Linear Filtering, Linear Processes, ARMA Processes

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Outline

Filtering theorem

Definitions, Notations

- $l^1(\mathbb{Z})$ is the set of sequences $(\alpha_k)_{k\in\mathbb{Z}}$ with values in $\mathbb{R}$ such that

$$\sum_{k\in\mathbb{Z}} |\alpha_k| < \infty$$

- **Linear Filtering**: Let $\mathrm{X} = (X_t)_{t\in\mathbb{Z}}$ be a stochastic process with values in $\mathbb{R}$. The following sequence of variable

$$Y_t = \sum_{k\in\mathbb{Z}} \alpha_k X_{t-k}, \ \ \forall\, t \in \mathbb{Z}$$

is called a linear filter of $(X_t)_{t\in\mathbb{Z}}$ with weights $(\alpha_k)_{k\in\mathbb{Z}}$ and we denote

$$\mathrm{Y} = F_\alpha \mathrm{X}$$

.

- When the sequence $(\alpha_k)_{k\in\mathbb{Z}}$ is with finite support, i.e, $\mathrm{card}\{k,\ \alpha_k \neq 0\} < \infty$, $\mathsf{Y} = F_\alpha \mathsf{X}$ exists and it's called a **moving-average** process associated to X.

Example: running averages

- Assume that we observe a time series (x_t)
- 7 days running average : for all $t > 7$

$$z_t = \frac{1}{7}\sum_{k=0}^{6} x_{t-k} = \frac{1}{7}\left(x_t + x_{t-1} + \ldots + x_{t-6}\right)$$

$$\alpha_k = \begin{cases} 1/7 & \text{if } 0 \le k \le 6 \\ 0 & \text{Otherwise} \end{cases}$$

Theorem 1

Let $(\alpha_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ and let $(X_t)_{t\in\mathbb{Z}}$ be a stochastic process bounded in L^p for $p \geq 1$, i.e.,

$$\sup_{t\in\mathbb{Z}}\mathbb{E}\left(|X|^p\right) < \infty.$$

Let

$$\forall\, t \in \mathbb{Z},\ \forall\, m,\ n \in \mathbb{N},\quad Y_{t,m,n} = \sum_{k=-m}^{k=n} \alpha_k X_{t-k}$$

Then, $\left(Y_{t,m,n}\right)_{m,n\geq 1}$ converge almost surely and in L^p when $(m,n) \to \infty$ to a random variable $Y_y \in L^p$, i.e.,

$$Y_{t,m,n} \to Y_t \in L^p \ \text{ and } \ \lim_{m,n,\to\infty} \mathbb{E}\left(|Y_{t,m,n} - Y_t|^p\right) = 0$$

Proof : Omitted.

Theorem 2: Filter of a stationary process

Let $(\alpha_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ and let $\mathrm{X} = (X_t)_{t\in\mathbb{Z}}$ be a stationary stochastic process, with second order, with mean $\mu_X = \mathbb{E}(X_t)$ and ACF $t \mapsto \gamma_X(t)$. Then the filter $F_\alpha \mathrm{X}$, defined by

$$Y_t = (F_\alpha \mathrm{X})_t = \sum_{k\in\mathbb{Z}} \alpha_k X_{t-k}, \ \ \forall\, t \in \mathbb{Z},$$

exists. It's a stationary process with second order, with mean and ACF functions as follow:

$$\mu_Y = \mu_X \sum_{k\in\mathbb{Z}} \alpha_k \ \text{ and } \ \gamma_Y(h) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \alpha_j \alpha_k \gamma_X(h+j-k), \ \ \forall\, t \in \mathbb{Z}$$

Proof of Theorem 2

- Since X is stationary and with second order, $||X_t^2|| = \mathbb{E}(\left|X_t^2\right|) = \gamma_X(0)^2 + |\mu_X|^2 < \infty$, then X is a bounded process. We can then apply Theorem 1 with $p = 2$ and show that the process F_αX exists and it belongs to L^2.

