Problems Time Series

Exercise 1 *Indicate whether each of the following statements is true or false, by circling your answer.*

Let $\mathbf{X} = (X_t)_{t\in\mathbb{Z}}$ be a stochastic process. If $\mathbf{X}$ is a stationary process we denote by $h \mapsto \gamma(h)$ the autocovariance of (X_t).

TRUE FALSE 1. *If $\mu(t) = E(X_t), \ \forall t$, then the autocovariance function of $\mathbf{X}$ is $\gamma(s,t) = \mathbb{E}(X_t X_s) - \mu(t)\mu(s)$.*

TRUE FALSE 2. *If $\mathbf{X}$ is a stationary process, then the correlation between X_0 and X_1 is equal to $\dfrac{\gamma(0)}{\gamma(1)}$*

TRUE FALSE 3. *Assume that $\mathbf{X}$ is a sequence of independent variables such that of t is even $X_t \sim$ Exp(1) and if t is odd $X_t \sim \mathcal{N}(1,1)$. Then $\mathbf{X}$ is strongly stationary.*

TRUE FALSE 4. *If $\mathbf{X}$ is a weakly stationary Gaussian stochastic process, then $\mathbf{X}$ is strongly stationary.*

TRUE FALSE 5. *Assume that $\mathbf{X}$ is a white noise, then $Y_t = X_{t+1} + 0.5X_t - 0.3X_{t-1}$ is a causal process.*

TRUE FALSE 6. *Autoregressive processes, AR(p), are invertible processes.*

TRUE FALSE 7. *Assume that $\mathbf{X}$ is a white noise and assume that there exists $\alpha \in l^1(\mathbb{Z})$ such that $F_\alpha X_t = X_t - \frac{1}{2}X_{t-20}$, then $\alpha_1 = 0$ and $\alpha_{20} = \frac{1}{2}$.*

TRUE FALSE 8. *The sum of two uncorrelated stationary processes is a stationary process.*

TRUE FALSE 9. *Assume that $\mathbf{X}$ satisfies the following equation $X_t = .3X_{t-1} + Z_t + 0.1Z_{t-1}$, then $\mathbf{X}$ is a stationary causal process but non-invertible.*

TRUE FALSE 10. *Assume that $\mathbf{X}$ is an AR(p) then $\gamma(h) \xrightarrow[h\to\infty]{} 0$.*

Exercise 2 *Prove the following result: Let* $\mathbf{Z}$ *be a WN*$(0, \sigma^2)$*, and let* α*,* $\beta \in l^1(\mathbb{Z})$.

$$\textit{If } F_\alpha \mathbf{Z} = F_\beta \mathbf{Z} \;\Rightarrow\; \alpha = \beta$$

Exercise 3 *Let* $(X_t)_{t \in \mathbb{Z}}$ *be a* ***moving-average*** *model of order* 1*, MA(1):*

$$X_t = Z_t + \theta Z_{t-1}$$

Prove that

1. $\mu_X(t) = 0$, $\forall\, t$
2. $\gamma_X(s,t) = \begin{cases} 0 & \textit{if } |s-t| > 1 \\ \theta\sigma^2 & \textit{if } |s-t| = 1 \\ (\theta^2+1)\sigma^2 & \textit{if } |s-t| = 0 \end{cases}$

Exercise 4 *Moving Average processes*

Let $(X_t)_{t \in \mathbb{Z}}$ *be a* ***moving-average*** *model of order* 1*, MA(1):*

$$X_t = Z_t + 0.3 Z_{t-1} + 0.1 Z_{t-2}$$

Compute the ACF of $(X_t)_{t \in \mathbb{Z}}$

Exercise 5 *Let* $(X_t)_{t \in \mathbb{Z}}$ *and* $\varphi \in \mathbb{R}$.

1. *Find a stationary solution of the AR(1) equation* $X_t = \varphi X_{t-1} + Z_t$*;*
2. *Express the ACF of the solution when* $|\varphi| < 1$*;*
3. *Solve the ARMA(1, 1) equation* $2X_t = X_{t-1} + 2Z_t + Z_{t-1}$*;*
4. *Show that there exists* $0 \leq \rho < 1$ *and* $C > 0$ *such that*

$$\forall\, h \in \mathbb{Z},\;\; |\gamma_X(h)| \leq C\rho^{|h|},\;\; \forall\, h \in \mathbb{Z};$$

Exercise 6 *ARMA processes*

Let us consider the ARMA(2,1) equation

$$X_t - \frac{1}{4} X_{t-2} = Z_t - Z_{t-1}$$

1. *Compute a stationary solution.*
2. *Is this solution causal? Justify?*
3. *Is this solution invertible? Justify?*
4. *Compute the ACF γ_X of X;*
5. *Compute proj$(X_s, H_{t-1,1})$ for $(s,t) = (100, 3)$*

Exercise 7 *Let $(Z_k)_{k\in\mathbb{Z}}$ be i.i.d. r.v. of normal law $\mathcal{N}(0,1)$, and let a, b be two reals.*

1. *Compute the process $\Delta_3 S_t$ where $S_t = \cos\left(\frac{2\pi}{3}t\right) + Z_t$ and where $\Delta_3 = 1 - B^3$.*
2. *Is a white noise? Justify your answer and precise weak or strong;*
3. *If $p \in \mathbb{N}$, $(Z_k^p)_{k\in\mathbb{Z}}$ a white noise ? Justify your answer, precise weak or strong, and compute if possible the mean and autocovariance;*
4. *Compute the autocovariance of $(X_k)_{k\in\mathbb{Z}}$ where $X_k = aZ_{k-1} + b_{Z\,k+1}$;*
5. *In which case $(X_k)_{k\in\mathbb{Z}}$ is causal? invertible? (justify your answers).*

Exercise 8 *Let us consider the following ARMA equation:*

$$X_t - 2X_{t-1} = Z_t - \frac{5}{2}Z_{t-1} + Z_{t-2}$$

1. *Compute the rational fraction of the equation, prove that it admits a unique stationary solution; and compute it;*
2. *Find $\alpha \in l^1(\mathbb{Z})$ such that $Z = F_\alpha X$ and $\alpha_k = 0$ for any $k \notin \mathbb{N}$.*