STAT 5370 – Decision Theory

Homework 1

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Contents

Problem 1

In this problem we will consider developing a Bayesian model for Poisson iid data; i.e., our observed data will consist of $Y_1, ..., Y_n \overset{iid}{\sim} Poisson(\lambda)$. Recall, a random variable Y is said to follow a Poisson distribution, with mean paramater λ, if its pmf is given by

$$p(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!} I(y \in \{0, 1, 2, ...\}) \tag{1}$$

Note, the Poisson model is often used to analyze count data.

(a) For the Poisson model, identify the conjugate prior. This should be a general class of priors.

If we assume the Gamma distribution as a prior for the Poisson model: $p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda} \propto \lambda^{\alpha-1}e^{-\beta\lambda}$; where $\alpha > 0$ (shape paramater) and $\beta > 0$ (rate paramater) then this choice of prior is also the conjugate prior for the Poisson model.

(b) Under the conjugate prior, derive the posterior distribution of $\lambda|y$. This should be a general expression based on the choice of the hyper-parameters specified in your prior.

To derive the posterior distribution we need the liklihood function associated with the Poisson model:

$$L(\lambda|y) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^y}{y!} = \frac{e^{-n\lambda}\lambda^{\sum\limits_{i=1}^{n} y_i}}{\prod\limits_{i=1}^{n} y!} \tag{2}$$

In general, the posterior distribution is defined as:

$$p(\lambda|y) = \frac{p(y|\lambda)p(\lambda)}{\int\limits_A p(y|\lambda)p(\lambda)d\lambda} \propto \underbrace{p(y|\lambda)}_{\text{L}(\lambda)}\underbrace{p(\lambda)}_{\text{Prior}} \tag{3}$$

With a Poisson DGP and a Gamma prior the derived posterior distribution is defined by:

$$\begin{aligned} p(\lambda|y) &= e^{-n\lambda}\lambda^{\sum\limits_{i=1}^{n} y_i} \cdot \lambda^{\alpha-1}e^{-\beta\lambda} \\ &= \lambda^{\sum y_i + \alpha - 1} \cdot e^{-(n-\beta)\lambda} \end{aligned} \tag{4}$$

The posterior distribution of λ under the Gamma prior is $Gamma(\sum y_i + \alpha, n + \beta)$, where α and β are the hyperparamaters of the Gamma distribution.

(c) Find the posterior mean and variance of $\lambda|y$. These should be general expressions based on the choice of the hyper-parameters specified in your prior.

The mean and variance of the Gamma distribution is given by the following expressions: $E(\lambda) = \frac{\alpha}{\beta}$; $Var(\lambda) = \frac{\alpha}{\beta^2}$.

Then the posterior mean and variance of $\lambda|y$ is:

$$
\begin{aligned}
E(\lambda|y) &= \frac{\alpha}{\beta} = \frac{\sum y_i + \alpha}{n + \beta} \\
Var(\lambda|y) &= \frac{\alpha}{\beta^2} = \frac{\sum y_i + \alpha}{(n + \beta)^2}
\end{aligned} \tag{5}
$$

(d) Obtain the MLE of λ. Develop and discuss a relationship that exists between the MLE and posterior mean identified in (c).

Starting from the Poisson distribution, $p(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!}$, we obtain the liklihood function:

$$
L(\lambda|y_1, y_2, ..., y_n) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} \tag{6}
$$

Then, taking log of equation 6 and differnetiating w.r.t λ we can obtain an expression for $\hat{\lambda}$:

