions. To do this, we first define actual reaction functions. A function F^i mapping from the space of q_j into the space of q_i ($i, j = 1, 2$ and $i \neq j$) is said to be an actual reaction function if it satisfies the condition that $q_i = F^i(q_j)$ if and only if $q_i = \phi^i(\tilde{\theta}^i(q_1, q_2))$ ($i, j = 1, 2$ and $i \neq j$). Then we adopt the following definition of consistency of perceived reaction functions. A perceived reaction function ψ^i is said to be *consistent around an equilibrium* (q_1^0, q_2^0) *at* θ_i^0 where $\theta_i^0 = \tilde{\theta}_i(q_1^0, q_2^0)$ if there exist open neighborhoods $N(q_i^0)$ of q_i^0 ($i = 1, 2$) such that for every $q_i \in N(q_i^0)$

$$(5.4) \quad F^j(q_i) = \psi^i(q_i, \theta_j^0), \qquad (i, j = 1, 2 \text{ and } i \neq j).$$

This condition means that firm i's perceived reaction function over the behavior of firm j ($j \neq i$) at an equilibrium parameter θ_i^0 agrees with the actual reaction function of firm j in a neighborhood of q_j^0. When functions are differentiable, then (6.4) implies the following consistency of conjectural variations: for every $q_i \in N(q_i^0)$

$$(5.5) \quad \frac{dF^j(q_i)}{dq_i} = \frac{\partial \psi^i(q_i, \theta_i^0)}{\partial q_i}, \qquad (k, j = 1, 2 \text{ and } i \neq j).$$

Our task in this section is to examine the above consistency condition and to find out the extent to which this condition narrows down the set of equilibrium outcomes. First let us examine a very simple model where the inverse demand function is given by $P(Q) = b - aQ$, both firms have an identical cost function of the form $C(q_i) = cq_i$ and conjectural variations are identical and constant denoted by δ. Therefore perceived reaction functions can be written as:

$$q_j = \psi^i(q_i, \theta_i) = \theta_i - \delta q_i .$$

The profit function fo firm i becomes:

$$\pi^i(Q_i, \theta_i) = (b - c - a\theta_i)\, q_i - a(1 - \delta)\, q_i^2 .$$

The first-order condition of the profit maximization yields the following best response function:

$$q_i = \phi^i(\theta_i) = (1/2) \frac{1}{a(1-\delta)} (b - c - a\theta_i) .$$

Now the consistency of a parameter θ_i requires that $q_j = \tilde{\theta}^i(q_1, q_2) - \delta q_i$ and this implies that

$$\tilde{\theta}^i(q_1, q_2) = q_j + \delta q_i, \qquad (i, j = 1, 2 \text{ and } i \neq j).$$

Thus the equilibrium condition can be written as:

$$q_i^0 = (1/2) \frac{1}{a(1-\delta)} (b - c - a\theta_i^0)$$

and

$$\theta_i^0 = q_j^0 + \delta q_i^0, \quad \text{i.e.,}$$

$$(5.6) \quad q_i^0 = (1/2)\frac{1}{a(1-\delta)}(b - c - a q_j^0 - a\delta q_i^0), \quad (i, j = 1, 2 \text{ and } i \neq j).$$

The system (6.6) has a unique symmetric solution given by:

$$(5.7) \quad q_i^0 = \frac{1}{3-\delta}\,\frac{b-c}{a}.$$

Now actual reaction functions can be obtained from

$$q_j = \phi^j(\bar{\theta}^j(q_1, q_2)) = (1/2)\frac{1}{a(1-\delta)}\{b - c - a(q_i + \delta q_j)\}$$

by solving for q_j. This yields:

$$(5.8) \quad q_j = F^j(q_i) = \frac{1}{2-\delta}\left(\frac{b-c}{a} - q_i\right).$$

Using (5.7) we have

$$\theta_i^0 = q_j^0 + \delta q_i^0 = \frac{1+\delta}{3-\delta}\,\frac{b-c}{a}$$

and perceived reaction functions are given by:

$$(5.9) \quad q_j = \psi^i(q_i, \theta_i^0) = \theta_i^0 - \delta q_i.$$

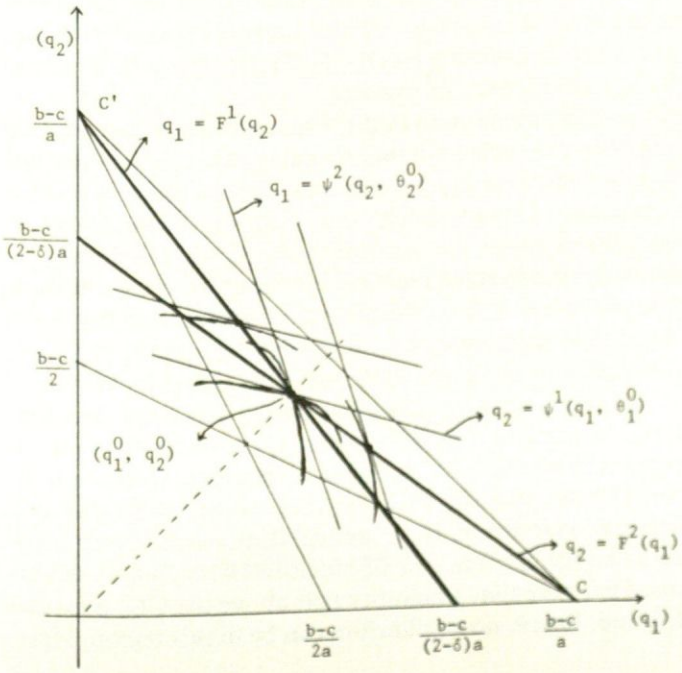

Fig. 5.1

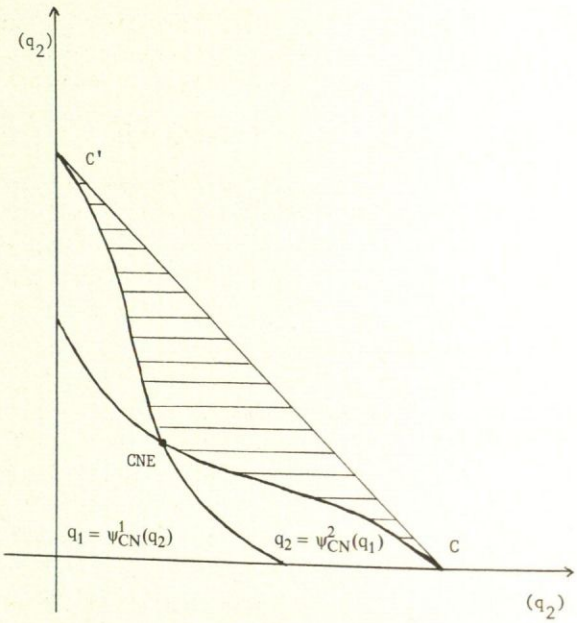

Fig. 5.2

and

Now it is straightforward to conclude that (5.8) and (5.9) agree with each other if and only if $\delta = 1$, i.e., firms have the competitive conjectural variation. In Figure 5.1, we illustrate the above result. There (q_1^0, q_2^0) is an equilibrium pair of quantities which is not a competitive outcome and, hence, $F^2(q_1)$ and $\psi^1(q_1, \theta_1^0)$ through (q_1^0, q_2^0) do not coincide. When $\delta = 1$, then $\psi^i(q_i, \theta_i^0)$ and $F^j(q_i)$ all collapse to the line CC'. It may be noted that the above result turns out to be true even without the assumption of the linearity of the inverse demand function provided that the marginal cost is constant, $P'(Q) + (1/2)QP''(Q) \neq 0$. This and some other result will be left as exercises for readers.

The above result may give the impression that the assumption of consistency of perceived reaction functions is so powerful that it enables us to select the competitive outcome as the only consistent outcome at least for the class of models permitted under our assumptions. Unfortunately, this is not so, for the above result is largely due to the assumption of the constancy of conjectural variations. Once we allow the nonlinearity of perceived reaction functions, then we can show that the set of equilibrium outcomes with consistant perceived reaction functions becomes very large. To see this, suppose that Cournot-Nash reaction functions $q_i = \psi^i_{CN}(q_j)$, $(i, j = 1, 2, \text{ and } i \neq j)$ are given as shown in Figure 5.2. Moreover assume that firms have identical constant marginal costs. Then the line CC' signifies the set of (q_1, q_2) such that $q_1 + q_2 = Q_C$ where Q_C is the total competitive output. If admissible perceived reaction functions are assumed to be continuous and downward sloping in the quantity variable, then we can show that any outcome in the shaded region bounded by C, C′ and CNE (Cournot-Nash equilibrium) can be obtained as an equilibrium pair of quantities with consistent perceived reaction functions. First note that a quantity pair above the CC' is associated with negative profits and, hence, no equilibrium can be in this region. Next,

note that the set of $(q_1, q_2) \geqq 0$ lying below the Cournot-Nash reaction function ψ^i_{CN} is associated with positively sloped isoprofit curve of $\tilde{\pi}^i$. Therefore quantity pairs lying in this region will not be compatible with a downward sloping perceived reaction function of firm i over the behavior of firm j ($j \neq i$). Thus the shaded region bounded by C, C′ and CNE becomes the only region which remains to be examined. Quantity pairs on the line CC′ are obtained as equilibrium outcomes with competitive conjectural variations which are consistent since the graph of perceived reaction functions in this case are exactly the line CC′. Now choose a quantity pair (q_1^0, q_2^0) inside of or in the interior of the shaded region. For the sake of simplicity, we assume that the profit function $\tilde{\pi}^i(q_1, q_2)$ is strictly quasi-concave in (q_1, q_2). Let the line through (q_1^0, q_2^0) supporting $\tilde{\pi}^2(q_1, q_2)$ at (q_1^0, q_2^0) be denoted by $q_1 = F^1(q_2)$. Then there exists a neighborhood U of (q_1^0, q_2^0) such that every point (q_1', q_2') on the line $q_1 = F^1(q_2)$ lying in the neighborhood has a negatively sloped isoprofit curve of $\tilde{\pi}^1(q_1, q_2)$ through (q_1', q_2'). Therefore, if $(q_1', q_2') \in U \cap \{(q_1, q_2) \mid q_1 = F^1(q_2)\}$, supporting lines of $\tilde{\pi}^1(q_1, q_2)$ at (q_1', q_2') can be written as $q_2 = \psi^1(q_1, \theta_1')$ with $\theta_1' = q_2'$ such that ψ^1 is strictly decreasing in q_1. Note that points on the line $q_1 = F^1(q_2)$ can be parameterized by q_2 and, if $\theta_1' \neq \theta_1''$ or $q_2' \neq q_2''$, then two supporting lines with θ_1' and θ_1'' as parameters, do not intersect each other in sufficiently small neighborhoods of (q_1', q_2') and (q_1'', q_2'') where $q_1' = F^1(q_2')$ and $q_1'' = F^1(q_2'')$. Therefore $\psi^1(q_1, \theta_1)$ can be modified and extended to a parameterized family of non-intersecting curves through points in $\{(q_1, q_2) \geqq 0 \mid q_1 = F^1(q_2)\}$ so that if $(q_1', q_2') \in U \cap \{(q_1, q_2) \mid q_1 = F^1(q_2)\}$ is given, then (q_1', q_2') maximizes $\tilde{\pi}^1(q_1, q_2)$

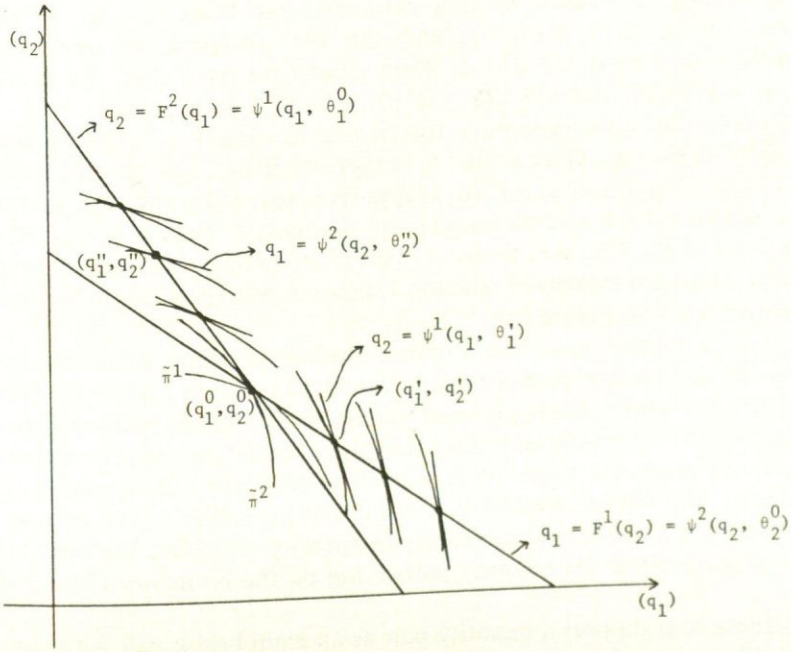

Fig. 5.3

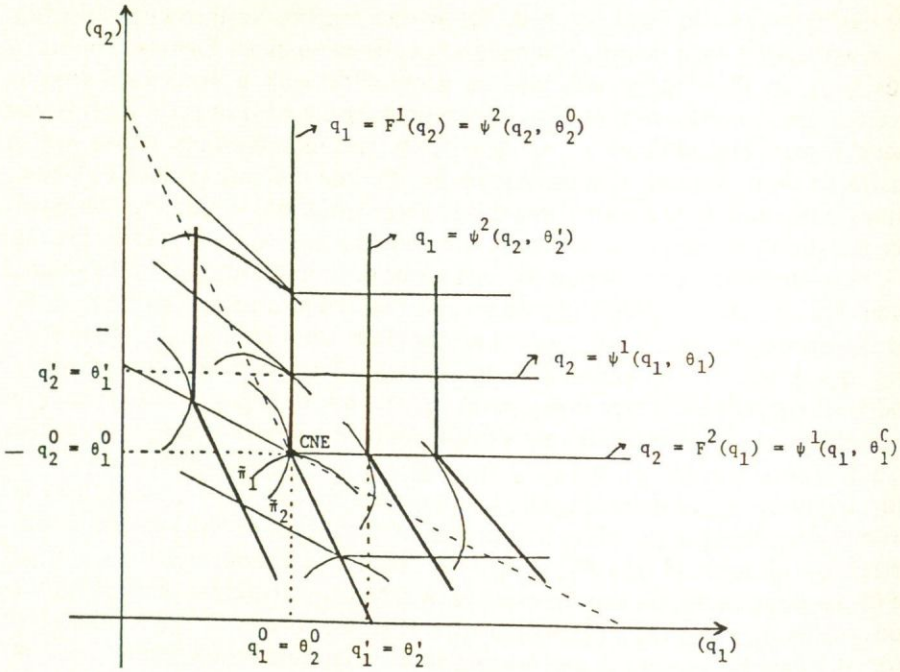

Fig. 5.4

subject to $q_2 = \psi^1(q_1, \theta_1')$ where θ_1' is a parameter satisfying $\theta_1' = q_2'$. Write $\tilde{\theta}^1(q_1, q_2) = q_2$, $\pi^1(q_1, \theta_1) = \tilde{\pi}^1(q_1, q_2)$ and the best response to maximize $\pi^1(q_1, \theta_1)$ with respect to q_1 be $\phi^1(\theta_1)$. Then clearly we have that, for every $(q_1', q_2') \in U$, $q_1' = F^1(q_2')$ if and only if $q_1' = \phi^1(\theta_1')$ with $\theta_1' = \tilde{\theta}^1(q_1', q_2')$.

Now we apply the same procedure for firm 2 to obtain $q_2 = F^2(a_1)$ and $q_1 = \psi^2(q_2, \theta_2)$ with $\theta_2 = q_1$. Since $\psi^1(q_1, \theta_1')$ supports $\tilde{\pi}^1(q_1, q_2)$ at $(q_2', q_2') \in U \cap \{(q_1, q_2) \mid q_1 = F^1(q_2)\}$ and $q_2 = F^2(q_1)$ supports $\tilde{\pi}^1(q_1, q_2)$ at (q_1^0, q_2^0), we get that $F^2(q_1) = \psi^1(q_1, \theta_1^0)$ in a neighborhood of q_1^0. Similarly $F^1(q_2) = \psi^2(q_2, \theta_2^0)$ in a neighborhood of q_2^0. Thus we can assert that (q_1^0, q_2^0) is an equilibrium pair of quantities with consistent perceived reaction functions. We provide an illustration of the above result in Figure 5.3.

We now give an illustration of how Cournot-Nash equilibrium quantities can be obtained as an equilibrium with consistent perceived reaction functions. This is done in Figure 5.4 where, for the ease of illustration, Cournot reaction functions are assumed to be linear in quantities. In this construction, perceived reaction functions are piecewise linear with kinks and they are nonincreasing in quantity variables. But with nonlinearity, it is possible to modify perceived reaction functions so that they become decreasing in quantity variables. We can also apply similar constructions for quantity pairs lying on the boundary C (or C′) and CNE.

Our construction to support a quantity pair as an equilibrium pair with consistent perceived reaction function relies on the strict quasi-concavity of profit

functions $\tilde{\pi}^i(q_1, q_2)$ in (q_1, q_2). If profit functions are not strictly quasi-concave, our construction can be modified by using kinked or nonlinear perceived reaction functions. Therefore our conclusion that any quantity pair in the region bounded by C, C′ and CNE can be supported as an equilibrium with consistent perceived reaction functions remains true even if profit functions are not strictly quasi-concave.

Chapter 9 Monopolistic Competition

1. Introduction

Undoubtedly, imperfect competition is one of the most interesting and yet one of the most difficult topics in economics. A cursory glance of any textbook on imperfect competition will likely give an impression of a showcase on various models of market structure without much unifying themes across various models. We believe that it is very difficult to avoid this impression simply because there are too many ways for a market to be imperfect. In particular, there are so many formulations possible regarding what each firm is supposed to know about consumers and also what each firm is supposed to think what other firms will be behaving. Of course, a fascinating direction to which we ought to proceed would be to find a way to rationalize why firms would form particular perceptions and conjectures over consumers and other firms and how firms would adopt strategies suitable for particular perceptions and conjectures. Until we succeed in doing this, we may have to content ourselves with a showcase of various models.

Another problem of imperfect competition arises from its inherent analytical difficulty. First of all, as Sonnenschein (1973) and Debreu (1974) have shown, the aggregate market demand function derived from the standard demand theory does not provide us with much qualitative restrictions suitable for the analysis of imperfect competition. Thus the analysis of imperfect competition necessitates much stronger restrictions on consumer's characteristics.

A closely related problem occurs because a firm in an imperfectly competitive market is usually assumed to maximize its profits subject to a market demand function. But then the firm's optimal strategy and its conditions for optimality would involve second and third derivatives of consumers' utility functions.

In view of these difficulties, much of the analysis of imperfectly competitive markets have been confined to manageable models which aim at interesting economic implications rather than more generality.

In this chapter we discuss three approaches to modeling imperfect competition. In section 2 we present a traditional model of monopolistic competition with product differentiation. In Section 3 we discuss Hotelling's model of product differentiation and in Section 4 we present a modification of that model.

2. A Traditional Model of Monopolistic Competition with Product Differentiation

a) Introduction

In this section, we will provide an explicit formulation of monopolistic competition with product differentiation in the spirit of Chamberlain (1933). We shall first examine a short-run equilibrium of monopolistic competition and, in particular, we shall indicate the well-known differences between a price strategy equilibrium and a quantity strategy equilibrium. We then proceed to examine Chamberlain's celebrated tangency equilibrium or a long-run equilibrium of monopolistic competition with free entry. A long-run equilibrium under monopolistic competition will be quite similar to a Cournot equilibrium with free entry of Section 4 in Chapter 8. The major distinction of Chamberlain's model from Cournot's model, of course, lies in the assumption on the nature of differentiated products and the resulting demand functions. As in Chamberlain, we will consider a symmetric case where all products enter into a utility function in a symmetrical fashion and cost conditions in producing each differentiated product are also symmetrical. We will also assume that only one firm produces one product and we will not make an inquiry regarding the choice of products and the sources of the product differentiation in this section.

b) Demand Functions and the Cost Function

Suppose that all consumers consuming a group of differentiated products can be lumped together as one consumer with a utility function of the form $U_n(q_1, q_2, \ldots, q_n, y) = V_n(q_1, q_2, \ldots, q_n) + y$ where q_i is the amount of the ith differentiated product consumed and y is the amount of a numeraire good consumed. The budget constraint of the consumer will be written as $\sum_{i=1}^{n} p_i q_i + y = M$ where p_i is the price of the ith differentiated product and the income M is assumed to be sufficiently large that a positive quantity of the numeraire good be consumed at an equilibrium.

The utility function V_n over a group of n differentiated products will be assumed to have a particular form:

$$(2.1)\quad V_n(q_1, q_2, \ldots, q_n) = (\alpha/\beta)\,Q - (1/2\beta)\frac{n^{r+1}\gamma}{(n-1)+n^{r+1}\gamma}Q^2 - (1/2\beta)\frac{n(n-1)}{(n-1)+n^{r+1}\gamma}\left(\sum_{j=1}^{n} q_j^2\right)$$

where $Q = \sum_{j=1}^{n} q_j$ and α, β and γ are positive constants. The above utility function is a modification of a function employed by Shubik and Levitan (1980, Chapters 6 and 7). Write $q = (q_1, q_2, \ldots, q_n)$ and define $s^2(q)$ by

$$(2.2)\quad s^2(q) = (1/n) \sum_{j=1}^{n} (q_i - \bar{q})^2$$

where $\bar{q} = (1/n) \sum_{j=1}^{n} q_j = Q/n$. Since $s^2(q) = (1/n) \sum_{j=1}^{n} q_j^2 - (Q/n)^2$, we can rewrite (7.1) as:

$$V_n(q) = (\alpha/\beta)\, Q - (1/2\beta)\, Q^2 - (1/2\beta) \frac{n^2(n-1)}{(n-1) + n^{r+1}\gamma}\, s^2(q)\,.$$

Define $V(Q, s, n)$ by:

$$V(Q, s, n) = (\alpha/\beta)\, Q - (1/2\beta)\, Q^2 - (1/2\beta) \frac{n^2(n-1)}{(n-1) + n^{r+1}\gamma}\, s^2\,.$$

If $r = 2$, then $\lim_{n\to\infty} V(Q, s, n) = (\alpha/\beta)\, Q - (1/2\beta)\, Q^2 - (1/2\beta)(s^2/\gamma)$, if $r > 2$, then $\lim_{n\to\infty} V(Q, s, n) = (\alpha/\beta)\, Q - (1/2\beta)\, Q^2$ and if $r < 2$, then $V(Q, s, n)$ diverges to minus infinity as n goes to infinity. Shubik and Levitan call γ a measure of the substitutability of the goods. With a minor difference, the utility function used by Shubik and Levitan is essentially a case with $r = 0$. (The last term of $V(Q, s, n)$ in Shubik and Levitan is $-(1/2\beta)\; n^2/(1+\gamma)\, s^2$.) It appears that the utility function with $r \geq 2$ has more desirable properties.

First-order conditions of the utility maximization subject to the budget constraint yields inverse demand functions as follows:

$$(2.3)\quad p_j = (\alpha/\beta) - (1/\beta) \frac{n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma}\, Q - (1/\beta) \frac{n(n-1)}{(n-1) + n^{r+1}\gamma}\, q_j\,.$$

If we define an average quantity produced by firms other than firm j by $\bar{q}_j = 1/(n-1)(Q - q_j)$, then (2.3) becomes:

$$(2.4)\quad p_j = (\alpha/\beta) - (1/\beta) \frac{n(n-1) + n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma}\, q_j - (1/\beta) \frac{(n-1)\, n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma}\, \bar{q}_j$$

$$= (\alpha/\beta) - (1/\beta)\, n\, q_j - (1/\beta) \frac{(n-1)\, n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma}\, (\bar{q}_j - q_j) \equiv P(q_j, \bar{q}_j, n)\,.$$

Summing (2.3) over j yields:

$$(2.5)\quad \bar{p} = (\alpha/\beta) - (1/\beta)\, Q \quad \text{or} \quad Q = \alpha - \beta\bar{p}$$

where $\bar{p} = (1/n) \sum_{j=1}^{n} p_j$. Using this fact, we can solve (2.3) for q_j to obtain

$$q_j = (1/n)\, \alpha - \left(\frac{1}{n} + \frac{n^r\gamma}{n-1}\right) \beta p_j + \frac{n^r\gamma}{n-1}\, \beta\bar{p}$$

and noting that $\bar{p} = \frac{n-1}{n}\, \bar{p}_j + \frac{1}{n}\, p_j$ and $\bar{p}_j = \frac{1}{n-1} \sum_{i \neq j} p_i$, we obtain the demand functions given by:

$$(2.6)\quad q_j = \frac{1}{n}\alpha - \left(\frac{1}{n} + n^{r-1}\gamma\right)\beta p_j + n^{r-1}\gamma\beta\bar{p}_j$$

$$= \frac{1}{n}\alpha - \frac{1}{n}\beta p_j + n^{r-1}\gamma\beta(\bar{p}_j - p_j)$$

$$\equiv X(p_j, \bar{p}_j, n)\,.$$

The above demand function is the perceived demand function which each firm employing a price strategy faces and it is Chamberlain's d-d demand curve. Chamberlain's ~~d-d~~ D-D demand curve can be defined as follows:

$$(2.7)\quad q_j = D(p, n) \equiv X(p, p, n) = \frac{1}{n}\alpha - \frac{1}{n}\beta p\,.$$

Clearly we have $|\partial X/\partial p_j|^{-1} = \dfrac{n}{(1+n^r\gamma)\beta} < |\partial D/\partial p|^{-1} = \dfrac{n}{\beta}$ and Figure 2.1 provides an illustration of relationship among (2.5), (2.6) and (2.7).

Let $C(q_j)$ be the total cost function for producing q_j which is identical for all differentiated products and $ac(q_j) = C(q_j)/q_j$ for $q_j > 0$. We will assume that the following conditions hold for the total average cost function $ac(q_j)$.

(2.8) (i) For every $q_j > 0$, $ac(q_j) > 0$ and twice differentiable; (ii) there exist $\mu > 0$ and $\eta > 0$ such that $\mu = ac(\eta)$, $ac'(q_j) < 0$ for $q \in (0, \eta)$, $ac'(\eta) = 0$ and $ac'(q_j) > 0$ for $q > \eta$; (iii) for every $q_j > 0$, $ac''(q_i) > 0$; and (iv) $\lim_{q_j \to 0} ac(q_j) > \alpha/\beta$.

In (2.8), η achieves the minimum value μ of the total average cost and we can define the number $\bar{n}$ of firms required to satisfy the demand for differentiated products when its price is μ and the output of each firm is η, i.e., $\eta = D(\mu, \bar{n})$ or $\bar{n} = (\alpha - \beta\mu)/\eta$. Although it is possible to carry out our analysis analogous to that in Section 5 on the Cournot equilibrium with free entry so that the number

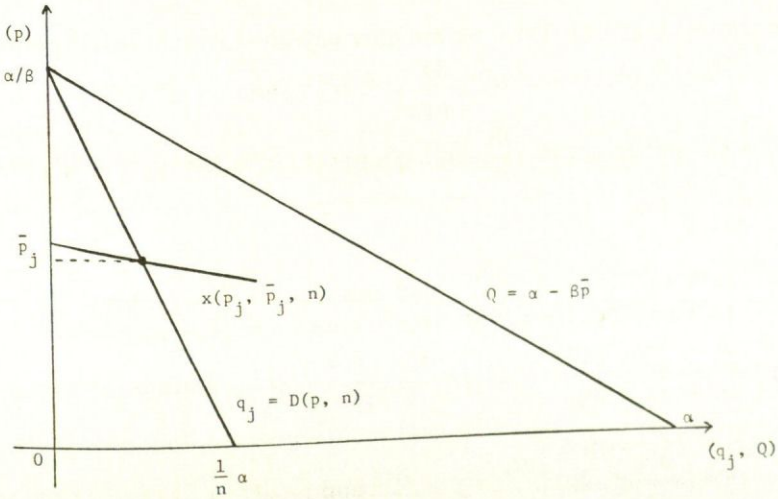

Fig. 2.1

of firms is required to take integral values; in order to simplify our analysis we shall disregard the discreteness of n and treat it as a real number.

c) Symmetric Short-run Equilibria with Differentiated Products

In this subsection, we shall examine symmetric short-run equilibria of monopolistic competition or oligopolistic equilibria with differentiated products when the number of firms is exogeneously fixed. We will not be concerned about the existence of an equilibrium because the existence crucially depends on the number of firms fixed. The equilibrium number of firms in a long-run equilibrium of monopolistic competition will be analyzed in the next subsection. With linear demand functions and a strictly convex total average cost function, it can be easily shown that a symmetric short-run equilibrium will exist if the number of firms is smaller than the equilibrium number of firms in the long run monopolistic competition. Our major interest in this subsection is to find out qualitative differences between an oligopoly equilibrium with price strategies and one with quantity strategies. We first define a symmetric short-run equilibrium with price strategies and one with quantity strategies, which will be respectively termed a *B-equilibrium* and a *C-equilibrium* for short, as follows:

(2.9) We call $((p_j^B), (q_j^B))$ a B-equilibrium if $((p_j^B), (q_j^B))$ satisfies: for every $j = 1, 2, \ldots, n$, (i) $\frac{\partial R}{\partial p_j}(p_j^B, \bar{p}_j^B, n) = C'(q_j^B)\frac{\partial X}{\partial p_j}(p_j^B, \bar{p}_j^B, n)$, (ii) $q_j^B = X(p_j^B, \bar{p}_j^B, n)$, and (iii) $p_j^B = \bar{p}_j^B$ and $q_j^B = \bar{q}_j^B$ where $R(p_j, \bar{p}_j, n) = p_j X(p_j, \bar{p}_j, n)$.

(2.10) We call $((p_j^C), (q_j^C))$ a C-equilibrium if $((p_j^C), (q_j^C))$ satisfies for every $j = 1, \ldots, n$, (i) $\frac{\partial}{\partial q_j}\tilde{R}(q_j^C, \bar{q}_j^C, n) = C'(q_j^C)$, (ii) $p_j^C = P(q_j^C, \bar{q}_j^C, n)$ and (iii) $p_j^C = \bar{p}_j^C$ and $q_j^C = \bar{q}_j^C$ where $\tilde{R}(q_j, \bar{q}_j, n) = q_j P(q_j, \bar{q}_j, n)$.

Because of the symmetry among firms, we can alternatively state that (p^B, q^B) is a B-equilibrium if $\frac{\partial R}{\partial p_j}(p^B, p^B, n) = C'(q^B)\frac{\partial X}{\partial p_j}(p^B, p^B, n)$ and $q^B = D(p^B, n)$, and (p^C, q^C) is a C-equilibrium if $\frac{\partial \tilde{R}}{\partial q_j}(q^C, q^C, n) = C'(q^C)$ and $p^C = F(q^C, n)$ where $F(q, n) = \frac{\alpha}{\beta} - \frac{1}{\beta}nq$.

Note that $\frac{\partial R}{\partial p_j} = X + p_j\frac{\partial X}{\partial p_j} = C'\frac{\partial X}{\partial p_j}$ if and only if $p_j + \left(\frac{\partial X}{\partial p_j}\right)^{-1} X = C'$,

$\left(\frac{\partial X}{\partial p_j}\right)^{-1} = \frac{(1/\beta)\, n}{1 + n^r\gamma}$ and $\frac{\partial P}{\partial q_j} = -(1/\beta)\frac{n(n-1) + n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma}$. Define

$$M^B(q, n) = \left\{p_j + \left(\frac{\partial X}{\partial p_j}\right)^{-1} X\right\}\Bigg|_{p_j = \bar{p}_j = F(q, n)} \quad \text{and}$$

$$M^C(q, n) = \left\{ P + q_j \frac{\partial P}{\partial q_j} \right\} \Bigg|_{q_j = \bar{q}_j = q}$$

which are nothing but marginal revenue functions for a B-equilibrium case and a C-equilibrium case respectively. We can easily show that:

$$M^B(q, n) = (\alpha/\beta)\left(n + \frac{n}{1 + n^r \gamma} \right) q$$

and

$$M^C(q, n) = (\alpha/\beta) - (1/\beta) \left\{ n + \frac{n(n-1) + n^{r+1}\gamma}{(n-1) + n^{r+1}\gamma} \right\} q \, .$$

Then we can obtain

$$\left| \frac{\partial M^C}{\partial q} \right| - \left| \frac{\partial M^B}{\partial q} \right| = (1 + n^r \gamma)^{-1} \{(n-1) + n^{r+1}\gamma\}^{-1} n^{2r+1} \gamma > 0 \, .$$

Therefore we must have that, at a B-equilibrium and a C-equilibrium, $q^C < q^B$ and $p^C > p^B$ as illustrated in Figure 2.2. Therefore we may say that a B-equilibrium outcome is more competitive than a C-equilibrium outcome.

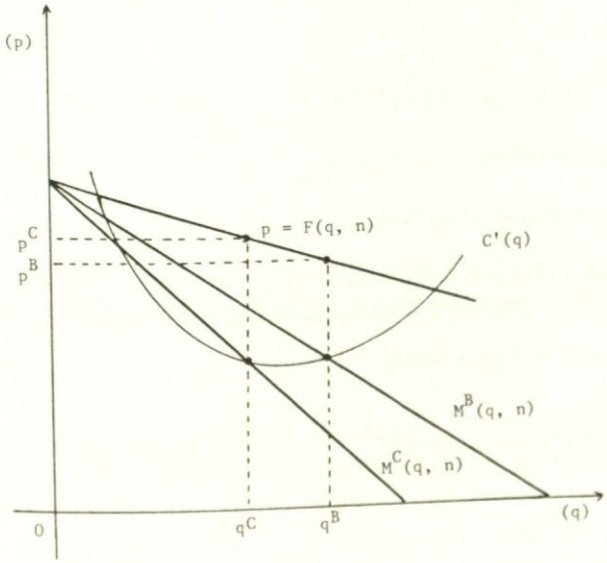

Fig. 2.2

d) Symmetric Long-run Equilibria of Monopolistic Competition

In this subsection, we shall examine various properties of the long-run equilibrium of monopolistic competition. First note that $\frac{\partial R}{\partial p_j} = q_j + p_j \frac{\partial X}{\partial p_j}$ and $C'(q_j) = ac(q_j) + q_j ac'(q_j)$. Therefore $\frac{\partial R}{\partial p_j} = C' \frac{\partial X}{\partial p_j}$ and $p_j = ac(q_j)$ will imply that

$ac'(q_j) = \left(\frac{\partial X}{\partial p_j}\right)^{-1}$ and similarly $\frac{\partial \tilde{R}}{\partial q_j} = C'$ and $p_j = ac(q_j)$ will imply that $ac'(q_j) = \frac{\partial P}{\partial q_j}$. Thus we can state conditions for a symmetric long-run equilibrium of monopolistic competition with price strategies and one with quantity strategies, called a *long-run B-equilibrium* and a *long-run C-equilibrium* respectively, as follows:

(2.11) (i) We call (p^B, q^B, n^B) a long-run B-equilibrium if (p^B, q^B, n^B) satisfies $ac'(q^B) = \left[\frac{\partial X}{\partial p_j}(p^B, p^B, n^B)\right]^{-1}$, $q^B = D(p^B, n^B)$ and $p^B = ac(q^B)$; and (ii) we call (p^C, q^C, n^C) a long-run C-equilibrium if (p^C, q^C, n^C) satisfies $ac'(q^C) = \frac{\partial P}{\partial p_j}$ (q^C, q^C, n^C), $p^C = F(q^C, n^C)$ and $p^C = ac(q^C)$.