- For the rest of the proof, we need to recall that the inner product in L^2 is continuous: if $U = \lim\limits_{n \to +\infty} U_n$ and $V = \lim\limits_{n \to +\infty} V_n$, then

$$\langle U, V \rangle = \lim_{n \to +\infty} \langle U_n, V_n \rangle = \left\langle \lim_{n \to +\infty} U_n, \lim_{n \to +\infty} V_n \right\rangle$$

Proof of Theorem 2

$$
\begin{aligned}
\mathbb{E}\left((F_\alpha X)_t\right) &= \left\langle 1, \sum_{k\in\mathbb{Z}} \alpha_k X_{t-k} \right\rangle = \left\langle 1, \lim_{n\to+\infty} \sum_{k=-n}^{k=n} \alpha_k X_{t-k} \right\rangle \\
&= \lim_{n\to+\infty} \left\langle 1, \sum_{k=-n}^{k=n} \alpha_k X_{t-k} \right\rangle \\
&= \lim_{n\to+\infty} \mathbb{E}\left(\sum_{k=-n}^{k=n} \alpha_k X_{t-k} \right)
\end{aligned}
$$

Proof of Theorem 2

$$\mathbb{E}\left((F_\alpha X)_t\right) = \lim_{n\to+\infty} \sum_{k=-n}^{k=n} \alpha_k \mathbb{E}(X_{t-k}) = \lim_{n\to+\infty} \mu_X \sum_{k=-n}^{k=n} \alpha_k$$

$$= \mu_X \sum_{k\in\mathbb{Z}} \alpha_k$$

Exercise 1

Prove that $\gamma_Y(t) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \alpha_j \alpha_k \gamma_X(t+j-k), \ \forall\, t \in \mathbb{Z}.$

Example: Filter of a WN

- Let $Z = (Z_t)_{t\in\mathbb{Z}}$ be a WN with mean 0 and variance σ^2,i.e., $Z \sim \mathrm{WN}(0,\sigma^2)$. Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$.
- Let $X = F_\alpha Z$ the filter associated to Z and α,
- According to Theorem 2, $\mu_X = \mu_Z \sum_{k\in\mathbb{Z}} \alpha_k = 0$
- $\forall\, t \in \mathbb{Z}$

$$\gamma_X(t) = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \alpha_j\alpha_k\gamma_Z(t+j-k) = \sigma^2 \sum_{j\in\mathbb{Z}} \alpha_j\alpha_{t+j}$$

- Such process is called **linear process.**

Example: ACF of a MA(q)

- Let Z $\sim$ WN$(0, \sigma^2)$. A time series X $= (X_t)_{t\in\mathbb{Z}}$ MA(q), $q \geq 1$, is defined as follows:

$$X_t = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k}$$

- X is a filter of Z associated to the sequence α equal to

$$\alpha_k = \begin{cases} 1 & \text{if} \quad k = 0 \\ \theta_k & \text{if} \quad k = 1, \ldots, q \\ 0 & \text{if} \quad k \notin \{0, 1, \ldots, \} \end{cases}$$

Example: ACF of a MA(q)

- Using Theorem 2, $\mu_X = 0$, $\gamma_X(0) = \sigma^2 \left(1 + \theta_1^2 + \ldots + \theta_q^2\right)$,
- and for $1 \leq h \leq q$, we have

$$\gamma_X(h) = \theta_h + \theta_1\theta_{h+1} + \ldots + \theta_{h-q}\theta_q,$$

and for $h > q$, $\gamma_X(h) = 0$.

Exercise 2

Let (Y_t) be a time series defined by

$$Y_t = \beta t + s_t + U_t$$

where $\beta \in \mathbb{R}$, where (s_t) is periodic function with periodicity 4, and where $U = (U_t)_{t\in\mathbb{Z}}$ is a stationary process

1. Is $(Y_t)_{t\in\mathbb{Z}}$ stationary?
2. Let's define $Z = F_\alpha Y$ where $\alpha_0 = 1$, $\alpha_4 = -1$ and $\alpha_k = 0$ for all $k \neq 0$ and 4. Show that (Z_t) is a stationary process and compute γ_Z in terms of γ_U.

Backshift Operator

- The operator B defined as for

$$B^k X_t = X_{t-k}, \;\; \forall\, t,\, k$$

  is called the backshift operator.

- The $\mathrm{X} = F_\alpha \mathrm{Z} \iff$

$$X_t = P_\alpha(B) Z_t$$

  where $P_\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^k$

Exercise 3

Let $\mathrm{X} = (X_t)$ be a time series defined by

$$X_t = \sum_{i=0}^{k} a_i t^i + Z_t, \ \ \forall\, t \in \mathbb{Z}$$

where (Z_t) is a $\mathrm{WN}(0, \sigma^2)$ and $a_0,\ a_1, \ldots,\ a_k \in \mathbb{R}$.