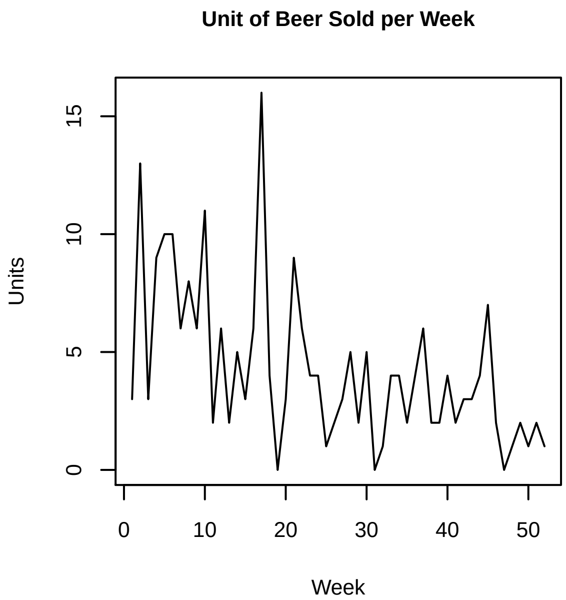
$$
\begin{aligned}
&\log L(\lambda|x_1, x_2, ..., x_n) = -n\lambda - \sum_{i=1}^{n} \log(y_i!) + \log(\lambda) \sum_{i=1}^{n} y_i \\
&\frac{\partial \log L(\cdot)}{\partial \lambda} = -n + \frac{\sum\limits_{i=1}^{n} y_i}{\hat{\lambda}} = 0 \\
&\hat{\lambda} = \frac{\sum\limits_{i=1}^{n} y_i}{n}
\end{aligned} \tag{7}
$$

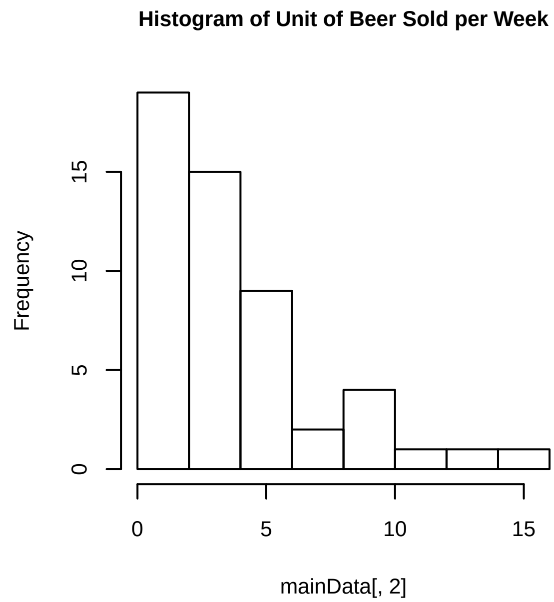
We can now see that when the hyperparamaters of the posterior mean (part c, equation 5) are zero the posterior mean collapses to the MLE estimator of λ. This shows that the MLE estimator of λ and the posterior mean of λ are related by a weighted average. Further, we can see that the MLE estimator for λ is $\sum_{i=1}^{n} y_i/n$ and the posterior mean are a combination of the weighted average of $\sum_{i=1}^{n} y_i/n$ and the prior mean.

(e) Write two separate R programs which can be used to find both a (1-α)100% equal-tailed credible interval and a (1-α)100% HPD credible interval for the Poisson model. These programs should take as arguments the following inputs: the observed data, prior hyper-parameters, and significance level.

