Digital Logic & Boolean Algebra

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Learning Objectives

1. Define and Identify Boolean Algebra Rules
2. Apply Boolean Algebra Rules to Simplify Boolean Expressions
3. Understand the basic logics of NOT, OR and AND operations
4. Use De Morgan's Theorem to simplify Boolean Expressions
5. Understand Boolean Functions and Standard Form

Digital Signal

- It is an **electrical signal** that has **two discrete values or levels**.
- These levels may be called as **LOW** level and **High** level.
- The signal will always be of one of the two levels.

- HIGH → LOGIC 1
- LOW → LOGIC 0

Digital (Logic) Circuit

- Signals in digital circuit operates using 2 discrete values and therefore said to be **Binary**. A binary digit, called a bit, has two values: 0 and 1

- Discrete elements of information are represented with groups of bits called **Binary Codes**.

- An n – bit binary code is group of n bits that assumes up to 2^n disticnt combinations of 1's and 0's, with each combination representing one element of the set that is being coded.

Boolean Algebra

- This characteristic of logic circuit allows the use of **Boolean Algebra** as a way of analyzing and designing digital circuits.

- Algebra associated with binary numbers is called Boolean Algebra.

- Variables used in Boolean Algebra are called **Boolean Variables**.
  - It is a variable that can only have one of two values: 0 or 1
  - Example: If X is a Boolean variable, then, X = 0 or X = 1

Boolean Algebra

- Boolean Algebra consists of Boolean variables and a set of logical operations

B. A = {0, 1 | AND, OR, NOT}

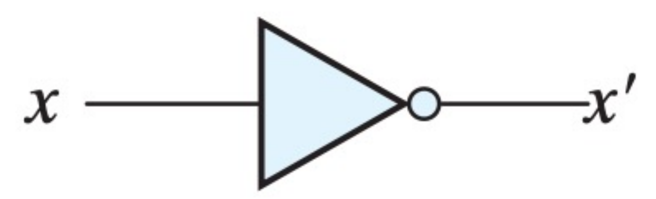
- Three basic logical operations - **AND**, **OR**, **NOT**

- Binary logic resembles binary arithmetic, and the operations AND and OR have similarities to multiplication and addition respectively

Basic Logic Operations

- NOT Operation (Complementation)

X	NOT (X)/$\overline{X}$
0	1
1	0

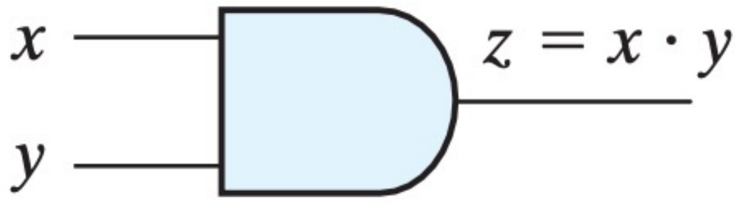


NOT gate or Inverter

Basic Logic Operations

- AND Operation (Logical Multiplication)

X	Y	X AND Y/ (X.Y)
0	0	0
0	1	0
1	0	0
1	1	1

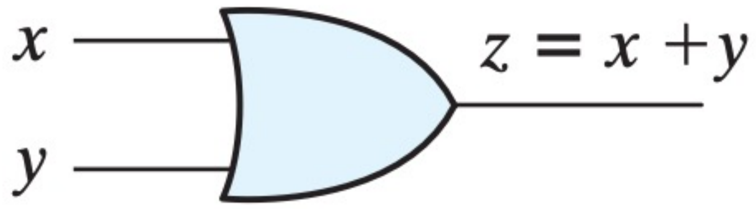


Two-input AND gate

Basic Logic Operations

- OR Operation (Logical Addition)