In this subsection, we will add an assumption that $\lim_{q \to 0} ac'(q) = -\infty$. This assumption is not very important, but it simplifies our analysis. Define $q^B(n)$ and $q^C(n)$ respectively by:

$$(2.12) \quad ac'(q^B(n)) = \left(\frac{\partial X}{\partial p_j}\right)^{-1} = -\frac{1}{\beta}\frac{n}{1+n^r\gamma}, \quad \text{and}$$

$$ac'(q^C(n)) = \left(\frac{\partial P}{\partial q_j}\right) = -\frac{1}{\beta}\frac{n(n-1)+n^{r+1}\gamma}{(n-1)+n^{r+1}\gamma}.$$

Also let $p^B(n)$ and $p^C(n)$ be defined by:

$$(2.13) \quad p^B(n) = F(q^B(n), n) \text{ and } p^C(n) = F(q^C(n), n).$$

It is easy to see that $q^B(n)$ and $q^C(n)$ are well-defined on $[0, \infty)$ since $a''(q) > 0$ for every $q > 0$ and $\lim_{q \to 0} ac'(q) = -\infty$. Note that $ac(q) = F(q, n)$ if and only if $ac(q) + \frac{1}{\beta} nq = \frac{\alpha}{\beta}$. In view of this relationship, we define

$$(2.14) \quad f^B(n) = ac(q^B(n)) + \frac{1}{\beta} nq^B(n), \quad \text{and}$$

$$f^C(n) = ac(q^C(n)) + \frac{1}{\beta} nq^C(n).$$

Since $ac'(q^B(0)) = ac'(q^C(0)) = 0$, $q^B(0) = q^C(0) = \eta$. Also $q^B(n) \in (0, \eta]$ and $q^C(n) \in (0, \eta]$ because $ac'(q^B(n)) \leqq 0$ and $ac'(q^C(n)) \leqq 0$. Therefore $f^B(0) = f^C(0) = ac(\eta) = \mu < \frac{\alpha}{\beta}$ and $\lim_{n \to \infty} f^B(n) = +\infty$ and $\lim_{n \to \infty} f^C(n) = +\infty$. Therefore for each $k = B$ and C, there exists $n^k > 0$ such that $f^k(n^k) = \frac{\alpha}{\beta}$ and this establishes the existence of a long-run k-equilibrium (p^k, q^k, n^k) with $p^k = p^k(n^k)$,

$q^k = q^k(n^k)$. Because $ac'(q^k) < 0$, we have that $p^k > \mu$ and $q^k < \eta$ indicating the Chamberlainian inefficiency of a long-run monopolistic competition.

By tedious computations, we can also show that

$$f^{B\prime}(n) = \{ac''\beta^2(1+n^r\gamma)^3\}^{-1} n^{r+1}\gamma\{(n-1)n^r\gamma - 1\} + \beta^{-1}q^B(n) .$$

Therefore if $r \geqq 2$ and $\gamma \geqq 1$, then $f^{B\prime}(n) > 0$ for $n \geqq 1$. Thus if $r \geqq 2$, $\gamma \geqq 1$ and $n^B \geqq 1$, then a long-run B-equilibrium will be unique. Although, for other cases, we are unable to assert the uniqueness, we will presume the uniqueness of a long-run k-equilibrium for k = B and C.

(2.15) *Lemma:* For every $n > 0$, (i) $0 < q^C(n) < q^B(n) < \eta$; and (ii) $f^C(n) < f^B(n)$.

$$\text{Since} \quad 0 < -ac'(q^B(n)) = -\left[\frac{\partial X}{\partial p_j}\right]^{-1} < -\left[\frac{\partial P}{\partial q_j}\right] = -ac'(q^C(n)) ,$$

we immediately obtain that $0 < q^C(n) < q^B(n) < \eta$. In order to show (ii), note that $ac(q)$ is a regular strictly convex function since $ac''(q) > 0$ for every $q > 0$. Therefore, for a given n, $ac(q^B(n)) > ac(q^C(n)) + ac'(q^C(n))(q^B(n) - q^C(n))$. Therefore we can obtain that:

$$f^B(n) - f^C(n) > \left(ac'(q^C(n)) + \frac{n}{\beta}\right)(q^B(n) - q^C(n)) .$$

But

$$ac'(q^C(n)) + \frac{n}{\beta} = \frac{1}{\beta}\{(n-1) + n^{r+1}\gamma\}^{-1}(n-1)n^r\gamma > 0$$

which, together with (i), implies that $f^C(n) < f^B(n)$.

(2.16) *Lemma:* For every $n > 0$, $f^B(n) < \mu + (1/\beta)\eta n$.

Again, by the regular strict convexity of $ac(q)$, we have that:

$$ac(\eta) > ac(q^B(n)) + ac'(q^B(n))(\eta - q^B(n)) .$$

Therefore

$$\mu + \frac{1}{\beta}n\eta - f^B(n) > \left(ac'(q^B(n)) + \frac{1}{\beta}n\right)(\eta - q^B(n))$$

and

$$ac'(q^B(n)) + \frac{1}{\beta}n = \frac{1}{\beta}\,\frac{n^{r+1}}{1+n^r\gamma} > 0$$

which yields the desired result.

These two lemmas together result in the following proposition on the numbers of firms in long-run equilibria of monopolistic competition.

(2.17) *Proposition:* At the long-run B-equilibrium and the long-run C-equilibrium, we have that (i) $\bar{n} < n^B < n^C$ where $\bar{n} = (\alpha - \beta\mu)/\eta$; and (ii) $n^B q^B < \bar{n}\eta$, $q^B < \eta$, and $n^C q^C < \bar{n}\eta$, $q^C < \eta$.

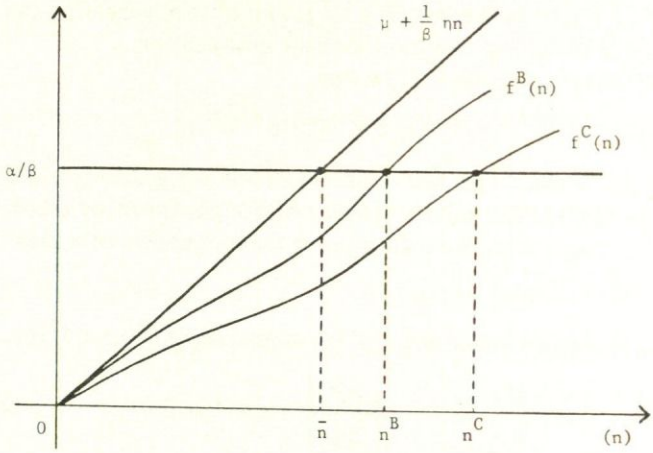

Fig. 2.3

The result $\bar{n} < n^B < n^C$ follows immediately from lemmas 2.15 and 2.16. See Figure 2.3 for the illustration of this result. The second result follows from the fact that $ac'(\eta) = 0 > ac'(q^B)$ and $0 > ac'(q^C)$. Hence

$$ac(\eta) = \mu = \frac{1}{\beta}(\alpha - \bar{n}\eta) < ac(q^B)$$

and, since $\alpha/\beta = f^B(n^B) = 1/\beta(q^B + n^B q^B)$, we can get $n^B q^B < \bar{n}\eta$, $q^B < \eta$, and similarly $n^C q^C < \bar{n}\eta$, $q^C < \eta$.

The above proposition shows the essence of Chamberlainian inefficiency, namely that the monopolistic competition results in the excessive entry of firms, each of which produces its output below the efficient scale. Moreover the above result indicates that the quantity strategic equilibrium results in more excessive entry than the price strategic equilibrium. We are unable to obtain further comparisons between q^B and q^C and between $n^B q^B$ and $n^C q^C$.

We now consider limiting properties of long-run equilibria under monopolistic competition. Since the number of firms is endogeneously determined at a long-run equilibrium, we have to change exogeneous parameters to examine limiting properties of long-run equilibria.

First suppose that the measure of substitutability gets larger, i.e., $\gamma \to \infty$. Since

$$\lim_{\gamma \to \infty} \left\{ -\frac{1}{\beta} \frac{n}{1 + n^r \gamma} \right\} = 0 \quad \text{and}$$

$$ac'(q^B) = \left(\frac{\partial X}{\partial p_j} \right)^{-1} = -\frac{1}{\beta} \frac{n}{1 + n^r \gamma}, \quad \lim_{\gamma \to \infty} q^B = \eta .$$

Therefore $\lim_{\gamma \to \infty} p^B = \mu$ and $\lim_{\gamma \to \infty} n^B = \bar{n}$. Therefore when the measure of substitutability gets larger, long-run B-equilibria converge to the competitive outcome. On the other hand, since

$$\lim_{\gamma\to\infty}\left\{-\frac{1}{\beta}\,\frac{n(n-1)+n^{r+1}\gamma}{(n-1)+n^{r+1}\gamma}\right\}=-\frac{1}{\beta},\qquad \text{we have that } \overline{\lim_{\gamma\to\infty}}\, q^B<\eta\,.$$

Therefore long-run C-equilibria will not approach the competitive outcome as γ gets larger.

Let us consider an alternative approach where an efficient scale of a firm becomes smaller just as in the analysis of a Cournot equilibrium with free entry. In order to carry out this approach, we define

$$ac_\varepsilon(q)=ac(q/\varepsilon)$$

and

$$\eta_\varepsilon=\varepsilon\eta$$

which is an efficient scale for the average cost function ac_ε. Let $(q_\varepsilon^B, p_\varepsilon^B, n_\varepsilon^B)$ and $(q_\varepsilon^C, p_\varepsilon^C, n_\varepsilon^C)$ be respectively a long-run B-equilibrium and a long-run C-equilibrium with a parameter ε. Since $ac_\varepsilon'(q)=1/\varepsilon\, ac'(q/\varepsilon)$, we have

$$ac'(q_\varepsilon^B/\varepsilon)=-\frac{\varepsilon}{\beta}\,\frac{n}{1+n^r\gamma}\bigg|_{n=n_\varepsilon^B}.$$

Let $\tilde{q}_\varepsilon^B=q_\varepsilon^B/\varepsilon$. Then we get

$$|ac'(\tilde{q}_\varepsilon^B)|=\frac{\varepsilon}{\beta}\,\frac{n}{1+n^r\gamma}\bigg|_{n=n_\varepsilon^B}<\frac{\varepsilon}{\beta}.$$

Therefore $\lim_{\varepsilon\to 0}\tilde{q}_\varepsilon^B=\eta$ and $\lim_{\varepsilon\to 0} ac(\tilde{q}_\varepsilon^B)=\mu$. Also since $ac_\varepsilon(\tilde{q}_\varepsilon^B)+(1/\beta)n^B\tilde{q}_\varepsilon^B=\alpha/\beta$, we have that $n^B q_\varepsilon^B=\alpha-\beta ac(\tilde{q}_\varepsilon^B)$ which yields $\lim_{\varepsilon\to 0} n^B q_\varepsilon^B=\alpha-\beta\mu=D(\mu)$. Therefore just as in the case of a Cournot equilibrium with free entry, a long-run B-equilibrium will converge to a competitive outcome.

In the case of a long-run C-equilibrium, we have that:

$$|ac'(\tilde{q}_\varepsilon^C)=\varepsilon\frac{n(n-1)+n^{r+1}\gamma}{(n-1)+n^{r+1}\gamma}\bigg|_{n=n_\varepsilon^C}<\varepsilon\left(\frac{1}{n^{r-1}\gamma}-\frac{1}{n^r\gamma}+1\right)\bigg|_{n=n_\varepsilon^C}.$$

Therefore $\lim_{\varepsilon\to\infty}\tilde{q}_\varepsilon^C=\eta$ provided that $r>1$. When $0\leqq r\leqq 1$, in order to guarantee $\lim_{\varepsilon\to 0}\tilde{q}_\varepsilon^C=\eta$, we need a condition that $\lim_{\varepsilon\to 0}\varepsilon n^{1-r}|_{n=n_\varepsilon^C}=0$. Therefore when $r>1$, then a long-run C-equilibrium also converges to a competitive outcome.

When these limiting properties hold, for sufficiently large γ or for sufficiently small η_ε, a long-run equilibrium under monopolistic competition will stay close to the competitive outcome. Therefore, as claimed by Stigler (1949) and Friedman (1953), various comparative static results should be similar between a long-run equilibrium of monopolistic competition and a long-run competitive market equilibrium. Note that a similar comment also applies to a Cournot equilibrium with free entry when an efficient scale is sufficiently small. Needless to say, these limiting properties alone do not diminish our interests in Cournot oligopoly or monopolistic competition.

3. Hotelling's Model of Product Differentiation or Spatial Competition

In this section, we will investigate the original Hotelling model of spatial competition which is perhaps the simplest model with explicitly treated product differentiation. In particular, we shall investigate a difficulty in the original Hotelling model regarding the existence of a spatial price equilibrium which was shown by D'Aspremont, Gubszewicz and Thisse (1979). This will motivate our analysis in the next section where the Hotelling model will be modified in order to avoid its difficulty.

In the Hotelling model, a product is characterized by a point on a line $[0, l]$ with length l. Each consumer has its best brand specification denoted by r and the consumer type r is assumed to be distributed uniformly over $[0, l]$ with the density $1/l$. Each consumer is assumed to buy only one unit of a differentiated product.

There are two firms, each of which chooses a brand and a price to charge for each unit of the brand. The choice of a brand and the production of each differentiated product are assumed to be costless. Suppose that a firm chooses its brand specification $a \in [0, l]$ and it sells the brand at price p. Then a consumer of type r will be assumed to obtain utility from consuming one unit of "a" by the amount $u(a, p, r) = -c|r-a| - p$. If firm 1 offers to sell brand "a" at p_1 and firm 2 offers to sell brand "$l-b$" at p_2, then a consumer of type r will choose one unit of the brand which satisfies $\max\{u(a, p_1, r), u(l-b, p_2, r)\} = -\min\{p_1 + c|r-a|, p_2 + c|r-(l-b)|\}$. We call $-u(a, p, r) = p + c|r-a|$ the cost for consumer r to buy the brand "a" at p.

Suppose that firm 1 chooses a brand $a \in [0, l]$ and firm 2 chooses a brand $l-b \in [0, l]$ with $a < l-b$ as shown in Figure 3.1. Let firm i's profit function be denoted by $\pi^i(p_1, p_2, a, b)$. Then it is easy to see that profit functions must satisfy the following conditions: (i) if $p_i + c(l-a-b) < p_j$, $(i, j = 1, 2; i \neq j)$, then $\pi^i(p_1, p_2, a, b) = p_i l$ and $\pi^j(p_1, p_2, a, b) = 0$, and if $|p_2 - p_1| < c(l-a-b)$, then $\pi^1(p_1, p_2, a, b) = (a+r)p_1$ and $\pi^2(p_1, p_2, a, b) = (l-a-r)p_2$ where $r = (1/2c)(p_2 - p_1) + (1/2)(l-a-b)$. See Figure 3.2(a) for case (i) with $i = 1$ and $j = 2$ and Figure 3.2(b) for case (ii) where kinked lines through p_i indicate a cost for each consumer of buying firm i's brand. When $p_2 = p_1 + c(l-a-b)$ (or $p_1 = p_2 + c(l-a-b)$), then consumers with their types in $[l-b, l]$ (or respectively in $[0, a]$) are indifferent between firm 1's brand and firm 2's brand. For these

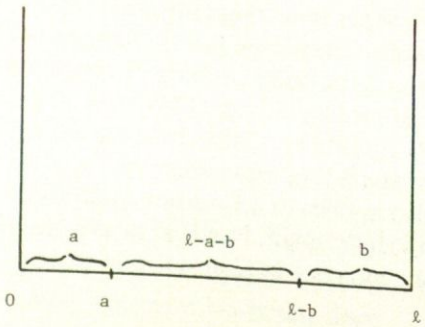

Fig. 3.1

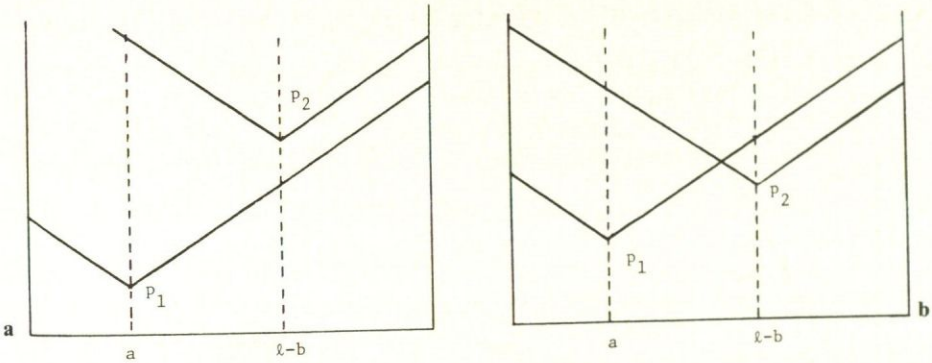

Fig. 3.2.a, b

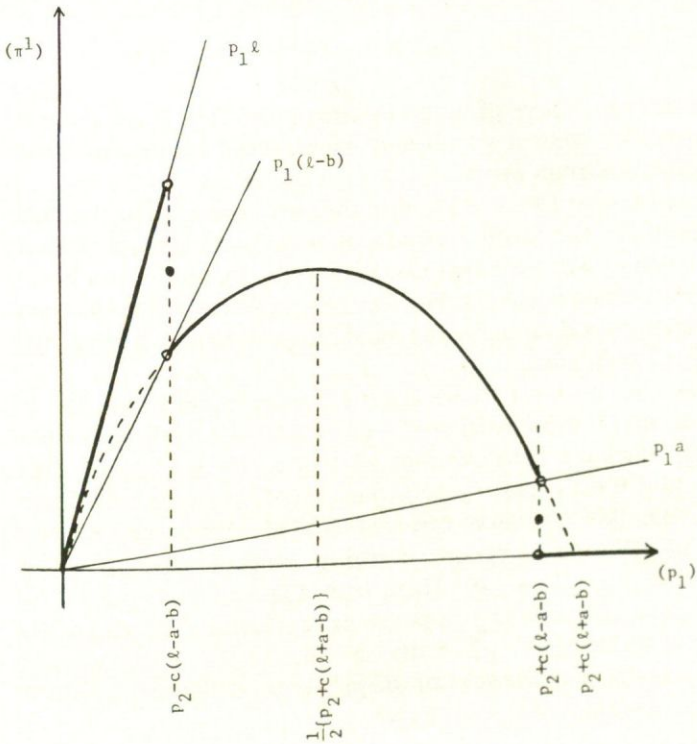

Fig. 3.3

cases, we will assume that firms evenly divide the market of indifferent consumers. Therefore, if $p_2 = p_1 + c(1-a-b)$ (or $p_1 = p_2 + c(1-a-b)$), then $\pi^1(p_1, p_2, a, b) = (1-\frac{1}{2}b)p_1$ and $\pi^2(p_1, p_2, a, b) = (\frac{1}{2})bp_2$ (or respectively $\pi^1(p_1, p_2, a, b) = (\frac{1}{2})ap_1$ and $\pi^2(p_1, p_2, a, b) = (1-\frac{1}{2}a)p_2$). Figure 3.3 gives us an illustration of firm 1's profit function for a given (p_2, a, b) which is drawn by a thick curve and lines. Note that the parabola in Figure 3.3 has a form $\pi^1 = (\frac{1}{2}c)p_1\{c(1+a-b)+p_2-p_1\}$.

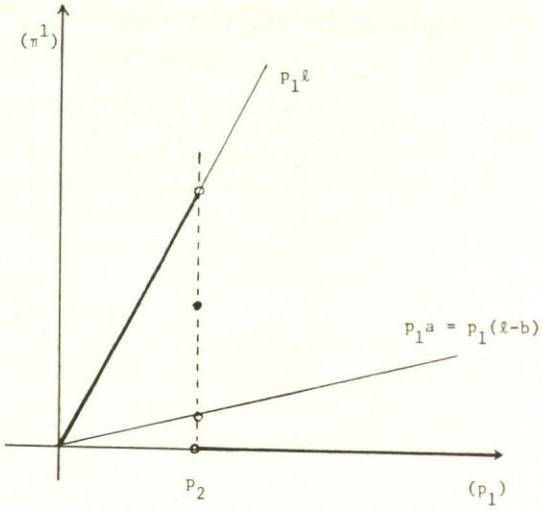

Fig. 3.4

We will now assume that choices of brand by firm 1 and firm 2 are given by "a" and "$1-b$" respectively with $a \leqq 1-b$ and we examine conditions under which a duopoly price equilibrium exists.

We begin with the case where $a+b=1$ so that the two firms choose the same brand. Under this situation, the profit function of firm 1 will have a form as shown in Figure 3.4 when $p_2>0$. Note that two lines $p_1(1-b)$ and p_1 a in Figure 3.3 will coincide and the parabola will not be effective for this case. It is clear that firm 1 will try to undercut p_2 and, as in the Bertrand duopoly model, a price equilibrium will be at $(p_1^*, p_2^*) = (0, 0)$.

Now let us assume that $a+b<1$. If we denote a price equilibrium pair by (p_1^*, p_2^*), then we claim that it must satisfy $|p_2^* - p_1^*| < c(1-a-b)$. Suppose that $|p_2^* - p_1^*| > c(1-a-b)$. We may consider that $p_2^* > p_1^* + c(1-a-b) > 0$. Then $\pi^2(p_1^*, p_2^*, a, b) = 0$ and if $0 < p_2 < c(1-a-b)$, then $\pi^2(p_1^*, p_2, a, b) > 0$. Therefore p_2^* can not be the best response price of firm 2. Now suppose that $|p_2^* - p_1^*| = c(1-a-b)$. Without loss of generality suppose that $p_2^* - p_1^* = c(1-a-b) > 0$, i.e., $p_1^* = p_2^* - c(1-a-b)$. Then, from Figure 3.3, it is clear that $p_1^* = p_2^* - c(1-a-b)$ can not be the best response price of firm. Therefore, at a price equilibrium, we must have $|p_2^* - p_1^*| < c(1-a-b)$.

If $|p_2^* - p_1^*| < c(1-a-b)$, then the peak of the parabola of Figure 3.3 must be the best response price. Therefore we must have:

$$p_1^* = (1/2)\, p_2^* + (1/2)\, c\,(1+a-b), \quad \text{and}$$

$$p_2^* = (1/2)\, p_1^* + (1/2)\, c\,(1+b-a)$$

and thus,

$$(3.1) \quad p_1^* = c\,\{1+(1/3)(a-b)\}, \quad \text{and}$$

$$(3.2) \quad p_2^* = c\,\{1+(1/3)(b-a)\}.$$

Note that $\pi^1(p_1^*, p_2^*, a, b) = (1/2c)(p_1^*)^2 = (1/2c)c^2 \ \{1+(1/3)(a-b)\}^2 = (c/2)\{1+(1/3)(a-b)\}^2$. On the other hand

$$\pi^1(p_1, p_2^*, a, b)\,|_{p_1 + p_2^* - c(l-a-b)} = l\{p_2^* - c(l-a-b)\} = (2/3)\,c\,l\,(a+2b).$$

Therefore $\pi^1(p_1^*, p_2^*, a, b) \geqq \pi^1(p_1, p_2^*, a, b)\,|_{p_1 = p_2^* - c(l-a-b)}$ if and only if $\{l+(1/3)(a-b)\}^2 \geqq (4/3)\,l\,(a+2b)$. Similarly we obtain that $\pi^2(p_1^*, p_2^*, a, b) \geqq \pi^2(p_1^*, p_2, a, b)\,|_{p_2 = p_1^* - c(l-a-b)}$ if and only if $\{l+(1/3)(b-a)\}^2 \geqq (4/3)\,l\,(2a+b)$. Therefore, we can state that (p_1^*, p_2^*) is a price equilibrium if and only if

(3.3) $\{l+(1/3)(a-b)\}^2 \geqq (4/3)\,l\,(a+2b)$, and

(3.4) $\{l+(1/3)(b-a)\}^2 \geqq (4/3)\,l\,(2a+b)$.

When choices of brand are symmetric in the sense that $a = b$, the above conditions reduce to $l \geqq 4a$ and $l \geqq 4b$ with $(p_1^*, p_2^*) = (cl, cl)$.

We nox examine the case $a = b$ in greater detail. Noting that the parabola becomes $(1/2c)\,p_1(cl + p_2 - p_1)$ independent of $a = b$, we can obtain

$$(1/2c)\,p_1(cl + p_2 - p_1)\,|_{p_1 = (1/2)(p_2 + cl)} - (p_1 l)\,|_{p_1 = p_2 - c(l-2a)}$$
$$= (1/8c)\,\{p_2^2 - 6cl\,p_2 + cl(9cl - 16ca)\}.$$

Define

$$F(p_2) = p_2^2 - 6cl\,p_2 + cl(9cl - 16ca).$$

Since $F(0) > 0$, F attains its minimum value at $p_2 = 3cl$ and $F(3cl) < 0$, there exists $p_2(a) \in (0, 3cl)$ such that $F(p_2(a)) = 0$. Therefore, if $p_2 \in [0, p_2(a)]$, the best response price of firm 1 is given by:

$$p_1 = (1/2)(p_2 + cl)$$

and, if $p_2 \in (p_2(a), 3cl]$, then the best response price of firm 1 is to slightly undercut $p_2 - c(l-2a)$ and we signify the response by writing

$$p_1 = p_2 - c(l-2a) - \varepsilon$$

for some small $\varepsilon > 0$. We also note that $F(cl) = 4c^2 l(l-4a)$ and thus $F(cl) \geqq 0$ if and only if $l \geqq 4a$. Therefore $p_2(a) \gtreqqless cl$ according to whether $l \gtreqqless 4a$. In Figure 3.6(a), we illustrate the existence of a unique price equilibrium for $l \geqq 4a$ which is at an intersection of firm 1's reaction function denoted by 11′ and firm 2's reaction function denoted by 22′. When $l < 4a$, the nonexistence of an equilibrium is illustrated in Figure 3.6(b).

If conditions (3.3) and (3.4) are satisfied, then the price equilibrium is given by (3.1) and (3.2) and profits at the price equilibrium are given by

$$\pi^1(p_1^*, p_2^*, a, b) = (c/2)\,\{l+(1/3)(a-b)\}^2, \quad \text{and}$$
$$\pi^2(p_1^*, p_2^*, a, b) = (c/2)\,\{l+(1/3)(b-a)\}^2.$$

Therefore, as argued by Hotelling, $\pi^1(p_1^*, p_2^*, a, b)$ increases as "a" increases and $\pi^2(p_1^*, p_2^*, a, b)$ increases as "b" increases. But increases in "a" and "b" will eventually lead the market into a region of the nonexistence of an equilibrium.

On the other hand, if $a = b$ and $a \leqq (1/4)l$, then a price equilibrium entails $p_i^* = cl$ and $\pi^i(p_1^*, p_2^*, a, b)\,|_{a=b} = (c/2)l^2$ which are independent of $a = b$ provided that $a \leqq (1/4)l$. Therefore if firms recognize their symmetry in their

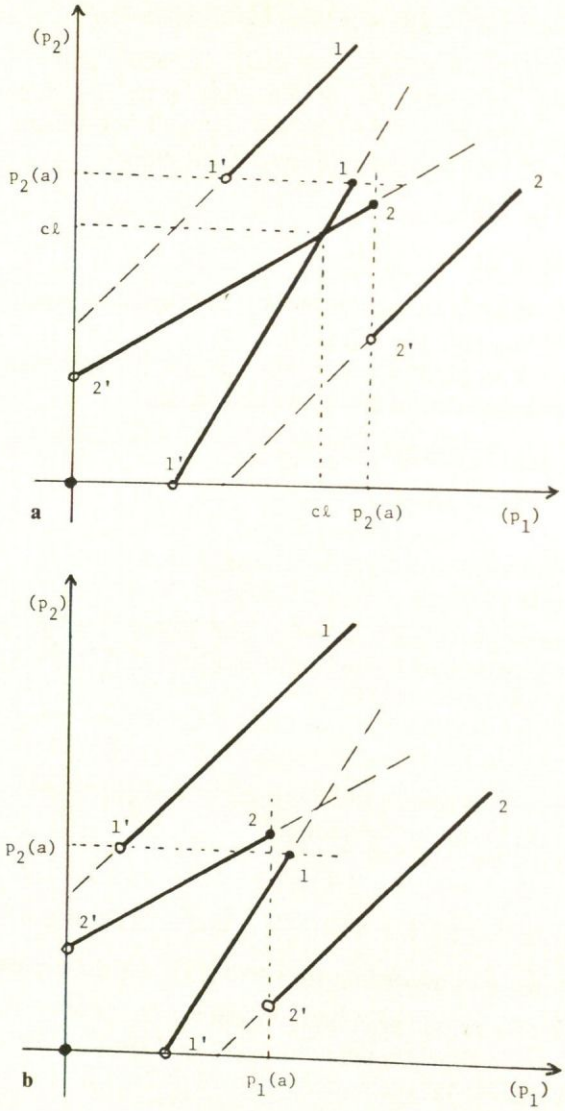

Fig. 3.6.a, b

choices of brand, then brand choices will be indeterminate for $a \in [0, (1/4)l]$ and there will be no tendency for firms toward the sameness of their brands contrary to Hotelling's claim. Note moreover that if $a = (1/2)l$, then an equilibrium exists with zero profit.

4. A Modified Hotelling Model of Product Differentiation and Monopolistic Competition

In this section, we will modify the original Hotelling model in two respects as in Salop (1979). First, we will introduce a reservation price due to the presence of an

outside numeraire good. This is a modification first introduced by Lerner and Singer (1937). Our next modification is to eliminate end points by considering the space of brand characteristics to be a circle with circumference $l > 0$. We could use $(-\infty, \infty)$ instead of a circle except for the difficulty that we need a countably infinite number of firms if they are evenly spaced. But then the distance between firms can be made arbitrarily small with a countably infinite number of firms and the entry of firms becomes meaningless. Throughout this section, we will focus on a symmetric equilibrium.

Let L denote the circle. If a consumer with type $r \in L$ considers buying a unit at price p from a firm with brand $a \in L$, then the utility of this consumer is assumed to be:

$$u(a, p, r) = \max\{0, v - c|r - a| - p\}$$

where $v > 0$ is the reservation price of the consumer to buy a unit of differentiated products.

We begin with a fixed number, say n, of firms, each of which produces a differentiated product with a given brand $a_i \in L$ $(i = 1, 2, \ldots, n)$. We assume that firm's brands are evenly spaced on L in the order of $a_1, a_2, \ldots, a_i, \ldots, a_n$ on L. Therefore $|a_{i+1} - a_i| = l/n$. We will first examine a symmetric price equilibrium when there are n firms with zero marginal costs and then we will examine a long-run symmetric equilibrium of monopolistic competition when firms have zero marginal cost and a positive fixed cost.

Suppose that all firms other than firm i are selling their products at price $\bar{p}$. We will derive firm i's demand curve for cases when (i) $v < \bar{p} + c(l/2n)$, (ii) $v = \bar{p} + c(l/2n)$, (iii) $\bar{p} + c/l/2n) < v < \bar{p} + c(l/n)$, and (iv) $\bar{p} + c(l/n) \leqq v$.

When $v < \bar{p} + c(l/2n)$, we illustrate a situation of firm i in Figure 4.1 where a portion of circle is stretched out on a line around $a_i \in L$. When

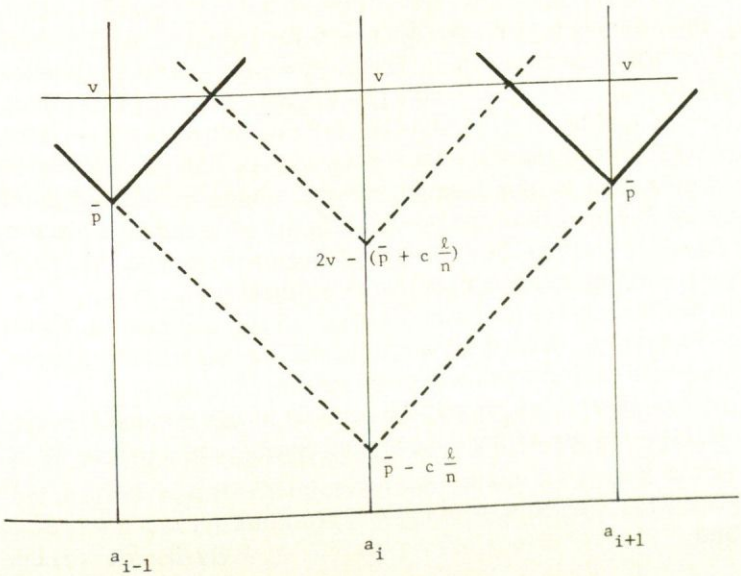

Fig. 4.1

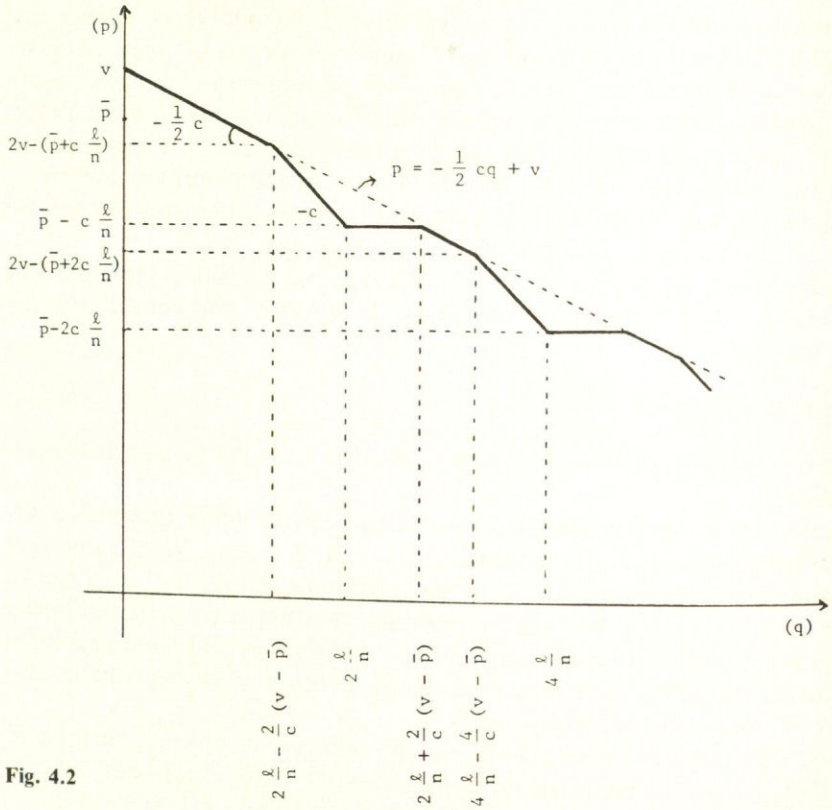

Fig. 4.2

$p \in [2v-(\bar{p}+c(l/n)), v]$, firm i obtains a market share $(1/c)(v-p)$ on each side of a_i. Hence we get $q = (2/c)(v-p)$ or $p = -(c/2)q + v$ with $p \in [2v-(\bar{p}+c(l/n)), v]$ or $q \in [0, 2(l/n) \quad -(2/c)(v-\bar{p})]$. On the other hand, if $p \in [\bar{p}-c(l/n), 2v-(\bar{p}+c(l/n))]$, then solving $p + cr = \bar{p} + c[(l/n) - r]$ for r yields a market share $r = (1/2c)\,[\bar{p} - p + c(l/n)]$ on each side of a_i. Therefore we get a demand function $q = (1/c) \quad [\bar{p} + c(l/n) - p]$ or $p = -cq + [\bar{p} + c(l/n)]$ for $p \in (\bar{p} - c(l/n), 2v - (\bar{p} + c(l/n)]$ or for $q \in [2(l/n) - (2/c)(v-\bar{p}), 2(l/n)]$. When $p = \bar{p} - c(l/n)$, then firm i can secure (l/n) and share $(1/c)(v-\bar{p})$ customers with its neighbor on each side of a_i. Computing in this fashion, we can obtain a demand curve illustrated in Figure 4.2. The region $p \in [2v-(\bar{p}+c(l/n)), v]$ is called the monopoly region, the region $[\bar{p}-c(l/n), 2v-(\bar{p}+c(l/n))]$ competitive and the region $[2v-(\bar{p}+2c(l/n)), \bar{p}-c(l/n)]$ supercompetitive by Salop (1979).

We can obtain demand curves for cases (ii), (iii) and (iv) as shown in Figure 4.3(a), (b) and (c) respectively. With these preparations, we can examine the possibility of price equilibria for a given n and given symmetric brands.