1. Show that $(1 - B)\mathrm{X}$ has a polynomial trend with degree $k - 1$
2. What happen to (X_t) when we apply the operator $(1 - B)^p$, $p \in \mathbb{N}^*$? (discuss according to p).

Exercise 4

Let (Z_t) be a $\mathrm{WN}(0, \sigma^2)$ and let

$$X_t = \sum_{i=0}^{t} \lambda^i (Z_{t-i} - Z_{t-i-1}).$$

1. Discuss, according to $\lambda \in \mathbb{R}$, the stationarity of (X_t)
2. Show, $\lambda \in (-1, 1)$, that there exists a stationary process (Y_t) such that

$$X_t - Y_t \xrightarrow[t \to +\infty]{L^2} 0$$

Causality and Invertibility

Let $Z = (Z_t)_{t\in\mathbb{Z}}$ be a stationary process, $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$, and let $X = F_\alpha Z$. X is called

- a **causal** process, if $\forall\, k < 0$, $\alpha = 0$:

$$X_t = \sum_{k=0}^{\infty} \alpha_k Z_{t-k}, \text{ and } \sum_{k=0}^{\infty} |\alpha_k| < \infty$$

- an **invertible** process if there exists a sequence $\beta = (\beta_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ such that $\forall\, k < 0$, $\beta = 0$ and $Z = F_\alpha X$:

$$Z_t = \sum_{k=0}^{\infty} \beta_k X_{t-k}, \text{ and } \sum_{k=0}^{\infty} |\beta_k| < \infty$$

Causality and Invertibility, Example

Let $Z = (Z_t)_{t \in \mathbb{Z}}$ be a WN.

- **Causal** processes: MA(q), $q \geq 1$:

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$$

- **Invertible** processes: AR(p)

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t$$

Composition and convolution

Theorem 5: Composition of filters

Let $\alpha,\ \beta \in l^1(\mathbb{Z})$ and let $\mathbf{X}$ be a stationary process, then

$$F_\alpha(F_\beta \mathbf{X}) = F_{\alpha * \beta}\mathbf{X},$$

where

$$(\alpha * \beta)_k = \sum_{j \in \mathbb{Z}} \alpha_j \beta_{k-j}$$

Composition of filters

- If α, $\beta \in l^1(\mathbb{Z})$, then $\alpha * \beta \in l^1(\mathbb{Z})$
- The convolution product in $l^1(\mathbb{Z})$ is
  - commutative:
  $$\alpha * \beta = \beta * \beta$$
  - is distributive with respect to addition,
  $$\alpha * (\beta + \gamma) = \alpha * \beta + \alpha * \gamma$$
  - identity element, $e = 1_0$, $\alpha * e = \alpha$

Lemma

Let α, $\beta \in l^1(\mathbb{Z})$ and the power series P_α defined on the unit circle of $\mathbb{C}$

$$P_\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^k.$$

Then for every α and β are in $l^1(\mathbb{Z})$,

- if $P_\alpha = P_\beta$, then $\alpha = \beta$
- $P_{\alpha * \beta} = P_\alpha P_\beta$

Theorem : Invertiblity of power series

Let $\alpha \in l^1(\mathbb{Z})$ such that $P_\alpha(z) = \sum_{k\in\mathbb{Z}} \alpha_k z^k$ is a polynomial. The following three statements are equivalent

1. α is invertible according to the convolution in $l^1(\mathbb{Z})$
2. If $P_\alpha(z) = 0$, then $|z| \neq 1$
3. $z \mapsto 1/P_\alpha(z)$ can be developped as a power of z in a crown in $\mathbb{C}$ containing the unit circle, i.e., $\exists r < 1 < R$ and $\beta \in l^1(\mathbb{Z})$ such that

$$\frac{1}{P_\alpha(z)} = \sum_{k\in\mathbb{Z}} \beta_k z^k, \ \ \forall r < |z| < R.$$

Theorem : Invertiblity of power series (2/2)

- Where one of the previous statements is satisfied, $\alpha^{-1} = \beta$,
- β is the inverse of α
- Furthermore,

$$\forall\, |z| \leq 1, \;\; P_\alpha(z) \neq 0 \iff \mathrm{Supp}(\beta) \subseteq \mathbb{N}$$