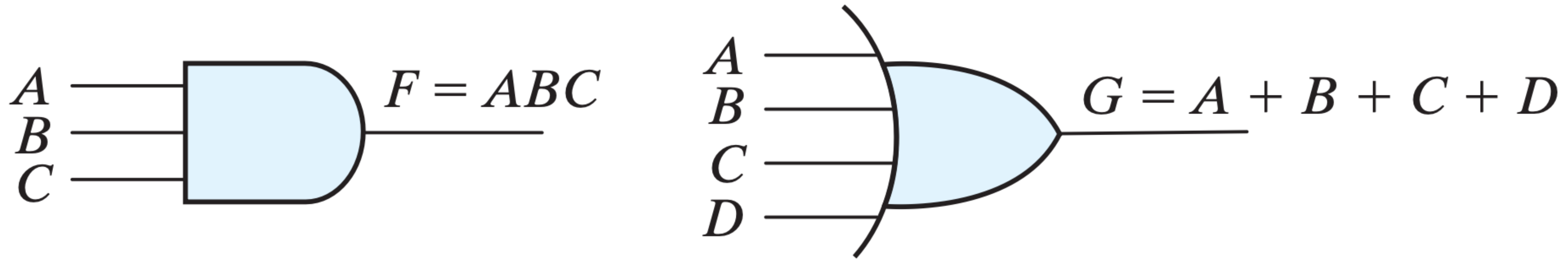
X	Y	X OR Y/ (X+Y)
0	0	0
0	1	1
1	0	1
1	1	1



Two-input OR gate

Logic Gates

- Example



(a) Three-input AND gate (b) Four-input OR gate

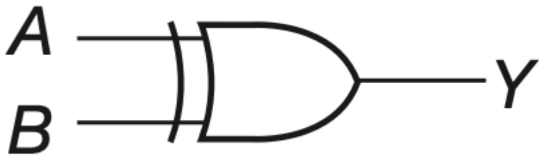
Logic Gates

NAND

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

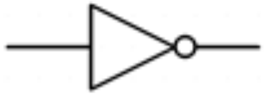
NOR

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0



XOR

A	B	Q
0	0	0
0	1	1
1	0	1
1	1	0

Logic Gate	Symbol	Description	Boolean
AND		Output is at logic 1 when, and only when all its inputs are at logic 1,otherwise the output is at logic 0.	$X = A \cdot B$
OR		Output is at logic 1 when one or more are at logic 1.If all inputs are at logic 0,output is at logic 0.	$X = A + B$
NAND		Output is at logic 0 when,and only when all its inputs are at logic 1,otherwise the output is at logic 1	$X = \overline{A \cdot B}$
NOR		Output is at logic 0 when one or more of its inputs are at logic 1.If all the inputs are at logic 0,the output is at logic 1.	$X = \overline{A + B}$
XOR		Output is at logic 1 when one and Only one of its inputs is at logic 1. Otherwise is it logic 0.	$X = A \oplus B$
XNOR		Output is at logic 0 when one and only one of its inputs is at logic1.Otherwise it is logic 1. Similar to XOR but inverted.	$X = \overline{A \oplus B}$
NOT		Output is at logic 0 when its only input is at logic 1, and at logic 1 when its only input is at logic 0.That's why it is called and INVERTER	$X = \overline{A}$

Logic Gates – HW – Exercise

Identify each of these logic gates by name, and complete their respective truth tables:

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	Output
0	
1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

A	B	Output
0	0	
0	1	
1	0	
1	1	

Boolean Algebra Rules

Law/Theorem	Law of Addition	Law of Multiplication
Identity Law	$x + 0 = x$	$x \cdot 1 = x$
Complement Law	$x + x' = 1$	$x \cdot x' = 0$
Idempotent Law	$x + x = x$	$x \cdot x = x$
Dominant Law	$x + 1 = 1$	$x \cdot 0 = 0$
Involution Law	$(x')' = x$	
Commutative Law	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative Law	$x+(y+z) = (x+y)+z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Distributive Law	$x \cdot (y+z) = x \cdot y+x \cdot z$	$x+y \cdot z = (x+y) \cdot (x+z)$
Demorgan's Law	$(x+y)' = x' \cdot y'$	$(x \cdot y)' = x' + y'$
Absorption Law	$x + (x \cdot y) = x$	$x \cdot (x + y) = x$

Boolean Algebra Rules

1. $X + 0 = X$
2. $X + 1 = 1$
3. $X.0 \quad = 0$
4. $X.1 \quad = X$
5. $X + X = X$
6. $X.X = X$
7. $X + \overline{X} = 1$
8. $X.\overline{X} = 0$
9. $\overline{\overline{X}} = X$