For case (i), $\bar{p} \in [2v-(\bar{p}+c(l/n)), v]$ and the demand in this monopoly region has the form $q = (2/c)(v-p)$. By setting the marginal revenue equal to zero we get

$$p^* = (1/2)v,$$

$$q^* = v/c, \quad \text{and}$$

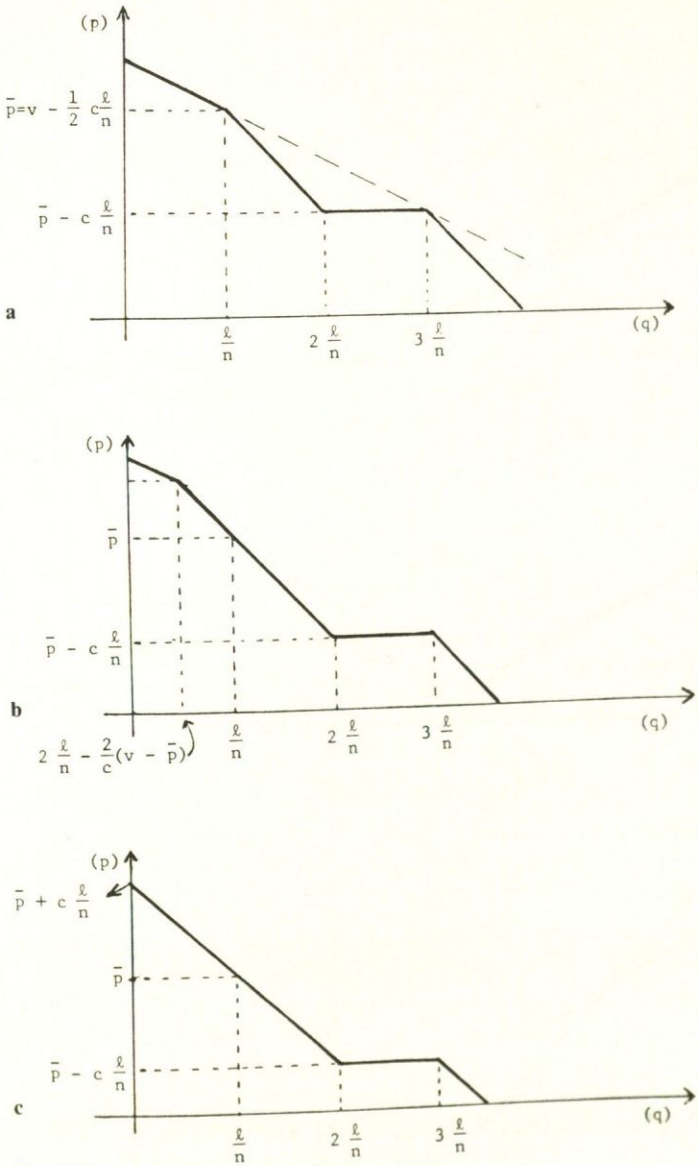

Fig. 4.3.a – c

$$\pi^* = p^* q^* = \frac{1}{2}\frac{v^2}{c}.$$

Since $v < [\bar{p} + c(l/2n)]\,|_{\bar{p}=p^*}$ must hold for case (i), we obtain a condition $n < A$ when $A = (cl/v)$. Also note that $2v - (p^* + c(l/n) > 0$ if and only if $n > (\frac{2}{3})A$ and that $p^* - c(l/n) < 0$. Therefore an equilibrium for case (i) can be divided into two cases (ia) $n < (\frac{2}{3})A$ and (ib) $(\frac{2}{3})A \leqq n < A$ which are illustrated in Figure 4.4(a) and (b) respectively.

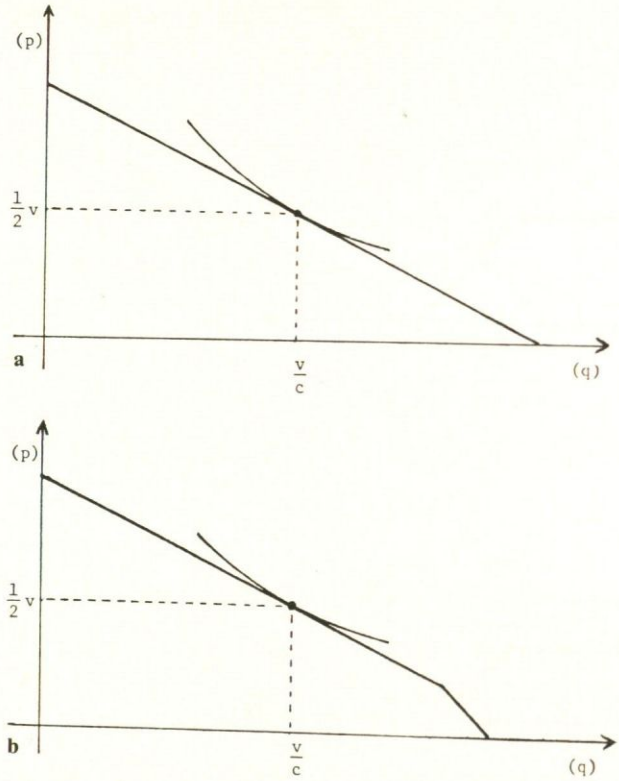

Fig. 4.4.a, b

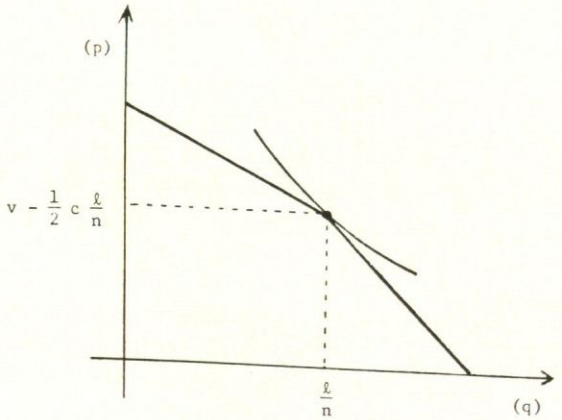

Fig. 4.5

If $A \leqq n \leqq \frac{3}{2}A$, then it is easy to check that an equilibrium must occur at a corner $p_n^* = v - \frac{1}{2}C(l/n)$ shown in Figure 4.5. In this case, we have

$$q_n^* = l/n \quad \text{and} \quad \pi_n^* = (l/n)\,[v - \tfrac{1}{2}c\,(l/n)]\,.$$

When $n > \frac{3}{2}A$, then an equilibrium occurs at $p_n^* = (cl/n)$, $q_n^* = (l/n)$ and $\pi_n^* = (cl^2/n^2)$. The demand curves will have slightly different shapes depending on whether or not $n > 2A$. See illustrations in Figure 4.6(a) and (b).

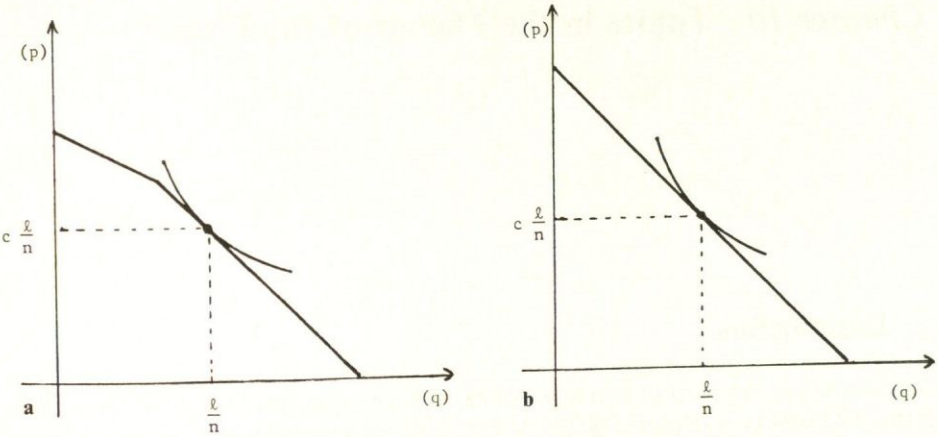

Fig. 4.6.a, b

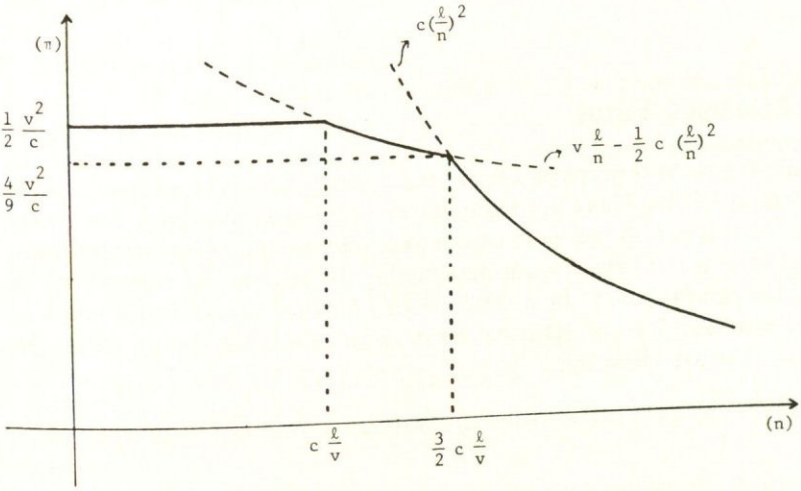

Fig. 4.7

We can now draw an equilibrium profit level as a function of n as shown in Figure 4.7. From this, we can obtain a long-run equilibrium of monopolistic competition when there exists a fixed cost F. If $F = \frac{1}{2}(v^2/c)$, then we will have a monopoly equilibrium where each firm has a share of monopoly market around its brand. When $\frac{4}{9}(v^2/c) \leqq F < \frac{1}{2}(v^2/c)$, then a long-run equilibrium will be achieved at a kink of each demand curve and, when $0 < F < \frac{4}{9}(v^2/c)$, then a long-run equilibrium will be achieved in Salop's competitive region. Of course when $F = 0$, infinitely many firms will enter and an equilibrium price will be achieved at zero marginal cost.

Chapter 10 Topics in the Theory of the Firm *

1. Introduction

In this chapter we present two types of extensions of the traditional model of the firm. The first type explores different modes of management and the second type looks at the dynamics of the firm's behavior.

First we look at labor managed firms. We then look at a regulated monopoly and a capitalist managed firm. Finally we study the dynamics of monopoly and duopoly.

2. Labor Managed Firms

One departure from the concept of a profit maximizing firm is to consider a firm managed by its workers. There are many interesting issues that arise here, e.g., the issue of incentives and the issue of controls. We assume that matters have been decided so that the firm aims at maximizing the returns per member of its labor force. Let the output, y, be a function of nonlabor inputs x and labor L. Let P be the output price and let q be the price of non-labor inputs. Then the returns per worker are given by:

$$(2.1) \quad R = \frac{P f(x, L) - qx}{L}.$$

If we maximize R, at an interior point and assume that f is smooth, then

$$(2.2)(a) \quad P f_x = q$$

and

$$(2.2)(b) \quad P f_L = \frac{P f - qx}{L}.$$

So that nonlabor inputs are paid the value of their marginal products and the value of the marginal product of labor is equated to the "surplus" per worker after other factors have been paid off. Substituting from (2.2)(a) into (2.2)(b), we get

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* Essentially all of the matrices exposited in this chapter appear here for the first time except for Section 3 which is based on El Hodiri and Takayama (1973). For the original formulation of the model of labor-managed firms, see Vanek (1970).

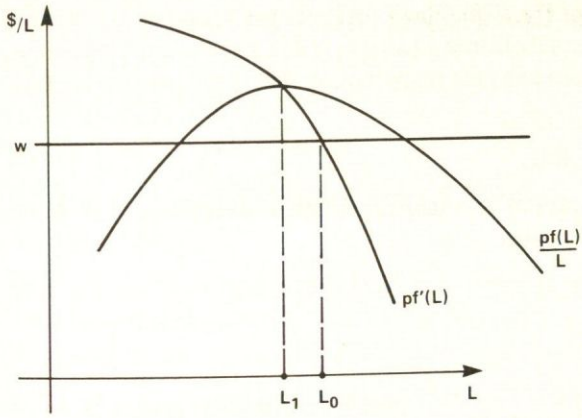

Fig. 10.1

$$(P f_L) = P f - (P f_x) x\,, \qquad \text{i.e.,}$$

(2.3) $\quad f = f_x x + f_L L\,.$

Thus output is always exhausted (!).

One of the commonly made assertions is that a labor managed firm would underutilize labor so that labor managed firms are employment restricting. Let us examine this assertion. Consider two firms; a profit maximizing firm and a labor managed firm. Let the input-output vector at maximum profits for the profit maximizing firm be y^0, x^0, L^0 and let those of the labor managed firm by y^1, x^1, L^1. Assume that f is concave. Then

$$P f^0 - P f^1 \leqq P f^1_x \cdot (x^0 - x^1) + P \overset{f}{\cancel{F}}{}^1_L (L^0 - L^1)$$

and

$$P f^1 - P f^0 \leqq P f^0_x \cdot (x^1 - x^0) + P f^0_L (L^1 - L^0)\,.$$

Adding the two inequalities and using (2.2) in addition to first order conditions, we have:

$$0 \leqq (P f^1_L - w)(L^0 - L^1)\,.$$

Thus we have employment restrictiveness if $P f^1_1 \geqq w$. But if $P f^1_L < w$ then, by (2.2), we have: $P f^1 < q x^1 + w L^1$ which says that the labor managed firm would make a net loss if it paid this members the going market wage. Hence it is reasonable to expect the labor managed firm to restrict employment. The ~~following~~ above diagram illustrates the contrast between a profit maximizing firm and a labor managed firm for the case of a single input.

3. Regulated Firms

Consider a price making firm which is subject to a regulatory rate of return constraint. We will show that the Averch-Johnson effect holds.

Let there be only two factors of production, labor and capital. Let y be the firms output and let the production function be given by: $y = f(K, L)$. Let $h(y)$ be

the average revenue function for the firm. Then total revenue is given by: $h(y)y$, which we write as: $g(K, L)$ after substituting for $y = f(K, L)$. Let r and w be the parametric rental of capital and wage rate respectively. Then the profit function $\pi(K, L)$ is given by:

$$(3.1)\quad \pi(K, L) = g(K, L) - rK - wL\,.$$

Suppose the rate of return on capital is constrained not to exceed, say, ϱ. Then the regulatory constraint has the form

$$(g(K, L) - wL)/K \leqq \varrho$$

i.e.,

$$(3.2)\quad \varrho K - g(K, L) - wL \geqq 0\,.$$

In order to study the effect of regulation we study two situations. The first is where at the profit maximizing K and L, (3.2) is ineffective, i.e., it holds as a strict inequality. Let the labor-capital pair in this case be denoted by $\mathring{L}$, $\mathring{K}$. The second case is where (3.2) is effective, i.e., it holds as an equation. Let the labor capital pair in this case be denoted by L^*, K^*.

Assume the g is concave, that $\mathring{L}$, $\mathring{K}$, L^*, K^* are all positive and that $\pi(0, 0) = 0$. In the first case, profit maximization is characterized by

$$(3.3)\quad \mathring{g}_L = w,\ \mathring{g}_K = r$$

where $\mathring{g}_L = \left.\dfrac{\partial g}{\partial L}\right|_{\mathring{L},\mathring{K}}$ and $\mathring{g}_K = \left.\dfrac{\partial g}{\partial K}\right|_{\mathring{L},\mathring{K}}$. In the second case we have

$$(3.4)\quad (1-\lambda)\, g_L^* = (1-\lambda)\, w$$

$$(3.5)\quad (1-\lambda)\, g_K^* = r - \lambda\varrho\,,$$

where λ is the Lagrange multiplier corresponding to (3.2), $\lambda \geqq 0$ and $q_L^* = \partial g/\partial L\,|_{L^*,K^*}$, $g_K^* = (\partial g/\partial K)\,|_{L^*,K^*}$. We now show that $\lambda < 1$. First note that both π^o and π^* are positive, where "o" denotes evaluation at $(\mathring{L}, \mathring{K})$ and "*" denotes evaluation at (L^*, K^*). Now $\pi^* + rK^* = \varrho K^*$ by the regulatory constraint. So $0 < \pi^* = (\varrho - r)\,K^*$. Since $K^* > 0$ we have $\varrho > r$. But if $\lambda = 1$, then by (3.5), $r = \varrho$ which is a contradiction. Thus by (3.4) we have:

$$(3.6)\quad g_L^* = w\,.$$

If $\lambda = 0$, then $\mathring{K} = K^*$ and $\mathring{L} = L^*$. If $\lambda > 0$ we show $K^* \geqq \mathring{K}$, i.e., that the firm overcapitalizes. By concavity of g, (3.2) and (3.6) we have:

$$(3.7)\quad \mathring{g} - B^* \leqq \mathring{g}_L^*(\mathring{L} - L^*) + \mathring{g}_K^*(\mathring{K} - K^*) = w(\mathring{L} - L^*) + \mathring{g}_K^*(\mathring{K}, K^*)\,.$$

We also have

$$(3.8)\quad g^* - \mathring{g} \leqq \mathring{g}_L(L^* - \mathring{L}) + \mathring{g}_K(K^* - K^0) = w(L^* - \mathring{L}) + r(K^* - \mathring{K})\,.$$

Adding (3.7) and (3.8), we get:

$$(3.9)\quad 0 \leqq (\mathring{g}_K^* - r)(\mathring{K} - K^*)\,.$$

Thus $K^* \geqq \mathring{K}$ if $\mathring{g}_K^* \leqq r$, which we now show.

Since $G(0, 0) = 0$, in view of the assumption that $\pi(0, 0) = 0$, it follows from the concavity of G and (3.6) that:

$$G^* \geqq G_L^* L^* + G_K^* K^* = wL^* + G_K^* K^* .$$

Hence

$$G^* - wL^* \geqq G_K^* K^* \quad \text{which, by 3.2., implies}$$

$$\varrho K^* \geqq G_K^* K^* .$$

Multiplying by λ and using (3.5) we have $0 \geqq \lambda(G_K^* - \varrho)K^* = G_K^* - r$, which is what we wanted to show. Thus $K^* \geqq \mathring{K}$.

4. A Capital Managed Firm

Let a firm be managed with the objective of maximizing returns on capital. If the firm is in a competitive industry, then its objective will be to maximize:

(4.1) $(P f(x, K) - qx)/K$,

where x is a vector of non-capital inputs, K is capital, f is the firm's production function, p is the price of output and q is the input price vector.

Assume that the firm's objective function has an interior maximum, then

(4.2) $P f_x = q$

(4.3) $P f_K = (P f - qx)/K$, i.e.,

(4.4) $P f = P f_K K + P f_x x$.

We can show that a capitalist firm under capitalizes (!) in exactly the same way that we proved that a labor managed firm is employment restricting.

Graphically, the case of a single input is illustrated below. In Figure 10.2 a profit maximizing firm will employ K_2 units while a "capitalist" firm will employ K_1 units. The curve VMP_K is the value marginal product of capital and the curve VAP_K is the value average product of capital. A profit maximizing firm will operate at the point where $r = VMP_K$ where r is the rate of rental for capital services.

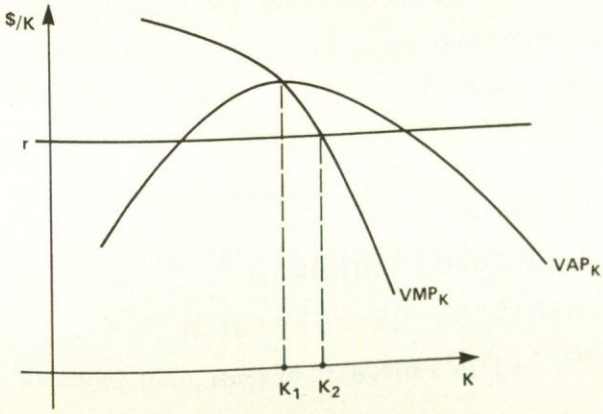

Fig. 10.2

5. Dynamics of Monopoly

Consider a firm which is a monopolist in the product market and which faces a perfectly competitive factor market. Let the time period under consideration start at time t_0 and end at time T. We assume that t_0 is fixed at $t_0 = 0$ and that t_1 is to be chosen optimally. Suppose that firm buys and keeps capital and that it can hire all it wants of another factor which we call labor.

The firm's output at time t, y(t), is given by:

$$(5.1) \quad y(t) = f(t, K(t), L(t)) .$$

Suppose the price that the firm can charge, its indirect demand function is given by:

$$(5.2) \quad p(t) = h(t, y(t)) .$$

Let q(t) be the price of capital goods at time t and let w(t) be the price of labor at time t. The firm's discounted net cash flow is given by

$$(5.3) \quad \int_0^{t_1} [P(t)\,y(t) - w(t)\,L(t) - q(t)\,I(t)]\,e^{-rt}\,dt ,$$

where I is the rate of gross addition to capital and ~~were~~ r is an appropriate rate of discount. The net addition to capital is given by:

$$(5.4) \quad \dot{K}(t) = I(t) - \alpha K(t), \qquad K(0) = K_0 ,$$

where α is the rate of depreciation. The firm maximizes the net discounted cash flow plus the present value of the scrap value of its capital at the end, i.e., it maximizes

$$(5.5) \quad V = q_{t_1} K_{t_1} e^{-rt_1} + \int_0^{t_1} (py - wL - qI)\,e^{-rt}\,dt.$$

Substituting from (5.1) – (5.4) in (5.5), the firm maximizes

$$(5.6) \quad q_{t_1} K_{t_1} e^{-rt_1} + \int_0^{t_1} \{h(f(K, L))\,f(K, L) - wL - q(\dot{K} + \alpha K)\}\,e^{-rt} .$$

Assume that h and f are continuously differentiable and that h f is concave in K and L. Then the first order conditions, Euler equations and transversality conditions, are necessary and sufficient for a global maximum. The Euler equations are given by

$$(5.7)(a) \quad \frac{d}{dt} q e^{-rt} = -[(h' + h) f_K - \alpha q]\,e^{-rt}$$

$$(5.7)(b) \quad (h' + h) f_L = w .$$

The transversality conditions are given by

$$(5.8) \quad P(t_1)\,y(t_1) - w(t_1)\,L(t_1) - q(t_1)\,\alpha k(t_1) = r q(t_1) k(t_1) .$$

Solving (5.7)(a) we have

$$(5.9) \quad q(t) e^{-rt} = q(t_1) e^{-rt_1 - \alpha(t_1 - t)} + \int_t^{t_1} (h' + h)\,f_K e^{-rs - \alpha(t - s)}\,ds.$$

Equation (5.7) states that labor gets paid its marginal value product. Equation (5.9) states that the price of new capital equals the imputed returns from the time the equipment comes on line to the end, with due consideration for depreciation, scrap value and discounting. Equation (5.8) states the intuitively appealing result, that a firm cashes itself out when it makes the same amount of money operating and maintaining equipment as it would by selling the equipment and putting the money in the bank.

6. A Case of Duopoly

In what is, perhaps, the earliest paper on differential games, C. F. Roos (1927), discusses the problem of oligopoly over time. Here, we got through his example for the case of duopoly where the cost is quadratic and the demand is linear.

Let there be two firms and let $y_i(t)$ be the output of firm i at time t. The price adjustment mechanism is given implicitly by the linear differential equation

$$(6.1)\quad y_1(t)+y_2(t)=ap(t)+b+h\dot{p}(t)\,,$$

where $p(t)$ is the market price at time t and where $\dot{p}(t)=dp/dt$. Let the cost functions, $C^i(y_i)$ be given by

$$(6.2)\quad C^i(y_i(t))=Ay_i^2(t)+By_i(t)+C\,.$$

Thus we are assuming the same cost structure for both firms. Finally let π_i be the present value of the net cash flow of firm i at a rate of discount r. Then we have

$$(6.3)\quad \pi_i=\int_0^{t_1}[Py_i-Ay_i^2-By_i-C]e^{at}\,dt\,,$$

where t_1 is the end planning period.

A Cournot solution of the above game involves deriving a reaction operator for each firm and then solving. The first firm solves the following problem of Lagrange in the calculus of variations:

$$(6.4)\quad \text{maximize}\int_0^{t_1}[Py_1-Ay_1^2-By_1-C]e^{-rt}\,dt$$

subject to

$$(6.5)\quad y_1+\bar{y}_2=ap+b+h\dot{p}\,.$$

Instead of using multipliers, we substitute for y_1, from (6.5). The firm's problem now is to maximize

$$\int_0^{t_1}[(P-B)(ap+b+h\dot{p}-\bar{y}_2)-A(ap+b+h\dot{p}-\bar{y}_2)^2-C]e^{-rt}\,dt\,.$$

The Euler equation for this last problem is

$$(6.6)\quad \frac{d}{dt}\{h(p-B)-2Ah(ap+b+h\dot{p}-\bar{y}_2)e^{-rt}\}$$
$$=e^{-rt}[(2ap+b+h\dot{p}-\bar{y}_2)-aB-2aA(ap+b+h\dot{p}-\bar{y}_2)]\,.$$

Rewriting (6.6), using (6.5), we have

$$(6.7)\quad \frac{d}{dt}\{[p-2Ay_1-\overset{B}{\cancel{V}}]e^{-rt}\} = \frac{a}{h}(p-2Ay_1-B)e^{-rt}+\frac{1}{h}y_1e^{-rt}.$$

Let $\overset{z}{\cancel{Z}}(t) = (p-2Ay_1-B)e^{-rt}$. Then (6.7) may be written as

$$(6.8)\quad \dot{z} = \frac{a}{h}z+\frac{1}{h}y_1e^{-rt}.$$

Solving (6.8) as a linear differential equation in z we get

$$(6.9)\quad z = e^{(at)/h}\left[z_* + \frac{1}{h}\int_0^t y_1 e^{-(r+a/h)s}\,ds\right].$$

But $z(t_1) = 0$ by the transversality condition, so

$$0 = z_* + \frac{1}{h}\int_0^{t_1} y_1 e^{-(r+a/h)s}\,ds.$$

Substituting in (6.9) we get

$$(6.10)\quad z(t) = -\frac{1}{h}e^{(at)/h}\int_t^{t_1} y_1 e^{-(r+a/h)s}\,ds.$$

Thus we have

$$(P-2Ay_1-B)e^{-rt}+\frac{1}{h}e^{(a/h)h}\int_t^{t_1} y_1 e^{-(r+a/h)s}\,ds = 0,$$

i.e.

$$(6.11)\quad P-2Ay_1-B+\frac{1}{h}\int_t^{t_1} y_1 e^{(r+a/h)(t-s)}\,ds = 0.$$

Similarly

$$(6.12)\quad P-2Ay_2-B+\frac{1}{h}\int_t^{t_1} y_2 e^{(r+a/h)(t-s)}\,ds = 0.$$

Adding (6.11) and (6.12) and using (6.1) we get:

$$(6.13)\quad 2P-2A(ap+b+h\dot{p})-2B+\frac{1}{h}\int_t^{t_1}(ap+b+h\dot{p})e^{(r+a/h)(t-s)}\,ds = 0.$$

Equation (6.13) is an equation in price alonge and can be solved for p. Roos (1927) gives the general expression for such solutions in the form

$$(6.14)\quad p = p_0 + C_1e^{m_1t}+C_2e^{m_2t}$$

where the expressions for C_1, C_2, m_1 and m_2 are rather complicated and won't be reproduced here.

Both Roos (1927) and Evans (1930) conjecture that the number of firms increase, the price path approaches the competitive price path. As of this writing, no complete proof of the conjecture exists.

Problems for Part III

1. Consider a monopolistic firm which produces a commodity "x" using labor as an input. Assume that (i) The production function of the firm is given by $f(L) = L^2$ where L is the input of labor. (ii) The firm faces an inverse demand function $p = b - ax$ where a, b are positive constants, p is the price of the commodity and x is the quantity demanded. (iii) The labor market is competitive and the hourly wage rate is given by $w > 0$.

   Questions:

   1. Find the revenue function $R(x)$ and the cost function $C(x)$ of the monopolists as functions of x.
   2. If the monopolist is to maximize its profit $\pi(x) = R(x) - C(x)$, then illustrate how the firm determines the level of output and the price of the commodity.
   3. Show how the level of output and the price charged would change if b increases.

2. Consider the following case involving tow producers, A and B:
   a. Producer A has the production function of the form $x = g(L) = 2\,L^{1/2}$ where x is the output and L is the input of labor. p_x denotes the price of x, and we the hourly wage rate of labor.
   b. Producer B is the sole purchaser of the product of producer A and produces the output y. The production function is denoted by $y = f(x)$. (For simplicity we assume that x is the only one input.) The price of y is denoted by p_y.
   c. The market for y and L are assumed to be competitive.
   d. Producer B is the monopsonist in the market for x.

   Questions:

   1. Given w and p_y, show how x, y, and p_x will be determined in the above situation.
   2. Show how x, y, and p_x will change if each one of w and p_y increases.
   3. Compare the solution of (1.) and the case where producer B is competitive in the market of x.

3. Consider a firm which produces one product "y" using two inputs "x_1" and "x_2". The firm faces competitive markets for inputs and a monopolistic market for the product.
   (i) The revenue function is denoted by $h(y, \alpha)$ where α is a shift parameter.
   (ii) The production function is denoted by $f(x_1, x_2)$.
   (iii) $y \geqq 0$, $x_i \geqq 0$ $(i = 1, 2)$ $\alpha \in R$.
   (iv) w_i $(i = 1, 2)$ is a unit price of x_i $(i = 1, 2)$.
   (v) h and f have continuous 2nd partial derivatives for $y > 0$, $\alpha \in R$ and $(x_1, x_2) > 0$.
   (vi) The Hessian of f is negative definite for $(x_1, x_2) > 0$ and $\partial f/\partial x_i > 0$ $(i = 1, 2)$ for $(x_1, x_2) > 0$.
   (vii) $\partial^2 R/\partial y \partial_\alpha > 0$ for $y > 0$, $\alpha \in R$.

Assume that the profit maximization problem you set up has a solution $(y, x_1, x_2) > 0$ and the 2nd order sufficient condition is satisfied at the solution.

a) *Find* the effect of α and y.
b) *Show* that either $\partial\tilde{x}_1/\partial\alpha > 0$ or $\partial\tilde{x}_2/\partial\alpha > 0$ where $\tilde{x}_i(\alpha)$ indicates an optimizing input function $(i = 1, 2)$.

Hint:

(i) Note that

$$D \equiv \det \begin{pmatrix} 0 & f_1, & f_2 \\ f_1, & f_{11}, & f_{12} \\ f_2, & f_{21}, & f_{22} \end{pmatrix} = -(f_2, -f_1) \begin{pmatrix} f_{11}, & f_{12} \\ f_{21}, & f_{22} \end{pmatrix} \begin{pmatrix} f_2 \\ -f_1 \end{pmatrix}.$$

(ii) For (b), check $f_2 \dfrac{\partial\tilde{x}_1}{\partial\alpha} + f_1 \dfrac{\partial\tilde{x}_2}{\partial\alpha}$.

(iii) Probably the easiest is to apply the comparative static analysis to four equations with variables λ, y_1, x_1, x_2 where λ is a Lagrange multiplier.

4. Consider a discriminating monopolist facing two isolated markets with linear inverse demand function $P_i = \alpha_i - \beta_i q_i$ $(i = 1, 2)$ and a total cost function $C(Q)$ where $Q = q_1 + q_2$.

(i) Find a condition for an optimum pricing policy of this monopolist. Interpret your result.

(ii) Compare results in (i) with the case of a monopolist who sets a uniform price $P_1 = P_2$ in two markets. Particularly examine what happens to the total quantity produced.

You may disregard the possibility of shuttingdown to answer this question.

5. Consider a discriminating monopolist as in the previous problem. Assume that $P_1 = \alpha - q_1$ and $P_2 = \frac{1}{3}\beta$. (The monopolist faces a competitive market in market 2.) The average cost function is given by:

$$AC(Q) = -Q + \frac{17}{12}\alpha \quad \text{for } Q\ [0, \alpha).$$

$$Q - \frac{7}{12}\alpha \quad \text{for } Q\ [\alpha, \infty).$$

Find an optimum strategy of this monopolist.

6. Consider a Cournot duopoly model with an inverse demand function $p = b - aQ$, $b > 0$, $a > 0$ where $Q = q_1 + q_2$ and with constant marginal costs $c_i > 0$ $(i = 1, 2)$.

(a) Show that the reaction function of firm i is given by:

$$\phi_i(q_j) = \max\left\{0, -\frac{1}{2}q_j + \frac{1}{2}\frac{b - c_i}{a}\right\}.$$

(b) Find a Cournot duopoly equilibrium when $b \geqq \max\{2c_1 - c_2, 2c_2 - c_1\}$.

(c) Find a Cournot duopoly equilibrium where either $b < 2c_2 - c_1$ or $b < 2c_1 - c_2$.

7. Suppose that there are two firms producing a homogeneous product. The inverse demand function for the product is given by $p = 200 - 2(q_1 + q_2)$ where p is the price of the product and $q_i (i = 1, 2)$ is the quantity of the product produced by firm i. Cost functions of firm 1 and firm 2 are given respectively by

$$c_1(q_1) = 120\,q_1 - (3/2)\,q_1^2 \quad \text{for} \quad 0 \leqq q_1 \leqq 40, \quad \text{and}$$

$$c_2(q_2) = 100\,q_2 - q_2^2 \quad \text{for} \quad 0 \leqq q_2 \leqq 50.$$

Questions:

a. Obtain the Cournot duopoly solution of the above market.

b. Examine the stability property of the Cournot equilibrium under the following adjustment process:

$$dq_1/dt = \phi_1(q_2) - q_1$$

and

$$dq_2/dt = \phi_2(q_1) - q_2$$

where $\phi_1(q_2)$, $\phi_2(q_1)$ are reaction functions of firm 1 and firm 2 respectively.

8. Consider a market where there are two firms producing a homogeneous product. The inverse demand function is given by $p = 100 - 2(q_1 + Q_2)$ where p is the price of the product and Q_i is the quantity sold by firm i (i = 1, 2). Cost functions of firm 1 and firm 2 are given respectively by: $C_1(Q_1) = 60\,Q_1 - (3/2)\,Q_1^2$ for $0 \leqq Q_1 \leqq 20$ and $C_2(Q_2) = 50\,Q_2 - Q_2^2$ for $0 \leqq Q_2 \leqq 25$.

(a) Find the Cournot equilibrium of the above market.

(b) Examine the stability property of the above market for an appropriate adjustment process.

9. Suppose that there are two firms producing a homogeneous product. The inverse demand function for the product is given by: $p = B - q$ where p and q are the price and the total quantity of the product respectively and B is a positive constant. The cost function of firm i (i = 1, 2) is given by: $c_i(q_i) = a_i\,q_i + q_i^2$ where q_i is the quantity produced by firm i and a_i is a constant.

Questions:

(a) Obtain the Cournot duopoly solution of the above market.

(b) Illustrate the stability of the Cournot solution.

10. Suppose that:

(a) There are two firms producing a homogeneous product.

(b) The inverse demand function is given by $p = -aQ + b$, $Q = q_1 + q_2$, $a > 0$, $b > 0$.

(c) Cost functions of firms 1 and 2 are given by $C_1 = C^1(q_1, \tau)$, $C_2 = C^2(q_2)$ respectively where τ is a parameter with the property

$$\frac{\partial C^1}{\partial \tau} < 0 \quad \text{and} \quad \frac{\partial^2 C^1}{\partial \tau \partial Q} = \frac{\partial MC^1}{\partial \tau} < 0 .$$

(d) Firm 1 maximizes the profit function $\pi^1(q_1, q_2, a, b, \tau)$ with respect to q_1 having q_2, a, b, τ as parameters. Firm 2 maximizes the profit function $\pi^2(q_2: q_1, a, b, \tau)$ with respect to Q_2 having q_1, a, b, τ as parameters. Let $q_1 = \phi^1(q_2: a, b, \tau)$ and $q_2 = \phi^2(q_1: a, b, \tau)$ be solutions of the above maximization by firms 1 and 2 respectively. These functions are called reaction functions.

(i) Obtain properties of reaction functions.

(ii) Consider the following adjustment process of each firm

$$\frac{dq_j}{dt} = \phi^i(q_j: a, b, \tau) - q_i) \qquad (i, j = 1, 2, i \neq j).$$

Find a sufficient condition for the local stability of a Cournot duopoly equilibrium.

(iii) Analyze how an equilibrium changes when a, b, or τ changes and also interpret your results.

11. Consider a Cournot duopoly model with an inverse demand function $p = b - aQ$, $b > 0$, $a > 0$, $Q = q_1 + q_2$ and with an identical marginal cost function $mc(q_i) = cq_i + d$ where $0 < d < b$, $bc < ad$ and $2a - c > 0$ will be assumed.

Obtain Cournot duopoly equilibria for the case $a > c$ and for the case $a < c$ and examine stability properties of equilibria.

12. Consider a Cournot duopoly with an inverse demand function $p = b - aQ$, $b > 0$, $a > 0$, $Q = q_1 + q_2$ and the total cost function $C_i(q_i) = cq_i + K_i$ $(i = 1, 2)$ where $K_i > 0$ is incurred only if firm i enters this market.

Examine properties of various possible equilibria depending upon K_i $(i = 1, 2)$.

