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A new method to simulate the triangular distribution

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ABSTRACT

A new method is developed to simulate the triangular distribution. The result is of interest from a practical as well as a theoretical viewpoint. The new method is surprisingly simple and is more efficient than the standard method of simulation by inversion of the cumulative distribution function. The method is also generalized to simulate the two-sided power distribution.

0. Scope and purpose

Triangular random variables are commonly used in risk analysis to quantify uncertainty. On the one hand, large scale simulations of are frequently undertaken, so a fast simulation method is important. On the other hand, a spreadsheet user may have to code his own generation method so simplicity is important. In this paper, we investigate the convenient and efficient simulation of observations from a triangular distribution. A new method is obtained that is surprisingly simple and efficient.

1. Introduction

In project evaluation and review (PERT) and risk analysis, the beta distribution is often used in theoretical analyses to specify risk and uncertainty. In practice, the triangular distribution is frequently used instead of the beta because a decision-maker's subjective viewpoints are more easily turned into parameter estimates, namely the minimum, maximum and most likely values [1–3]. In reliability analysis, the triangular distribution can be used when the failure rate of a component is not known with precision [4]. Due to its frequent use in applications, it is important to be able to efficiently simulate the triangular distribution.

The goal is to simulate the general triangular density, given in the middle panel of Fig. 1. We know that we can easily simulate the two special cases of the triangular given in the other two panels of Fig. 1. The left panel is the density of the minimum of two independent uniform variables and the right panel is the corresponding maximum. The general triangular looks like a linear combination of these left and right extreme triangular densities. Of course, the density of functions of several random variables cannot be obtained by simple geometric manipulation of density functions.

When comparing different methods to generate random variables there are competing factors to consider. Devroye [5] lists six factors:

1. Speed
2. Initialization (or set-up) time
3. Length of (compiled) code

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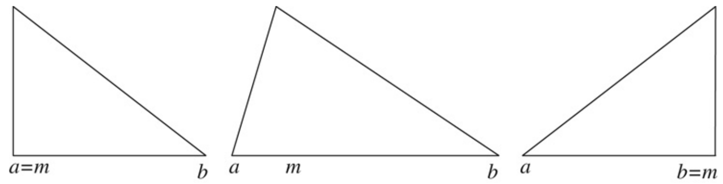

Fig. 1. Three Triangular Densities. The left and right panels are the cases that correspond to the minimum and maximum, respectively, of two uniform variates. The typical triangular density, with mode m, is shown in the center panel.

4. Portability (machine independence)
5. Range of applicability
6. Simplicity and readability.

He remarks that "Of these factors, the last one is perhaps the most neglected in the literature." We have found a new algorithm that scores well in all these measures and especially on its simplicity and intuitive nature.

The most common and general method for simulating from a non-uniform cumulative distribution F is the inversion method: namely, compute $F^{-1}(u)$ where u is a uniform (0, 1) variate. The result will produce a simulated value from the distribution F. This is the commonly used method to simulate the triangular distribution.

If it is not possible to invert F in closed form (or involves functions which are time- consuming to evaluate) then other methods to consider include acceptance/rejection and special purpose methods. In this paper, we will consider a special purpose method for the triangular distribution which is faster than using inversion.

2. The triangular distribution

The triangular distribution has three parameters: the lower limit a, the upper limit b and mode m. Fig. 1 shows the three possible cases with the general situation in the middle figure. The height is determined from the parameters since the area in each triangle must be 1 unit. The case on the left in Fig. 1 will be referred to as a left-triangular density and the one on the right as a right-triangular density. If $m = (a + b)/2$, then the density is called the symmetric triangular as it is symmetric about m.

The motivation for this paper is as follows. Suppose we obtain 2 independent variates u and v which are uniform on the interval $[a, b]$. Then we compute the following three quantities:

(i) minimum (u, v)
(ii) maximum (u, v)
(iii) $(u + v)/2$.

It is well-known that (i) corresponds to the left-triangular density, (ii) to the right-triangular density and (iii) to the symmetric triangular. We now want to find an equally simple method that will generate the arbitrary density shown in the middle of Fig. 1 and include these three instances as special cases.

We may restrict ourselves to the simulation of triangular variates on (0, 1): $a = 0, b = 1$ with arbitrary mode c, and then scale the results to any (a, b). If we obtain the variate t that simulates the triangular on (0, 1) then $a + (b - a)t$ will have the triangular distribution on (a, b) with mode $m = a + (b - a)c$.