POWERUP
WITH THE CHANGE

Boolean Algebra Rules

10. Cumulative law

    $A + B = B + A$

    $A . B = B . A$

11. Redundancy law / Absorption

    $A + A. B = A$

    $A .(A + B) = A$

12. Associative law

    $(A + B) + C = A + (B + C)$

    $(A . B). C = A. (B. C)$

13. Distributive law

    $A. (B + C) = A. B + A. C$

    $A + B. C = (A + B)(A + C)$

Boolean Algebra Rules

14. …

$$X + \overline{X}Y = X + Y$$

$$\overline{P}\,\overline{Q} + P = P + \overline{Q}$$

$$\overline{A} + A\overline{B} = \overline{A} + \overline{B}$$

Example simplification

The **function** looks like this:

$F(A, B, C) = (A+B)B' + B' + BC$

We use the **Distributivity Law** $[(A+B)B' = AB' + BB']$:

$\boxed{F(A, B, C) = AB' + BB' + B' + BC}$

Example simplification

$$\boxed{F(A, B, C) = AB' + BB' + B' + BC}$$

We use one of the **Complement Law**(BB' = 0) and afterwards the **Identity of Addition** (AB' + 0 = AB'):

$$F(A, B, C) = AB' + B' + BC$$

We take a **common factor B'** of the first two:

$$\boxed{F(A, B, C) = B'(A + 1) + BC}$$

Example simplification

$\boxed{F(A, B, C) = B'(A + 1) + BC}$

We use the **Annihilator of Addition** ($A + 1 = 1$) and afterwards the **Identity of Multiplication** ($B '* 1 = B'$):

$F(A, B, C) = B' + BC$

We use the **Complement Law** ($B' + BC = B' + C$):

$\boxed{\mathbf{F(A, B, C) = B' + C}}$

Exercises

1. Select the Boolean expression that is not equivalent to $x.x + x.x'$
   a. $x.(x + x')$
   b. $(x + x').x$
   c. x'
   d. x

2. Select the expression which is equivalent to $x.y + x.y.z$
   a. $x.y$
   b. $x.z$
   c. $y.z$
   d. $x.y.z$

Exercises

3. Select the expression which is equivalent to $(x + y).(x + y')$
   a. y
   b. y'
   c. x
   d. x'

4. Select the expression that is not equivalent to $x.(x' + y) + y$
   a. $x.x' + y.(1 + x)$
   b. $0 + x.y + y$
   c. $x.y$
   d. y

Exercises

- Using Boolean algebra, simplify the following expressions:

$$(\mathrm{i})\, A = \overline{X}Y\overline{Z} + \overline{X}YZ + X\overline{Y}Z + XYZ$$

$$(ii)\, X = AB + A(B + C) + B(B + C)$$

$$(iii)\, A = X + \overline{Y} + \overline{X}Y + (X + \overline{Y}).\overline{X}Y$$

Exercises

$$
\begin{aligned}
(i) A &= \overline{X}Y\overline{Z} + \overline{X}YZ + X\overline{Y}Z + XYZ \\
&= \overline{X}Y(\overline{Z} + Z) + XZ(\overline{Y} + Y) \\
&= \overline{X}Y + XZ
\end{aligned}
$$

$$
\begin{aligned}
(ii) X &= AB + A(B + C) + B(B + C) \\
&= AB + AB + AC + B.B + BC \\
&= AB + AC + B + BC \\
&= B(A + 1 + C) + AC \\
&= B + AC
\end{aligned}
$$

Exercises

$$(iii)\, A = X + \overline{Y} + \overline{XY} + (X + \overline{Y}).\overline{XY}$$

$$= X + \overline{Y} + \overline{X}Y$$

$$= X + \overline{Y} + Y$$

$$= 1$$

POWERUP
WITH THE CHANGE

Exercises

$$(iv)\ Z = ABC[AB + \overline{C}(BC + AC)]$$

$$(v)\ Y = (A + \overline{A})(AB + AB\overline{C})$$

$$(vi)\ Y = (A + \overline{B})(A + C)$$

$$(vii)\ Z = AB + ABC + ABCD + ABCDE$$

Boolean Algebra Rules

14. De Morgan Law

$$\overline{(A\,.\,B)} = \overline{A} + \overline{B}$$

$$\overline{(A+B)} = \overline{A}\,.\,\overline{B}$$

$$\overline{(A\,.\,B\,.\,C)} = \overline{A} + \overline{B} + \overline{C}$$

$$\overline{(A+B+C)} = \overline{A}\,.\,\overline{B}\,.\,\overline{C}$$

HW - Exercises

1. Use DeMorgan's Theorem, as well as any other applicable rules of Boolean algebra, to simplify the following expression so there are no more complementation bars extending over multiple variable:

2. $\overline{A + \bar{B}C}$
3. $\overline{(A + B)\,.\,(\bar{C} + D)}$
4. $\overline{\overline{\bar{X}\,.\,Y} + \overline{XY}}$
5. $\overline{(A + BC).\,(D + EF)}$
6. $\overline{X\bar{Y}\,\overline{(\bar{W}+\bar{Y})}}$
7. $\overline{\overline{A.\overline{AB}} + \overline{B.\overline{AB}}}$
7. $\overline{\overline{AB} + \overline{AC}}$
8. $\overline{\overline{XY\bar{Z}}Y}$
9. $\overline{\overline{J + K}JL}$

Boolean Functions

- Algebraic expression consisting of;
  - Binary values,
  - Constants 0 and 1,
  - Logic operation symbols

- Example:

$$F_1 = x + y'z$$

$$F_2 = x'y'z + x'yz + xy'$$

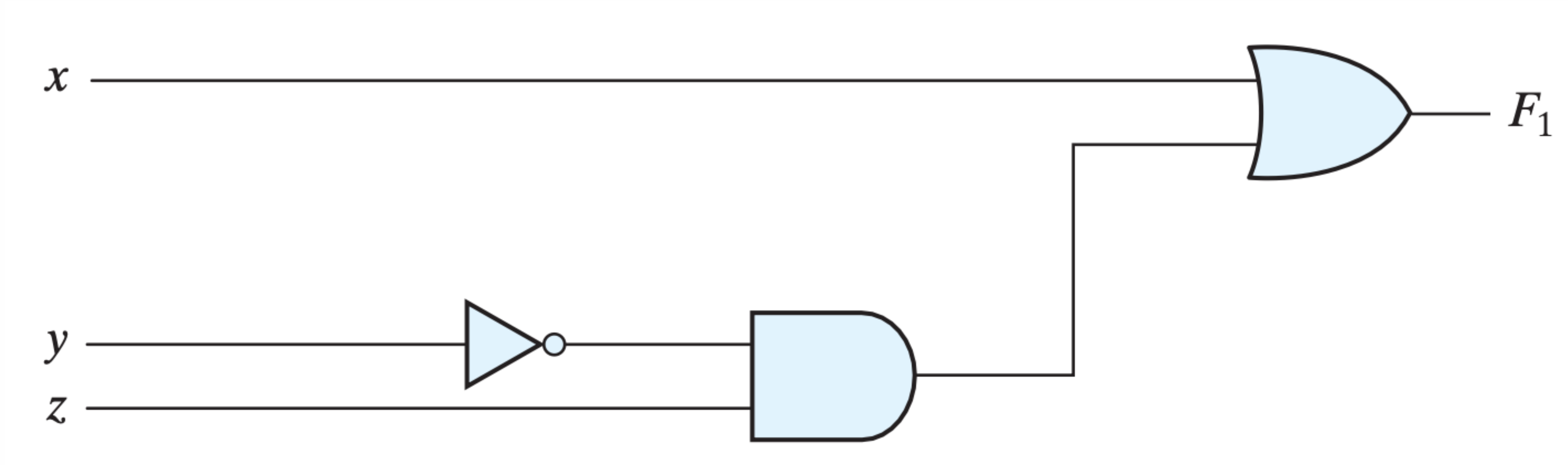
Boolean Functions

$$F_1 = x + y'z$$
$$F_2 = x'y'z + x'yz + xy'$$

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	0	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	0
1	1	1	1	0