13. Consider the case of a Cournot duopoly with an inverse demand function $p = \alpha - \beta Q$, $Q = q_1 + q_2$ and a cost function $C(q_i) = \gamma q_i$, $\gamma > 0$. Assume that each duopolist employs a discriminating price strategy with two prices. Find a Cournot equilibrium and compare it with the usual Cournot equilibrium and the competitive outcome. Work out two cases of discriminating pricing strategies: (a) Simultaneous discriminating strategies and (b) sequential discriminating strategies, as in the section on the discriminating monopoly in a single market.

14. Consider the case of Bertrand-Edgeworth duopoly model with capacity constraints. Suppose that capacity limits are given by $K_i > 0$ $(i = 1, 2)$ which are not necessarily symmetric. Since $K_1 \geqq Q_c$ and $K_2 \geqq Q_c$ result in the usual Bertrand outcome, assume that either $K_1 < Q_c$ or $K_2 < Q_c$.

Examine the existence and the nonexistence of pure strategy equilibria. In particular, show that a pure strategy equilibrium exists if and only if $K_1 + 2K_2 \leqq Q_c$ and $2K_1 + K_2 \leqq Q_c$.

15. Consider an oligopoly model with differentiated products. Assume that an inverse demand function for the product i produced by firm i is given by: $p_i = b - a_1 q_i - a_2 Q_i$ $(i = 1, 2, \ldots, n)$ where $Q_i = \sum_{j \neq i} q_j$, $b > 0$, $a_1 > a_2 > 0$.

The cost functions are symmetrical with $C(q_i) = cq_i$, $c > 0$.

(a) Find a quantity strategy equilibrium and a price strategy equilibrium. Compare these two equilibria and examine what will happen if n, the number of firms, increases.

(b) Suppose now that $C(q_i) = cq_i + F$ with $F > 0$. Examine longrun equilibria under quantity strategy and under price strategy.

16. Consider an oligopoly model where each firm possesses a perceived reaction function over other firms. Suppose that firm i's profit function is given by $\tilde{\pi}^i(q_i, Q_i) = q_i P(q_i + Q_i) - C(q_i)$ $(i = 1, 2, \ldots, n)$ where $Q_i = \sum_{j \neq i} q_j$, $P(\cdot)$ is an inverse demand function and $C(\cdot)$ is the identical total cost function and suppose further that firm i's perceived reaction function over other firms is given by $Q_i = \psi^i(q_i, \theta_i) = \delta q_i + \theta_i$ so that the conjectural variation is constant and identical among firms.

Note that the first-order condition of the profit maximization by firm j can be written as:

superscript

$$
\begin{aligned}
0 &= \frac{\partial \tilde{\pi}^j}{\partial q_j} + \delta \frac{\partial \pi^j}{\partial Q_j} = p(q_j + Q_j) + (1+\delta) q_j P'(q_j + Q_j) - C'(q_j) \\
&= P(q_i + Q_i) + (1+\delta) q_j P'(q_i + Q_i) - C'(q_j) \\
&= P(q_j + q_i + Q_{ji}) + (1+\delta) q_j P'(q_j + q_i + Q_{ji}) - C'(q_j)
\end{aligned}
$$

where $Q_i = Q - q_i$, $\quad Q_{ji} = Q - q_j - q_i \quad$ and $\quad Q = \sum_{i=1}^{n} q_i$.

(a) If we impose a symmetry condition $q = 1/(n-1) Q_i$, then firm j's first-order condition becomes

$$P(q_i + Q_i) + \frac{1+\delta}{n-1} Q_i P'(q_i + Q_i) - C'\left(\frac{1}{n-1} Q_i\right) = 0.$$

This defines $Q_i = F(q_i)$ which can be interpreted as a true reaction function of all firms other than firm i. Find $\delta^1(Q) = F'(q_i)\,|_{q_i = (1/n)Q}$.

(b) Alternatively, we can consider

$$P(q_j + q_i + Q_{ji}) + (1+\delta) q_j P'(q_j + q_i + Q_{ji}) - C'(q_j) = 0.$$

This defines a function $q_j = H(q_i, Q_{ji})$ which can be interpreted as a true reaction function of firm j given q_i and Q_{ji}.

Find

$$\delta^2(Q) = (n-1)\left.\frac{\partial H}{\partial q_i}\right|_{\substack{q_i = (1/n)Q \\ Q_{ji} = (n-2/n)Q}} .$$

(c) The conjectural variation δ can be said to be consistent if either $\delta^1(Q) = \delta$ or $\delta^2(Q) = \delta$ alternatively. Examine the consistency when $C''(q_i) = 0$ and/or $n = 2$. What can be said when $P(Q) = b + (a/Q)$ Investigate consistency conditions when $C''(q_i) \neq 0$.

Part IV The Economy

Introduction

In this part we provide a brief introduction to those economic considerations that involve all branches of the economy, in contrast to what we did in the earlier part where we were concerned only with individual agents or industries.

In Chapter 11, we study production possibility sets, activity analysis and equilibria of production sectors. In Chapter 12 we present a brief introduction to welfare economics and general equilibrium analysis.

Chapter 11 Production Possibility Set, Activity Analysis and Equilibria of Production Sectors

1. Introduction

A production possibility set and the surface of a production possibility set called a product transformation curve (or surface) is a very popular concept employed often in undergraduate textbooks and in applied economics. It provides us with a summary of production sectors incorporating the initial endowment of resources and technological constraints. The usefulness of the concept of a production possibility set originates in two basic assumptions or restrictions on an economy: (a) that commodities can be classified into producible commodities and nonproducible commodities called primary factors of production, and (b) that the supply of primary factors are inelastic or fixed. Thus given the supply of primary factors of production and given technology sets of production sectors, the set of feasible production can be projected onto the space of producible goods and this generates a production possibility set summarizing the endowment of primary factors and technological restrictions in the space of producible goods. This reduction of the endowment of primary factors and production sectors into a production possibility set enables us to lower the dimension of commodities to consider and also it enables us to link prices of producible goods to prices of primary factors or factor prices. The surface of a production possibility set is sometimes called either a production possibility frontier or a product transformation curve (or surface).

The analysis of this chapter is very closely related to "activity analysis" which is usually confined to the analysis of production activities with constant returns to scale and with a finite set of activities. Our analysis in this chapter retains the assumption of constant returns to scale, but allows an infinite set of activities. Whether the set of activities is finite or infinite, we can obtain almost the same results as anticipated by Koopmans (1951, Introduction, pp. 5 – 6).

2. The Concept of a Production Possibility Set

We suppose that, in the economy we consider, there are r industries or production sectors, n producible commodities and m nonproducible commodities with fixed supplies. Each production sector will be characterized by its technology set denoted by $T_i \subseteq R_+^{2n+m}$. A typical element of T_i will be denoted by $(x_i, v_i, z_i) \in T_i$ where $x_i \in R_+^n$, $v_i \in R_+^m$ and $z_i \in R_+^n$ are respectively inputs of producible goods, inputs of nonproducible goods and outputs of producible goods. Mostly in our

analysis below, T_i will be assumed to be a nonempty, closed convex cone in R^{2n+m}. Therefore T_i may be interpreted as industry's aggregate technology set for a constant cost industry with free entry obtained in Chapter 6 of Part II.

Let $\hat{v} \in R^m_+$, $\hat{v} > 0$ be a fixed supply of nonproducible goods or primary factors. Then $Y(\hat{v})$, a production possibility set under free disposability, can be defined as follows:

$$(2.1) \quad Y(\hat{v}) = \{y \in R^n \mid y \leqq \sum_{i=1}^{r} (z_i - x_i),\ \sum_{i=1}^{r} v_i \leqq \hat{v} \quad \text{and} \quad (x_i, v_i, z_i) \in T_i,\ (i = 1, 2, \ldots, r)\}.$$

Note that, in the definition of $Y(\hat{v})$ above, we do not require y to be nonnegative. Whether we impose a condition $y \in R^n_+$ or not depends on the nature of feasible consumption sets. If feasible consumption sets are assumed to be R^n_+, then this condition can be imposed on y. But in this chapter, we will carry out our analysis without introducing consideration of feasible consumption sets.

In our analysis to be followed, we shall exclude a possibility of joint production and we shall assume that the production sector i produces the ith producible good. Thus we will have n = r, and with a slight abuse of our notation, we will denote $z_i \in R_+$ for the output level of the ith producible good produced by production sector i. We sometimes call z_i the activity level of production sector i. Furthermore, in this section, we will assume that each technology set can be characterized by a production function F^i: $R^{n+m}_+ \to R_+$ and we write $z_i = F^i(x_i, v_i)$. Then the production possibility set can now be written as

$$(2.2) \quad Y(\hat{v}) = \{y \mid y \leqq \sum_{i=1}^{n} (z_i e_i - x_i),\ \sum_{i=1}^{n} v_i \leqq \hat{v} \quad \text{and} \quad F^i(x_i, v_i) - z_i \geqq 0,\ (i = 1, 2, \ldots, n)\}.$$

Throughout this chapter except when we state it otherwise, we shall maintain the following assumption on production functions.

(2.3) *Assumption:* For each $i = 1, 2, \ldots, n$, F^i: $R^{n+m}_+ \to R_+$ is continuous, positively homogeneous of degree one, strictly quasi-concave and nondecreasing in each variable on R^{n+m}_+.

Efficient aggregate production is closely related to individual firm's profit maximization as well as the aggregate profit maximization. Therefore if $(x_i^*, v_i^*, z_i^*) \in T_i$ $(i = 1, 2, \ldots, n)$ generates an efficient aggregate production, then there exists $(p^*, w^*) \geqq 0$ and $(p^*, w^*) \neq 0$ such that

$$p^* \cdot \sum_{i=1}^{n} (z_i^* e_i - x_i^*) - w^* \cdot \sum_{i=1}^{n} v_i^* \geqq p^* \cdot \sum_{i=1}^{n} (z_i e_i - x_i) - w^* \cdot \sum_{i=1}^{n} v_i$$

for every $(x_i, v_i, z_i) \in T_i$. This can be rewritten as

$$p^* \cdot \sum_{i=1}^{n} (z_i^* e_i - x_i^*) + w^* \cdot \left(\hat{v} - \sum_{i=1}^{n} v_i^*\right) \geqq p^* \cdot \sum_{i=1}^{n} (z_i e_i - x_i) + w^* \cdot \left(\hat{v} - \sum_{i=1}^{n} v_i\right).$$

We also have that for every $w \geqq 0$,

$$p^* \cdot \sum_{i=1}^{n} (z_i^* e_i - x_i^*) + w\left(\hat{v} - \sum_{i=1}^{n} v_i^*\right) \geqq p^* \sum_{i=1}^{n} (z_i^* e_i - x_i^*) + w^*\left(\hat{v} - \sum_{i=1}^{n} v_i^*\right)$$

since $w^*\left(\hat{v} - \sum_{i=1}^{n} v_i^*\right) = 0$ and $\hat{v} - \sum_{i=1}^{n} v_i^* \geqq 0$. Therefore $((x_i^*, v_i^*, z_i^*)_{i=1}^{n}, w^*)$ is a saddle point of $L((x_i, v_i, z_i)_{i=1}^{n}, w) = p^* \cdot \sum_{i=1}^{n} (z_i e_i - x_i) + w\left(\hat{v} - \sum_{i=1}^{n} v_i\right)$. If we let $y^* = \sum_{i=1}^{n} (z_i^* e_i - x_i^*)$, then we can conclude that y^* maximizes $p^* y$ for $y \in Y(\hat{v})^*$. Therefore the aggregate profit maximization can be reduced the maximization of the value of producible goods on the gross national product where factor prices do not appear explicitly. We shall carry out further investigation on the maximization of the gross national product in the next section.

3. The GNP Maximization and the Value Minimization of Resources

Let the GNP (gross national product) maximization and its maximum value function (if exists) be written by:

$$(3.1) \quad \phi(p, \hat{v}) = \max\left\{p \cdot y \,\middle|\, y = \sum_{i=1}^{n} (z_i e_i - x_i),\ \sum_{i=1}^{n} v_i \leqq \hat{v},\ F^i(x_i, v_i) - z_i \geqq 0,\ (i = 1, 2, \ldots, n)\right\}.$$

Suppose that for a given $p^* \geqq 0$, $p^* \neq 0$, $(y^*, (x_i^*, v_i^*, z_i^*))$ is a solution to the above GNP maximization problem. Then* for some $w^* \in R_+^m$, $((x_i^*, v_i^*)_{i=1}^{n}, w^*)$ is a saddle point of $L((x_i, v_i), w) =$

$$p^* \cdot \sum_{i=1}^{n} (F^i(x_i, v_i) e_i - x_i) + w\left(\hat{v} - \sum_{i=1}^{n} v_i\right), \quad \text{i.e.,}$$

(3.2)(a) for every $(x_i, v_i) \in R_+^{n+m}$, $(i = 1, 2, \ldots, n)$,
$L((x_i, v_i), w^*) \leqq L((x_i^*, v_i^*), w^*)$, and

(3.2)(b) for every $w \in R_+^m$, $L((x_i^*, v_i^*), w^*) \leqq L((x_i^*, v_i^*), w)$.

Let $a_i \in R_+^n$ and $b_i \in R_+^m$ be input coefficient vectors of producible goods and nonproducible goods respectively such that $F^i(a_i, b_i) = 1$.

We now define a unit cost function $c_i: R_+^{n+m} \to R$ for production sector i by:

$$(3.3) \quad c_i(p, w) = \min\{p \cdot a_i + w \cdot b_i \mid F^i(a_i, b_i) \geqq 1\}$$

and also unit cost minimizing input coefficients of producible and nonproducible inputs will be respectively denoted by $a_i(p, w)$ and $b_i(p, w)$. Of course, we have $c_i(p, w) = p \cdot a_i(p, w) + w \cdot b_i(p, w)$. For a given $p \in R_+^n$, $p \neq 0$, let us define the

* See Theorem (5.5), mathematical appendix.

value minimization of resources or the endowment of primary factors and its minimum value function by:

(3.4) $\psi(p, \hat{v}) = \min_{w \in R^m_+} \{w \cdot \hat{v} \mid c_i(p, w) \geqq p_i, \quad (i = 1, 2, \ldots, n)\}$.

Let $w^* \in R^m_+$ be a solution to the above minimization problem for a given $p^* \in R^n_+$, $p^* \neq 0$. Since unit cost functions are concave, there exist $z^* = (z_i^*) \in R^n_+$ such that (w^*, z^*) is a saddle point of

$$K(w, z) = -w \cdot \hat{v} + \sum_{i=1}^{n} z_i(c_i(p^*, w) - p_i^*), \quad \text{i.e.,}$$

(3.5)(a) for every $w \in R^m_+$, $K(w, z^*) \leqq K(w^*, z^*)$,

(3.5)(b) for every $z \in R^n_+$, $K(w^*, z^*) \leqq K(w^*, z)$.

We will show that the following duality relationship exists between the GNP maximization problem (3.1) and the resource-value minimization problem (3.5).

(3.6) *Proposition:* Assume that for a given $p^* \in R^n_+$, $p^* \neq 0$, both (3.1) and (3.4) have solutions.

(i) If $((x_i^*, v_i^*), w^*)$ is a saddle point of $L((x_i, v_i), w)$ and if we let $z_i^* = F^i(x_i^*, v_i^*)$, then (w^*, z^*) is a saddle point of $K(w, z)$; (ii) if (w^*, z^*) is a saddle point of $K(w, z)$ and if $c_i(p^*, w)$ has a support plane $b_i(p^*, w^*) \cdot (w - w^*) = 0$ at w^*, then $((x_i^*, v_i^*), w^*)$ is a saddle point of $L((x_i, v_i), w)$ where $(x_i^*, v_i^*) = (a_i(p^*, w^*), b_i(p^*, w^*))z_i^*$, $(i = 1, 2, \ldots, n)$; (iii) $\phi(p^*, \hat{v}) = \psi(w^*, \hat{v})$.

To show (i), let $((x_i^*, v_i^*), w^*)$ be a saddle point of $L((x_i, v_i), w)$ and define $z_i^* = F^i(x_i^*, v_i^*)$. By setting $(x_j, v_j) = (x_j^*, v_j^*)$ for $j \neq i$ in (3.2)(a), we obtain $p_i^* F^i(x_i, v_i) - p^* \cdot x_i - w^* \cdot v_i \leqq p_i^* F^i(x_i^*, v_i^*) - p^* \cdot x_i^* - w^* \cdot v_i^*$ for every $(x_i, v_i) \in R^{n+m}_+$. Hence if $z_i^* = 0$, then $p^* \cdot x_i^* + w^* \cdot v_i^* = 0$. For $z_i^* > 0$, define $a_i^* = x_i^*/z_i^*$, $b_i^* = v_i^*/z_i^*$. The above inequality also implies that $p^* \cdot a_i^* + w^* \cdot b_i^* \leqq p^* \cdot a_i + w^* \cdot b_i$ for every $(a_i, b_i) \in S_i$ where $S_i = \{(a_i, b_i) \mid F^i(a_i, b_i) \geqq 1\}$. Thus $a_i^* = a_i(p^*, w^*)$ and $b_i^* = b_i(p^*, w^*)$. If $z_i^* = 0$ for some i, then set $a_i^* = 0$ and $b_i^* = 0$. Then (3.2)(a) and (3.2)(b) can be rewritten respectively as

(3.7)(a) for every $(a_i, b_i, z_i) \in S_i \times R_+$,

$$-w^* \cdot \hat{v} + \sum_{i=1}^{n} z_i^*(p^* \cdot a_i^* + w^* \cdot b_i^* - p_i^*)$$

$$\leqq -w^* \cdot \hat{v} + \sum_{i=1}^{n} z_i(p^* \cdot a_i + w^* \cdot b_i - p_i^*), \quad \text{and}$$

(3.7)(b) for every $w \in R^m_+$,

$$-w \cdot \hat{v} + \sum_{i=1}^{n} z_i^*(p^* \cdot a_i^* + w \cdot b_i^* - p_i^*)$$

$$\leqq -w^* \cdot \hat{v} + \sum_{i=1}^{n} z_i^*(p^* \cdot a_i^* + w^* \cdot b_i^* - p_i^*).$$

By the cost minimization condition, we have $c_i(p^*, w) = p^* a_i(p^*, w) + w b_i(p^*, w) \leqq p^* \cdot a_i^* + w \cdot b_i^*$. Thus (3.7)(b) yields

$$K(w, z^*) = -w \cdot \hat{v} + \sum_{i=1}^{n} z_i^* (c_i(p^*, w) - p_i^*) \leqq -w \cdot \hat{v} + \sum_{i=1}^{n} z_i^* (p^* \cdot a_i^* + w \cdot b_i^*)$$
$$\leqq K(w^*, z^*).$$

On the other hand, setting $(a_i, b_i) = (a_i^*, b_i^*)$ in (3.7)(a) yields $K(w^*, z^*) \leqq K(w^*, z)$ for every $z \in R_+^n$. This proves (i).

To prove (ii), let us suppose that (w^*, z^*) is a saddle point of $K(w, z)$ and let us define $(a_i^*, b_i^*) = (a_i(p^*, w^*), b_i(p^*, w^*))$ and $(x_i^*, v_i^*) = (a_i^*, b_i^*) z_i^*$. We note that it suffices to prove (3.7)(a) and (3.7)(b).

By the cost minimization condition, for every $(a_i, b_i) \in S_i$, $p^* \cdot a_i^* + w^* \cdot b_i^* \leqq p^* \cdot a_i + w^* \cdot b_i$. Hence (3.5)(b) implies that

$$K(w^*, z^*) \leqq K(w^*, z) = -w^* \cdot \hat{v} + \sum_{i=1}^{n} z_i (p^* \cdot a_i^* + w^* \cdot b_i^* - p_i^*)$$
$$\leqq -w^* \cdot \hat{v} + \sum_{i=1}^{n} z_i (p^* \cdot a_i + w^* \cdot b_i - p_i^*)$$
$$\text{for every} \quad (a_i, b_i, z_i) \in S_i \times R_+ .$$

This proves (3.7)(a).

Let $I = \{i \mid p_i^* = c_i(p^*, w^*)\}$ and $W(p^*) = \{w \in R_+^m \mid c_i(p^*, w) \geqq p_i^*$ for every $i \in I\}$. Then w^* maximized $-w \cdot \hat{v}$ on $W(p^*)$. By the assumption for (ii), $c_i(p^*, w)$ has support plane $b_i^* \cdot (w - w^*) = 0$ at $w = w^*$. Since $c_i(p^*, w^*) = p^* \cdot a_i^* + w^* \cdot b_i^*$, we can write the tangent plane as the set of w satisfying $p^* \cdot a_i^* + w \cdot b_i^* = p_i^*$. If we define $\tilde{W}(p^*) = \{w \in R_+^m \mid p^* \cdot a_i^* + w \cdot b_i^* \geqq p_i^*$ for every $i \in I\}$, then w^* also maximizes $-w \cdot \hat{v}$ on $\tilde{W}(p^*)$. Since any supporting plane to $W(p^*)$ at w^* also supports $\tilde{W}(p^*)$ at w^*, we must have that for every $w \in R_+^m$,

$$-w \cdot \hat{v} + \sum_{i=1}^{n} z_i^* (p^* \cdot a_i^* + w \cdot b_i^* - p_i^*) \leqq -w^* \cdot \hat{v}$$
$$+ \sum_{i=1}^{n} z_i^* (p^* \cdot a_i^* + w^* \cdot b_i^* - p_i^*) .$$

This proves (3.7)(b).

To prove (iii), note that at saddle points, we must have

$$w^* \cdot \left(\hat{v} - \sum_{i=1}^{n} v_i^*\right) = 0 \quad \text{and} \quad \sum_{i=1}^{n} z_i^* (c_i(p^*, w^*) - p_i^*) = 0 . \quad \text{Therefore}$$

$$\phi(p^*, \hat{v}) = \sum_{i=1}^{n} p^* \cdot (z_i^* e_i - x_i^*) = \sum_{i=1}^{n} z_i^* (p_i^* - p^* a_i^*) = \sum_{i=1}^{n} z_i^* w^* \cdot b_i^*$$
$$= \sum_{i=1}^{n} w^* v_i^* = w^* \cdot \hat{v} = \psi(p^*, \hat{v}).$$

This concludes our proof.

Note that when $c_i(p^*, \cdot)$ is differentiable with respect to w at $w^* \gg 0$, then by the envelope theorem, $\dfrac{\partial c_i(p^*, w^*)}{\partial w} = [b_i(p^*, w^*)]'$. Hence $c_i(p^*, w)$ has a support plane $b_i^* \cdot (w - w^*) = 0$ at w^*. It should be noted that $W(p^*) \subseteq \tilde{W}(p^*)$ and we have shown that if w^* minimizes $w \cdot \hat{v}$ for $w \in W(p^*)$, then w^* minimizes $w \cdot \hat{v}$ for $w \in \tilde{W}(p^*)$. The above envelope property is very important to obtain this conclusion.

It is possible to regard the above duality relationship as a particular version of duality in homogeneous programming by Eisenberg (1961). If we rewrite our maximization problem as: $\underset{(x_i, v_i)}{\text{maximize}}\ p^* \cdot (F^i(x_i, v_i) e_i - x_i)$ subject to

$$\left\{ (v_i) \in R_+^{nm} \mid w \cdot \sum_{i=1}^{n} v_i \leqq w \cdot \hat{v} \quad \text{for all} \quad w \in R_+^m \right\}.$$

Then the objective function of this maximization problem is positively homogeneous of degree one in $(x_i, v_i)_{i=1}^n$. Since $w \cdot \hat{v}$ is also positively homogeneous of degree one in w, the dual minimization problem according to Eisenberg becomes: $\underset{w}{\text{minimize}}\ w \cdot \hat{v}$ subject to

$$\left\{ w \in R_+^m \mid w \cdot \sum_{i=1}^{n} v_i \geqq \sum_{i=1}^{n} p^* \cdot (F^i(x_i, v_i) e_i - x_i) \quad \text{for all} \quad (x_i, v_i) \in R_+^{n+m}, \quad (i = 1, 2, \ldots, n) \right\}.$$

For this particular case the constraints for the minimization problem can be further reduced to $\{w \in R_+^m \mid c_i(p^*, w) \geqq p_i^*, (i = 1, 2, \ldots, n)\}$ using unit cost functions.

When the GNP function ϕ: $R_+^n \times R_+^m \to R$ is differentiable in the interior of $R_+^n \times R_+^m$, then by the envelope theorem, we have that for $(p, \hat{v}) > 0$, $\dfrac{\partial \phi}{\partial p}(p, \hat{v}) = [y(p, \hat{v})]'$ and $\dfrac{\partial \phi}{\partial v}(p, \hat{v}) = w$ where $y(p, \hat{v})$ is an aggregate supply function of producible goods as a solution to the GNP maximization problem provided that a supply function of producible goods exists. As we will show in the next section, $y(p, \hat{v})$ can be a set-valued map.

4. A Structure of Production Possibility Sets

At the end of the last section, we have noted that an aggregate supply map $y(p, \hat{v})$ may be multivalued. In order to find out dimensions of an aggregate supply map, we have to investigate further properties of a structure of production possibility sets.

Suppose that we are given an efficient point y^* in $Y(\hat{v})$ and suppose that $y^* = \sum_{i=1}^{n} (z_i^* e_i - x_i^*)$, $\sum_{i=1}^{n} v_i^* \leqq \hat{v}$ and $z_i^* = F^i(x_i^*, v_i^*)$, $(i = 1, 2, \ldots, n)$. Let p^* be an efficient price associated with y^*, i.e., $p^* \cdot y^* = \phi(p^*, \hat{v})$ and let w^* be a dual solution so that $w^* \cdot \hat{v} = \psi(p^*, \hat{v})$. We assume that $z^* = (z_i^*)_{i=1}^n \gneqq 0$ and $w^* \gneqq 0$.

Define an $(n \times n)$ matrix $A(p^*, w^*)$ and an $(m \times n)$ matrix $B(p^*, w^*)$ respectively by $A(p^*, w^*) = [a_1(p^*, w^*), a_2(p^*, w^*), \ldots, a_n(p^*, w^*)]$ and $B(p^*, w^*) = [b_1(p^*, w^*), b_2(p^*, w^*), \ldots, b_n(p^*, w^*)]$. Then we have $y^* = [I - A(p^*, w^*)]z^*$, $B(p^*, w^*) z^* = \hat{v}$ and $p^*[I - A(p^*, w^*)] = w^* B(p^*, w^*)$.

Now suppose that there exist $\bar{y} \in Y(\hat{v})$ and $\bar{z} \in R^n_+$ satisfying $\bar{y} = [I - A(p^*, w^*)]\bar{z}$ and $B(p^*, w^*)\bar{z} = \hat{v}$. Then we have that $p^* \cdot \bar{y} = p^*[I - A(p^*, w^*)]\bar{z} = w^* B(p^*, w^*)\bar{z} = w^* \cdot \hat{v} = \psi(p^*, \hat{v})$. Therefore by (3.6) (iii), $p^* \cdot \bar{y} = \phi(p^*, \hat{v})$, i.e., $\bar{y} \in y(p^*, \hat{v})$. Define the following sets as follows:

$$L(p^*, w^*) = \{u \in R^n \mid B(p^*, w^*)u = 0\},$$

$$Z(p^*, w^*) = \{z^*\} + L(p^*, w^*),$$

$$Z_+(p^*, w^*) = R^n_+ \cap Z(p^*, w^*), \quad \text{and}$$

$$\tilde{y}(p^*, w^*) = \{y \mid y = [I - A(p^*, w^*)]z \text{ for some } z \in Z(p^*, w^*)\}.$$

(4.1) *Theorem:* If $\varrho(I - A(p^*, w^*)) = n$ and $\varrho(B(p^*, w^*)) = r$ where ϱ indicate the rank of a matrix, then the dimension of $y(p^*, \hat{v})$ is $\max\{n - r, 0\}$.

To prove the above theorem, first note that $L(p^*, w^*)$ is the kernel of the linear transformation $B(p^*, w^*)$ and so that the dimension of $L(p^*, w^*)$ is $\max\{n - r, 0\}$. Since $[I - A(p^*, w^*)]$ is nonsingular the dimension of $\tilde{y}(p^*, w^*)$ is also $\max\{n - r, 0\}$. Since

$$y(p^*, \hat{v}) = \{y \mid y = [I - A(p^*, w^*)]z \quad \text{for some} \quad z \in Z_+(p^*, w^*)\}$$

and z^* is in the interior of $Z_+(p^*, w^*)$ relative to $Z(p^*, w^*)$, the dimension of $y(p^*, \hat{v})$ must also be $\max\{n - r, 0\}$.

As an example, suppose that $n > m$, i.e., the number of producible goods is greater than the number of productive factors and suppose further that $\varrho(B(p^*, w^*)) = m$. Then the dimension of the aggregate supply map becomes $n - m$ and the surface of the production possibility set will consist of $(n - m)$ dimensional flat portions. In particular, when $m = 1$, the surface will be $(n - 1)$-dimensional and this result is sometimes called the nonsubstitution theorem of Samuelson. (See Samuelson (1951) and Arrow (1951) among others.) With $m = 1$, it is possible to give more complete and global characterization of the structure of a production possibility set to which we shall now turn.

It will turn out to be notationally more convenient to use technology sets instead of production functions for our discussions below. We shall begin by defining the following sets derived from technology sets.

$$T_i^0 = \{(x_i, z_i) \mid (x_i, v_i, z_i) \in T_i \quad \text{and} \quad v_i = 0\},$$

$$Y_i^0 = \{y_i \in R^n \mid y_i \leqq z_i e_i - x_i \quad \text{for some} \quad (x_i, z_i) \in T_i^0\},$$

$$Y^0 = \sum_{i=1}^{n} Y_i^0 = \{y \mid y = \sum_{i=1}^{n} y_i \quad \text{for some} \quad y_i \in Y_i^0 \quad (i = 1, 2, \ldots, n)\},$$

$$Y_i(\hat{v}) = \{y_i \mid y_i \leqq z_i e_i - x_i, v_i \leqq \hat{v} \quad \text{for some} \quad (x_i, v_i, z_i) \in T_i\}.$$

Note that Y_i^0 is the set of net outputs by sector i without using the primary factor and $Y_i(\hat{v})$ is the set of net outputs by sector i when the whole supply of the

primary factor is devoted to the production of sector i. Then we shall prove the following theorem providing a characterization of the structure of the production possibility set.

(4.2) *Theorem:* Assume that for each $i = 1, 2, \ldots, n$, (i) T_i is a cone in R^{n+2}, and (ii) if $(x_i, v_i, z_i) \in T_i$ and $(x_i^0, z_i^0) \in T_i^0$, then $(x_i + x_i^0, v_i, z_i + z_i^0) \in T_i$. Then

$$Y(\hat{v}) = Y^0 + \sum_{i=1}^{n} \{\lambda_i Y_i(\hat{v}) \mid \lambda_i \geqq 0, \sum_{i=1}^{n} \lambda_i = 1\} \quad \text{where}$$

$$\sum_{i=1}^{n} \lambda_i Y_i(\hat{v}) = \{y \mid y = \sum_{i=1}^{n} \lambda_i y_i \quad \text{for some} \quad y_i \in Y_i(\hat{v}), \quad (i = 1, 2, \ldots, n)\}.$$

Remark: If C is a cone in R^n, then for every $x, x' \in C$, $x + x' \in C$ if and only if C is a convex set. With this property in mind, notice that the above condition (ii) in Theorem (4.2) requires an additivity only between T_i and T_i^0. Therefore the above theorem holds without the convexity of T_i. When $m > 1$, we can obtain a similar characterization as shown in Otani (1973), but the convexity of T_i can not be easily dispensed with.

To prove the theorem, we first show that

$$Y^0 + \sum_{i=1}^{n} \{\lambda_i Y_i(\hat{v}) \lambda_i \geqq 0, \sum_{i=1}^{n} \lambda_i = 1\} \subseteq Y(\hat{v}).$$

Suppose that $y = y^0 + \sum_{i=1}^{n} \lambda_i y_i$ for some $\lambda_i \geqq 0$,

$$\sum_{i=1}^{n} \lambda_i = 1, \; y^0 \in Y^0 \text{ and } y_i \in Y_i(\hat{v}), \; (i = 1, 2, \ldots, n).$$

Then there exist $y_i^0 \in Y_i^0$, $(x_i^0, z_i^0) \in T_i^0$, $(x_i, v_i, z_i) \in T_i$ such that

$$y^0 = \sum_{i=1}^{n} y_i^0, \quad y_i^0 \leqq z_i^0 e_i - x_i^0, \quad y_i \leqq z_i e_i - x_i \quad \text{and} \quad v_i \leqq \hat{v}.$$

Since T_i is a cone, we have that $(\lambda_i x_i, \lambda_i v_i, \lambda_i z_i) \in T_i$ and $\lambda_i y_i \leqq \lambda_i z_i e_i - \lambda_i x_i$. By the condition (ii), we get

$$(\lambda_i x_i + x_i^0, \lambda_i v_i, \lambda_i z_i + z_i^0) \in T_i \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i v_i \leqq \sum_{i=1}^{n} \lambda_i \hat{v} = \hat{v}.$$

Since $y \leqq \sum_{i=1}^{n} \{(\lambda_i z_i + z_i^0) e_i - (\lambda_i x_i - x_i^0)\}$, we have shown that $y \in Y(\hat{v})$.

To prove the converse, i.e.,

$$Y(\hat{v}) \subseteq Y^0 + \sum_{i=1}^{n} \{\lambda_i Y_i(v) \mid \lambda_i \geqq 0, \sum_{i=1}^{n} \lambda_i = 1\},$$

fix $y \in Y(\hat{v})$. Then there exist $(x_i, v_i, z_i) \in T_i$, $(i = 1, 2, \ldots, n)$ such that

$$y \leqq \sum_{i=1}^{n} (z_i e_i - x_i) \quad \text{and} \quad \sum_{i=1}^{n} v_i \leqq \hat{v}.$$

Let $y_i = z_i e_i - x_i$, $(i = 1, 2, \ldots, n)$, $I_0 = \{i \mid v_i = 0\}$ and $I_1 = \{i \mid v_1 > 0\}$. Then clearly $y_0 \sum_{i \in I_0} y_i \in Y^0$. Without loss of generality, assume that $I_1 \neq \phi$. For each $i \in I_1$, define $\lambda_i = v_i / (\sum_{i \in I_1} v_i)$. Clearly $\lambda_i > 0$ for $i \in I_1$, $\sum_{i \in I_1} \lambda_i = 1$ and $v_i \leqq \lambda_i \hat{v}$ since $\sum_{i \in I_1} v_i = v_i / \lambda_i \leqq \hat{v}$. Define $(x_i', v_i', z_i') = (1/\lambda_i)(x_i, v_i, z_i)$ for $i \in I_1$. Then $y_i' = z_i' e_i - x_i' \in Y_i(\hat{v})$ since $(x_i', v_i', z_i') \in T_i$ and $v_i' \leqq \hat{v}$. Thus we have found that $y_0 \in Y^0$, $y_i' \in Y_i(\hat{v})\,(i \in I_1)$ and $y = y_0 + \sum_{i \in I_1} \lambda_i y_i'$. Hence we can conclude that $y \in Y_0 + \sum_{i=1}^{n} \{\lambda_i Y_i(\hat{v}) \mid \lambda_i \geqq 0, \sum_{i=1}^{n} \lambda_i = 1\}$.

In order to find out the usefulness of the above characterization of the structure of a production possibility set, we shall consider two examples.