Two methods that are currently used to generate a triangular distribution on (0, 1):

(i) *Inversion method.* This can be used for all triangular distributions since the inverse CDF can be expressed in closed form. The density (f) and cumulative distribution (F) are given by:

$$\begin{aligned} f(x) &= \begin{cases} 2x/c, & \text{if } 0 \le x \le c \\ 2(1-x)/(1-c), & \text{if } c \le x \le 1 \end{cases} \\ F(z) &= \begin{cases} z^2/c, & \text{if } 0 \le z \le c \\ 1-(1-z)^2/(1-c), & \text{if } c \le z \le 1. \end{cases} \end{aligned} \tag{1}$$

Therefore,

$$F^{-1}(u) = \begin{cases} \sqrt{cu}, & \text{if } 0 \le u \le c \\ 1-\sqrt{(1-c)(1-u)}, & \text{if } c \le u \le 1. \end{cases} \tag{2}$$

If u is a uniform [0, 1] variate then $F^{-1}(u)$ will be triangular on [0, 1]. We want to find a simulation method that is more efficient than using (2). In particular, we would like to avoid the square root and the decision of which of the two terms in F^{-1} to use.

(ii) *Acceptance/rejection method.* In the case of the triangular, this is not competitive with the inversion method but is widely used for pedagogical purposes. Consider the arbitrary density shown in the center of Fig. 1. Imbed this in a convenient region such as the rectangle shown in Fig. 2 that consists of the bottom side of the triangle plus the dashed lines. As shown in

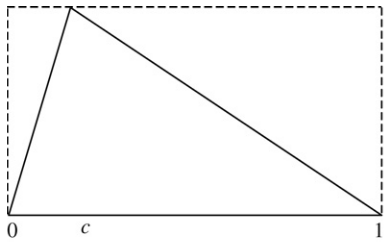

Fig. 2. The Triangular Density on (0, 1) with mode c. The rectangle (dashed line) contains the region between the horizontal axis and the triangular density as a subset.

Fig. 2, we assume we are using the interval (0, 1) and so the height of the triangle and rectangle is 2 units. Let x be distributed uniformly on (0, 1) and y uniformly on (0, 2) such that x and y are independent. Then (x, y) is uniformly distributed over the interior of the rectangle in Fig. 2 with joint density 1/2. We then check to see if (x, y) is in the interior of the triangle: $y \leq f(x)$ where f is the density function of the triangular on (0, 1). If so, we use the value of x as our simulated value of the triangular distribution. If not, we generate more (x, y) pairs until we eventually succeed.

The reason that this method works is that we will stop with a pair (x, y) that is clearly uniformly distributed over the interior of the triangle. The pair must have joint density 1 since this is the area under the triangle. Therefore, for given x in (0, 1), the density of x is obtained by integrating the joint density over all y, from 0 to $f(x)$. Since the joint density is 1 this yields $f(x)$, the triangular density.

3. The MINMAX method

We now introduce a new special purpose algorithm to simulate the triangular distribution. This new method, which we call the MINMAX method, will generate a triangular variate on [0, 1] with mode c by using

$$(1-c)\mathrm{MIN}(u, v) + c\mathrm{MAX}(u, v) \tag{3}$$

where u and v are independent random variables distributed uniformly on [0, 1]. If c is 0, 1 or 0.5 then this method reduces to (i)–(iii), respectively, in Section 2. Expression (3) claims that an arbitrary triangular distribution can be generated by a weighted average of left and right triangular variates. It is important to note that $\mathrm{MIN}(u, v)$ and $\mathrm{MAX}(u, v)$ are dependent random variables so we cannot choose other methods to simulate the left and right triangular distributions and then use a linear combination of them via (3). See the Appendix for the proof that (3) generates a triangular variable.

4. Random vectors in the plane

If it is possible to generate points uniformly over the region between a given density f and the horizontal axis then the x-coordinate of these points will be distributed with density f (see [6]). The acceptance/ rejection method is a repetitive method to do this when we cannot sample the region exactly but must approximate it. In this section we will show the connection to the MINMAX method.