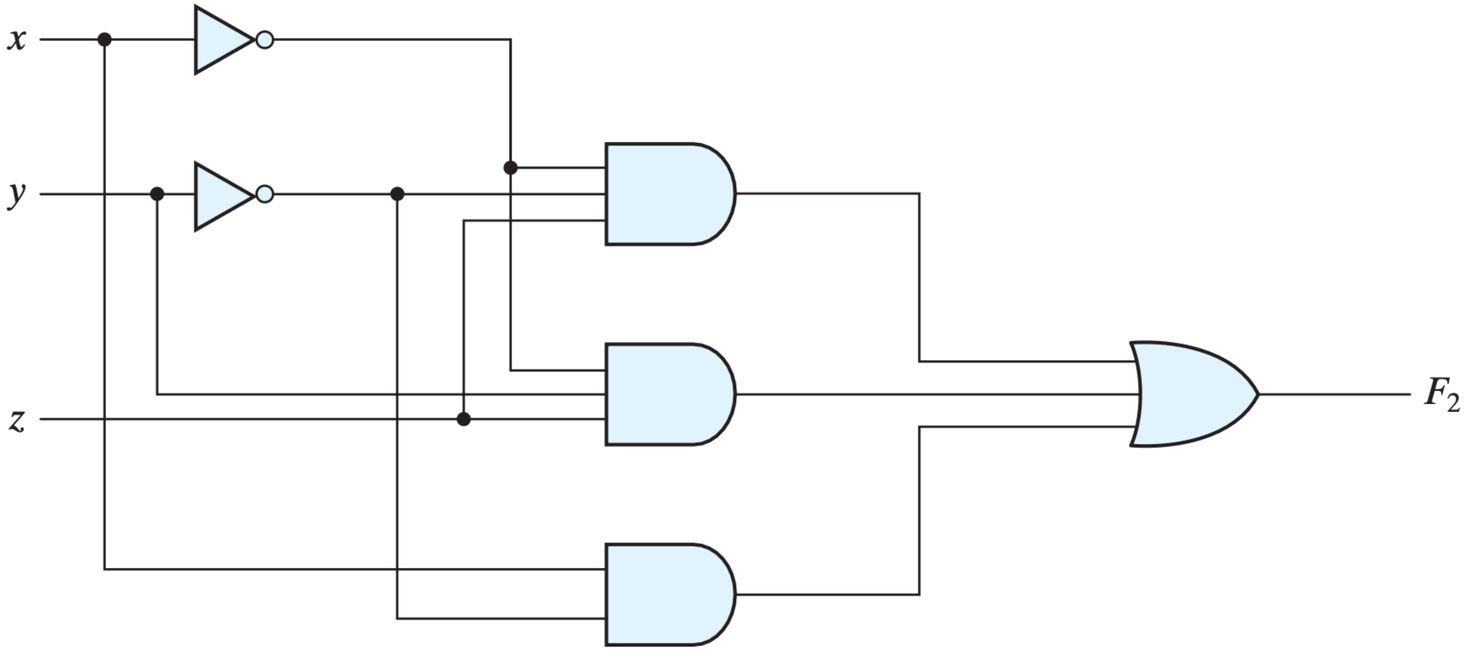
Boolean Functions

$$F_1 = x + y'z$$



Boolean Functions

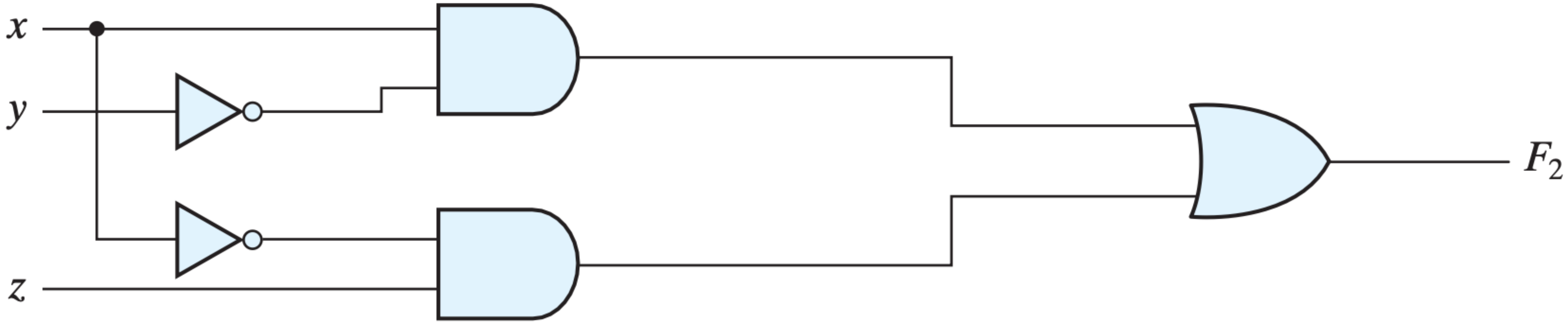
$$F_2 = x'y'z + x'yz + xy'$$



(a) $F_2 = x'y'z + x'yz + xy'$

Boolean Functions

$$F_2 = x'y'z + x'yz + xy' = x'z + xy'$$



(b) $F_2 = xy' + x'z$

Minterms and Maxterms

			Minterms		Maxterms	
x	**y**	**z**	**Term**	**Designation**	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

Minterms and Maxterms

- Example

x	y	z	Function f_1	Function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Minterms and Maxterms

- Example – as a sum of Minterms

$$f_1 = x'y'z + xy'z' + xyz = m_1 + m_4 + m_7$$

$$f_2 = x'yz + xy'z + xyz' + xyz = m_3 + m_5 + m_6 + m_7$$

NIBM POWERING GREAT MINDS

Minterms and Maxterms

- Example – as a product of Maxterms

$$f_1 = (x + y + z)(x + y' + z)(x + y' + z')(x' + y + z')(x' + y' + z)$$
$$f_1 = M_0 . M_2 . M_3 . M_5 . M_6$$

$$f_2 = (x + y + z)(x + y + z')(x + y' + z)(x' + y + z)$$
$$f_2 = M_0 . M_1 . M_2 . M_4$$