(4.3) *Example:* We consider a case where each technology set is characterized by a single activity with fixed coefficients and the number of producible goods is just two. Let $(a_{1i}, a_{2i}, b_i)'$ be a fixed vector of a single activity of sector i with $a_{ji} \geqq 0$ and $b_i > 0$. Then technology set of sector i can be defined by:

$$T_i = \{(x_i, v_i, z_i) \mid (x_i, v_i) \geqq (a_{1i}, a_{2i}, b_i) z_i, z_i \geqq 0\}.$$

If we let $A = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}$ and $B = (b_1, b_2)$, then the production possibility set becomes:

$$Y(\hat{v}) = \{y \in R^2 \mid y \leqq (I - A)z, Bz \leqq \hat{v} \quad \text{for some} \quad z \in R^2_+\}.$$

Since the primary factor is indispensible for each sector, i.e., $b_i > 0$ $(i = 1, 2)$, we can neglect Y^0. Therefore by Theorem (4.2),

$$Y(\hat{v}) = \{\lambda Y_1(\hat{v}) + (1 - \lambda) Y_2(\hat{v}) \mid \lambda \in [0, 1]\} \quad \text{where}$$

$$Y_1(\hat{v}) = \left\{ y \in R^2 \mid y \leqq \begin{pmatrix} 1 - a_{11} \\ -a_{21} \end{pmatrix} (\hat{v}/b) \right\} \quad \text{and}$$

$$Y_2(\hat{v}) = \left\{ y \in R^2 \mid y \leqq \begin{pmatrix} -a_{12} \\ 1 - a_{22} \end{pmatrix} (\hat{v}/b_2) \right\}.$$

The set $Y_1(\hat{v})$ is illustrated in Figure 4.1. and the production possibility set is illustrated in Figure 4.2. In Figure 4.2., the triangle OPQ is formed by taking convex combinations between points in $Y_1(\hat{v})$ and points in $Y_2(\hat{v})$. If we define a row vector $c \in R^2$ by $C = B(I - A)^{-1}$ provided that $c \geq 0$, then $p^* = c$ will support efficient points on PQ and $y(p^*, \hat{v})$ consists of the line segment PQ. We further note that there exists $y^* \in Y(\hat{v})$ with $y^* \gg 0$ if and only if $(I - A)$ satisfies the Hawkins-Simon condition, i.e., all northwest principal minors of $(I - A)$ are positive or $1 - a_{11} > 0$ and $(I - A) = (1 - a_{11})(1 - a_{22}) - a_{21} a_{12} > 0$. Since the absolute slope of OP is $a_{21}/(1 - a_{11})$ and the absolute slope of OQ is $(1 - a_{22})/a_{12}$, the Hawkins-Simon condition for this example is that the absolute slope of OQ be greater than the absolute slope of OP. The Hawkins-Simon condition with $b \gg 0$

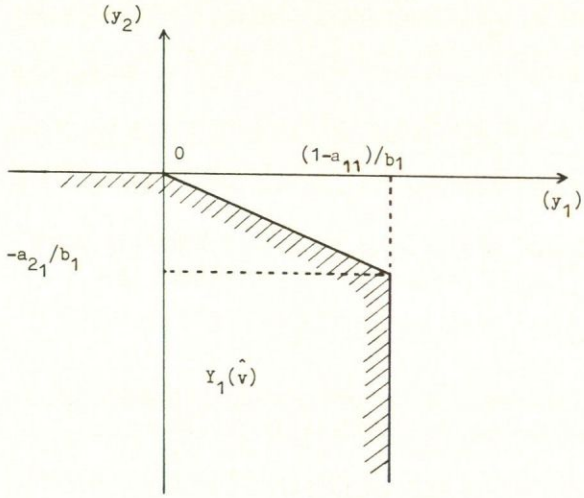

Fig. 4.1

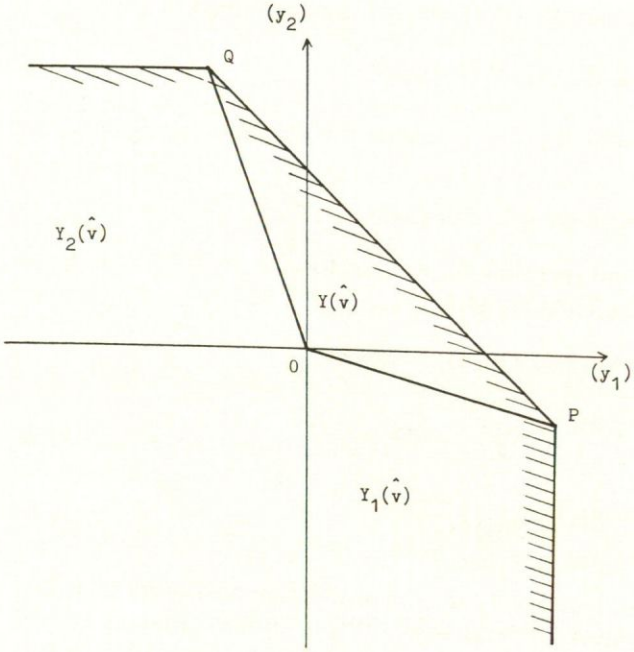

Fig. 4.2

also implies that $c > 0$. See Nikaido (1968, Chapter II) for the Hawkins-Simon condition and various related results.

(4.4) *Example:* We shall again assume that $n = 2$, but we shall consider smooth production functions with constant elasticity of substitution given by $F^i(x_{ji}, v_i) = (\alpha_i x_{ji}^{-\varrho_i} + \beta_i v_i^{-\varrho_i})^{-1/\varrho_i}$ $(i = 1, 2; j \neq i)$ where $\alpha_i > 0$, $\beta_i > 0$, $\alpha + \beta_i = 1$

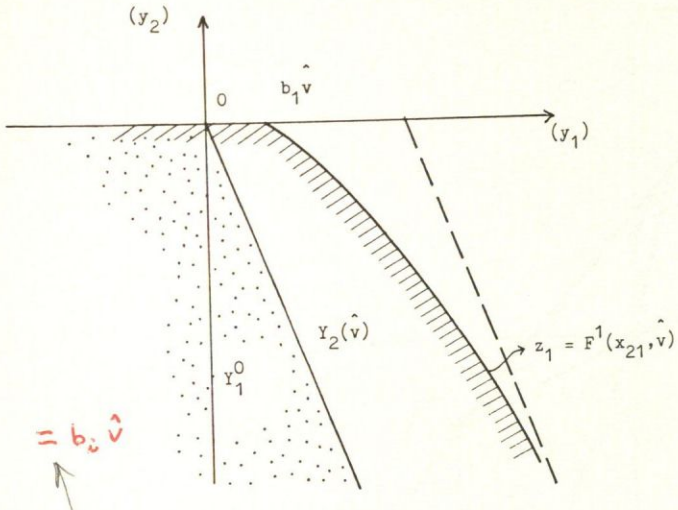

Fig. 4.3

and $-1 < \varrho_i < 0$. It is well-known that the elasticity of substitution σ_i is given by $\sigma_i = 1/(1+\varrho_i)$. Hence $-1 < \varrho_i < 0$ entails $\sigma_i > 1$. When the constant elasticity of substitution is greater than one, then a positive output is possible with only one input. Hence, in particular, the primary factor is not directly indispensable. Let $a_1 = (\alpha_i)^{-1/\varrho_i}$ and $b_i = (\beta_i)^{-1/\varrho_i}$ $(i = 1, 2)$. Then we can show that $F^i(x_{ji}, 0) = a_i x_{ji}$, $F^i(0, \hat{v})$ and $\lim_{x_{ji} \to \infty} \dfrac{\partial F^i(x_{ji}, \hat{v})}{\partial x_{ji}} = a_i$. Therefore Y_i^0 and $Y_i(\hat{v})$ become

$$Y_1^0 = \left\{ y \mid y \leqq \begin{pmatrix} z_1 \\ -x_{21} \end{pmatrix}, \quad z_1 = a_1 x_{21} \right\},$$

$$Y_2^0 = \left\{ y \mid y \leqq \begin{pmatrix} -x_{12} \\ z_2 \end{pmatrix}, \quad z_2 = a_2 x_{12} \right\},$$

$$Y_1(\hat{v}) = \left\{ y \mid y \leqq \begin{pmatrix} z_1 \\ -x_{21} \end{pmatrix}, \quad z_1 = F^1(x_{21}, \hat{v}) \right\}, \quad \text{and}$$

$$Y_2(\hat{v}) = \left\{ y \mid y \leqq \begin{pmatrix} -x_{12} \\ z_2 \end{pmatrix}, \quad z_2 = F^2(x_{12}, \hat{v}) \right\}.$$

We give an illustration of Y_1^0 and $Y_1(\hat{v})$ in Figure 4.3. In Figure 4.3, note that the asymptote to the frontier of $Y_1(\hat{v})$ given by the curve $y_1 = z_1 = F^1(x_{21}, \hat{v})$ has the same slope as the frontier line of Y_1^0 given by $y_1 = a_1 x_{21}$. Depending on the nature of $Y^0 = Y_1^0 + Y_2^0$, i.e., the productivity of production sectors without the input of the primary factor, we can consider three cases.

Case (i): Assume that $Y^0 \cap R_+^2 = \{0\}$ and $Y^0 \cap (-Y^0) = \{0\}$. See Figure 4.4 for the illustration of this case. In this case, the production possibility frontier will have a one-dimensional portion PQ generated by an activity OP of sector 1 and an activity OQ of Sector 2. Therefore, along PQ, the production will appear as if outputs are generated by fixed coefficients production of Example (4.3), i.e., the nonsubstitution theorem holds along PQ.

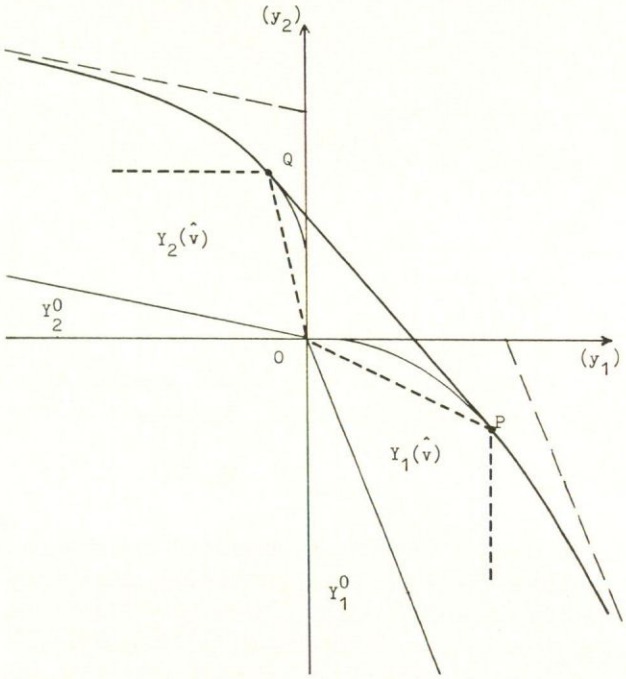

Fig. 4.4

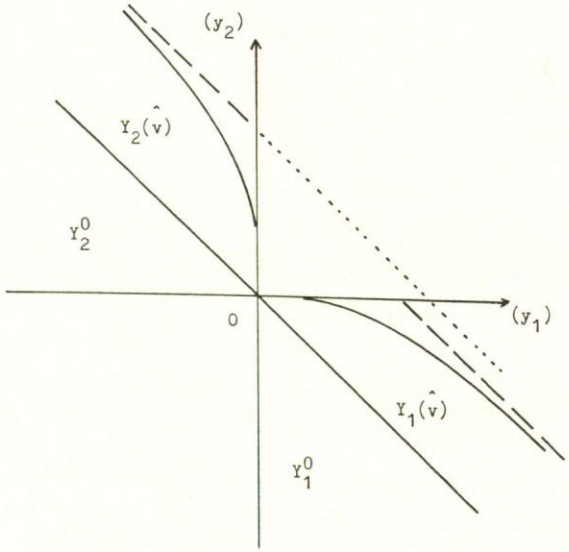

Fig. 4.5

Case (ii): When there exists $y^* \in Y^0 \cap R^2_+$ such that $y^* \neq 0$, then Y^0 will include the whole nonnegative quadrant. Therefore this model of production can be considered economically unreasonable.

Case (iii): Suppose now that $Y^0 \cap R^2_+ = \{0\}$, but $Y^0 = -Y^0$. See Figure 4.5 for an illustration of this case. In this case, even if production functions are smooth,

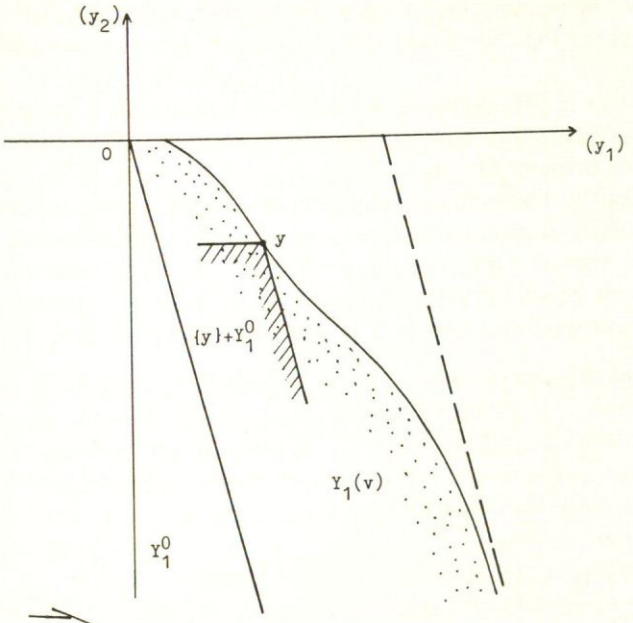

Fig. 4.6

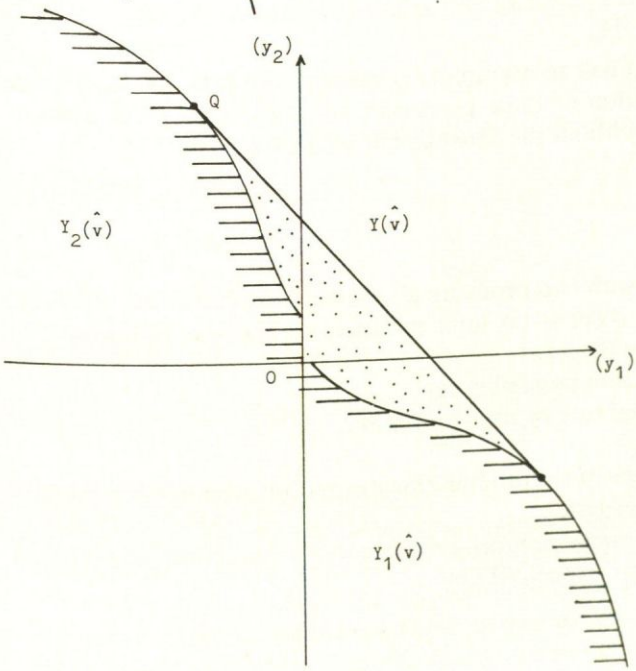

Fig. 4.7

the production possibility set will not be closed. This case can also be considered economically unreasonable because, in order to get closer to the northeast boundary of $Y(\hat{v})$, activity levels z_i $(i = 1, 2)$ of both sectors must become infinitely large.

Thus we have seen that only case (i) possesses economically reasonable properties. It should be noted that although the primary factor is not directly in-

dispensable for each sector, the primary factor for case (i) is indirectly indispensable for efficient outputs along PQ. See Otani (1973) for more general analysis concerning the above point.

When $Y_i(\hat{v})$ includes Y_i^0 as in this example, there is no need to add Y^0 separately as in Theorem (4.2). We need to add Y_i^0 separately, when there are some sectors which do not use the primary factor.

As we have noted previously, Theorem (4.2) does not require the convexity of technology sets or the concavity of production functions. Note that condition (ii) of Theorem (4.2) implies that if $y \in Y_i(\hat{v})$ and $y' \in Y_i^0$, then $y + y' \in Y_i(\hat{v})$ or equivalently if $y \in Y_i(\hat{v})$, then $\{y\} + Y_i^0 \subseteq Y_i(\hat{v})$. Also if $F^i(x_{ji}, v_i)$ is continuously differentiable on the positive quadrant and if it is positively homogeneous of degree one, then

$$F^i(x_{ji}, 0) = a_i x_{ji}$$

where $a_i = F(1, 0)$, and

$$\lim_{x_{ji} \to \infty} \frac{\partial F^i(x_{ji}, \hat{v})}{\partial x_{ji}} = \lim_{x_{ji} \to \infty} \frac{\partial F^i(1, \hat{v}/x_{ji})}{\partial x_{ji}}$$

$$= \frac{\partial F^i(1, 0)}{\partial x_{ji}} = a_i.$$

Thus the frontier of $Y_i(\hat{v})$ has an asymptote parallel to the function of Y_i^0. See Figure 4.6 for an illustration of these properties and Figure 4.7 illustrates the nonsubstitution theorem without the convexity of technology sets.

Problems

1. Consider an economy with two production sectors, two producible goods (x_1 and x_2). Assume that there is no joint production. For the following two cases, Case A and Case B:

   (a) Find the production possibility set.

   (b) Find the marginal rate of transformation.

   For Case A alone:

   (c) Examine the shape of the product transformation schedule (by examining the derivative properties).

   Case A: There is a single non-producible good. Production functions of the two sectors are given respectively by

   $$f^1(v_1) = v_1^\alpha \quad \text{and}$$
   $$f^2(V_2) = v_2^\beta$$

   where v_i is an input of the primary factor and $0 < \alpha < 1$ and $\beta > 1$. The resource constraint is given by

   $$v_1 + v_2 \leqq v, \quad v_i \geqq 0 \quad (i = 1, 2).$$

   Case B: There are two primary factors, say, labor and capital. Production functions of the two sectors are given by:

$f^1(K_1, L_1) = K_1^{\alpha} L_1^{1-\alpha}$ and

$$f^2(K_2, L_2) = \min\left\{\frac{K_2}{a}, \frac{L_2}{b}\right\}$$

where K_i, L_i are inputs of capital and labor to sector i and $1 > \alpha > 0$, $a > 0$, $b > 0$. Resource constraints are given by:

$$K_1 + K_2 \leqq \bar{K}$$
$$L_1 + L_2 \leqq \bar{L}$$
$$K_i \geqq 0, \quad L_i \geqq 0 \quad (i = 1, 2).$$

2. Consider an economy with two sectors, two producible goods and two non-producible goods (say, labor and capital). The technology of sector i is characterized by the vector of input coefficients needed to produce one unit of commodity i which is given by (a_{1i}, a_{2i}) and (b_{Li}, b_{Ki}), where a_{ji} is the input of producible good j $(j = 1, 2)$ and b_{ji} is the input of non-producible good j $(j = L, K)$.

Let z_i be the activity level (or gross output) of sector i and L and K be the total stocks of labor and capital available.

Assume

i) $b_{K1}/b_{L1} < K/L < b_{K2}/B_{L2}$ and

ii) the Hawkins-Simon condition on the input-output matrix.

Define:

$$\tilde{B} = \begin{pmatrix} \beta_{K1}, & \beta_{K2} \\ \beta_{L1}, & \beta_{L2} \end{pmatrix} = \begin{pmatrix} b_{K1}, & b_{K2} \\ b_{L1}, & b_{L2} \end{pmatrix} \begin{pmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}^{-1}$$

or $\tilde{B} = B(I - A)^{-1}$

$$\text{where } B = \begin{pmatrix} b_{K1} & b_{K2} \\ b_{L1} & b_{L2} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and I is a (2×2) identity matrix.

(a) Show that $\beta_{K1}/\beta_{L1} < \beta_{K2}/\beta_{L2}$. (*Hint:* Look at det $\tilde{B}$ and det B). Also explain the economic meaning of coefficients β_{ji} $(j = L, K)$, $(i = 1, 2)$.

(b) Let p_i $(i = 1, 2)$ be the price of producible good i and w and r be the wage rate of labor and the rental rate of capital respectively. Find (w, r) corresponding to various (p_1, p_2). (Find imputed prices of labor and capital corresponding to various prices of final goods.) You can use the following facts in the "Hint."

(*Hint:* Let $c^i(p_1, p_2, w, r)$ be the cost function of sector i, i.e., $c^i(p_1, p_2, w, r) = a_{1i}p_1 + a_{2i}p_2 + b_{Li}w + b_{Ki}r$. Then $z_i > 0$ implies that $p_i = c^i(p_1, p_2, w, r)$ and $p_i < c^i(p_1, p_2, w, r)$ implies that $z_i = 0$. Also if there is an unemployed factor, then the price of the factor must be zero. You can get a good insight if you draw the production possibility set.)

3. Consider an economy with two production sectors having production functions $f^i(L_i, K_i)$, $i = 1, 2$. Consider the following maximization problem with parameters y_1, L, K:

$$\text{Maximize } y_2 = f^2(L_2, K_2)$$

s.t.

$$f^1(L_1, K_1) - y_1 \geqq 0$$

$$L - L_1 - L_2 \geqq 0$$

$$K - K_1 - K_2 \geqq 0 \quad \text{and}$$

$$L_i \geqq 0, K_i \geqq 0, y_i \geqq 0 \quad (i = 1, 2).$$

Let the Lagrangean function of this problem be

$$f^2(L_2, K_2) + p(f^1(L_1, K_1) - y_1) + w(L - L_1 - L_2) + r(K - K_1 - K_2).$$

Assuming interior solutions and appropriate differentiability conditions on f^i, we write the solution of the above problem as:

$$L_i = \tilde{L}_i(y_1, L, K)$$

$$K_i = \tilde{K}_i(y_1, L, K)$$

and the objective function as

$$y_2 = T(y_1, L, K) = f^2[\tilde{L}_2(y_1, L_1, K_1), \quad \tilde{K}_2(y_1, L, K)].$$

(a) Obtain the first order conditions of the above maximization problem.
(b) Show that: (without applying the envelope theorem)

$$\frac{\partial T}{\partial y_1} = -p, \quad \frac{\partial T}{\partial L} = w, \quad \text{and} \quad \frac{\partial T}{\partial K} = r.$$

(c) Give concise economic interpretations of results (a) and (b).

4. Consider an economy where there are two producible commodities, say, x_1 and x_2, and two primary commodities, say, L and K. We assume that:
(a) The production function for x_i is given by

$$x_i = \min\left\{\frac{L_i}{a_i}, \frac{K_i}{b_i}\right\} \quad (i = 1, 2).$$

(b) The supplies of primary commodities are fixed at $\bar{L} > 0$, $\bar{K} > 0$.

(c) Assume that $\dfrac{b_1}{a_1} < \dfrac{\bar{K}}{\bar{L}} < \dfrac{b_2}{a_2}$.

Questions:

(i) Obtain input demand functions for L_i and K_i for $w > 0$ and $r > 0$ where w is the price of L and r is the price of K.

(ii) Describe the production possibility set in the space of producible commodities.

(iii) Let $p_1 = p$ and $p_2 = 1$ where p_i is the price of x_i. (p is the price of x_1 in terms of x_2 and x_2 is the numeraire commodity.) Utilizing the production pos-

sibility set in (ii), obtain the aggregate supply function of x_i as a function of p along the production possibility frontier.

(iv) Obtain w and r as functions of p. *Hint:* Use the following fact when constant returns to scale prevail. If $x_i > 0$, then the price of x_i = the average cost of x_i. If the price of x_i is less than the average cost of x_i, then $x_i = 0$. (To prove the above fact you can use the "duality theorem" in linear programming.)

(v) Assume $x_1 > 0$ and $x_2 > 0$. Show how x_1 and x_2 will change if either $\bar{L}$ or $\bar{K}$ increases.

5. Suppose that there are four commodities x_i $(i = 1, 2, \ldots, 4)$ and two primary factors, say labor and capital. The supply of labor and capital is fixed at $L > 0$ and $K > 0$ respectively. Let p_i be the price of commodity i. Production coefficients of labor and capital required in producing a unit of commodity i are denoted by b_{iL}, b_{iK} respectively. We assume that:

(a) $p_2 = 1/2$, $p_3 = p_4 = 1$

(b) $(b_{L1}, b_{K1}) = (1, 1/2)$, $(b_{L2}, b_{K2}) = (1, 1)$

$(b_{L3}, b_{K3}) = (1, 2)$ and $(b_{L4}, b_{K4}) = (1, 3)$.

(c) $1/2 < \dfrac{K}{L} < 2$.

(d) Output levels are determined in order to maximize the value of total output, i.e., $p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$.

Questions:

(i) Obtain optimum patterns of production for various p_2.

(ii) Determine imputed prices of labor and capital for each pattern of production.

Hint: Do not use any computational method since p_2 is not given. You can solve the problem by a simple diagram using the dual problem.

Chapter 12 Introduction to Welfare Economics and General Equilibrium Analysis

1. Introduction

In the earlier chapters we dealt with the behavior of individual agents be they consumers or producers. We also dealt with markets for individual goods. In this chapter we discuss the simultaneous behavior of all agents. The discussion attempts to clarify the relationship between what is "good" for the group and what is "good" for its individual constituents. When we speak of this relationship we are in the realm of welfare economics. This will be done in Section 2. In our discussion we will assume that "good for the society" means Pareto efficient. What is "good for individuals" will mean a general competitive equilibrium. In the absence of externalities, in addition to some other less striking assumptions, it is shown that the two "good" coincide. We call attention to the shallowness of the philosophic import of this result which may be stated as follows: If every member of society is an island into themselves, then what is best for society coincides with what is best for the individual. (How else can it be?)

In Section 3 we discuss the question of existence of competitive equilibrium. In both sections we assume that preferences are representable by utility functions which are continuously differentiable up to and including the second order. We also assume that technology sets are described by equally smooth production functions.

2. Production, Pareto Efficiency, and Competitive Equilibrium

Let $I = \{1, 2, 3, \ldots, n\}$, $J = \{1, \ldots, m\}$, $L = \{1, 2, \ldots, l\}$ be the index sets of consumption agents, goods and firms respectively. The preferences of consumer i are represented by $u^i: X_i \to R_+$, where X_i is the consumption set of consumer i, for $i \in I$. We shall think of X_i as R^m_+. In stating that u^i depends only on x_i we already have assumed away externalities in consumption.

Let the technology of firm r be given by:

$$Y_r = \{y_r \in R^m \mid f^r(y_r) \leqq 0\}, \; r \in L,$$

where y_r is a netput vector, i.e., a vector whose components are differences between outputs and inputs. We are assuming away externalities in production as well. Let ω_{ij} be the initial endowment of consumer i in good j and let θ_{ir} be the share of consumer i in the profits of firm r. Such profits, π_r, are given by $\pi_r = py_r$ if p is a price vector and y_r is a netput vector.

More natural to have $\geq$

a) Efficiency in Production

Let $y = \sum_r y_r$ be a given aggregate netput vector. We say that y is feasible if $f^r(y_r) \leqq 0$ so each individual firm's contribution is technologically feasible and if $y + \sum_i \omega_i \geqq 0$, where ω_i is the initial endowment vector for consumer i. We say that $\hat{y}$ is an *efficient aggregate production* (EAP) if it is feasible and if there does not exist another feasible y such that $y \geq \hat{y}$ in the sense of vector ordering introduced earlier. Thus, $\hat{y}$ is EAP if it can't be improved upon by producing more output or utilizing less input since $y \geq \hat{y}$ means $y_j \geqq \hat{y}_j$ for all j with strict inequality for at least one j and since y_j is negative if j is a net input and non-negative if j is a net output.

Now suppose $\hat{y}$ is EAP then, by the scalerization lemma*, it solves the following mathematical programming problem:

$$\text{maximize} \sum_j \alpha_j \sum_r y_{rj} \text{ subject to:}$$

$$-f^r(y_r) \geqq 0, \qquad r \in L$$

$$\sum_r y_r + \sum_i \omega_i \geqq 0,$$

where $\alpha_j > 0$ for all $j \in J$.

By the first order necessary conditions we conclude the existence of multipliers ν_r and μ_j that:

(i) $\alpha_j - \nu_r \hat{f}^r_j + \mu_j = 0, \quad r \in L, \quad j \in J$

(ii) $\nu_r \hat{f}^r = 0, \quad \mu \cdot \left(\sum_r y_r + \sum_i \omega_i \right) = 0$

(iii) $\nu_r \geqq 0, \quad \mu \geqq 0,$

where $\hat{f}^r_j = \partial f^r / \partial y_{sj} |_{y_s = \hat{y}_r}$.

But (i) implies that $\nu_r > 0$ provided $\hat{f}^r_y \neq 0$, i.e., provided the vector $(\hat{f}^r_1, \ldots, \hat{f}^r_m) \neq 0$. This condition is easily satisfied, think of the case where $f^r(y_r) = y^r_1 - g(y^r_2, \ldots, y^r_m)$. Then we would have, by (ii), $f^r(\hat{y}_r) = 0$. Thus if $\hat{y}$ is AEP then its constituent netput vectors are on the frontiers of the technology sets of the firms provided these sets are nontrivial.

Suppose furthermore that Y_r are convex sets; then we have:

$$\text{(iv)} \quad \sum_j \alpha_j \sum_s \hat{y}_{sj} + \sum_j \mu_j \left(\sum_r \hat{y}_{rj} - \sum_i \omega_{ij} \right) \geqq \sum \alpha_j \sum_r y_{rj} + \sum_j \mu_j \left(\sum_r y_{rj} - \sum_i \omega_{ij} \right)$$

$$\text{for all } y_r \in Y_r, \; r \in L.$$

Inequality (iv) holds in particular if we set $y_r = \hat{y}_r$ for $r \neq r_0$. Thus we have

$$\text{(v)} \quad \sum_j (\alpha_j + \mu_j) \hat{y}_{r_0 j} \geqq \sum_j (\alpha_j + \mu_j) y_{r_0 j} \quad \text{for all} \quad y_{r_0 j} \in Y_{r_0}.$$

---

* See remark following Theorem 5.10 in mathematical appendix.

Let the price of good j, $\hat{p}_j$ be $\alpha_j + \mu_j$. Then (v) states that firm r_0 maximizes its profits at $\hat{y}_{r_0}$ subject to technology. We have thus shown, under convexity, that AEP implies profit maximization. Thus a centrally planned economy can decentralize the implementation of an AEP plan by signaling prices $\hat{p}$ to the firms. There is no need to have an individual signal for each firm. On the other hand, if efficiency is not aggregate, i.e., if the definition of efficiency was in terms of the array $(y_{11}, \ldots, y_{1m}, \ldots, y_{l1}, \ldots, y_{lm})$, then the signal will have to be individualized since the α's in the scalerization procedure will be firm and commodity specific.

b) Pareto Efficiency and Equilibrium

A consumption $x = (x_1, \ldots, x_n) \in R_+^{nm}$ is said to be feasible if it can be sustained by a feasible production, i.e., if there exist netput vectors y_r such that

(i) $f^r(y_r) \leqq 0, \quad r \in L$ and

(ii) $\sum_r y_r + \sum_i \omega_i - \sum_i x_i \geqq 0$.

Let x' and x'' be feasible consumption vectors. Then x' is *Pareto superior* to x'' if $u^i(x_i') \geqq u^i(x_i'')$ for all $i \in I$ and if $u^{i_0}(x_{i_0}') > u^{i_0}(x_{i_0}'')$ for some $i_0 \in I$. We say that a feasible $\hat{x}$ is *Pareto efficient* if there does not exist another feasible consumption which is Pareto superior to $\hat{x}$.

Suppose now that $\hat{x}$ is Pareto efficient. Then for each $i_0 \in I$, $\hat{x}$ solves:

$M(i_0)$: max $u^{i_0}(x_{i_0})$ subject to

(1) $u^i(x_i) \geqq u^i(\hat{x}_i)$ for all $i \in I, \quad i \neq i_0$;

(2) $f^r(y_r) \leqq 0$; and

(3) $\sum_r y_r + \sum_i \omega_i - \sum_i x_i \geqq 0$.

Let us check the rank condition, i.e., here the constraint qualification, for problem $M(i_0)$. Without loss of generality assume $i_0 = 1$. The relevant matrix is presented below with the columns representing derivatives of the constraints with respect to $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_l$ in that order. Define A(1) by

$$A(1) = \begin{pmatrix} 0 & \hat{u}^2_{x_2} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \hat{u}^n_{x_n} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \hat{f}^1_{y_1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \hat{f}^1_{y_2} & \ldots & 0 \\ & \vdots & & & \vdots & & & \\ 0 & 0 & \ldots & & 0 & 0 & \ldots & \hat{f}^l_{y_l} \\ -1 & -1 & \ldots & & -1 & 1 & \ldots & 1 \end{pmatrix}$$

If we assume, reasonably, that there is at least one good, one firm and one consumer then the maximum rank of the matrix A(1) equals the number of its rows. Assume further that none of the vectors $\hat{u}^i_{x_i}$, $\hat{f}^r_{y_r}$ is a zero vector where $i = 2, \ldots, n$

and $r = 1, \ldots, l$. Then the matrix $A(1)$ has full rank. To illustrate this step of our discussion, consider the case where $m = 2$, $n = 3$, $l = 1$. Then $A(1)$ has the form:

$$\begin{pmatrix} 0 & 0 & \hat{u}_1^2 & \hat{u}_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{u}_1^3 & \hat{u}_2^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{f}_1^1 & \hat{f}_2^1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

The rows of the above matrix are clearly linearly independent provided $(\hat{u}_1^2, \hat{u}_2^2) \neq 0$, $(\hat{u}_1^3, \hat{u}_2^3) \neq 0$ and $(\hat{f}_1^1, \hat{f}_2^1) \neq 0$. In view of the normality of each of our $M(1)$ optimization problems we may take the maximand multiplier in the pre-Lagrangian to be 1. Thus we have, by the first order necessary conditions:

(4) $\hat{u}_j^1 = \mu_j^1, \quad j \in J,$ $= \mu_j^1$

(5) $\nu_i^1 \hat{u}_j^i, \quad j \in J, \quad i \neq 1, \quad i \in I,$

(6) $\gamma_r^1 \hat{f}_j^r = \mu_j^1, \quad j \in J, \quad r \in L,$

where ν_i^1, γ_r^1 and μ_j^1 are the multipliers corresponding to constraints of type (1), (2) and (3) respectively and where each multiplier is non-negative and its product with the corresponding constraint is zero.

Repeating this process for each i_0 and adding the resulting equations, we get

(7) $\nu_i \hat{u}_j^i = \mu_j$

(8) $\gamma_r \hat{f}_j^r = \mu_j$ ν_i

where $\eta_i = \sum_{i_0 \in I} \nu_i^{i_0}$, $\gamma_r = \sum_{i_0 \in I} \gamma_r^{i_0}$ and $\mu_j = \sum_{i_0 \in I} \mu_j^{i_0}$. Notice that $\nu_i > 0$ for all $i \in I$ since $\nu_{i_0}^{i_0} = 1$ by normality. Thus we have

(7′) $\hat{u}_j^i = \left(\dfrac{1}{\nu_i}\right) \mu_j .$

To collect (harvest) the above results we state a proposition.

(1) *Proposition:* Let $(\hat{x}, \hat{y})$ be an interior (no zero components) Pareto efficient. Suppose further that $\hat{u}_{x_i}^i \neq 0$ and $\hat{f}_{y_r}^r \neq 0$ for all $i \in I$ and all $r \in R$. Then there exist numbers ν_i, μ_j and γ_r, ($i \in I$, $j \in J$ and $r \in R$) such that (7) and (8) above are satisfied and such that

(9) $\gamma_r \hat{f}^r = 0, \quad \gamma_r \geqq 0, \quad r \in L$

(10) $\mu_j \left(\sum_r y_r + \sum_i \omega_i - \sum_i x_i \right) = 0, \quad \mu_j \geqq 0, \quad j \in J$

(11) $\nu_i > 0$ for all $i \in I$.

We now turn to competitive equilibrium. The concept describes an allocation x, y and a price p. The only thing "competitive" about it is that prices are taken parametrically by all agents.

Suppose each consumer is allotted a share θ_{ri} of the profits π_r of firm r. Then the budget set of consumer i is given by:

(12) $B^i(p) = \{x_i \in X_i \mid p x_i \leqq p\omega_i + \sum_r \theta_{ri} \pi_r\}$.

We are now ready to introduce a formal definition. The triple (x^*, y^*, p^*), with $p^* \geqq 0$, is a *competitive equilibrium* if

1) Each consumer maximizes utility subject to the budget constraint, i.e., for all $i \in I$, x_i^* solves:

$\max u^i(x_i)$ subject to $x_i \in B^i(p^*)$.

2) Each firm maximizes profits subject to technology, i.e., for all $r \in L$, y_r^* solves:

$\max p^* y_r$ subject to $f^r(y_r) \leqq 0$.

3) For each good, supply is not less than demand, if supply does exceed demand then the good in question is a free good, i.e., for all $j \in J$ we have

$$\left(\sum_r y_{rj}^* + \sum_i \omega_{ij} - \sum_i x_{ij}^*\right) = E_j^* \geqq 0 \quad \text{and} \quad p_j^* E_j^* = 0 .$$

We assume now that (x^*, y^*) is an interior competitive equilibrium, $p^* \geq 0$ and that $f^{*r}_{y_r} \neq 0$ for all $r \in R$. By the first order necessary conditions (for (1) and (2)) we have: There exist λ_i and σ_r such that

(13) $u^{*i}_j = \lambda_i p_j^*$, $\quad i \in I$, $\quad j \in J$

(14) $\sigma_r f^{*r}_j = p_j^*$, $\quad r \in R$, $\quad j \in J$

(15) $\sigma_r f^{*r} = 0$.

Conditions (13), (14) and (15) should remind us of conditions (7′), (8) and (9) above. What we have proved is the following remark.