Devroye ([5], Ch. 11) shows how to sample points uniformly within a given triangle in the plane. Let s, w, z be points in the plane that are not collinear and therefore will form the vertices of a triangle. This sampling method is given by:

1. Generate two independent uniform [0, 1] variates u and v.
2. Let $\mathrm{MIN} = \mathrm{minimum}(u, v)$ and $\mathrm{MAX} = \mathrm{maximum}(u, v)$.
3. Define the result as $p = s\mathrm{MIN} + w(1 - \mathrm{MAX}) + z(\mathrm{MAX} - \mathrm{MIN})$.

This results in a point p in the plane selected uniformly over a triangle with vertices s, w, z.

We now consider a special case of this method and show how it reduces to the MINMAX method. Set $s = (1, 0)$, $w = (0, 0)$ and $z = (c, 2)$. This defines a triangle with the base oriented along the x-axis from 0 to 1. The modal point of the density is at z. Generating points uniformly on the interior of this triangle is what we had accomplished with the acceptance/ rejection method (after a pair was accepted). But in this case we do not need to use a rectangle to enable us to simulate the points; we can use the triangle itself. The x coordinate of p will give us the variate of interest: it will have a triangular density on [0, 1] with mode $0 \leq c \leq 1$. To verify this, the x coordinate of p generated from step 3 is: $\mathrm{MIN} + c(\mathrm{MAX} - \mathrm{MIN}) = (1-c)\mathrm{MIN} + c\mathrm{MAX}$. This shows that (3) will produce observations from a triangular distribution.

5. A special purpose algorithm

Inversion and acceptance/ rejection approaches are general purpose methods and are not expected to be the most efficient for any given distribution. As a special purpose algorithm to simulate the triangular distribution consider the following:

$$c + (u - c)\sqrt{v} \tag{4}$$

Table 1
Elapsed time (seconds) to generate one million triangular variates for three algorithms using loop structure or MATLAB's vector operations

	Loop	Vector
Inversion method	2.91	0.329
One line method	0.42	0.183
MINMAX method	2.10	0.094

(Computer: Core 2 Duo, 2.4 GHz).

with independent uniform [0, 1] variates u and v. This "One Line method" has been used [7] to simulate the triangular distribution. Clearly this will be faster than inversion and will provide a stiffer challenge for the MINMAX algorithm.

It is interesting to compare the formula for the One Line method with the MINMAX. Expression (4) can be written as

$$\left(1-\sqrt{v}\right)c+\sqrt{v}u. \tag{5}$$

Both (3) and (5) are convex linear combinations; the random variable $\sqrt{v}$ plays the role in (5) of the constant c in (3). In (3), two points are obtained inside the interval [0, 1] and then the algorithm's value is a point between them determined by c. On the other hand, in (5), we start with a simulated value u along with fixed c. The algorithm's value is then a point between them determined by the random variable $\sqrt{v}$. This starts with a uniform variate u and then moves it toward c. This will concentrate the density near c as desired.

6. Efficiency of the simulation

Using the MATLAB 7.3 package, we simulated one million triangular values using the inversion method, the One Line method and the MINMAX method. The built-in uniform random number generator was used. First, each method was implemented with a straightforward loop with the results stored in an array. Then the process was repeated using vector operations available in MATLAB which eliminates the need for an explicit loop structure. The elapsed times, in seconds, using a desktop personal computer (Core2 Duo, 2.4 GHz), are given in Table 1. These times include the time to store the results in an array for later use. We averaged the times over 100 replications.

The MINMAX method must generate twice as many uniform variates as the inversion method, however no square root is computed and no conditional statement is required. Built-in min and max functions are used. Table 1 indicates that with or without a loop MINMAX is faster than inversion. And in the vector case it is twice as fast as the One Line method (0.094 vs. 0.183).

7. Extension of the method to the TSP

At this point it would be interesting to see what happens when we generalize this method to more than two uniform variables. Let $u_1, \ldots, u_n$ be I.I.D. uniform (0, 1) variables. Then consider the random variable generated by

$$(1-c)\mathrm{MIN}(u_1,\ldots,u_n)+c\,\mathrm{MAX}(u_1,\ldots,u_n). \tag{6}$$

In statistics, the special case $c = 0.5$ is known as the midrange. In that context the distribution of (6) has been known for a long time [8]. Evidently it has not been examined for $c \neq 0.5$.