Examples (1/3)

$$\alpha = (\alpha_k), \quad \alpha_k = \begin{cases} 1 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 0, 1 \end{cases}, \ P_\alpha(z) = 1 - z$$

$$\frac{1}{P_\alpha(z)} = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \ \ \forall\, |z| < 1$$

Then

$$\alpha^{-1} = 1_{\mathbb{N}}(k) \notin l^1(\mathbb{Z}).$$

Examples (2/3)

$$\alpha = (\alpha_k), \quad \alpha_k = \begin{cases} 2 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 0, 1 \end{cases}, \; P_\alpha(z) = 2 - z$$

$$\frac{1}{P_\alpha(z)} = \frac{1}{2 - z} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} z^k, \;\; \forall\, |z| < 2$$

Then

$$\alpha^{-1} = 2^{(k+1)} 1_{\mathbb{N}}(k) \in l^1(\mathbb{Z}).$$

Examples (3/3)

$$\alpha = (\alpha_k), \quad \alpha_k = \begin{cases} 1/2 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 0, 1 \end{cases}, \; P_\alpha(z) = \frac{1}{2} - z$$

$$\frac{1}{P_\alpha(z)} = \frac{1}{\frac{1}{2} - z} = \frac{2}{1 - 2z} = \sum_{k=0}^{\infty} 2^{k+1} z^k, \quad \forall\, |z| < 1/2$$

Examples (3/3)

$$\begin{aligned}
\frac{1}{P_\alpha(z)} &= \frac{1}{\frac{1}{2} - z} \\
&= \left(-\frac{1}{z}\right) \times \frac{1}{1 - \frac{1}{2z}} \\
&= \sum_{k=-\infty}^{-1} -2^{-k+1} z^k, \;\; |z| > \frac{1}{2}
\end{aligned}$$

Then

$$\alpha^{-1} = -2^{k+1} 1_{k \leq -1}(k) \in l^1(\mathbb{Z}).$$

Theorem 5: Identifiability

Let $\mathcal{S}$ be the set of stationary processes with second order.

1. Let Z be a WN$(0, \sigma^2)$, for all α, $\beta \in l^1(\mathbb{Z})$

$$\text{If } F_\alpha Z = F_\beta Z \ \Rightarrow \ \alpha = \beta$$

2. Let $\alpha \in l^1(Z)$, for all X, Y $\in \mathcal{S}$:

$$\text{If } F_\alpha X = F_\alpha Y \ \Rightarrow \ X = Y$$

Exercise 5

Prove Theorem 5

ARMA Equation

Definition (V1)

A time series $X = (X_t)_{t\in\mathbb{Z}}$ is called an **ARMA** process of order $(p, q) \in \mathbb{N} \times \mathbb{N}$ if X satisfies the following equation:

$$\forall t \in \mathbb{Z}, \quad X_t = \sum_{k=1}^{p} \varphi_k X_{t-k} + Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k}$$

where

- $\varphi = (\varphi_1, \dots, \varphi_p) \in \mathbb{R}^p$,
- $\theta = (\theta_1, \dots, \theta_q) \in \mathbb{R}^q$
- $Z = (Z_t)_{t\in\mathbb{Z}}$ is a $\mathrm{WN}(0, \sigma^2)$

$$X \sim \mathrm{ARMA}(p, q)$$

Definition (V2)

A time series $\mathrm{X} = (X_t)_{t\in\mathbb{Z}}$ is called an **ARMA** process of order $(p,q) \in \mathbb{N}\times\mathbb{N}$ if X satisfies the following equation:

$$\Phi(B)\mathrm{X} = \Theta(B)\mathrm{Z}$$

where

- B is the backshift operator
- $\Phi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$
- $\Theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$
- $\mathrm{Z_t} = (Z)_{t\in\mathbb{Z}}$ is a $\mathrm{WN}(0,\sigma^2)$

Definition (V3)

A time series $\mathsf{X} = (X_t)_{t\in\mathbb{Z}}$ is called an **ARMA** process of order $(p, q) \in \mathbb{N} \times \mathbb{N}$ if X satisfies the following equation:

$$F_{\alpha_\varphi}\mathsf{X} = F_{\alpha_\theta}\mathsf{Z}$$

where $([\alpha_\varphi]_k)_{k\in\mathbb{Z}}$ and $([\alpha_\theta]_k)_{k\in\mathbb{Z}}$ are sequences defined as follows

$$[\alpha_\varphi]_k = \begin{cases} 1 & \text{if } k = 0 \\ -\varphi_k & \text{if } 1 \le k \le p \\ 0 & \text{otherwise} \end{cases} \quad \text{and } [\alpha_\theta]_k = \begin{cases} 1 & \text{if } k = 0 \\ \theta_k & \text{if } 1 \le k \le q \\ 0 & \text{otherwise} \end{cases}$$