Minterms and Maxterms

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

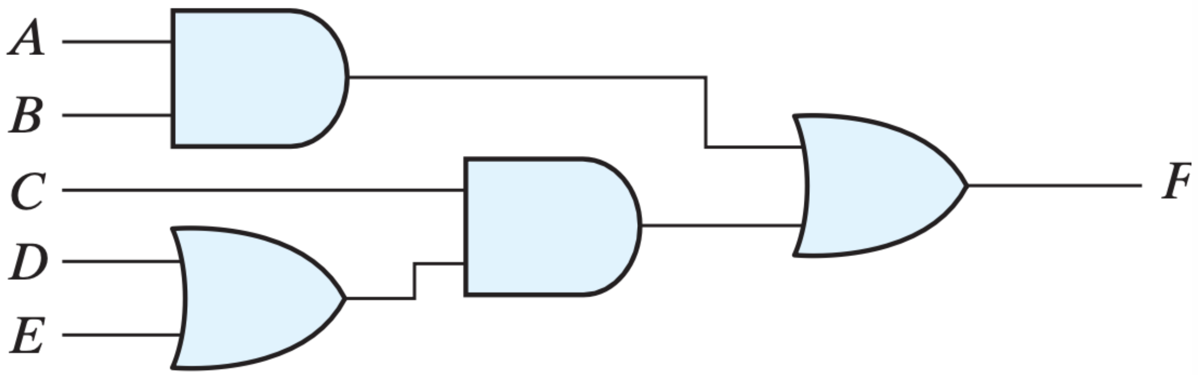
Minterms

Maxterms

Standard Form

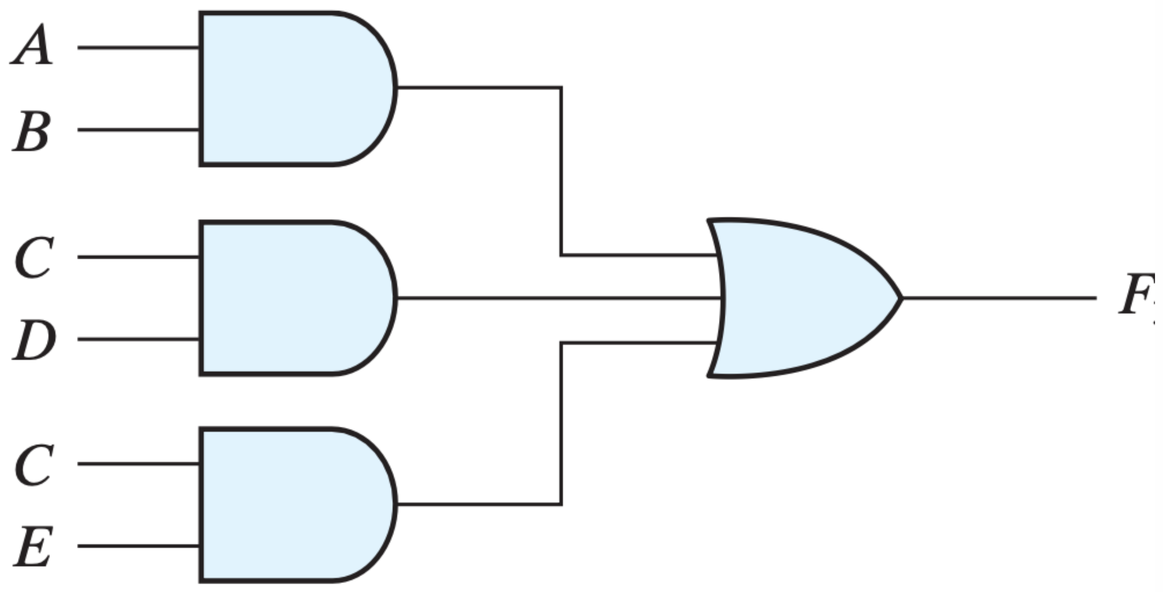
- Results in a **two-level structure** of gates
  - Sum of Products
  - Product of Sums

- Example: $\mathrm{F} = AB + C(D + E)$



Standard Form

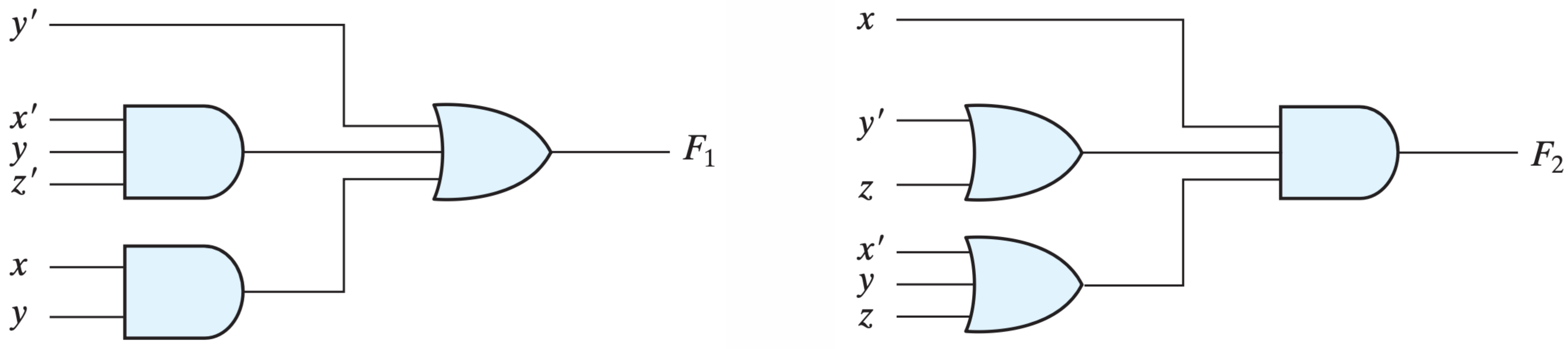
- Example:$\mathrm{F} = AB + C(D + E) = AB + CD + CE$