We first note that MIN and MAX have Beta distributions with densities $f_{\mathrm{MIN}}(x) = n(1-x)^{n-1}, f_{\mathrm{MAX}}(x) = nx^{n-1}$ and in the case $n = 2$ they reduce to the right and left triangular densities. As we found with case $n = 2$, it turns out that the case (6) also will generate a variable with a density that looks like it is a simple combination of the densities of MIN and MAX even though they are dependent random variables. The density of the random variable defined by (6) is

$$f(x)=\begin{cases}nx^{n-1}/c^{n-1}, & 0<x\le c\\ n(1-x)^{n-1}/(1-c)^{n-1}, & c\le x<1\end{cases} \tag{7}$$

with the proof given in the Appendix. This family of distributions is called the standard Two-Sided Power distribution (TSP) introduced by van Dorp and Kotz [9] and then generalized in [10] to any interval (a,b). It includes the uniform ($n = 1$) and triangular ($n = 2$). (In (6) and (7) we require that n be an integer whereas in the definition of the TSP it may be any positive real value.)

The TSP distribution has been used as a model of interest rate changes and in general as a flexible family similar to the beta distribution (see [9]). Moments and theoretical properties of this family of distributions can be found in [11].

An example density is shown in Fig. 3 for $n = 3$ and $c = 0.25$. For n larger than 2, the MINMAX method rapidly loses its advantage in simulation compared to the inversion method. However, the result that (6) has a TSP distribution is still a very interesting and totally unexpected theoretical result.

8. Conclusions

We have created a special purpose generation method called the MINMAX method for random variates from a triangular distribution. We have seen that it is faster than the standard inversion method, simpler to implement and easy to understand.

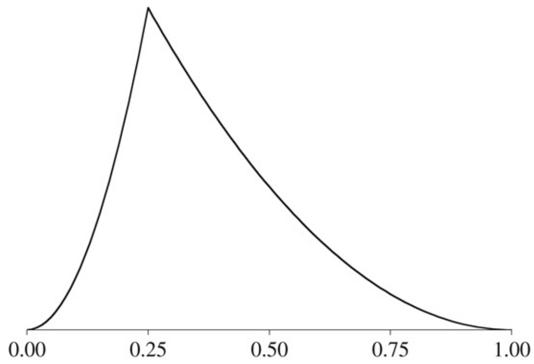

Fig. 3. An example of the standard two-sided power (TSP) distribution with $n = 3$ and $c = 0.25$. On each side of the mode c, the density function is a parabola.

It is quite surprising that the method has not previously been discovered. We showed that the new MINMAX method (for $n = 2$) could be deduced from Devroye's triangle method. The MINMAX method was generalized to simulate the two-sided power (TSP) distribution for integer n.

The MINMAX method is an improvement over inversion as the standard variate generation method for the triangular distribution. It is also faster than the special purpose One Line method (4) when using vector operations. The fact that the MINMAX method also is easy to understand makes it an interesting pedagogical example of a special purpose simulation algorithm.

Acknowledgments

The authors wish to thank anonymous reviewers for bringing Refs. [8,12] to our attention as well as the class of TSP distributions.

Appendix. Derivation of the MINMAX Method

We will derive the distribution of the variable in Eq. (6). We start with n independent uniform (0, 1) variables and define the variables MIN and MAX over these n values. The joint density of $z_1 = \text{MIN}$ and $z_2 = \text{MAX}$ is $\varphi(z_1, z_2) = n(n-1)(z_2 - z_1)^{n-2}$ on $0 < z_1 < z_2 < 1$ [12]. Then we define new variables $y = (1-c)z_1 + cz_2$ and $w = z_1$. The joint density of these new variables is $g(y, w) = \varphi(z_1, z_2)\,|J|$ and the absolute value of the Jacobian of the inverse transformation can be calculated to be $|J| = 1/c$. Then $g(y, w) = n(n-1)(y-w)^{n-2}/c^{n-1}$ on the triangular region $w < y < (1-c)w + c$ for $0 < w < 1$. We will now integrate out w leaving the density f of the variable y.

For $0 < y \le c$: $f(y) = \int_0^y g(y, w)\mathrm{d}w = \frac{ny^{n-1}}{c^{n-1}}$. For $c \le y < 1$: the variable w can range between $(y-c)/(1-c)$ and y. Therefore, $f(y) = \int_{(y-c)/(1-c)}^{y} n(n-1)(y-w)^{n-2}/c^{n-1}\mathrm{d}w$ and after some algebra this reduces to $\frac{n(1-y)^{n-1}}{(1-c)^{n-1}}$.

This verifies the density of the variable in (6) is given by the TSP distribution in (7). The case $n = 2$ provides a direct proof of the validity of MINMAX method (3).

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