AR and MA processes

A time series $X = (X)_{t\in\mathbb{Z}}$ is called

- an AR(p) if X satisfies

  $$\Phi(B)X = Z$$

  where $\Phi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$ and $\Theta(z) = 1$.

- a MA(q) if X can be written as follows

  $$X = \Theta(B)Z$$

  where $\Theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$ and $\Phi(z) = 1$.

where $Z \sim WN(0, \sigma^2)$

Existence of the AR(1)

Theorem 6: AR(1) Equation: $X_t = \varphi_1 X_{t-1} + Z_t$.

- If $|\varphi_1| = 1$, there's no stationary process solution of AR(1).
- If $|\varphi_1| < 1$, AR(1) has a unique causal solution:

$$X_t = \sum_{k=0}^{\infty} \varphi_1^k Z_{t-k},$$

- If $|\varphi_1| > 1$, AR(1) has a unique non-causal solution:

$$X_t = -\sum_{k=1}^{\infty} \varphi_1^{-k} Z_{t+k} = -\sum_{k=-\infty}^{-1} \varphi_1^k Z_{t-k},$$

Proof (1/4)

- Let (X_t) be a stationary solution of second order of AR(1)
- We prove by recurrence on r, that for all $r \geq 0$

$$X_t = \sum_{k=0}^{r} \varphi_1^k Z_{t-k} + \varphi_1^{r+1} X_{t-r-1} \qquad (*)$$

- Since $|\varphi_1| < 1$, $(\varphi_1^k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ and $Z_t \in L^2$, by using Theorem 1, Chapter 2, Slide 12),

$$\sum_{k=0}^{r} \varphi_1^k Z_{t-k} \xrightarrow[r\to+\infty]{L^2} \sum_{k=0}^{+\infty} \varphi_1^k Z_{t-k}$$

Proof of $(*)$ (2/4)

- $r = 0$, $X_t = \varphi_1^0 Z_t + \varphi_1^{0+1} X_{t-0-1}$
- Assume $(*)$ is true for $r \geq 0$, according to AR(1) equation: $X_{t-r-1} = Z_{t-r-1} + \varphi_1 X_{t-r-2}$ Then

$$
\begin{aligned}
X_t &= \sum_{k=0}^{r} \varphi_1^k Z_{t-k} + \varphi_1^{r+1} X_{t-r-1} \\
&= \sum_{k=0}^{r} \varphi_1^k Z_{t-k} + \varphi_1^{r+1} \left(Z_{t-r-1} + \varphi_1 X_{t-r-2} \right) \\
&= \sum_{k=0}^{r+1} \varphi_1^k Z_{t-k} + \varphi_1^{r+2} X_{t-r-2}
\end{aligned}
$$

Proof (3/4)

- Since (X_t) is a stationary and second order process and $|\varphi_1| < 1$

$$\varphi_1^{r+1} X_{t-r-1} \xrightarrow[r \to +\infty]{L^2} 0.$$

- Then

$$X_t = \sum_{k=0}^{+\infty} \varphi_1^k Z_{t-k}$$

- If $|\varphi_!| > 1$, prove by recurrence that

$$X_t = -\sum_{k=1}^{r} \varphi_1^{-k} Z_{t+k} + \varphi_1^{-r} X_{t+r}, \quad \forall\, r \geq 1 \qquad (**)$$

- $|\varphi_1| = 1$, AR(1)= Random Walk

Proof of $(**)$ (4/4)

- $r = 1$, $X_t = \varphi_1^{-1} X_{t+1} - \varphi_1^{-1} Z_{t+1}$
- Assume $(**)$ is true for $r \geq 1$, according to AR(1) equation:

$$X_{t+r} = \varphi_1^{-1} X_{t+r+1} - \varphi_1^{-1} Z_{t+r+1}$$

Then

$$\begin{aligned} X_t &= -\sum_{k=1}^{r} \varphi_1^{-k} Z_{t+k} + \varphi_1^{-r} X_{t+r} \\ &= -\sum_{k=1}^{r} \varphi_1^{-k} Z_{t+k} + \varphi_1^{-r} \left(-\varphi_1^{-1} Z_{t+r+1} + \varphi_1^{-1} X_{t+r+1} \right) \\ &= -\sum_{k=1}^{r+1} \varphi_1^{-k} Z_{t+k} + \varphi_1^{r+1} X_{t+r+1} \end{aligned}$$

Existence of the ARMA process

Theorem 7

Let Φ and Θ be the polynomials associated to the ARMA(p, q) Equation.