(b) $AB + CD + CE$

Standard Form

- Example: $F_1 = y' + xy + x'yz'$ and $F_2 = x(y' + z)(x' + y + z')$

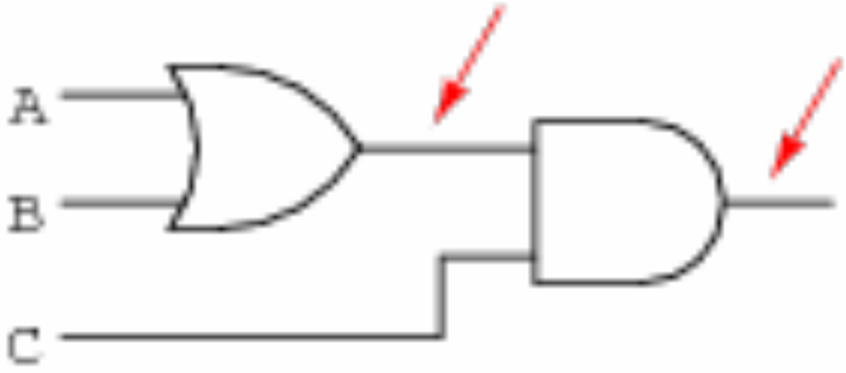


(a) Sum of Products

(b) Product of Sums

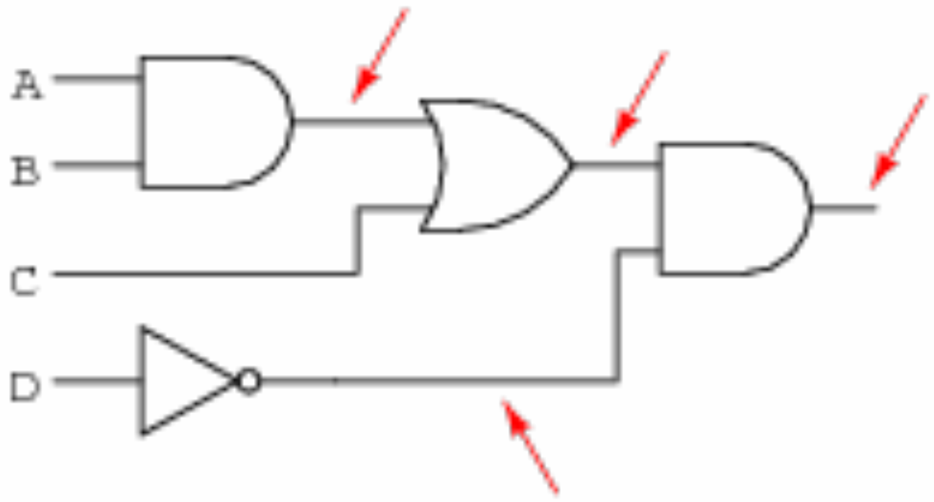
Exercises

1. Convert the following logic gate circuit into a Boolean expression, writing Boolean sub-expressions next to each gate output in the diagram:



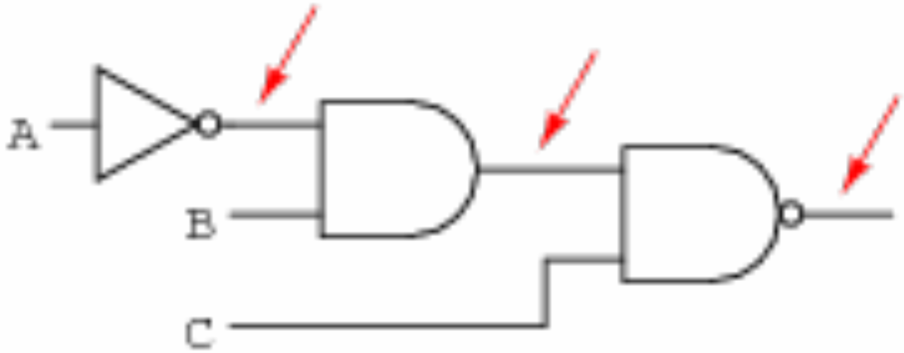
Exercises

2. Convert the following logic gate circuit into a Boolean expression, writing Boolean sub-expressions next to each gate output in the diagram:



Exercises

3. Convert the following logic gate circuit into a Boolean expression, writing Boolean sub-expressions next to each gate output in the diagram:



Exercises

4. Following Boolean expression is written down on a paper and handed over to you, to build a gate circuit:

$$A\bar{B} + \bar{C}(A + B)$$

Draw a logic gate circuit for this function

Exercises

5. Implement the following Boolean expression in the form of a digital logic circuit:

$$\overline{(\overline{AB} + C)}B$$

Form the circuit by making the necessary connections between the pins of 74LS37 integrated circuits (IC’s)

Exercises

Note: IC used – 74LS37