(1) *Remark:* If (x^*, y^*, p^*) is a competitive equilibrium where x^*, y^* are interior and $p^* \geq 0$, then there are multipliers such that the necessary conditions for Pareto efficiency are satisfied. On the other hand, if $(\hat{x}, \hat{y})$ is Pareto efficient, then there are prices such that the necessary conditions for competitive equilibrium are satisfied.

To see the validity of the first part of the remark, take $\nu_i = 1/\lambda_i$ (note that $\lambda_i > 0$) and $\mu_j = p_j^*$ and you get (7′), (8) and (9). To see the second part, do the reverse and get (13), (14) and (15).

The relationship between competitive equilibrium and Pareto efficiency goes far beyond the "market conditions" expressed in (13) – (15) as they relate to (7) – (9). If we assume nonsatiation, then every competitive equilibrium is Pareto efficient. Just like that with no further assumptions! Naturally one has to recall the fundamental assumption of absence of externalities. The converse of the above statement is not true in general. Technical and substantive assumptions are needed. Some type of convexity is required. In addition, one has to be willing to redistribute the initial endowments and the profit shares. We now state a global result and a local result establishing the implications of Pareto optimality.

(2) *Proposition:* If, in addition to the assumptions of Proposition 1, the utility functions u^i are concave and the functions f^r are convex, then there exist prices p^*

and a redistribution of profit shares and initial endowment such that (x^*, y^*, p^*) is a competitive equilibrium.

To prove the proposition, take $p_j^* = \mu_j$ in (7′), (8) – (11). Then by (7′) and the concavity of u^i, consumer i maximizes u^i, at x_i^*, subject to $p^* x_i \leqq p^* x_i^*$. Now let ω_i^1 and θ_{ri}^1 be such that:

$$(+) \quad p^* x_i^* = p^* \omega_i^1 + \sum_r \theta_{ri}^1 \pi_r(p^*).$$

Note that (+) is possible in view of (10) and our choice of p^*. By convexity of f^r and by (8), firm r maximizes $p^* y_r$ at y_r^* subject to $f^r \leqq 0$. That supply $\geqq$ demand follows from the feasibility of $\hat{x}$. The free good proviso follows from (10). This completes the proof of Proposition 2.

We now state a local result.

(3) *Proposition:* If in addition to the assumptions of Proposition 1 we have:

(a) $\sum_i \sum_k \hat{u}_{jk}^i \xi_j \xi_k < 0$ for all $\xi \neq 0$ with $\hat{u}_x \xi = 0$,

(b) $\sum_j \sum_k f_{jk}^r \eta_j \eta_k > 0$ for all $\eta \neq 0$ with $\hat{f}_y^r \eta = 0$.

Then the conclusion of proposition 2 holds in an neighborhood of $(\hat{x}, \hat{y})$.

To prove proposition 3 take $p^* = \mu$ in (7′) – (11). Thus we get (13) – (15). Condition (a) of this proposition states that the second order sufficient conditions for a local maximum at $\hat{x}_i$ of u^i subject to $p^* x_i \leqq p^* \hat{x}_i$ are satiesfied. Condition (b) yields sufficient conditions for a local maximum at $\hat{y}_r$ of $p^* y_r$ subject to $f^r \leqq 0$. Now intersect all of the neighborhoods implied by all of these local maxima and get the neighborhood promised in the statement of the proposition. The rest of the proof is exactly the same as that of Proposition 2.

c) Existence of Equilibria

In this section we provide an introduction to the study of existence of general competitive equilibria. We often encounter economists who claim that the proof of the existence of an equilibrium belongs to the realm of mathematicians. Here we would like to spend a brief moment to reflect upon the rational why we economists have to carry out this task of the proof of the existence of an economic equilibrium. In Economics, models we formulate theoretically are written down and stated on paper. If, in stead, we have a model of an airplane, then certainly we can actually construct the model and in fact constructing the model into an actual model airplane is the best way to find out whether or not we are missing some parts and also to check whether or not these parts we have fit together. But unfortunately we can not actually construct an economic model. Yet we also have to make sure that the model we formulate on paper can be theoretically constructed so that we can find out its consistency, i.e., whether or not we are missing some descriptions of an economy and whether or not constituencies of the economy fit together into our perception of an economy. We carry out this

examination on the possibility of actual construction by means of the proof of the existence of an economic equilibrium. This is not quite a problem for mathematics per se, although mathematicians may be very good at solving the problem. This is because mathematicians will be more interested in problems of the existence of a solution without much interpretation. We confine our attention to the case of pure exchange. To simplify matters, even further, we modify the equilibrium definition so that markets are cleared. We further assume that demand functions are single valued. Finding an equilibrium, finding a market clearing price, the amounts to solving a system of equations simultaneously. In view of the homogeneity of degree zero, of demand functions we will search for our equilibria is a set of normalized prices. Since $p > 0$ we know that $\sum_{j \in J} P_j > 0$. Define

$$S = \{P \in E^n_{++} \mid \sum p_j = 1\}.$$

We obtain p by defining $p_j = P_j / \sum p_j^1$. By homogeneity of the demand functions we know that $h(P) = h(p)$. Let $E(p) = h(p) - \omega$, where $\omega = \sum_i \omega_i$. Our problem is to find $p^* \in S$ such that $E(p^*) = 0$. Mathematically, we are trying to find a solution, in S, of the m equations $E(p) = 0$.

What is known so far is that there are several types of sufficient conditions that guarantee the existence of such a solution. We review two types of conditions that provide an exact solution and a condition that provides an approximate solution.

The first type of sufficient conditions for an exact solution is popular in physics but quite unpopular in economics. Suppose for some reason that $E(p)$ is the gradient of some function $F(p)$.

Suppose furthermore that $F(p)$ is concave (convex). Then the question of existence of $\bar{p} \in S$ such that $E(\bar{p}) = 0$ is equivalent to the question of existence of a maximum (a minimum) of the function F on S. Since F is plenty continuous and since S is compact, the answer is yes. So why then is such a simple approach rejected by economists? The answer lies in investigating conditions on E that would guarantee the existence of the right kind of F. A set of such conditions would assume that E is C^1, the Jacobian of E is symmetric and negative semi-definite (positive semi-definite). If there is only one consumer or n clones of one, if no good were a Giffen good and if the demand functions are C^1, then we have all of the conditions that we need. The problem is that the case of one consumer, or n clones thereof, is not interesting. If we allow at least two distinct consumers then, as Sonnenschein (1972) has shown, there is not much we can do in terms of restricting individual behavior to get the right aggregate excess demand function.

We now discuss a type of sufficient conditions that has been accepted by economists. It uses Brouwer's fixed point theorem. In fact all proofs of existence of equilibrium use a variant of a fixed point theorem which proves that such a theorem is essential.

We now state an existence theorem for competitive equilibria. It is not the best available theorem but we hope that the compensating clarity of representation is realized.

Proposition: Suppose for each consumer i we have: $\omega^i > 0$, u^i is strictly monotone, strictly quasi-concave and continuously differentiable. Then there exists p^* which is an equilibrium, i.e., such that $E(p^*) = 0$.

The proof follows Cassels (1981). By hypothesis given $p > 0$ the consumer's optimum, max u^i subject to $px^i = p\omega^i$, exists and is unique so we have a demand function. Furthermore that function $h^i(p)$ is continuous for $p > 0$. In order to apply Brouwer's fixed point theorem we have to allow some prices to be zero so that, instead of S, we use

$$\bar{S} = \{p \geqq 0 \mid \sum_{j \in J} p_j = 1\}.$$

The set $\bar{S}$ is a closed and bounded subset of the finite dimensional R^n so it is compact. But if a price is zero, then demand may explode and h looses the continuity which we need. We overcome this by pretending, truthfully, that ω is bounded above by a finite, but possibly very large, vector $t > 0$, i.e.

(i) $\omega < t$.

We assume further, falsely, that the consumer is aware of that bound. So he maximizes subject to the budget equation and subject to

(ii) $x^i \leqq t$.

The utility maximizing function thus begotten is not a demand function. We use this bogus demand function to get a bogus equilibrium. Then we turn around and show that the true demand function coincides with the false one at the equilibrium and we get what we want.

Let $\tilde{h}^i(p) = \text{argu max } u^i(x^i)$ over $px^i = p\omega^i$ and $x^i \leqq t$. Then we have:

(*) $\sum_i p(\tilde{h}^i(p) - \omega^i) = 0$

for all $p \in S$. *We claim that there exists* p^* *with:*

(iii) $\sum_i \tilde{h}^i(p^*) \leqq \omega$.

We prove our claim later. We use it to show that p^* yields an equilibrium under the true h. By (iii) and (i) we have

(iv) $\tilde{h}^i(p^*) < t$.

Thus the constraint (ii) is ineffective at $\tilde{h}^i(p^*)$ and so can be, locally and globally, ignored. Thus $\tilde{h}^i(p^*)$ maximizes u^i subject to the budget equation and no other constraints. By strict quasi-concavity, which implies uniqueness of optimum, $\tilde{h}^i(p^*) = h^i(p^*)$. Furthermore, strict monotonicity of u^i implies that (iii) holds as an equation so that $E(p^*) = h(p^*) - \omega = 0$.

Now we must prove (iii). Consider the funtion $g: \bar{S} \to E^n$ given by

(v) $g^j = (p_j + s_j)/G, \quad j = 1, \ldots, n,$ where

$s_j = \max\left(\sum_i \tilde{h}^{ij}(p) - \omega, 0\right)$ and $G = 1 + \sum_j s_j$. Clear g is continuous and maps $\bar{S}$

to $\bar{S}$. Thus, by Brouwer's fixed point theorem, there exists p^* such that $g(p^*) = p^*$, i.e., such that $(p^* + s)/G = p^*$ so:

$$p^* + s = \left(1 + \sum_j s_j\right) p^* = p^* + (\sum s_j)\, p^* .$$

Thus $s_j = (\sum_j s_j) p_j^*$. Since $s_j \geqq 0$ have two cases: $(\sum_j s_j) p_j^* = 0$ or $(\sum s_j) p_j > 0$. In the first case, if $p_j^* > 0$, then $\sum_j s_j = 0$ and so $s_j = 0$. Hence we have (iv), and if $p_j^* = 0$ then $s_j = 0$ and we get (iv). In the second case we must have, since $\sum_j s_j \geqq 0$, $p_j^* > 0$ and $\sum_j s_j > 0$. Thus $s_j > 0$. But then $\sum p_j s_j > 0$ which contradicts (*).

Problems

1. Consider an exchange economy with two commodities, say "x" and "y", and two traders, say, Mr. A and Mr. B. Mr. A has (strict) preference ordering given by:

   $(x, y)\, P_A\, (x', y')$ iff $(x + y > x' + y'')$ or
   $(x + y = x' + y'$ and $x > x')$.

   Mr. B has a preference ordering given by

   $(x, y)\, G_B\, (x', y')$ iff $\min\{x, y\} \geqq \min\{x', y'\}$.

   (a) Can these two preference orderings be representably by utility functions?
   (b) Can demand functions generated by these two preference orderings be representable by utility functions?
   (c) Illustrate the set of Pareto optimum allocations of this economy.

2. Consider again an exchange economy with two goods, "x" and "y", two traders, Mr. A and Mr. B. Suppose that Mr. A and Mr. B have an identical preference ordering given by

   $(x, y)\, G\, (x', y')$ iff $\min\{x, y\} \geqq \min\{x', y'\}$.

   Let $(\bar{x}_A, \bar{y}_A) = (1, 0)$ and $(\bar{x}_B, \bar{y}_B) = (0, 1)$ be the initial endowment vectors of Mr. A and Mr. B respectively.
   (a) Illustrate the set of Pareto optima of this economy.
   (b) Illustrate the set of competitive equilibria of this economy.

3. Consider an exchange economy consisting of two consumers with utility functions $u_i(x_{i1}, x_{i2})$.

   Show what would be the set of Pareto optimum allocations in the following cases, assuming positive total initial endowment of both goods, i.e., $(w_1, w_2) > 0$.

   (i) $u_1(x_{11}, x_{12}) = x_{11} + 1/2\, x_{12}$ and $u_2(x_{21}, x_{22}) = 2x_{21} + x_{22}$.

   (ii) $u_1(x_{11}, x_{12}) = \text{Min}\{x_{11}, x_{12}\}$ and $u_2(x_{21}, x_{12}\} = \text{Min}\left\{\frac{x_{21}}{2}, x_{22}\right\}$.

(iii) $u_1(x_{11}, x_{12}) = \text{Min}\{x_{11}, x_{12}\}$ and $u_2(x_{21}, x_{22}) = x_{11} + x_{12}$.
(iv) $u_1(x_{11}, x_{12}) = x_{11}$ and $u_2(x_{21}, x_{22}) = x_{22}$.
(v) $u_1(x_{11}, x_{12}) = x_{11} + x_{12}$ for $x_{11} < w_1$ and $x_{12} < w_2$.
$u_1(x_{11}, x_{12}) = -1$ if $x_{11} \geqq w_1$ or $x_{12} \geqq w_2$ and
$u_2(x_{21}, x_{22}) = 1$ for every $(x_{11}, x_{12}) \geqq 0$.

4. Consider an exchange economy consisting of two consumers, say Mr. A and Mr. B. Preferences of Mr. A and Mr. B are given respectively by $(x_A, y_A) P_A (x'_A, y'_A)$ if and only if $[x_A > x'_A]$ or $[x_A = x'_A$ and $y_A > y'_A]$, and $(x_B, y_B) P_B (x'_B, y'_B)$ if and only if $[y_B > y'_B]$ or $[y_B = Y'_B$ and $x_B > x'_B]$. Assume that initial holdings of Mr. and Mr. B are $(0, w_A)$ and $(w_B, 0)$ respectively where $w_A > 0$ and $w_B > 0$.

Questions:
a. What is the set of individually rational* allocations?
b. What is the set of pareto optimum allocations?

5. Consider an exchange economy with two commodities and two consumer agents. Consumer agent i is characterized by (X_i, G_i) where X_i is a consumption set and G_i is a preference ordering on X_i. Suppose that $X_i = R^2_+$ (the nonnegative quadrant of R^2) and $x_i \in X_i$ denotes $x_i = (x_{i1}, x_{i2})$. Assume that $x_1 G_1 x'_1$ if and only if $\min\{x_{11}, x_{12}\} \geqq \min\{x'_{11}, x'_{12}\}$ and $x_2 G_2 x'_2$ if and only if $x_{21} + x_{22} \geqq x'_{21} + x'_{22}$. The total initial endowment vector is given by (12, 4).

(a) Find the set of Pareto optimum allocations.

(b) Suppose that the distribution of the initial endowment is $W_1 = (12, 0)$ and $W_2 = (0, 4)$. Find the set of competitive equilibria.

(c) Suppose now that the distribution of the initial endowment is $W_1 = (8, 4)$ and $W_2 = (4, 0)$. What would be the set of competitive equilibria.

6. Suppose that there are two consumers, Mr. 1 and Mr. 2. The utility function of Mr. 1 is $u_1(x_{11}, x_{12}) = \alpha \log x_{11} + (1-\alpha) \log x_{12}$ for $x_{11} \geq 0$, $x_{12} \geq 0$ and the utility function of Mr. 2 is $u_2(x_{21}, x_{22}) = x_{21} + \beta x_{22}$ for $x_{21} \geqq 0$, $x_{22} \geqq 0$.

(i) Obtain the set of Pareto optimum points with the total initial endowment $(w_1, w_2) > 0$.

(ii) Suppose that the initial endowment of Mr. 1 is $(w_1, 0)$ and the initial endowment of Mr. 2 is $(0, w_2)$. Obtain the completitive equilibrium.

7. Consider the following simple economy with a laborer and a landowner. We assume that:

(a) The production function is given by $y = AL^{1/2}$ where A is a constant parameter, y is the output of wheat, L is the input of labor and land is a fixed input implicit in the production function.

(b) The laborer has a utility function given by

$(u(x, z) = \alpha \log x + (1-\alpha) \log z$

* An allocation is said to be *individually rational* if each individual is at least as well off as he would be at a consumption vector equal to his initial endowment.

where α is constant, x is the consumption of wheat and z denotes leisure.

(c) $z = \bar{L} - L$ where L is labor supply and $\bar{L}$ is the total labor hours.

(d) Let the price of wheat be one and let W be the hourly wage rate.

(e) There is no cost of leisure.

(f) The land-owner obtains rent income in terms of wheat and he consumes it.

Questions:

(i) Find a competitive equilibrium of the above economy.

(ii) Show what would happen to the wage rate and the rent income when $\bar{L}$ increases (or the increase of labor force) and also when A increases (or an increase in productivity).

8. Consider the following simple economy.

(i) There is only one consumer with the utility function

$$u(x, \bar{L} - L) = x^{\alpha}(\bar{L} - L)^{\beta}$$

where x is the amount of the aggregated consumption goods, L is the supply of labor hours and $\bar{L}$ is the total labor hours available.

(ii) There is one production sector with the production function

$$x = f(L) = \begin{cases} 0 & \text{if } 0 < L < C \\ L - C & \text{if } C \leqq L \end{cases}$$

where C is a fixed constant (the minimum requirement of labor in production).

Questions:

(a) Find the optimum allocation of labor hours, i.e., the allocation which maximizes the consumer's welfare.

(b) Let the wage rate = 1 and p = the price of the aggregated consumption goods. Then show that p = 1 is the price compatible with the Pareto optimum point but the profit of the production sector is negative.

9. Consider the following simple economy:

(i) There are two final commodities, say commodity 1 and commodity 2.

(ii) Production functions of commodities are given by $y_1 = L_1$ and $y_2 = L_2^{1/\beta}$ where y_i is the output of commodity i; $L_i \geqq 0$ is the input of labor to produce y_i and β is a constant with $\beta \geqq 1$.

(iii) The supply of labor is fixed at $L > 0$.

(iv) There is a single consumer with his utility function given by $u(x_1, x_2) = \alpha_1 \log x_1 + \alpha_2 \log x_2$ where α_1, α_2 are positive constants and x_i is the consumption of commodity i.

Questions:

(a) Obtain a competitive equilibrium of the above economy.

(b) Discuss the stability of a competitive equilibrium obtained in (i).

10. Consider the following economy with two final commodities, say "x" and "y", two production sectors, say, sector x and sector y, and two consumers.

(a) Consumer i $(i = 1, 2)$ has the following utility function: $u^i(x^i, y^i) = y^i + \alpha^i \log x^i$ where $x^i \geqq 0$, $y^i \geqq 0$ and α^i is a positive constant.

(b) Consumer i obtains a fixed proportion θ_i of the produced national income where $\theta_i > 0$ and $\theta_1 + \theta_2 = 1$.

(c) Production functions of sector x and sector y are given respectively by: $x = L_x^{1/2}$ and $y = (1/\beta) L_y$ where x and y denote the output of "x" and the output of "y", L and L_y are respectively inputs of labor to produce (x) and (y).

(d) There is a fixed supply of labor denoted by L. Assume that $L > (1/2)\beta(\alpha_1 + \alpha_2)$.

Questions:

(i) Find a competitive equilibrium.

(ii) Find the set of all Pareto optimum allocations.

11. Consider the following simple economy:

(i) There are two commodities, say "x" and "y" produced by two production sectors.

(ii) Production functions of "x" and "y" are given respectively by $x = aL_x (a > 0)$ and $y = bL_y^{1/2} (b > 0)$ where x and y are the output of "x" and the output of "y" respectively and L_x and L_y are the input of labor to produce "x" and "y" respectively.

(iii) The supply of labor is fixed at $L > 0$.

(iv) There is a single consumer with this utility function given by $u(C_x, C_y) = \log C_x + \log C_y$ where C_x and C_y are the consumption of (X) and (Y) respectively.

(v) All the wage income and the profit of two sectors go to the consumer.

Questions:

(a) Obtain the product transformation schedule and the marginal rate of transformation.

(b) Obtain the marginal rate of substitution of the consumption.

(c) Obtain a competitive equilibrium of this economy.

12. Consider an economy consisting of one producer and one consumer. Assume that:

(i) The production function of the firm is given by $f(L) = 0$ for $0 \leqq L \leqq c^2/4$ and $f(L) = 2L^{1/2} - c$ for $L > c^2/4$ where L is the input of labor and c is a positive constant.

(ii) The utility function of the consumer is given by $u(x, L) = \alpha \log x + (1 - \alpha) \log (\bar{L} - L)$ where x is the consumption of the product, L is the supply of labor, $\bar{L}$ is the maximum labor hours available and α is a positive constant.

(iii) Both the producer and the consumer are price takers. Let p be the price of the product and $w = 1$ be the wage rate.

Questions:

(a) Obtain the supply function of the product by the firm, and also write down the demand function of the product by the consumer.

(b) Find an equilibrium price and a condition which guarantees its existence.

13. Consider an economy with a laborer and a capitalist where there is one consumption good produced by a labor input. Assume that:

(i) The production function is given by $f(L) = 0$ for $0 \leqq L < A^3/4$ and $f(L) = 2L^{1/2} - A$ for $L > A^2/4$ where L is a labor input and A is a positive constant.

(ii) The utility function of the laborer is given by $u(L, x) = (\bar{L} - L) + \alpha \log x$ where x denotes the aggregate consumption good consumed by the laborer and $\bar{L}$ is the maximum labor supply feasible.

(iii) The capitalist obtains the profit in terms of the aggregate consumption good and he consumes it.

(a) Find the shut-down point of the production function.

(b) Derive supply functions and demand functions and find a competitive equilibrium of this economy if it exists.

Mathematical Appendix

Introduction

In this appendix we collect together some of the significant mathematical results used in the text. The idea has never been to attempt the impossible feat of teaching all the required mathematics but to set the tools out for the record. Because we chose the conversational narrative form of presentation we had to include some topics that are not specifically used in the text and we had to leave some concepts out. We hope that the appendix will provide a convenient reference for the reader who knows the material and that if offers an incentive for serious study to those who do not.

1. Relations and Functions

Let X and Y be sets. The Cartesian, cross, product of X and Y written as $X \times Y$ is defined as the set of ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A relation $\mathscr{R}$ from X to Y, written $\mathscr{R}: X \to Y$ is a nonempty subset of $X \times Y$. We write $y \in \mathscr{R}(x)$ sometimes and write $(x, y) \in \mathscr{R}$ or $x \mathscr{R} y$ some other times. The inverse of a relation $\mathscr{R}$, written as $\mathscr{R}^{-1}$ is a subset of $Y \times X$ given by: $(y, x) \in \mathscr{R}^{-1}$ if and only if $(x, y) \in \mathscr{R}$. A *relation on* X is a relation from X to X, i.e., a subset of $X \times X$. A relation $\mathscr{R}$ on X is a partial ordering of X if $\mathscr{R}$ is *reflexive*, i.e. $x \mathscr{R} x$, and *transitive*, i.e., $x \mathscr{R} y$ and $y \mathscr{R} z$ imply $x \mathscr{R} z$. The set X is then said to be partially ordered by $\mathscr{R}$. Let X be partially ordered by $\mathscr{R}$. If in addition $\mathscr{R}$ is *antisymmetric*, i.e. $x \mathscr{R} y$ and $y \mathscr{R} x$ imply $x = y$, and if $\mathscr{R}$ is *complete*, i.e. for all x and y in X either $x \mathscr{R} y$ or $y \mathscr{R} x$, then we say that $\mathscr{R}$ is a *total order* on X. A relation ϱ on X is an *equivalence relation* if it is reflexive, transitive and *symmetric*, i.e. $x \varrho y$ implies $y \varrho x$. Let ϱ be a equivalence relation on X, the set of points in X equivalent to a point y, i.e., the set $\{x \in X \mid x \varrho y\}$ is called the equivalent class of y. Example: The set of real numbers is totally ordered by the inequality "$\geqq$", the equality "$=$" is an equivalence relation on this set. The set of ordered pairs of real numbers (x, y) is partially ordered by the inequality "$\geqq$" defined by $(x, y) \geqq (x', y')$ iff $x \geqq x'$ and $y \geqq y'$ but is not completely ordered by it.

Let f be a relation from X to Y. Suppose $(x, y) \in f$ and $(x, y') \in f$ imply $y = y'$. Then f is said to be a function form X to Y and we write $x \to y = f(x)$. The domain of f, D(f), is the set $\{x \in X \mid (x, y) \in f\}$ and the range of f, R(f), is the set $\{y \in Y \mid (x, y) \in f\}$. Let $f: X \to Y$ be a function with $D(f) = X$. The inverse of f, written as f^{-1}, is the inverse relation of f, i.e., it is defined by $(y, x) \in f^{-1}$ iff $(x, y) \in f$. If f^{-1} is a function, then we say the f is *1 – 1* or *injective*. If $R(f) = Y$, then f is

onto or *surjective*. If f is injective and surjective, then it is *bijective*. Example: $f(x) = x^3$ is bijective with $X = Y = R$ = the set of real numbers, $f(x) = x^2$ from R to R is neither injective nor surjective, finally $f(x) = 2x$ from positive integers to positive integers is injective but not surjective. Let X, Y, Z be sets and let f: $X \to Y$ and g: $Y \to Z$ be functions. The *composition* of f and g written $g \circ f$ is given by the rule $x \to (g \circ f)(x) = g(f(x))$. A special function from X to X is the *identity function* denoted by id. The identity function is given by: $x \to id(x) = x$. Note that $f \circ f^{-1} = id$.

2. Linear Spaces

Let X be a set. Suppose addition on and multiplication by real numbers are defined on $X \times X$ and $R \times X$ respectively and with values in X. Then we say X is closed under these operations. Suppose the two operations, called *linear operations,* are *commutative, associative* and *distributive,* and suppose that both operations have unique neutral elements, then we say that X is a real linear space. In other words, if we have: For all x, y, z in X and all $\alpha, \beta \in R$, $x+y = y+x$, $(x+y)+z = x+(y+z)$, there uniquely exists $0 \in X$ such that $0+x = x$, $\alpha(x+y) = \alpha x + \alpha y$, $(\alpha+\beta)x = \alpha x + \beta x$ and $1x = x$, then X is a real linear space. Elements of X are occasionally called vectors. The set of n-tuples of real numbers with addition and multiplication defined componentwise is a real linear space. It is the one used most often in this book.

A subset V of a real linear space X is said to be a *linear subspace* of X if it is closed under the same linear operations as X, i.e., if $(\alpha x + \beta y) \in V$ for all x, $y \in V$ and all $\alpha, \beta \in R$. Let A be a subset of a linear space X. A *linear combination* of Ais an element $x \in X$ given by $x = \sum_{i=1}^{n} \alpha_i y_i$ where $y_i \in A$ and $\alpha_i \in R$ for $i = 1, \ldots, n < \infty$. A linear combination of A is *nontrivial* if the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \neq 0$; otherwise, the linear combination is trivial. A subset A of a linear space X is *linearly independent* if whenever 0 is a linear combination of A it is a trivial linear combination, i.e., $0 = \sum_{i=1}^{n} \alpha_i y_i$, $y_i \in A$ implies $\alpha_i = 0$ for all i. A set that is not linearly independent is *linearly dependent*. The largest linearly independent subset of X is called a *Hamel basis* of X. Such a set always exists as a matter of faith. The *dimension* of a space X, dim X, is the number of elements in its Hamel basis. Even though a Hamel basis is not unique, the dimension of the space is. Example: Let X be the set of ordered pairs of real numbers. The sets $\{(0, 1), (1, 0)\}$ and $\{(-1, 1), (1, 1)\}$ are Hamel basis of X, Dim X = 2. Let A be a subset of X. A *translation* of A by x is the set $\{x + a \mid a \in A\}$. If A is a subspace of X, then a translation of A is called an *affine subspace* of X. Let x and y be two elements of X. The *line* through x and y is the set: $\{\alpha x + (1-\alpha) y \mid \alpha \in R\}$. The *line segment* connecting x and y is set: $\{\alpha x + \beta y \mid \alpha \geqq 0,\ \beta \geqq 0,\ \alpha + \beta = 1\}$, which we write as [x, y]. When $0 < \alpha < 1$ and $0 < \beta < 1$, i.e., when we wish to exclude x and y, then we denote the line segment by (x, y)*. A set is *convex* if it contains the

* It will be clear from the context whether we mean an interval or an ordered pair by (x, y).

line segment [x, y] connecting any two points x, y in it. Let X and Y be linear spaces. A function f: $X \to Y$ is *additive* if $f(x+y) = f(x) = f(y)$ for all $x, y \in X$; it is *homogenous* if for all $x \in X$ and $\alpha \in R$ we have $f(\alpha x) = \alpha f(x)$. If f is additive and homogeneous it is said to be *linear*.

Let $x, y \in X$ be given. Suppose $\dim X = n < \infty$. We define the *inner product* $x \cdot y$ of x and y by: $x \cdot y = \sum_{i=1}^{n} x_i y_i$. The inner product is commutative and distributive and homogenous, i.e., for all x, y, Z in X and all $\alpha, \beta \in R$ we have: $x \cdot y = y \cdot x$, $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(\alpha x) \cdot (\beta y) = (\alpha\beta) x \cdot y$.

A function f: $X \to R$ is called a *functional*. If $\mathrm{Dim}\, X = n < \infty$ and we fix a basis for x, then *linear functionals* can be written as inner product, i.e., we can write: $f(x) = a \cdot x$ where $a \in X$. We can think of a linear function as a vector of linear functionals. For instance, let $\dim X = n$ and let $\mathrm{Dim}\, Y = m$ and fix basis for X and Y. Let f: $X \to Y$ be linear. Then f is m linear functionals; $f^1(x) = (f^1(x), \ldots, f^m(x))$. We then can write: $f^i = a_i \cdot x = \sum_{j=1}^{n} a_{ij} x_j$, $i = 1, \ldots, m$.

We define a matrix as an array of numbers, for example

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

has m rows and n columns. The sum of two matrices of similar dimensions is the matrix whose elements are sums of respective elements, i.e., $C = A + B$ is given by $(c_{ij}) = (a_{ij} + b_{ij})$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. The product of two matrices A, which is $m \times k$, and B, which is $k \times n$, is the $m \times n$ matrix C with $c_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$ where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. A linear function f can be represented, relative to given basis of X and Y by: $f(x) = AX$, where A is $m \times n$ and $x \in X$ is taken to be an $n \times 1$ matrix.

Let A be a *square matrix*, i.e., $m = n$, then we can define a real number det A. Operationally, let i be any row of A, then $\det A = \sum_j (-1)^{i+j} a_{ij} \operatorname{cof} A_{ij}$, where $\operatorname{cof} A_{ij}$ = the determinant of the square matrix obtained from A by eliminating i^{th} row and the jth column.

The *rank* of a matrix is the order of the largest square submatrix of it which has a nonzero determinent. If A is a square matrix of order n and there exists a unique matrix A^{-1} such that $A^{-1}A = I$, where I is such that $a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, then we say that A is invertible and its inverse is A^{-1}, I is called the identity matrix. A square matrix is *invertable* iff its rank equals its order. The *transpose* of a matrix A, written A′, is given $a^*_{ij} = a_{ji}$ so we interchange rows and columns. A matrix is *symmetric* if $A^* = A$. Consider the function Q: $X \to R$ defined by: $Q(x) = x^* A x = \sum_{i,j} a_{ij} x_i x_j$ where A is a square symmetric matrix. The function Q is called a *quadratic form*. A quadratic form is *positive definite* if $Q(x) > 0$ for

all $x \neq 0$. If $Q(x) \geqq 0$ for all x, then Q is positive semi-definite. A quadratic form is negative definite (semi-definite) if $-Q$ is positive definite (semi-definite).

Let A be a square matrix of order n. The principle minors of A of order s are the determanents of the square submatrices of A of order s given by:

$$A_s = \begin{pmatrix} i_1 i_2 \ldots i_s \\ i_1 i_2 \ldots i_s \end{pmatrix}, \quad i_1 < i_2 < \ldots < i_s,$$

and where the matrix A_s is:

$$A_s = \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_s} \\ a_{i_s i_1} & a_{i_s i_2} & \cdots & a_{i_s i_s} \end{pmatrix}.$$

A necessary and sufficient condition for Q to be positive semi-definite is non-negative that *all* principal minors of order s to be $s = 1, 2, \ldots, n$. The principal minors successive of A (northeast minors of A) are the determinants of the square submatrices of A given by $\begin{pmatrix} 1 & 2 & \ldots & s \\ 1 & 2 & \ldots & s \end{pmatrix}$, $s = 1, \ldots, n$. It is not enough to check only the successive principal diagonal minors in order to establish positive (negative) semi-definiteness. For instance the successive principal minors of $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ are all non-negative and yet the quadratic form defined by it is not positive semi-definite since $Q = -x_2^2 < 0$ for $x_2 \neq 0$. A necessary and sufficient condition for a quadratic form Q to be positive definite is that successive principal minors are positive. If the principal successive minors are positive, then all principle diagonal minors are positive.

Let B be an $m \times n$ matrix of rank $m < n$. We sometimes need to investigate the definiteness (semi-definiteness) of Q subject to: $Bx = 0$. We then say we are investigating a *constrained quadratic form*. In preparation for characterization of these forms, we define the matrix:

$$\Delta = \begin{pmatrix} 0 & \vdots & B \\ \cdots & \cdots & \cdots \\ B' & \vdots & A \end{pmatrix},$$

where 0 is a square matrix of zeros of order m.

A necessary and sufficient condition for the constrained quadratic form to be positive definite is that the successive principal minors of Δ of order s have the sign of $(-1)^m$, where $s = 2m+1, \ldots, m+n$. A necessary and sufficient condition for the constrained quadratic form to be positive semi-definite is that the principal minors of order s of the matrix either have the sign of $(-1)^m$ or vanish, $s = 2m+1, \ldots, m+n$. For negative definiteness and semi-definiteness, replace "sign of $(-1)^m$" by "sign of $(-1)^{m+s}$" in the respective condition.

3. Calculus

Before we proceed further, we need to discuss "being close." IF we make this concept definite then we have a topological structure. In this book we deal nearly exclusively with a finite dimensional real linear space where *distance* between two

points x and y, written $d(x, y)$, is defined by: $d(x, y) = [(x-y) \cdot (x-y)]^{1/2}$. We say that N is a *neighborhood* of x if there exists $\varepsilon > 0$ such that $N = \{y \mid d(x, y) < \varepsilon\}$. With distances defined as above our n-dimensional linear space is called a *Euclidean n-space* and is denoted here by R^n. In R^n the *length* of an element x, denoted by $\|x\|$ is given by $\|x\| = d(x, 0)$.

A functional $f: R^n \to R$ is continuous at x_0 if for all $\varepsilon > 0$ there exist $\delta > 0$ such that $d(x_1 x_0) < \delta$ implies $d(f(x), f(x_0)) < \varepsilon$. If f is continuous at all $x \in D(f)$, then we say f is continuous. Let $F: R^n \to R^m$ be given. We can think of F as a vector of functionals; $F = (f^1, \ldots, f^m)$. F is continuous if all of its component functionals is continuous. A real functional $f: R^n \to R$ is said to be *(Frechét) differentiable* at x_0 if there exist two functionals h and ϕ, both depending on x_0, such that:

$$f(x_0 + a) - f(x_0) = h(x_0; a) + \phi(x_0; a)$$

where h is linear in a and where $\lim_{a \to 0} \left(\frac{1}{\|a\|}\right)(\phi(x_0; a)) = 0$. The linear functional h is the *Frechét differential* of f at x_0. Frechét differentials are often simply called differentials.

A function $f: R^n \to R$ is *Gateaux differentiable* at x_0 in the direction of a if the limit

$$\lim_{t \to 0} \frac{f(x_0 + ta) - f(x_0)}{t} = g(x_0, a)$$

exists. Let l_i be a vector of zeros except that the ith component is l. If the Gateaux differential at x_0 in the direction of l_i exists then we call it the partial derivative of f at x_0 with respect to the ith component of x. We denote such a *partial derivative* by $\mathring{f}_i$ or by $(\partial f/\partial x_i)_{x = x_0}$. If $\mathring{f}_i$ exists for each $i = 1, \ldots, n$, we denote by $\mathring{f}_i$ the vector whose components are $\mathring{f}_i$. If f is Frechét differentiable, then it is Gateaux differentiable. We, then, further have: $h(x_0; a) = g(x_0, s) = \mathring{f}_x \cdot a$.

A Frechét differentiable function is continuous. A continuous function need not be differentiable. A Gateaux differentiable function need not be Frechét differentiable. For instance the function:

$$f(x_1, x_2) = \begin{cases} x_1^2 x_2^2 / x_1^2 + x_2^2, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

is not continuous at 0 and, hence, is not Frechét differentiable. Yet it is Gateaux differentiable at zero in all directions.