1. If Φ doesn't have a unit root, **then** ARMA(p, q) has a unique stationary solution, given by $F_\alpha Z$ where

$$\alpha = \alpha_\theta * \alpha_\varphi^{-1}$$

2. If the Equation ARMA(p, q) has a linear process $F_\alpha Z$ as a solution, then any unit root of Φ is also a unit root of Θ.

Proof (1)

- Since Φ has no unit root, by applying Theorem 4 (L2, Slide 35), $\exists\, \beta \in l^1(\mathbb{Z})$, such that
$$\beta = \alpha_\varphi^{-1}.$$
Then
$$\mathsf{X} = F_{\beta * \alpha_\varphi} \mathsf{X} = F_\beta \left(F_{\alpha_\varphi} \mathsf{X} \right) = F_\beta \left(F_{\alpha_\theta} \mathsf{Z} \right) = F_{\beta * \alpha_\theta} \mathsf{Z}$$

Proof (2)

- $F_\alpha Z$ is a solution of ARMA(p, q), then

$$F_{\alpha_\varphi} X = F_{\alpha_\theta} Z \Rightarrow F_{\alpha_\varphi}(F_\alpha Z) = F_{\alpha_\theta} Z \Rightarrow F_{\alpha * \alpha_\varphi} Z = F_{\alpha_\theta} Z$$

- According to Theorem 5, $\alpha * \alpha_\varphi = \alpha_\theta$.

  Then $P_\alpha(z)\Phi(z) = \Theta(z)$.

Causal and Invertible solution of ARMA(p, q)

Theorem 8

Let Φ and Θ be two polynomials associated to ARMA(p, q). Assume that Φ and Θ don't have common zeroes. Then

- The solution of ARMA(p, q) is causal $\iff \forall z, |z| \leq 1,$ $\Phi(z) \neq 0$.
- The solution of ARMA(p, q) is invertible $\iff \forall z, |z| \leq 1,$ $\Theta(z) \neq 0$.

Proof: Exercise

Example 2(1/2)

Z $\sim$WN$(0, \sigma^2)$, and ARMA$(1, 1)$ Equation:

$$X_t = .5X_t + Z_t + 2Z_{t-1}$$

- $\Phi(z) = 1 - .5z$ has 2 as a root, with modulus > 1, Then the equation above admit a unique causal solution.
- $\Theta(z) = 1 + 2z$ has $-1/2$ as a root, with modulus < 1, Then the solution is not invertible.
- The inverse of α_φ is the sequence $\beta \in l^1(\mathbb{Z})$ are the coefficients of the power series expansion of $\dfrac{\Theta(z)}{\Phi(z)}$ in $|z| < 2$,

$$\frac{\Theta(z)}{\Phi(z)} = \frac{1 + 2z}{1 - .5z} = 1 + \sum_{k=1}^{\infty} 5(1/2)^k z^k$$

Example 1(2/2)

- The solution is

$$X_t = Z_t + \sum_{k=1}^{\infty} 5(1/2)^k Z_{t-k}$$

- By applying Theorem 2,

$$\gamma_X(h) = \sigma^2 \left(1_{h=0} + 5(1/2)^{|h|} 1_{h \neq 0} + (25/3)(1/2)^{|h|} \right)$$

Example 2(1/4)

- Consider the following ARMA Equation

$$X_t = .4X_{t-1} + .21X_{t-2} + Z_t + .6Z_{t-1} + .09Z_{t-2}$$

- The polynomials Φ and Θ are

$$\Phi(z) = 1 - .4z - .21z^2 = (1 - .7z)(1 + .3z),$$

$$\Theta(z) = 1 + .6z + .09z^2 = (1 + .3z)^2.$$

- The ARMA can be reset as follows

$$X_t = .7X_{t-1} + Z_t + .3Z_{t-1}.$$

Example 2(2/4)