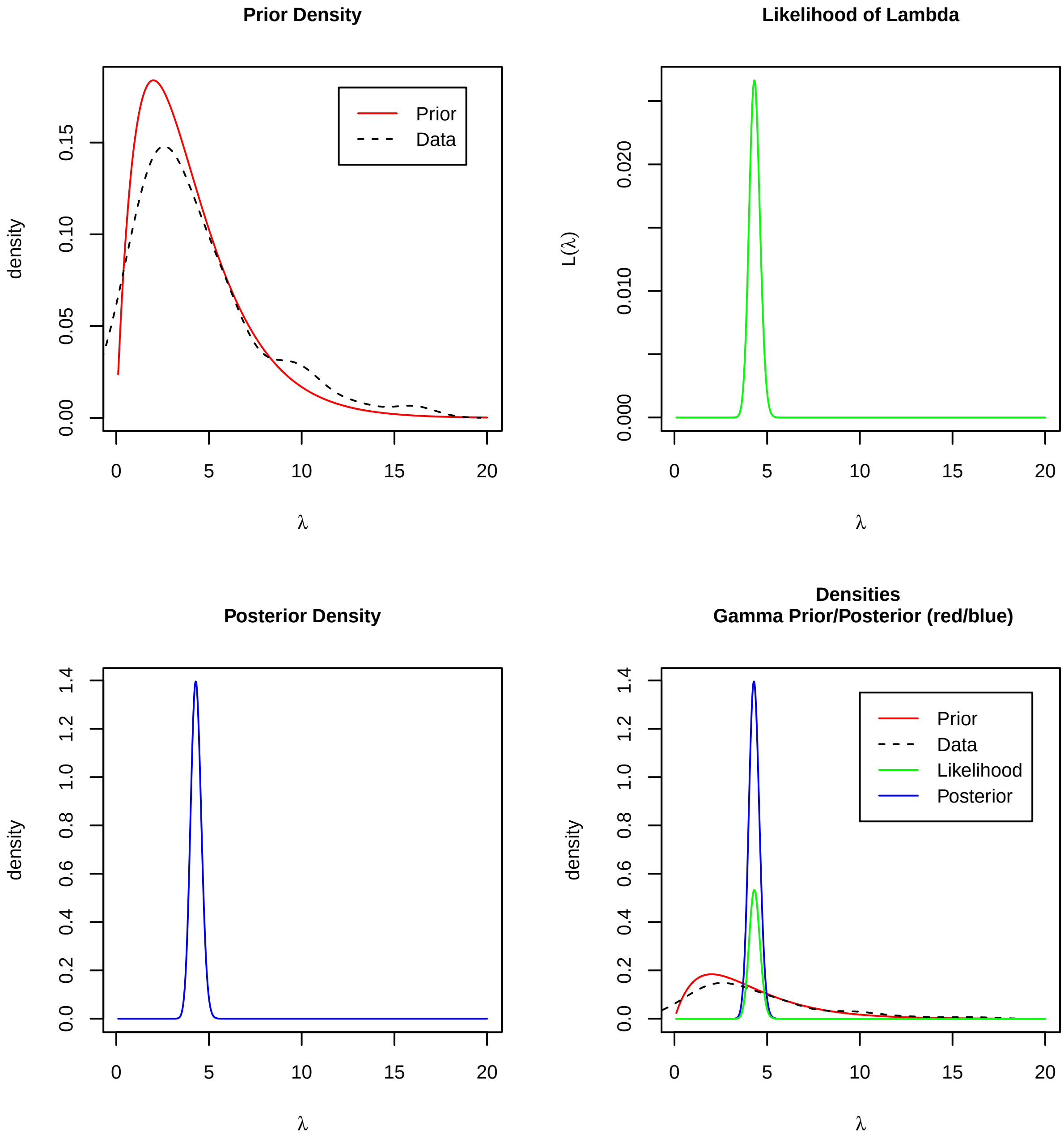
Please see Appendix

(f) Find a data set which could be appropriately analyzed using the Poisson model. This data set should be of interest to you, and you should discuss, briefly, why the aforementioned model is appropriate; e.g., consider independence, identically distributed, etc. etc. You will also need to provide the source of the data.





(g) Analyze the data set you have selected in (e). Provide posterior point estimates of λ, credible intervals, etc. etc. Your analysis should be accompanied by an appropriate discussion of your findings.




```
## [1] 3.761782 4.883812
```

Appendix - Code for Problem 1, part e.

R code for the Equal Tail Credible Interval:

```
equalTail_CI <- function(y, a1, b1, alpha){

  n <- length(y)
  qgamma(c(0.025, 0.975), a1, b1)

}
```