If the Gateaux differential (or the directional derivative) exists in all directions and is continuous in x, then the function is Frechét differentiable. If the partial derivatives exist and are continuous, then we say that the function is continuously differentiable.

The function $F: R^n \to R^m$ is differentiable if each of the component functionals, f^i, is differentiable. The matrix valued function

$$\mathring{F}_x = \begin{pmatrix} \mathring{f}_x^1 \\ \vdots \\ \mathring{f}_x^m \end{pmatrix}$$

is the *Jacobian* of F at x_0.

The first differential provides a linear approximation of differentiable functions. Another way of stating this is the *mean value theorem:* Let x and y be given points and let [x, y] be the line segment connecting them. Suppose f is continuous on [x, y] and let f be differentiable on (x, y) which is the line segment connecting x and y excluding x and y. Then there exists a number $\gamma \in (0, 1)$ such that: $f(y) - f(x) = \tilde{f}_x \cdot (x - y)$, where $\tilde{f}_x$ is the Frechét derivative of f at $\tilde{x} = x + sy$.

The second Frechét differential is a quadratic approximation of a function, if it exists. More concretely, a function $f: R^n \to R$ has a *second Frechét differential* at x_0 if there exist a linear function h, a quadratic form Q and a function θ all depending on x_0 such that:

$$f(x_0 + a) - f(x_0) = L(x_0; a) + Q(x_0; a) + \theta(x_0; a), \quad \text{where} \lim_{\|a\| \to 0} \theta(x^0; a)/\|a\|^2 = 0 .$$

Since $x \in R^n$ we can write Q as $Q = a\mathring{H}a'$, where $\mathring{H}$ is the $n \times n$ matrix given by:

$$\mathring{H} = \mathring{f}_{xx} = \left[\frac{\partial^2 f}{\partial x_i \, \partial x_j} \right] \Bigg|_{x = x_0} .$$

The matrix $\mathring{f}_{xx}$ is called the Hessian of the function f at x_0. If f_{xx} is continuous in x then $\mathring{f}_{xx}$ is symmetric.

Higher order derivatives may be defined by repeatedly differentiating derivatives of lower orders. If the derivatives of order r of f exist, then we say $f \in D^r$. If these derivatives are continuous, then we write $f \in C^r$.

Let $F: R^{n+m} \to R^n$ be given, so that $F(x, y) = (F^1, \ldots, F^n)$, where $x \in R^n$, $y \in R^m$ and $F^i: R^{n+m} \to R$, $i = 1, \ldots, n$. Suppose F is additively separable in x and y and linear in x, i.e., F(x, y) can be written as $F(x, y) = AX - G(y)$, where A is an $n \times n$ matrix and where $G: R^m \to R^n$. A necessary and sufficient condition for solving: $F(x, y) = 0$ is that A is nonsingular. In fact we can write $x = A^{-1}G(y) = \xi(y)$. Note that $\xi(y)$ is unique and is as differentiable as G(y). Furthermore, obviously, $F(\xi(y), y) = 0$ for all y.

Things are not that simple in the case where F(x, y) is not linear in x and additively separable in y. First, we don't have necessary conditions for "solvability." We don't even have sufficient conditions for solvability. The implicit function theorem provides conditions for extending a "solution" uniquely i.e. starting from a point $(\mathring{x}, \mathring{y})$ where $F(\mathring{x}, \mathring{y}) = 0$, it provides conditions for the existence of a (locally unique) solution $x = \xi(y)$ such that $F(\xi(y), y) = 0$ is valid neary $\mathring{y}$.

(3.1) *Implicit Function Theorem:* Suppose $F(\mathring{x}, \mathring{y}) = 0$ for some $(\mathring{x}, \mathring{y})$. Suppose F is C^1 in x and C in y over some neighborhood of $(\mathring{x}, \mathring{y})$. Suppose further that the Jacobian $\mathring{F}_x$ of F with respect to x at $(\mathring{x}, \mathring{y})$ is nonzero. Then there exists a neighborhood $N(y_0)$ of y_0 and a unique function $\xi: N(y_0) \to R^n$ such that $\xi(y)$ is continuous on $N(y_0)$ and $F(\xi(y), y) = 0$ on $N(y_0)$. If F is C^q in x and y, then ξ is also C^q. In this case, for $q \geqq 1$, we have $\mathring{\eta}_y = {}^0\bar{F}_x^1 \mathring{F}_y$, where $\mathring{\xi}_x$ and $\mathring{F}_y$ are matrices of partial derivatives of η with respect to x and F with respect to y, respectively, evaluated at $(\mathring{x}, \mathring{y})$.

An example where the theorem's conditions are not all satisfied is given by $F(x, y) = x^3 + y^3 - 3axy$ with $a \neq 0$ (here $n = m = 1$). Take $(\mathring{x}, \mathring{y}) = (0, 0)$. Then

$\mathring{F}_x = 0$ and it is not possible to solve for x as a unique function of y as seen by the graph:

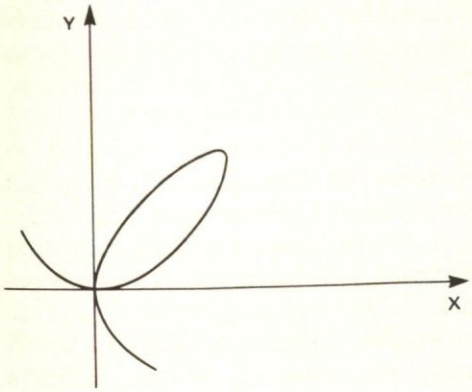

Fig. A.2.1

The reader should not worry too much since no self respecting marginal utility function has derivatives like that (!).

One immediate application of differentiation is the characterization of homogeneous functions. Let f: $D \to R$ be given where D is an open cone in R^n. The function f is said to be positively homogeneous of degree s if and only if $f(\lambda x) = \lambda^s f(x)$ for all $x \in D$, $\lambda > 0$.

(3.2) *Euler's Formula.* Suppose f is C^1 on D. Then f is positively homogeneous of degree s if and only if $f_x \cdot x = s f(x)$ for all $x \in D$.

(We note that most economics books emphasize the trivial direction of the implication in the above theorem ($\Rightarrow$) and fail to mention the other direction.)

4. Concavity and Generalization

Let f: $D \to R$ be given where D is a convex subset of R^n. The function f is concave if a line strung between any two points on its graph never lies above that graph. Formally: The function f is *concave* if and only if for any x and y and for any $\theta \in [0, 1]$ we have:

(4.1) $f(\theta x + (1-\theta) y) \geqq \theta f(x) + (1-\theta) f(y)$.

If $(-f)$ is concave, then f is a *convex function.* If a function is both convex and concave then it is *affine.* In what follows, whenever we discuss differentiability, we take D to be an open subset of R^n.

Our first theorem states that a concave function lies below its tangent always.

(4.2) *Theorem:* First Order Characterization of Concavity. Let f be differentiable. f is concave if and only if

$$f(x) - f(\mathring{x}) \leqq \mathring{f}_x \cdot (x - \mathring{x}) \quad \text{for all } x \text{ and } \mathring{x}.$$

The following graph illustrates the theorem.

We next have a *second order characterization of concavity.*

(4.3) *Theorem:* Second Derivative Characterization of Concavity. Let f be C^2. f is concave if and only if $\sum_i \sum_j f_{ij} a_i a_j \leqq 0$ for all x and for all a, where (f_{ij}) is the Hessian of f at x.

If, in (4.1) $x \neq y$ and the inequality is strong for $\theta \in (0, 1)$, then f is said to be *strictly concave.* A function f which is C^1 is strictly concave if and only if for all $x \neq \mathring{x}$ we have:

$$f(x) - f(\mathring{x}) < \mathring{f}_x \cdot (x - \mathring{x}) .$$

The *second order characterization of strict concavity* is not as nice. It is true that if $a f_{xx} a' < 0$ for all x and for all $a \neq 0$, then f (which is assumed to be C^2) is strictly concave. However, it is not true that if f is strictly concave then $a f_{xx} a' < 0$ for all x for all $a \neq 0$. The function $f(x) = -x^4$ has a zero Hessian at $x = 0$ and is certainly strictly concave. It is true however that if $f \in C^2$ and strictly concave, then the set of points where the Hessian of f is singular is relatively small.

As we have seen, concavity has been used by economists to study sufficient conditions for maxima and to study sensitivity analysis in optimization theory. The condition of concavity is too strong and hence various generalizations have been introduced to provide enough concavity to perform the task at hand but still not require full concavity. The first of these is quasi-concavity.

A function f is *quasi-concave* if and only if for all x and y we have

(4.4) $f(x) \geqq f(y)$ implies that $f(\theta x + (1-\theta) y) \geqq f(y)$ for all $\theta \in [0, 1]$.

So f is quasi-concave if and only if the upper contour sets are convex sets. If f is concave, then it is quasi-concave. The converse is not true. For instance $f(x) = x^3$ is quasi-concave but it not concave. Also $g(x) = x_1 x_2$ is quasi-concave but not concave. Both f and g can be concavified, i.e. can be transformed into concave functions via a monotone increasing transformation e.g. $\phi(y) = y^{1/10}$ would work for both. Thus if they were ordinal utility functions, the transformed functions would represent the same preferences. It is not true in general that a quasi-concave function can be concavified but some can. We will return to this later but for now we characterize quasi-concavity of differentiable functions. First, first order characterization.

(4.5) *Theorem:* First Order Characterization of Quasi-Concavity. Let f be C^1. Then f is quasi-concave if and only if $f(x^2) - f(x^1) \geqq 0$ implies $f_x^1 \cdot (x^2 - x^1) \geqq 0$ for all x^1 and x^2.

Next we state a second order characterization of quasi-concavity.

(4.5) *Theorem:* Second Order Characterization of Quasi-Concavity. Let f be C^2. Then f is quasi-concave if and only if $\sum_i \sum_j f_{ij} a_i a_j \leqq 0$ *for all* x *and for all* a *with* $f_x \cdot a = 0$.

The theorem states that if we restrict ourselves (our increments really) to a level curve of f (an isoquant or an indifference surface), then f behaves in a concave manner.

We define *strict quasi-concavity* by making the *inequalities* in (4.4) strong, i.e. f is strictly quasi-concave iff for all $x \neq y$ and for $\theta \in (0, 1)$ we have: $f(x) > f(y)$ implies $f(\theta x + (1-\theta) y) > f(y)$. If f is C^1, then a *first order character-*

ization of strict quasi-concavity is that f is strictly quasi-concave if and only if $f(x^2) - f(x^1) > 0$ implies that $f_x^1 \cdot (x^2 - x^1) > 0$ for all x^1 and all x^2 with $x^1 \neq x^2$.

check

Second order type conditions for strict quasi-concavity parallel those of strict concavity. If f is C^2 and $a\, f_{xx} a' < 0$ for all $a \neq 0$ with $f_x a = 0$ and for all x, then f is strictly quasi-concave. The converse is not true as one can see by considering the function $f(x) = x_1 x_2$. The Hessian of f is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ so the second differential, $a\, f_{xx} a'$, is: $2\, a_1 a_2$. The restriction of a to $f_x \cdot a = 0$ yields $x_2 a_1 = -x_1 a_2$. If either x_1 or x_2 is zero, then a is restricted only by $a \neq 0$. Take, say, $a_1 = 0$ and get $2 a_1 a_2 = 0$. But f is strictly quasi-concave.

We now define *pseudo concavity* of C^1 functions. A C^1 function f is pseudo concave if $f(x^2) - f(x^1) > 0$ implies $f_x^1 \cdot (x^2 - x^1) > 0$, for all x^1 and all x^2 with $x^1 \neq x^2$. The last conditional sentence is equivalent to:

If $f_x^1 \cdot (x^2 - x_1) \leqq 0$, then $f(x^1) \geqq f(x^2)$. Finally a C^1 function is strictly pseudo concave if and only if $f(x^2) - f(x^1) \geqq 0$ implies $f_x^1 \cdot (x^2 - x^1) > 0$ for all x^2 and x^1 with $x^1 \neq x^2$. Strict pseudo-concavity is equivalent to:

If $f_x^1 \cdot (x^2 - x^1) \leqq 0$, then $f(x^1) > f(x^2)$ where x^1 and x^2 are arbitrary except for $x^1 \neq x^2$.

We have mentioned some of the relationships between all of these types of concavity. We summarize the relationships in the following schema. The requisite degree of differentiability is assumed. A solid arrow indicates implication and dotted are indicators of "does not apply."

imply

strict concavity $\rightarrow$ concavity $\rightarrow$ strict pseudo concavity $\rightarrow$ pseudo concavity $\rightarrow$
strict quasi-concavity $\rightarrow$ quasi-concavity.

Using information about quadratic forms we characterize quasi-concavity of C^2 functions. *Let* f *be* C^2, *then* f *is quasi-concave iff for all* x, *the principal minors of the matrix*

$$\tilde{H} = \begin{pmatrix} 0 & f_x \\ f_x' & f_{xx} \end{pmatrix}$$

starting with the third either vanish or have the sign of $(-1)^{k+1}$. One could verify easily that $f(x) = x_1 x_2$ is quasi-concave whenever $x_1 x_2 \geqq 0$ but is never concave. For $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$\tilde{H} = \begin{pmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{pmatrix},$$

the principal minor of H of order 1 are zeros, but the principal minors of order 2 is -1. Thus f is not concave. To verify quasi-concavity, we check the signs of principal minors of $\tilde{H}$ from order 3 on. There is only one; the determinant of $\tilde{H}$. It is $2 x_1 x_2$ which is positive if x_1 and x_2 have the same sign and is zero if either is zero. Thus f is quasi-concave.

We state that a sufficient, but not necessary, *second order condition* for f to be *strictly concave* is that the successive principal minors of f_{xx} alternate in sign with the first being negative. This must hold for all x. Finally, a sufficient, but not necessary, *second order condition* for f to be *strictly quasi-concave* is that for

all x the successive principal minors of $\tilde{H}$ of order k, starting with $k = 2$, have the sign of $(-1)^{k+1}$.

We conclude this section with some remarks about continuity and differentiability of concave functions. In case $n = 1$, i.e. $f: D \to R$, $D \subseteq R$, good things happen. If f is concave on an interval I then it is continuous on the interior of that interval and differentiable at all, except a countable number of points of I. For $n \geqq 2$ it gets complicated. Let $f: A \to R$ be convex on a convex *open* subset A of R^n. Then f is continuous on A. As far as differentiability is concerned, a concave function on an open convex subset A of R^n is Frechét-differentiable almost everywhere on A. At those points of differentiability, the function f is C^1. If a concave function is Gataux differentiable at a point in A, then it is Frechét differentiable there. Thus requiring only Gataux differentiability does not add to the number of points of "differentiability."

5. Optimization in R^n

Let $f: R^n \to R$ be given. We study the characterization of extreme of f. We say that $\hat{x}$ provides a *local maximum* of f if there exists a neighborhood $N(\hat{x})$ such that $f(\hat{x}) \geqq f(x)$ for all $x \in N(\hat{x})$. If $N = R^n$, then $\hat{x}$ is said to provide a *global maximum* of f. If we specify $\hat{x} \neq x$ and have $f(\hat{x}) > f(x)$, then these maxima are called strict maxima. If you are into minima then study maxima of $(-f)$. We won't. Let f be C^2. Then we say that $\mathring{x}$ is a *critical point* of f if $\mathring{f}_x \cdot a = 0$ for all a (in that case $f_x = 0$). If $a\,\mathring{f}_{xx} a' \neq 0$ for all $a \neq 0$, then $\mathring{x}$ is said to be a regular critical point of f. A *first order necessary condition* for $\hat{x}$ to locally maximize f is that $\hat{x}$ be a *critical* point of f. If f is concave and $\hat{x}$ is a critical point of f, then $\hat{x}$ globally maximizes f. Quasi-concavity won't do, for consider x^3 at $\hat{x} = 0$. Pseudo concavity gets us there, since $\hat{f}_x = 0$ would then imply that $f(\hat{x}) \geqq f(x)$ for all $x \neq \hat{x}$. Strict pseudo concavity yields a strict global maximum at a critical point. A second order necessary condition for a local maximum at $\hat{x}$ is that $a\,\hat{f}_{xx} a' \leqq 0$ for all a. Thus, if $\hat{x}$ provides f with a local maximum for f, then the principal minors of $\hat{f}_{xx}$ of order k either vanish or have the sign of $(-1)^k$, $k = 1, \ldots, n$. If $\hat{x}$ is a critical point of f and if $a\,\hat{f}_{xx} a' < 0$ for all $a \neq 0$, then $\hat{x}$ provides a local maximum for f. A critical point $\hat{x}$, thus, locally maximizes f if the successive principal minors of $\hat{f}_{xx}$ alternate in sign with the first being negative and none is zero. Suppose all critical points of f are regular. Then this last condition is both necessary and sufficient for $\hat{x}$ to be a local maximum of f.

If we constrain x to $x \geqq 0$, then the first order necessary condition for $\hat{x}$ to be a local maximum is $\hat{f}_x \leqq 0$, $\hat{x}_i \hat{f}_i = 0$. This can be derived from the unconstrained case by setting $z_i^2 = x_i$ and maximizing $f(z_1^2, z_2^2, \ldots, z_n^2)$ and using both the first and second order conditions.

We now come to our last "unconstrained" problem: finding saddle points. Let $\phi(u, v)$ be a real valued function of $u \in R^r$ and $v \in R^s$. We say that $(\mathring{u}, \mathring{v})$ is a *local saddle point (LSP)* of ϕ if there exists a neighborhood $N(\mathring{u}, \mathring{v})$ such that:

$$\phi(u, \mathring{v}) \leqq \phi(\mathring{u}, \mathring{v}) \quad \text{for all} \quad (u, \mathring{v}) \in N$$

and

$$\phi(\mathring{u}, \mathring{v}) \leqq \phi(\mathring{u}, v) \quad \text{for all} \quad (\mathring{u}, v) \in N\,.$$

If $N = R^{r+s}$, then we simply say that $(\mathring{u}, \mathring{v})$ is a saddle point (SP) of ϕ. If we restrict ourselves to R_+^{r+s}, then we speak of *nonnegative local saddle points* (NNLSP) and of nonnegative saddlepoints (NNSP).

Let ϕ be C^1. If $(\mathring{u}, \mathring{v})$ is an NNSP (and hence an NNSLP), then we have

$$\phi_{\mathring{u}} \leqq 0\,, \quad \mathring{u}_i \mathring{\phi}_{u_i} = 0 \quad \text{for} \quad i = 1, \ldots, r\,.$$
$$\mathring{\phi}_v \geqq 0\,, \quad \mathring{v}_j \mathring{\phi}_{v_j} = 0 \quad \text{for} \quad j = 1, \ldots, s\,.$$

The above conditions are called *nonnegative quasi-saddle point conditions* (NNQSP). We have just stated that NNSP implies NNQSP. The converse is true under concavity-convexity of ϕ.

(5.1) *Theorem:* If $(\mathring{u}, \mathring{v})$ is NNSP for ϕ and if ϕ is concave in u for every fixed v and convex in v for every fixed u, then $(\mathring{u}, \mathring{v})$ is NNSP of ϕ.

We now move on to the really constrained problem. Let f^α: $R^n \to R$, $\alpha = -m, \ldots, -1, 0, 1, \ldots, l$ be real valued functions of x. The problem we wish to study is:

Optimization Problem: maximize f^0 subject to $f^\alpha(x) = 0$ for $\alpha < 0$ and
$$f^\alpha(x) \geqq 0 \quad \text{for} \quad \alpha > 0\,.$$

Let $C = \{x \mid f^\alpha(x) = 0 \text{ for } \alpha < 0 \text{ and } f^\alpha(x) \geqq 0 \text{ for } \alpha > 0\}$. The set C is called the constraint set for our problem. We restate the problem as: maximize f^0 on C. We say that $\hat{x} \in C$ is a *local constrained maximum* of f^0 if there exists a neighborhood N of $\hat{x}$ such that $f(\hat{x}) \geqq f(x)$ for all $x \in N \cap C$. If N is R^n, then $\hat{x}$ is a *global constrained maximum.* Assume now that all f^α are C^2. A first order necessary condition for $\hat{x}$ is given below.

(5.2) *Theorem* (Karush (1939)): *First order necessary condition* 1. If $\hat{x}$ is a local constrained maximum, then there exists a vector $\lambda = (\lambda_{-m}, \ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_l) \neq 0$ such that $\hat{\psi}_x = 0$ where $\psi(x, \lambda) = \sum_\alpha \lambda_\alpha f^\alpha$ and where $\lambda_\alpha \geqq 0$ for $\alpha \geqq 0$ and $\lambda_\alpha f^\alpha(\hat{x}) = 0$ for $\alpha \neq 0$.

The problem with the above theorem is that it is possible for λ_0 to be zero. In which case we get a characterization of a maximum which is independent of the maximand. For instance (see Bliss (1946)) consider maximizing $f(x) = x_1 + x_2$ subject to $x_1^2 + x_2^2 = 0$. Clearly the maximum occurs at zero so $\hat{x} = (0, 0)$. $\psi(x, \lambda) = \lambda_{-1}(x_1^2 + x_2^2) + \lambda_0(x_1 + x_2)$.

$$\hat{\psi}_x = \begin{pmatrix} 0 + \lambda_0 \\ 0 + \lambda_0 \end{pmatrix}.$$

Thus $\lambda_0 = 0$. This is indicative of what can happen. The derivative characterization addresses the question: what happens if we change things a little near $\hat{x}$. In the above example, there is no room for movement. The set C is the singleton $\{0\}$. To avoid this and situations as embarrassing sufficient conditions, we're devised to assure that $\lambda_0 \neq 0$. These conditions travel under the names: "constraint qualification" and "normality." In the smooth case they all are equivalent to the condition we state shortly; "the rank condition." Let $\mathring{x}$ be given. Define $A(\mathring{x}) = \{\alpha \neq 0 \mid f^\alpha(\mathring{x}) = 0\}$. If $\alpha > 0$ and $\alpha \in A(\mathring{x})$, the f^α is effective at $\hat{x}$. Let

$\# A(\mathring{x})$ the number of elements of the set α. Let $F(x; \mathring{x})$ be the vector valued function whose component functions are $f^\alpha(x)$ with $\alpha \in A(\mathring{x})$.

(5.3) *The Rank Condition:* The constraint functions f^α, $\alpha \neq 0$, are said to satisfy the rank condition if the Jacobian matrix of F at $\mathring{x}$ has the rank $= \# A(\mathring{x}) < n$.

If $\hat{x}$ is a local constrained maximum and if the rank condition is satisfied at $\hat{x}$ then λ_0 in the last theorem is positive. In fact we can take $\lambda_0 = 1$. We write:

$$L(x, \lambda) = f^0 + \sum_{\alpha \neq 0} \lambda_\alpha f^\alpha(x) .$$

The first order necessary conditions are now: $\hat{L}_x = 0$, $\lambda_\alpha \geqq 0$ for $\alpha > 0$, $\lambda_\alpha f^\alpha(\hat{x}) = 0$. These conditions are the more familiar ones in economics. The reason for neglecting the rank condition by many writers is that it is trivially satisfied by most reasonable models of economic behavior. For instance in the theory of consumers behavior all it means is that the price vector is not the zero vector, an assumption which is not hard to satisfy. But one msut be careful.

A special case of our problem was studied by Kuhn and Tucker (1951). Suppose $\alpha \geqq 0$ and that x is constrained to R^n_+. The problem then is:

(*KT*): maximize $f^0(x)$ subject to $f^\alpha(x) \geqq 0$, $\alpha = 1, \ldots, l$ and subject to $x \geqq 0$. Since the problem appears often in economics, we state the first order necessary conditions for it. Let us define:

$$K(x, \mu) = f^\alpha + \sum_{\alpha = 1}^{l} \mu_\alpha f^\alpha , \quad \text{where} \quad \mu \geqq 0 .$$

It can be seen *that if* $\hat{x}$ *solves* KT *and the rank condition is satisfied, then there exist* $\hat{\mu} \geqq 0$ *such that* $(\hat{x}, \hat{\mu})$ *is an NNQSP of* K. Simply take μ_α here to be equal to λ_α and note that $\hat{K}_{\mu_\alpha} = \hat{f}_\alpha$ which is $\geqq 0$ since $\hat{x}$ satisfies the constraints.

First order sufficient conditions for the Kuhn-Tucker problem are now stated. If $K(x, \mu)$ is concave and if $(\hat{x}, \hat{\mu})$ is NNQSP of K then $(\hat{x}, \hat{\mu})$ is NNSP of K. But then, one can see, that $\hat{x}$ solves the KT problem. Notice that K is a non-negative linear combination of f^α, $\alpha = 0, 1, \ldots, l$. Thus, if f^α are concave, $\alpha = 0, 1, \ldots, l$, we know that K is concave in x. But K is linear in μ so it is convex in μ.

Things are not that nice in the general case. The concavity of f^α for all α would not guarantee the concavity of L since λ^α, $\alpha < 0$, can be negative as well as positive. Thus the best we can do by way of first sufficient *order sufficient conditions* is:

(5.4) *Theorem:* (Uzawa (1958)). Suppose f^α are affine for $\alpha < 0$ and concave for $\alpha \geqq 0$. Let $\hat{x}$ satisfy the first order necessary conditions and let $\hat{x} \in C$. Then $\hat{x}$ is a global constrained maximum.

Assuming L is pseudo-concave would provide a sufficient condition for a critical point of L to be a global constrained maximum. But L is *not* part of the data and so conditions on it are not satisfactory.

Before we move on to second order conditions, we make two further comments. The first concerns the relation between maximizing L and solving the optimization problem and the second concerns quasi-concavity. People often

talk as if solving the optimization problem is equivalent to finding an unconstrained maximum of the Lagrangian L. The truth is that if L is concave in x, then solving the optimization problem implies finding an SP of L which implies maximizing L in x. On the other hand, if $\hat{x} \in C$ maximizes L, then $\hat{x}$ solves the optimization problem. No conditions whatsoever are needed for this part. An equivalence theorem, too expensive, may be stated as follows:

(5.5) *Theorem:* Let $\hat{x}$ in C be given. Assume the rank condition is satisfied at $\hat{x}$. Let f^α be linear for $\alpha < 0$ and concave for $\alpha \geqq 0$. Then $\hat{x}$ solves the optimization problem if and only if there exist $\hat{\lambda}$ such that $(\hat{x}, \hat{\mu})$ is SP of L with $\lambda^\alpha \geqq 0$ if $\alpha > 0$ and $\lambda^\alpha \hat{f}^\alpha = 0$.

About quasi-concavity. It is not true that a critical point of a quasi-concave function provides a maximum of that function. Indeed $\hat{x} = 0$ is a critical point of $f(x) = x^3$ but is not a maximizing point of this quasi-concave function. On the other hand, $\hat{x} = (1, 1)$ is a critical point of L for $f^0 = x_1 x_2$ and $f^1 = 2 - x_1 - x_2$ where $\lambda^1 = 1$. The function L is not quasi-concave in x over the set: $\{x \mid x \geqq 0\}$ even though f^0 and f^1 are. Furthermore $\hat{x}$ is a global maximum for this problem. This indicates that establishing the quasi-concavity is not a very productive line of investigation. To introduce our next first order sufficient condition, we need to define a property of sets. We say that a set Z is *star shaped* at $\mathring{x} \in Z$ if $x \in Z$ implies $\theta \mathring{x} + (1 - \theta) x$ is in Z for all $0 \leqq \theta \leqq 1$. A convex set is star shaped but there are star shaped sets which are not convex.

(5.6) *Theorem:* Suppose $\hat{x} \in C$ satisfies the first order necessary conditions. Suppose f^0 is quasi-concave, f^α is affine for $\alpha < 0$ and the set: $\{x \mid f^\alpha(x) \geqq 0 \text{ for } \alpha > 0\}$ is star shaped at $\hat{x}$. Then $\hat{x}$ is a global constrained maximum.

In particular if f^α is affine for all $\alpha \neq 0$ and if f^0 is quasi-concave, then $\hat{x} \in C$, which satisfies the first order necessary conditions, is a global maximum.

We now move on to second order conditions. Of course we assume f^α is C^2 for all α. The second order conditions relate to the behavior of the second derivatives of the Lagrangian L in the appropriate directions of change. These are implied by the constraints on the variables.

(5.7) *Theorem: Second order necessary conditions.* Let $\hat{x}$ be a local constrained maximum. Assume the rank condition is satisfied. Then $\hat{L}_{xx} a' \leqq 0$ *for all* a *with* $\hat{f}^\alpha_x \cdot a = 0$ for all $\alpha \in A(\hat{x})$, where $\hat{L}_{xx}$ is the Hessian of the Lagrangian and where $A(\hat{x})$ is the set of indices of effective, at $\hat{x}$,constraints.

In terms of bordered Hessians, the conditions of the theorem state that the principal minors of order k, $k = 2(\# A(\hat{x})) + 1, \ldots, k + n$ of the matrix

$$B = \begin{pmatrix} 0 & \hat{F}_x \\ \hat{F}'_x & \hat{L}_{xx} \end{pmatrix}$$

either vanish or have the sign of $(-1)^{k + \# A(\hat{x})}$, where the matrix $\hat{F}_x$ is the Jacobian of the effective constraints at $\hat{x}$.

By way of an example, suppose $f^0(x) = x_1 x_2 x_3$ and $f^1 = 3 - (x_1 + x_2 + x_3)$ and there are no equality constraints. Then $\hat{x} = (1, 1, 1)$, $A(\hat{x}) = \{1\}$ and B is given by:

$$B = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

We have to check the principal minors of orders 3 and 4. Each of the ones of order 3 is 2 and the order 4 one is -3 and that is as it should be.

The second order necessary condition combined with the first order condition for the case of linear constraints are the same as the second order characterization of quasi-concavity of f, except that the latter has to hold for all x. We will come back to that in a little while.

(5.8) *Theorem: Second order sufficient conditions.* Suppose $\hat{x} \in C$. Let $\hat{x}$ satisfy first order necessary conditions and suppose $a\,\hat{L}_{xx}a' < 0$ for all $a \neq 0$ with $\hat{f}^{\alpha}_{x} \cdot a = 0$ for all $\alpha \in A(\hat{x})$. Then $\hat{x}$ is a local constrained maximum.

In terms of the matrix B above, the second order sufficient condition states that the successive principal minors of B, starting with the $2(\#A(\hat{x}))+1^{st}$, alternate in sign with the first one having the sign of $(-1)^{3(\#A(\hat{x}))+1}$. We can verify that $\hat{x} = (1, 1, 1)$, in the last example, indeed satisfies the second order sufficient condition and is a local constrained maximum.

We note that if the constraints are linear, then the sufficient conditions combined with first order necessary conditions resemble the sufficient conditions for strict quasi-concavity of f^0 except for the fact that the latter must hold for all x.

Our final topic is that of *vector maxima.* We abuse the notation by using f^0 to denote a vector valued function; $f^0: R^n \to R^k$. We write $f^0(x) = (f^{01}(x), \ldots, f^{0k}(x))$ where $f^{0i}: R^n \to R$. Since R^k is not completely ordered by "$\geqq$", we must choose a specific definition of vector maxima. The definition most often used in economics is in the sense of Pareto. We will characterize vector maxima in the sense of Pareto (*VMP*). The problem is to maximize f^0 (now a vector) subject to $f^{\alpha} = 0$ for $\alpha < 0$ and $f^{\alpha} \geqq 0$ for $\alpha > 0$ just as before. A point $\hat{x} \in C$ (as before) is a local VMP if there exists a neighboorhood N of $\hat{x}$ such that there does not exist $x' \in N \cap C$ with $f^0(x') \geq f^0(\hat{x})$, i.e., such that $f^{0i}(x') \geqq f^{0i}(\hat{x})$ for all i $(= 1. \ldots, k)$ with strict inequality for at least one i. If $N = R^n$, then $\hat{x}$ is simply VMP.

(5.9) *Remark:* If $\hat{x}$ is VMP, then it solves all of the following k maximization problems: M(i): max f^{0i} subject to a) $x \in C$ and b) $f^{0j}(x) \geqq f^{0j}(\hat{x})$ for $i \neq j$, $i = 1, \ldots, k$.

The remark makes it easy to provide necessary conditions for VMP. For suppose $\hat{x}$ is VMP and let all f^{α} be C^2. Suppose further that the rank condition for each M(i) is satisfied. Then there exist multipliers λ^i_{α}, $\alpha \neq 0$ and multipliers v^i_j, $j \neq i$ such that $\hat{L}^i_x = 0$, $v^i_j \geqq 0$, $\lambda^i_{\alpha} \geqq 0$ (for $\alpha > 0$) and $\lambda^i_{\alpha} f^{\alpha}(\hat{x}) = 0$, where $L^i = f^{0i} + \sum_{j \neq i} v^i_j f^{0j} + \sum_{\alpha \neq 0} \lambda^i_{\alpha} f^{\alpha}$. Adding up these conditions we get our next theorem.

(5.10) *Theorem: Necessary conditions for* VMP. Assume that $\hat{x}$ is a VMP and that the rank condition holds for each M(i). Then there exist positive numbers μ_i, $i = 1, \ldots, k$, and λ_{α} for $\alpha \neq 0$ such that

$\hat{L}_x = 0, \lambda_\alpha \geqq 0$ for $\alpha > 0$ and $\lambda_\alpha f^\alpha(\hat{x}) = 0$, where

$$L = \sum_{i=1}^{k} \mu_i f^{0i} + \sum_{\alpha \neq 0} \lambda_\alpha f^\alpha .$$

I

The numbers μ_i are simply the sums of multipliers, over i, μ_j^i and since $\mu_i^i = 1$ and $\mu_j^i \geqq 0$ the μ^i's are positive. Of f^{0i}, for all i, and f^α, for $\alpha > 0$, are concave and if f^α are affine for $\alpha < 0$ then $\hat{x}$ is VMP and rank conditions imply L is maximized in x at $\hat{x}$.

Next we move to envelope theorems. We provide two versions of the envelope theorem and related properties of the maximand function evaluated at the maximizing points.

(5.11) *Lemma* (The Envelope Theorem): Let $H: X \times A \to R$ and $f: X \times A \to R$, $X \subseteq R^n$, $A \subseteq R^m$ be continuously differentiable in an open neighorhood $V \times U$ of $(\bar{x}, \bar{\alpha}) \in X \times A$. Assume further that the problem to maximize $H(x, a)$ subject to $f(x, a) \geqq 0$ and has a C^1-solution $\xi: U \to V$ satisfying: $\bar{x} = \xi(\bar{a})$, for every $a \in U$ $f(\xi(a), a) = 0$ and there exists $\lambda(a) > 0$ such that $H_x(\xi(a), a) + \lambda(a) f_x(\xi(a), a) = 0_n$ where the subscript "x", indicates the derivative with respect to x and 0_n is an $(n \times 1)$ zero vector.

Then, for every $a \in U$, we have that

$$F_a(a) = \{H_a(x, a) + \lambda(a) f_a(x, a)\}|_{x = \xi(a)}$$

where $F(a) = \max_x \{H(x, a) \mid f(x, a) \geqq 0\}$ and the subscript "a" indicates the derivative with respect to "a".

The proof is quite simple. Since $F(a) = H(\xi(a), a)$ and $f(\xi(a), a) = 0$, for every $a \in U$, we have that:

$$F_a(a) = \xi_a(a)[H_x(x, a)]' + H_a(x, a)$$

$H_x(x, a)\, \xi_a(a)$

and

$$\xi_a(a)[f_x(x, a)]' + f_a(x, a) = 0_m$$

$f_x(x, a)\, \xi_a(a)$

where x is evaluated at $\xi(a)$.

Therefore the above two relations and the first-order condition of the Lagrangean function yield:

$f_x(x, a)\, \xi_a(a)$

$$F_a(a) = -\lambda(a)\xi_a(a)[f_x(x, a)]' + H_a(x, a)$$
$$= H_a(x, a) + \lambda(a) f_a(x, a)$$

where $x = \xi(a)$.

When the objection function is linear in the parameter vector "a" so that $H(x, a) = H(x) + a \cdot x$ with a slight abuse of the notation H, we can obtain a more specific version of the Envelope Theorem.

(5.12) *Lemma:* (The Envelope Theorem): Suppose that $X \subseteq R^n$, $H: X \to R$, $a \in D$ and $F(a) = \max_{x \in X} \{H(x) + a \cdot x\}$ is well-defined on an open convex neighborhood U of $\bar{a}$. Then (i) F is a convex function on U; and (ii) if the above maximization

problem admits a continuous solution function ξ: $U \to X$, then F is C^1 on U and for every $a \in U$, $F_a(a) = \xi(a) \cancel{= \xi(a)}$ where F_a indicates the derivative of F with respect to a.