- The roots of Φ and Θ are respectively 10/7 and -10/3
- We can then compute causal and invertible solutions for the above ARMA equation.
- The solution (X_t) can be written then

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \qquad \text{and} \qquad Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \qquad t \in \mathbb{Z},$$

Example 2(3/4)

- where, for all $|z| \leq \min(10/3, 10/7) = 10/7$

$$\sum_{j=0}^{\infty} \psi_j z^j = \psi(z) = \frac{\theta(z)}{\phi(z)} = \frac{1 + .3z}{1 - .7z} = (1 + .3z) \sum_{j=0}^{\infty} (.7z)^j,$$

and

$$\sum_{j=0}^{\infty} \pi_j z^j = \pi(z) = \frac{\phi(z)}{\theta(z)} = \frac{1 - .7z}{1 + .3z} = (1 - .7z) \sum_{j=0}^{\infty} (-.3z)^j$$

Example 2(4/4)

- Then, for all $j \in \mathbb{N}$

$$\psi_0 = 1 \qquad \text{and} \qquad \psi_j = (.7 + .3)(.7)^{j-1} = (.7)^{j-1},$$

and

$$\pi_0 = 1 \qquad \text{and} \qquad \pi_j = (-1)^j(.3 + .7)(.3)^{j-1} = (-1)^j(.3)^{j-1}.$$

- Hence,

$$X_t = Z_t + \sum_{j=1}^{\infty}(.7)^{j-1} Z_{t-j} \qquad \text{and} \qquad Z_t = X_t + \sum_{j=1}^{\infty}(-1)^j(.3)^{j-1} X_{t-j}.$$

Exercise

Let (Z_t) be a WN. For each of the following equations, is there a stationary process solution? If the solution exist, can you say if it's a causal or invertible solution?

1. $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$;
2. $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$;
3. $X_t + 0.6X_{t-2} = Z_t + 1.2Z_{t-1}$;
4. $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.

Solution (1/2)

1. ☑ Stationary ☑ Causal ☑ Invertible

```
> abs(polyroot(c(1,.2,-.48)))
[1] 1.666667 1.250000
```

2. ☑ Stationary ☐ Causal ☑ Invertible

```
> abs(polyroot(c(1,1.9,.88)))
[1] 0.9090909 1.2500000
> abs(polyroot(c(1,.2,.7)))
[1] 1.195229 1.195229
```

Solution (2/2)

3. ☑ Stationary ☑ Causal ☐ Invertible

```
> abs(polyroot(c(1,0,.6)))
[1] 1.290994 1.290994
> abs(polyroot(c(1,1.2)))
[1] 0.8333333
```

4. ☑ Stationary ☑ Causal ☑ Invertible

```
> abs(polyroot(c(1,1.8,.81)))
[1] 1.111111 1.111111
```

Exercise

Let $\mathrm{X} = (X_t)_{t\in\mathbb{Z}}$ be the solution of the following ARMA$(2,1)$ process

$$(1 - B + B^2/4)X_t = (1 + B)Z_t$$

where $\mathrm{Z} = (Z_t)_{t\in\mathbb{Z}}$ is a WN$(0, \sigma^2)$

1. Show that the solution of the above equation is causal and can be written as follows:
$$X_t = \sum_{k\geq 0} \psi_k Z_{t-k}.$$
2. Is X invertible?
3. Compute the coefficients $(\psi_k)_{k\in\mathbb{Z}}$
4. Compute the ACF of X.
5. Verify the previous results using `R`

Exercise

1. What are the polynomials Φ and Θ in the following ARMA process

$$X_t - 3X_{t-1} = Z_t - \frac{10}{3} Z_{t-1} + Z_{t-2}$$

   where $Z = (Z_t)_{t \in \mathbb{Z}}$ is a $WN(0, \sigma^2)$
2. Show that this ARMA Equation has a unique stationary solution.
3. Is the solution causal?
4. Compute this solution in terms of Z
5. Is the solution invertible?

Exercise

Let

- $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ be two centred independent processes, i.e., $\forall\, t,\ s\ Y_t$ and X_s are independent.
- $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ and $\eta = (\eta_t)_{t\in\mathbb{Z}}$ be two WN$(0, \sigma^2)$.

Assume that X and Y are ARMA(1,1) processes:

$$X_t = \varphi_1 X_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} \text{ and } Y_t = \varphi_2 Y_{t-1} + \eta_t + \theta_2 \eta_{t-1}$$

1. Show that if $\max(|\varphi_1|, |\varphi_2|) < 1$ then X and Y are causal.
2. Show that if $\max(|\eta_1|, |\eta_2|) < 1$ then X and Y are invertible.
   Assume now that $\max(|\varphi_1|, |\varphi_2|) < 1$ and $\max(|\eta_1|, |\eta_2|) < 1$.
3. Show that $\forall\ s,\ t$, $\text{Cov}(\epsilon_s, \eta_t) = 0$
4. Compute the ACF of $Z_t = X_t Y_t$.

Theorem, ACF of an ARMA process

Let Φ and Θ be two polynomials. Assume that

$$\forall\, |z| \leq 1, \;\; \Phi(z) \neq 0.$$

Let X the stationary causal process solution of the following ARMA Equation

$$\Phi(B)\mathsf{X} = \Theta(B)\mathsf{Z}$$

where Z $\sim$WN$(0, \sigma^2)$.
Then $\exists\; 0 \leq \rho < 1$ and $C > 0$ such that

$$\forall\, h \in \mathbb{Z}, \;\; |\gamma_X(h)| \leq C\rho^{|h|}$$

Proof

- X is causal, then $\exists \psi \in l^1(\mathbb{Z})$ such that $X_t = \sum_{k \in \mathbb{N}} \psi_k Z_{t-k}$
- $\psi \in l^1(\mathbb{Z})$, then $\exists\ 0 \leq \rho < 1$ and $C' > 0$ such that $\forall k \in \mathbb{N}$

$$|\psi_k| \leq C' \rho^k$$

- X is a linear process, then

$$\gamma_X(h) = \sigma^2 \sum_{j \in \mathbb{N}} \psi_k \psi_{k+h}$$

- Then

$$\begin{aligned} |\gamma_X(h)| &\leq \sigma^2 \sum_{j \in \mathbb{N}} (C')^2 \rho^{2j+|h|} \\ &\leq \sigma^2 \frac{(C')^2}{1-\rho^2} \rho^{|h|} \end{aligned}$$

Box-Pierce and Ljung-Box Tests

Estimation of the Autocorrelation

- Let $x_1, \ldots, x_T$ be a realisation (observations) of a time series $\mathsf{X} = (X_t)_{t \in \mathbb{Z}}$.

- $\forall, k = 1, \ldots, T-1$

$$\widehat{\gamma}_X(k) = \frac{1}{T} \sum_{t=k+1}^{T} (x_t - \overline{x}_T)(x_{t-k} - \overline{x}_T),$$

  and

$$\widehat{\rho}_X(k) = \frac{\sum_{t=k+1}^{T} (x_t - \overline{x}_T)(x_{t-k} - \overline{x}_T)}{\sum_{t=1}^{T} (x_t - \overline{x}_T)^2}.$$

Theorem

Assume that $x_1, \ldots, x_T$ is a sequence of i.i.d random variables with second order, i.e., $\mathbb{E}(x_k^2) < \infty,\ \forall\, k = 1, \ldots, T.$ Then

- $\widehat{\rho}_X(k),\ k = 1, \ldots, T$ are approximately independent
- $\widehat{\rho}_X(k),\ k = 1, \ldots, T$ follow approximately a Normal distribution with mean 0 and variance $\frac{1}{T}$.

Hypothesis testing (2/2)

- Statistical test: for all $h \geq 1$

$$H_0^h \,:\, \rho_1 = \rho_2 = \ldots = \rho_h = 0$$

vs

$$H_1^h \,:\, \exists\, k \leq h \;\rho_k \neq 0$$

- The test statistic:

$$Q(h) = T \sum_{j=1}^{h} \widehat{\rho}_j^2 = \sum_{j=1}^{h} \left(\frac{\widehat{\rho}_j - 0}{1/\sqrt{T}} \right)^2$$

Hypothesis testing (1/2)

- The test statistic (small samples):

$$Q(h) = T(T+2)\sum_{j=1}^{h} \frac{\widehat{\rho}_j^2}{T-k}$$

- Distribution of the test statistic: under H_0^h,

$$Q(h) \sim \chi^2(h)$$

- If the test is used on residuals obtained from a model with m parameters, then, under H_0^h,

$$Q(h) \sim \chi^2(h-m)$$