The proof of (i) is identical to that of the concavity of the minimum expenditure function of demand theory in Chapter 2. The proof of (ii) can be found in Nikaido (1968, pp. 298 – 300) and Hurwicz and Uzawa (1971, pp. 120 – 122) although their result does not contain H(x) in their objective function. We will provide the proof of (ii) to make sure that the presence of H(x) does not cause any problem.

Let e_j be the j-th unit vector, $a \in U$, and $h \in R$ be small enough so that $a + he_j \in U$. Since F is the maximum value function, we have that:

$$\begin{aligned} F(a) &= H[\xi(a)] + a \cdot \xi(a) \\ &\geqq H[\xi(a+he_j)] + a \cdot \xi(a+he_j) \end{aligned}$$

and

$$\begin{aligned} F(a+he_j) &= H[\xi(a+he_j)] + (a+he_j) \cdot \xi(a+he_j) \\ &\geqq H[\xi(a)] + (a+he_j) \cdot \xi(a) . \end{aligned}$$

These two inequalities give us respectively the following inequalities.

$$H[\xi(a+he_j)] - H[\xi(a)] + a \cdot \xi(a+he_j) - a \cdot \xi(a) \leqq 0$$

and

$$H[\xi(a+he_j)] - H[\xi(a)] + a \cdot \xi(a+he_j) - a \cdot \xi(a) \geqq h\,\{\xi_j(a) - \xi_j(a+he_j)\}$$

Since $\lim_{h \to 0} \{\xi_j(a) - \xi_j(a+he_j)\} = 0$ by the continuity of ξ, we get that*

$$\begin{aligned} 0 &= \lim_{h \to 0} \{\xi_j(a) - \xi_j(a+he_j)\} \\ &\leqq \lim_{h \to 0} \frac{1}{h} \{H[\xi(a+he_j)] - H[\xi(a)]\} \\ &\quad + \lim_{h \to 0} \frac{1}{h} \{a \cdot \xi(a+he_j) - a \cdot \xi(a)\} \leqq 0 . \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{h \to 0} \frac{1}{h} \{H[\xi(a+he_j)] - H[\xi(a)]\} \\ &\quad + \lim_{h \to 0} \frac{1}{h} \{a \cdot \xi(a+he_j) - a \cdot \xi(a)\} = 0 . \end{aligned}$$

Now then

$$\begin{aligned} \lim_{h \to 0} \frac{1}{h} \{F(a+he_j) - F(a)\} &= \lim_{h \to 0} \frac{1}{h} \{H[\xi(a+he_j)] - H[\xi(a)]\} \\ &\quad + \lim_{h \to 0} \frac{1}{h} \{a \cdot \xi(a+he_j) - a \cdot \xi(a)\} + \lim_{h \to 0} \xi_j(a+he_j) = \xi_j(a) . \end{aligned}$$

Therefore F has continuous partial derivatives for each $a \in U$ and $\partial F(a)/\partial a_j = \xi_j(a)$.

When the differentiability of ξ is further assumed, then we get the following extremely important properties of the maximizer function ξ. We shall omit

* We assume that $h > 0$. Similarly argument applies when $h < 0$.

proofs of these, since they are already proved in the text in connection with the expenditure minimization problem in demand theory.

(5.13) *Lemma:* If the maximization problem of Lemma (~~6.5~~ 5.12) admits a C^1-solution ξ: $U \to R^n$ in an open neighborhood U of $\bar{a}$, the nfor every $a \in U$, the $(n \times n)$ matrix $\xi_a(a) = (\partial \xi_i(a)/\partial a_j)$ is symmetric and positive semi-definite.

(5.14) *Lemma:* In addition to the assumption of Lemma (5.13), if $H(x) \equiv 0$ and $X = \{x \in R^n \mid f(x) \geqq 0\}$, then (i) U can be chosen as an open cone around the ray through $\bar{a}$, $\xi(a)$ is homogeneous of degree 0 in a and $F(a)$ is homogeneous of degree one in a; (ii) if f is regularly strict quasi-concave in x at $\bar{x} = \xi(\bar{a})$, then for every $a \in U$, we have that for any $\eta \in R^n$ with $a \cdot \eta = 0$ and $\eta \neq 0$, $\eta' \xi_a(a) \eta > 0$.

6. Optimization in More General Spaces

We start with *unconstrained* problems in *calculus of variations*. Let T be a subset of the extended real line. We choose among functions y: $T \to R^n$ together with their derivatives y′ which are assumed to exist almost everywhere on T. Our problem is to maximize the functional:

$$(6.1) \quad J = g(t_0, y_0, t_1, y_1) + \int_{t_0}^{t_1} f(t, y, y')\, dt\,,$$

Nonlinear Point

where t_0, y_0 are initial time and position and where t_1, y_1 are terminal time and position with none, one or more to be chosen optimally such that $[t_0, t_1] \subseteq T$ and where $y(t_i) = y_i$, $i = 0, 1$.

A *global solution* to the above problem is a 6-tuple $\hat{z} = (\hat{t}_0, \hat{y}_0, \hat{t}_1, \hat{t}_1, \hat{y}', \hat{y})$ such that $J(\hat{z}) \geqq J(z)$ for all z. A local solution is defined in terms of neighborhoods. A strong local solution involves comparing $\hat{z}$ with a large set of z's. We say that $\hat{z}$ is a *strong local solution* if $J[\hat{z}] \geqq j[z]$ for all z with

$$d_0(\hat{z}, z) = \| (\hat{t}_0, \hat{y}_0, \hat{t}_1, \hat{y}_1) - (t_0, y_0, t_1, y_1 \| + \sup_{t \in T} \| \hat{y}(t) - y(t) \| < \varepsilon, \quad \text{some } \varepsilon > 0.$$

A weak local solution involves comparing $\hat{z}$ to a smaller set of z's; $\hat{z}$ is a *weak local solution* if $J[\hat{z}] \geqq J[z]$ for all z with

$$d_1(\hat{z}, z) = d_0(\hat{z}, z) + \sup_{t \in T} \| \hat{y}' - y' \| < \varepsilon, \qquad \text{for some } \varepsilon > 0\,.$$

We first characterize weak local solutions of the case with no restrictions on z. We assume that g and f are C^2. Let $\hat{z}$ be a weak local solution. Then we have

$$(6.1) \quad \hat{f}_y - \frac{d}{dt} \cdot \hat{f}_{y'} = 0$$

$$(6.2) \quad \hat{g}_{y_1} + \hat{f}_{y'} \big|_{t = t_1} = 0$$

$$(6.3) \quad \hat{g}_{y_0} - \hat{f}_{y'} \big|_{t = t_0} = 0$$

$$(6.4) \quad \hat{g}_{t_1} + (\hat{f} - \hat{y}' \hat{f}_{y'}) \big|_{t = \hat{t}_1 \cancel{= 0}} = 0$$

$$(6.5) \quad \hat{g}_{t_0} - (\hat{f} - \hat{y}' \hat{f}_{y'}) \big|_{t = \hat{t}_0 \cancel{= 0}} = 0$$

Condition (6.1) is known as the Euler equation and conditions (6.2) – (6.5) are known as the transversality conditions. If y', y_1, y_0 is *restricted to be non-negative,* then the corresponding equation becomes an inequality; $\leqq 0$ with equality if the optimal value of the relevant variable is positive. *Nonnegativity constraints on y* however are entirely different since they imply restrictions on y' as well as the end points y_0 and y_1. Suppose now that $y \geqq 0$ is the only constraint. Then we must have $y_0 \geqq 0$ and $y_1 \geqq 0$ so (6.2) and (6.3) become inequalities, with equation if the relevant variable is positive at the optimum. To state a modification of (6.2), note that if $\hat{y}$ is positive for all t, then no modification is needed. Let $\bar{t}$ be a point where $\hat{y}(\bar{t}) = 0$. Then $y(t) \geqq 0$ is assured by having $y'(t) \geqq 0$ for $t \geqq \bar{t}$. The modification is then: $\hat{f}_y - (d/dt)\hat{f}_{y'} \leqq 0$ with equality if $\hat{y}' > 0$.

We now go back to the unconstrained case. A further first order necessary condition is given by the Weierstrass inequality. If $\hat{z}$ is a weak local solution then, in addition to (6.1) – (6.5) we have

$$(6.6) \quad E(t, \hat{y}, \hat{y}', y') = f(t, \hat{y}, \hat{y}') - f(t, \hat{y}, y') - (y' - \hat{y}')\hat{f}_{y'} \leqq 0 \qquad \text{for all } y' .$$

We can state (6.1) – (6.6) in terms of the Hamiltonian. Let $p = -f_{y'}$. Define: $H(t, y, y', t) = f + y'p$.

Conditions (6.1) – (6.6) can be stated as:

$$(6.1') \quad p' = -H_y$$

$$(6.2') \quad \hat{g}_{y_1} - p(t_1) = 0$$

$$(6.3') \quad \hat{g}_{y_0} + p(t_0) = 0$$

$$(6.4') \quad \hat{g}_{t_1} + \hat{H}(t_1) = 0$$

$$(6.5') \quad \hat{g}_{t_0} - \hat{H}(t_0) = 0$$

$$(6.6') \quad \hat{y}' \text{ maximizes } H(t, \hat{y}, y', p) \qquad \text{over all } y'.$$

Conditions (6.1′) – (6.6′) look like optimal control conditions. In fact the connection is deeper but we won't get into it here.

A second order necessary condition is given by the Legender inequality: if $\hat{z}$ is a weak local solution then

$$(6.7) \quad \sum_i \sum_j \hat{f}_{y_i' y_j'} \gamma_i \gamma_j \leqq 0 \qquad \text{for all } \eta .$$

Condition (6.7) can be derived from (6.6′). A further second order necessary condition is given by the Jacobi inequality. We state it for the fixed end point case.

$$(6.8) \quad \int_{t_0}^{t_1} \left[\sum_i \sum_j \hat{f}_{y_i y_i} \eta_i \eta_j + \sum_i \sum_j \hat{f}_{y_i' y_j'} \eta_i' \eta_j' \right] dt + d^2 g \,|_{z = \hat{z}} \leqq 0$$
$$\text{for all } \eta \text{ with } \eta(t_0) = \eta(t_1) = 0 .$$

Sufficient conditions for this problem are harder to state in view of the nature of the space. *First order sufficient conditions* for a global solution are the easiest to state and verify. If (6.1) – (6.6) hold at $\hat{z}$ and if g and f are concave, then $\hat{z}$ is a global solution. It is not true, alas, that strengthening the Weierstrass inequality or the Legendre inequality would in addition to (6.1) – (6.6) be sufficient for $\hat{x}$ to

be a weak local solution. In fact, strengthening the Legendre inequality merely implies the Weierstrass inequality. One needs to assume the absence of conjugate points. This added requirement is hard to verify and is not stated here. Finally if the Jacobi condition holds as a strict inequality for all $\eta \neq 0$ with $\eta(t_0) = \eta(t_1) = 0$ in addition to (5.1) – (5.6), then $\hat{z}$ is a weak local solution.

We next move to a problem in optimal control.

We consider a system whose motion is given by:

(6.9) $\dot{x} = f(t, x, u, b)$,

where $x: T \to R^{n_1}$, $u: T \to R^{n_2}$, $b \in R^{n_3}$, $T \subseteq R_+$, and $f: R^{n_1+n_2+n_3+1} \to R^{n_1}$. It is customary to refer to x as state variable, u as control variable and b as control parameter. For instance x could be a stock variable, e.g. the stock of capital, u could by a decisional flow, e.g. gross investment, and b could be a policy parameter which is to be chosen but which does not depend on t. Even though t is often thought of as time, one can find examples where t represents income, space or even the index identifying a firm or a consumer. Technically we take u to belong to a set U of measurable functions on T with values in R^{n_2} and x belongs to the set $\mathscr{H}$ of absolutely continuous functions on T with values in R^{n_1}.

There are several types of constraints that may be imposed by the nature of the problem being modeled. We distinguish two types of constraints: instantaneous or path constraints and isoperimetric constraints.

Path constraints may have the form:

(6.10.a) $\phi^\alpha(t, x, u, b) = 0, \quad \alpha = -1, -2, \ldots, -m_1$

(6.10.b) $\phi^\alpha(t, x, u, b) \geqq 0, \quad \alpha = 1, 2, \ldots, m_2$

where $\phi^\alpha: R^{n_1+n_2+n_3+1} \to R^{m_1+m_2}$. In preparation for stating the isoperimetric constraints we define:

(*) $L^\beta = g^\beta(t_0, x_0, t_1, x_1, b) + \int_{t_0}^{t_1} h^\beta(t, x, u, b)\, dt$,

where $\quad -l_1 \leqq \beta \leqq l_2$; $g^\beta: R^{2n_1+n_3+2} \to R^{l_1+l_2+1}$, $h: R^{n_1+n_2+n_2+1} \to R^{l_1+l_2+1}$.

The variables t_0, x_0 and t_1, x_1 are the initial and terminal times and states. The constraints are stated as

(6.11.a) $L^\beta = 0 \quad$ for $\quad \beta = -1, 2, \ldots, -l_1$

(6.11.b) $L^\beta \geqq 0 \quad$ for $\quad \beta = 1, 2, \ldots, l_2$.

The problem is to maximize L^0, defined by (*), subject to the constraints (6.9) – (6.11) where $u \in U$.

Several types of problems can be transformed to the above form. For instance the calculus of variations problems in $(t, y, \dot{y})$ are accommodated by writing $\dot{y} = u$. Another example is that of "functions of integrals" where instead of L^β we have:

(+) $\psi^\beta[g^\beta(t_0, x_0, t_1, x_1, b);\ \int_{t_0}^{t_1} h^\beta(t, x, u, b)\, dt]$, $\quad$ where $-l_1 \leqq \beta \leqq l_2$.

In this case we introudce the variable $S(t)$

$$(++) \quad \dot{S}^{\beta} = h^{\beta}(t, x, u, u), \; -l_1 \leqq \beta \leqq l_2 .$$

We then agument (6.9) by $(++)$. Then $(+)$ becomes

$$(+++) \quad \psi^{\beta}[g^{\beta}(t_0, x_0, t_1, b); \quad (S_1 - S_0)] ,$$

where $S_i = S(t_i)$, $i = 0, 1$. The problem is now in standard form.

We assume that the functions f, ϕ, g and h are continuously differentiable up to and including the second order as function to and from Euclidean spaces. Let $z^0 = (x^0, u^0, b^0)$ be given. Let $\bar{\alpha}$ designate any index α where $\phi^{\alpha}(t, z^0) = 0$. We now state the, often misunderstood, rank condition:

$$(6.12) \quad \text{(R) } \textit{The matrix } (\mathring{\phi}^{\bar{\alpha}}_{u_j}) \textit{ where } \mathring{\phi}^{\bar{\alpha}}_{u_j} = \frac{\partial}{\partial u_j} \phi^{\bar{\alpha}} \bigg|_{z^0}, \quad \textit{has full rank.}$$

Condition (R) is not sufficient for "normality" as we will see later on. It is stated here because it is, like differentiability, a convenient regularity condition which ~~made~~ makes it easy to use finite space proofs of characterization theorems along the lines of classical calculus of variations.

We provide, below a review of characterization theorems. In all of these sections $[t_0, t_1]$ is to be optimally chosen from T and the optimal t_1 may be infinite or finite.

Before we state the characterization theorems, let us clear up some common misunderstandings. The Lagrangian for our problem should be of the form:

$$(6.13) \quad L = \sum_{\beta} \lambda_{\beta} L^{\beta} + \Gamma^1(f - \dot{x}) + \Gamma^2(\phi) .$$

where λ_{β} are constants with $\lambda_{\beta} \geqq 0$ if $\beta \geqq 0$, and where Γ^1 and Γ^2 are linear functionals in the respective spaces. The form (6.13) is *not* what we shall use here nor is it the form used in the optimal control calculus of variations literature. We use, instead, our three functions related to the function L defined in (6.13) even though we don't specify the relationship here. The form (6.13) is the right one to use if a functional analysis path is followed, e.g. by specializing Hurwicz (1958) theorems. What we use here are three functions which we term: The path Lagrangian, the Hamiltonian and the end point Lagrangian. The path Lagrangian is defined by:

$$(6.14) \quad F(\lambda, \mu, q, t, z) = \sum_{\beta} \lambda_{\beta} h^{\beta} + \sum_{i=1}^{n} q_i (f^i - \dot{x}_i) + \sum_{\alpha} \mu_{\alpha} \phi^{\alpha}$$

where λ_{β} are constants and q and μ are functions of time to be specified later. The Hamiltonian is defined by:

$$(6.15) \quad H(\lambda, \mu, q, t, z) = \sum_{\beta} \lambda_{\beta} g^{\beta} + \sum_{i} q_i f^i + \sum_{\alpha} \mu_{\alpha} \phi^{\alpha} .$$

Finally, the end point Lagrangian is given by:

$$(6.16) \quad G(\lambda, \mu, q, t_0, t_1, x_0, x_1, b) = \sum_{\beta} \lambda_{\beta} g_{\beta}$$

The standard first order necessary conditions, see e.g. Hestenes (1965), assert the existence of constants λ_β, $-l_1 \leqq \beta \leqq l_2$, not all zero and functions q, μ as continuous as $\hat{u}$ (the optimal control) with (λ_0, q, μ) never being a zero vector such that

(6.17.1) $\dot{q} = -\hat{H}_x$ where $\hat{H}_{x_i} = \frac{\partial}{\partial x} H \Big|_{z=\hat{z}}$.

(6.17.2) $\hat{H}_u = 0$, where $\hat{H}_{u_j} = \frac{\partial H}{\partial u_j}\Big|_{z=\hat{z}}$.

(6.17.3) $\mu_\alpha \geqq 0$, $\mu_\alpha \hat{\phi}^\alpha = 0$ for $\alpha > 0$.

(6.17.4) $H(\lambda, 0, q, \hat{x}, \hat{u}, \hat{b}) \geqq H(\lambda, 0, q, \hat{x}, u, \hat{b})$ for all u satisfying

$$\phi^\alpha(\hat{x}, u, \hat{b}) \begin{cases} = 0 & \text{for } \alpha < 0 \\ \geqq 0 & \text{for } \alpha > 0 \end{cases}$$

(6.17.5) $\hat{H} = \int_{t_0}^{0} \frac{\partial}{\partial t} \hat{H}\, dt$

(6.17.6.1) $q(t_1) = \hat{G}_{x_1}$, $q(t_0) = -\hat{G}_{x_0}$, where $\hat{G}_{x_r} = \frac{\partial G}{\partial x}\Big|_{x_s = \hat{x}_r}$, $r = 0, 1$.

(6.17.6.2) $\hat{G}_0 = \hat{H}(t_0)$, $\hat{G}_{t_1} = -\hat{H}(t_1)$, where $G_{r_t} = \frac{\partial G}{\partial t_s}$, $r = 0, 1$.

(6.17.6.3) $\lambda_0 \geqq 0$, $\lambda_\beta \geqq 0$, $\lambda_\beta \hat{L}^\beta = 0$ for $\beta > 0$.

(6.17.6.4) $\hat{G}_b + \int_{t_0}^{t_1} \hat{H}_b\, dt = 0$, where $G_{b_i} = \frac{\partial G}{\partial b_i}$, $H_{b_i} = \frac{\partial H}{\partial b_i}$.

Several remarks are now in order. First we note that (6.17.1) and (6.17.2), if written in terms of L, would be the Euler-Lagrange equations. The statement (6.17.4) is the Pontriagin maximum principal or, if written in terms of L, the Weierstrass inequality. One should note that (6.17.4) is a statement of a constrained maximization problem, for any given t, in a finite dimensional space. If some constraint qualification is satisfied, then (6.17.4) implies (6.17.2). Such a constraint qualification is indeed provided by the rank condition stated in the introduction. It is easy to see that the magic of this rank condition is not potent enough to bestow normality (that $\lambda_0 \neq 0$) on the entire problem. A continuum of counter examples is provided by Caratheodory rigidity (see Young (1969) and Pars (1962)). A simpler class of counter examples is provided as follows: Let $f(t, x, u) = 0$ in (6.9) and let constraints (6.10) be trivial so they don't restrict our choices; e.g., let them be of the form $u = u$. Then certainly the rank condition is satisfied. The problem becomes a finite dimensional problem. The Slater's (see Kuhn and Tucker (1951)) counter example is now a counter example to any derivative characterization of optimality. If the reader still has doubts, then a reference to the method of proof of the Euler-Lagrange equation (e.g. Bliss (1946)) should make things even more obvious. In Bliss (1946) and in Pars (1962) proofs a parameterization takes place. The problem is transformed into an end point problem

which is finite dimensional. The rank condition is used in this first parameterization. The new problem is a constrained maximization problem in a finite space. To characterize its solution one may perform and second parameterization. But then one may need a further rank condition; an end point rank condition as it were. If the path rank condition was enough then the one at the end would not be needed.

Conditions (6.17) are only necessary. To make them sufficient, see Mangasarian (1966), one needs to further assume:

(C) *Assumption* C: 1) Either the functions f are linear in x, u and b *or* the multipliers q are nonnegative always and f are concave. 2) Either the functions ϕ^α, $\alpha < 0$, are linear *or* $\mu_\alpha \geqq 0$ and ϕ^α concave. 3) The functions ϕ^α, $\alpha > 0$, are concave. 4) Either the functions g^β, h^β, $\beta < 0$, are linear *or* $\lambda_\beta \geqq 0$ and h^β is concave. 5) The functions g^β and h^β are concave for $\beta \geqq 0$. The proof of this type of sufficiency theorem, a trivial extension of Magasarian (1966), may be found in El-Hodiri (1971).

Higher order necessary conditions fall into two types. The simpler types are analogous to the Clebsch necessary conditions in the calculus of variations. They are as follows:

(6.18.1) $\pi \hat{H}_{uu} \pi' \leqq 0$ for all π with:

(6.18.2) $\hat{\phi}_u^{\bar{\alpha}} \pi = 0$,

where $\hat{H}_{uu}$ is the Hessian of H at $\hat{z}$ and where $\bar{\alpha}$ a signifies the fact that $\phi^{\bar{\alpha}} = 0$. Thus conditions were derived by Berkovitz (1961). Strengthening the Clebsch condition, (6.18), by making the inequality strict and assuming (6.17.2) and (6.17.3) to hold does not suffice to make $\hat{z}$ optimal. It merely implies that (6.17.4), the maximum principle, holds. There are cases in which the maximum principle is sufficient but the conditions are difficult to verify. The conditions are analogous to those that assure the sufficiency of the Weierstrass condition in the calculus of variations.

The second type of second order necessary conditions are of the Jacobi-Mayer-Bliss type (see Bliss (1946)). They all assume normality which is hard to verify and they themselves are cumbersome. The difficulty is in the equations that describing the tangent space where the increments are allowed to be; the counterpart of (6.18.2) above. Further, the condition is in terms of the second variation of $\mathscr{L}$. The interested reader is referred to Berkovitz (1961).

Nonnegativity constraints on "variables" are of special interest to economists. If the variables in question are u, x_0 or x_1, then nothing major needs to be done. Simply change (6.17.2) and (6.17.6) as follows:

(6.17.2′) $\hat{H}_u \leqq 0, \quad \hat{u}_j \hat{H}_{u_j} = 0, \quad j = 1, \ldots, n_2.$

(6.17.6.1) $\hat{G}_{x_1} - q(t_1) \leqq 0, \quad \hat{x}_{1i}(\hat{G}_{x_1} - q(t_1)) = 0, \quad i = 1, \ldots, n_1.$

$\hat{G}_{x_0} + q(t_0) \leqq 0, \quad \hat{x}_{0i}(\hat{G}_{x_0} + q(t_0)) = 0, \quad i = 1, \ldots, n_1.$

The same can be said about constraints $b \geqq 0$.

On the other hand, if the constraints are of the form $x(t) \geqq 0$, then the above theory *can not* handle them. Such constraints fall in the category of "bounded state variables." More generally, suppose we have constraints of the form

(6.19) $R^\gamma(t, x) \geqq 0, \quad \gamma = 1, 2, \ldots, s,$

where R^γ are continuously differentiable of order 2. Then the rank condition can not be satisfied. Russak (1970) has a partial solution to this problem. Suppose, at some point, say $\bar{t}$, $R(\bar{t}, x(\bar{t})) = 0$. Call $\bar{t}$ a contact point. Then if $(d/dt)\, R(t, x(t))\,|_{\bar{t}} < 0$ the constraint (6.19) will be violated. A way to prevent that is to have $(d/dt)\, R(t, x) \geqq 0$. But his means $R_t + R_x \cdot \dot{x} \geqq 0$ where the subscripts denote partial derivatives. Substituting for x from (6.9) we have

(6.19′) $R_t^\gamma + R_x^\gamma \cdot f(t, x, u, b) \geqq 0.$

If we were to move in fits and starts, from contact point to contact point, then we may think of (6.19.′) as our constraint. The rank condition now has a chance of being satisfied. Assume that it is. Then we introduce an additional multiplier $\nu(t)$ and modify H to:

(6.20) $H^I = H + \sum_\gamma \nu^\gamma (R_t + R_x^\gamma \cdot f(t, x, u, b)).$

The multipliers $\nu(t)$ are such that conditions (6.17) hold with H^I replacing H and

(6.21.1) $\nu \geqq 0, \quad \dot{\nu} \leqq 0, \quad \dot{\nu}\hat{R}^\gamma = 0$

(6.21.2) $\nu(\hat{t}_0)\, R(\hat{t}_0, \hat{x}_0) = \nu(\hat{t}_1)\, R(\hat{t}_1, \hat{x}_1) = 0.$

Note that, contrary to popular beliefs, it is $\dot{\nu}$ that vanishes when the constraint is ineffective along the path. This is not surprising since we are "controlling" R through its time serivatives. Now, see also Berkovitz (1962), we can deal with nonnegativity constraints on the state variables. In that case

$$H^I = \sum_\beta \lambda_\beta h^\beta + \sum_i (q_i + \nu_i) f^i + \sum_\alpha \mu_\alpha g^\alpha,$$

with $\nu \geqq 0$, $\dot{\nu} \leqq 0$, $\dot{\nu}\hat{x} = 0$, $\nu(t_0)\,\hat{x}_0 = \nu(t_1)\,\hat{x}_1 = 0$. One can see then that it is not a simple case of turning some equations in (6.17) to inequalities. However, in the absence of constraints (6.10) and (6.11) it indeed is almost as simple as that! The Euler equation for the otherwise unconstrained calculus of variations problem becomes an inequality. Consider the problem of maximizing $\int_{t_0}^{t_1} f^0 t, y, \dot{y})\, dt$ subject to the constraint: $y \geqq 0$ which we dealt with in Section 1. In terms of contrasts let $u = \dot{y}$ and $x = y$; then the problem is: maximize $\int_{t_0}^{t_1} f^0(t, x, u)\, dt$ subject to $\dot{x} = u$, $x \geqq 0$. Our H^I has the form

$$\lambda_0 f^0 + (q + \nu)\, u.$$

Clearly $\lambda_0 > 0$ so take it equal to 1. The modified condition (6.17) gives

$$\dot{q} = -f_x^0, \quad (q + \nu) = -f_u^0.$$

Taking time derivative of the second equation we get $\dot{q} = -\dot{\nu} - (d/dt)\, f_u^0$. Thus $f_x^0 = \dot{\nu} + (d/dt)\, f_u^0$. So $f_x^0 \leqq (d/dt)\, f_u^0$ with equality is $x > 0$. Reverting back to the calculus of variations problem, we have: $f_y^0 \leqq (d/dt)\, f_{\dot{y}}$ with equality if $y > 0$. What happens to the Euler equation parallels what happens in the finite dimensional case but only in this very simple case.

The simple control problem assumes a scalar criterion functional and that adjustments in the system are instantaneous. In dynamic welfare economics (see. e.g. El-Hodiri-Takayama (1971)) one must deal with vector criteria. We present here a characterization of optimal control with a vector criterion with the maximum defined in the sense of a pareto optimum. The scalarization procedure has been presented by Halkin and Neustadt (1966). An earlier and more general scalarization is due to Hurwicz (1958). What is presented here is due to El-Hodiri (1971) and follows a suggestion by Hurwicz during conversations with El-Hodiri around 1962. The timing of effect on the state of the system in relation to the timing of controls arises naturally in economics whenever time lags appear. For instance, in the case of investment models that recognize the time profiles, of construction of investment projects a la Frisch (1955). The characterization of optimality for this case is a slight modification of Halaney (1968).

First we deal with vector criteria. Let L^0 defined by (*) be a vector of functionals so $L^0 = (L^{01}, L^{02}, \ldots, L^{0r})$. We define a Pareto optimal solution $\hat{z}$ to be such that there does not exist another z that satisfies constraints 1) – 3) such that

$$L^{0s}(z) \geqq L^{0s}(\hat{z}) \quad \text{for} \quad s = 1, \ldots, r$$

with strict inequality for a least one s. Clearly if $\hat{z}$ is Pareto optimal then it solves all of the following scalar optimal control problems:

$\mathscr{C}(s_0)$: max L^{0s_0} subject to 1) – 3) and to

$$L^{0s}(z) \geqq L^{0s}(\hat{z}) \quad \text{for} \quad s \neq s_0 .$$

Writing down the conditions (6.17) for each of the problems $\mathscr{C}(s_0)$ with multipliers designated by a superscript s_0 and then summing these conditions over s_0, we get conditions exactly like (6.17) with the Hamiltonian and end point Lagrangian ~~are~~ respectively:

$$H^{II} = \sum_s \omega_s f^{0s} + \sum_{\beta \neq 0} \lambda_\beta h^\beta + \sum_i q_i f^i + \sum_\alpha \mu_\alpha \phi^\alpha$$

$$G^{II} = \sum_s \omega_s g^{0s} + \sum_{\beta \neq 0} \lambda_\beta g^\beta$$

where λ_β, q_i and μ_α are the sums over s_0 of the multipliers for the $\mathscr{C}(s_0)$ problems. Note that $\omega \neq 0$ and $\omega_s \geqq 0$. To get all $\omega_s > 0$ we need normality for each $\mathscr{C}(s_0)$. Conditions for this are awkward to state in general but in the usual welfare economics problem it is rather straightforward in view of the, usually assumed, absence of externalities (see El-Hodiri-Takayama (1971)).

We now turn to noninstantaneous adjustment. The processes we model here are processes with memory. We take the simple case where that is modeled by a system of differential equations with retarded arguments, where end points are fixed and where the constraints and objective functional have lagged variables. The optimality conditions are due to Halaney (1968).

Let (6.9) be replaced by:

$$(6.9') \quad \dot{x}(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_{\sigma_1}), u(t), u(t-\delta_1), u(t-\delta_2), \ldots, u(t-\delta_{\sigma_2}))$$

where τ's and δ's are positive numbers. Let (6.10) be replaced by

$$(6.10') \quad \phi^{\alpha}(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_{\sigma_1}), u(t), u(t-\delta_1, \ldots, u(t-\delta_{\sigma_2}))$$
$$= 0 \quad \alpha < 0$$
$$\geqq 0 \quad \alpha > 0.$$

Finally let (*) be replaced by

$$(**) \quad L = \int_{t_0}^{t_1} h^{\beta}(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_{\sigma_1}), u(t), u(t-\delta_1), \ldots, u(t-\delta_{\sigma_2}))\, dt\,.$$

The problem is to maximize L^0 subject to (6.9′), (6.10′) and (6.11) with L^{β} defined by **. Let $H^0 = H$, $H^{\tau_r} = H$ with t replaced by τ_r in all the variables and let $H^{\delta_r} = H$ with t replaced by $t + \delta_r$. In this way each variable becomes current once. Define

$$H^{II} = H^0 + \sum_{r=1}^{\sigma_1} H^{\tau_r} + \sum_{r=1}^{\sigma_2} H^{\delta_r}.$$

The necessary conditions for optimality are those stated in (6.17.1) – (6.17.4) with H replaced by H^{II}.

Finally we deal with a case which for some reason is popular with economists; the case where the end times are fixed with the terminal time being infinite. The reader interested only in the asymptotic behavior of the optimal path is referred to the excellent paper by Feinstein and Luenberger (1981). We have some comments about the path. Again, contrary to popular perception Pontriagin, etc. (1961) dealt with the case of infinite fixed terminal time. They modified the transversality condition by requiring that it be stated as a limit at $t_1 \to \infty$. Pontriagin, etc. (1961) had fixed control set and did not present sufficient conditions.

7. A Maximum Theorem

In this section let $X = R^n$ and $Y = R^n$. Let f be a map that assigns to each element of a subset D of X a nonempty subset of Y. We say that f is a *multifunction* from X to Y with domain D and range $A = \cup\{(f(x))\}x \in D$. We say that the multifunction f is *lower continuous at x* if for every open set P in Y with $P \cap f(x) \neq \phi$ there exists a neighborhood N(x) such that: $z \in N(x) \to f(z) \cap P \neq \phi$. For example, the multifunction:

$$f(x) = \begin{matrix} [0, 1] & \text{if} \quad x \neq 0 \\ [0, 2] & \text{if} \quad x = 0 \end{matrix}$$

is not lower continuous at x = 0. A multifunction f is said to be *upper* continuous at x if for every open set P in Y with $f(x) \subseteq P$ there exists a neighborhood N(x) such that $z \in N(x) \to f(z) \subseteq P$. For example the multifunction:

$$g(x) = \begin{matrix} [0, 1] & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0 \end{matrix}$$

is not upper continuous at x = 0. Note that, in the above two examples, f is upper but not lower continuous and g is lower but not upper continuous. If the

images of points under f are singletons, then f is upper continuous if and only if it is lower continuous.

We say that a multifunction is continuous at a point x if it is upper and lower continuous at x. In the above examples neither f nor g is continuous at $x = 0$. However the multifunction h is given by:

$$h(x) = [0, 1] \quad \text{for all} \quad x \in R,$$

is continuous at $x = 0$. A multifunction is *upper (lower) continuous* if it is upper (lower) continuous at every point x in its domain. A multifunction is *continuous* if it is upper and lower continuous*.

We now are ready to state the maximum theorem which is the reason for introducing the present section. By way of motivation, consider the following optimization problem:

Let X and Y be as above and let $f: X \to R$ be a function which is real valued. Suppose X is our decision space and Y is our parameter space. Let $g: X \times Y \to R^k$ be a function with vlaues in R^k. Our problem is: maximize $f(x; y)$subject to $g(x; y) \geqq 0$. We are interested in the nature of dependence of solutions to the optimization problem on the parameters. To this end define the mapping G as follows:

$$G(y) = \{x \in X \mid g(x, y) \geqq 0\}.$$

Assuming $G(y)$ is a multifunction relative to some subset D of Y, our question may be phrased as follows: Let the function h and the mapping k be defined by:

$$h(y) = \max f(x) \text{ subject to } x \in G(y).$$

$$k(y) = \{x \in G(y) \mid f(x) = h(y)\}.$$

The maximum theorem concerns the continuity properties of h and k.

(7.1) *A Maximum theorem:* Let $f: X \to R$ be a continuous real valued function and let G be a multifunction from Y to X which is continuous and which is *compact valued,* i.e. for all $x \in D$, $G(x)$ is closed and bounded. Then the real valued function h defined by:

$$h(y) = \max f(x) \text{ subject to } x \in G(Y)$$

is continuous. Furthermore, the multifunction k defined by:

$$k(y) = \{x \in g(y) \mid f(x) = h(y)\}$$

is upper continuous and compact valued.

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* The reader should note the differences in definitions used here and in Berge (1963) which we, otherwise, follow.

8. Notes on the Literature

Section 1 is normally included in the standard introductory chapter in many analysis books, for a reference on the subject, see e.g. Kurtowski (1962). There are many references on linear spaces, for abstract concepts see Halmos (1958) and for concrete concepts see Gantmacher (1959) and Shillov (1971). On Section 3 see Fleming (1977) and McShane and Botts (1959). On Section 4, see Fenchell (1953), Valentine (1964) and Roberts and Varberg (1973). On Section 5 see Hestenes (1975) and Bliss (1946). See also Stoer and Witzgall (1970) and El Hodiri (1973). 1971 On Section 6 see Hestenes (1966) and Young (1969). On Section 7 see Berge (1963).

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∇F instead of F' - F_x

$\nabla^2 F$ " " F'' F_{xx}

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This text presents a rigorous and reasonably complete statement of microeconomic theory as it exists today. It starts with a unified treatment of consumers' demand, followed by an exposition of the theory of production as well as costs and profit functions of competitive firms. Market structures are then studied in detail and finally a brief introduction to general equilibrium and welfare economics is presented.

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ISBN 3-540-17994-1
ISBN 0-387-17994-1