Introduction to Time Series

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Outline

What’s a time series?

Types of Data

- Cross-sectional data
- Time Series data
- Panel/Longitudinal Data

Cross-sectional data

- This kind of data is obtained by collecting several observations from a sample of individuals extracted from a given population.
- Time doesn't play any role in the analysis.
- Examples:
  - SAT scores of high school students in a particular year.
  - Gross domestic product (GDP) of a set of countries for a specific year.
  - Data from a Survey on the opinion of a sample of people.

Cross-sectional data

- Formally
  - $\{w_1, \ldots, w_n\}$ represents the sample of the observed individuals
  - $\{x_1, \ldots, x_n\}$ represents the observed data, for all $i = 1, \ldots, n$

$$x_i = X(w_i)$$

  where $X = (X^1, \ldots, X^d)$ is a random vector representing the set of variables of interest.

- The analysis of cross-sectional data aims to estimate the probability distribution of the random vector. This task can be performed by either visualizing or estimating statistical parameters such as central tendency, dispersion, and many other statistics.

- Statistical analysis methods: PCA, MCA, Regression models...

Time series data

- A time series data is an observation of an individual for a span of regular times.

- Time series can be heights of ocean tides measured at fixed instant of the day. It can also be the daily closing value of the Dow Jones Industrial Average.

- Formally, a time series data is a sequence $(x_t)_{t \in T}$ where $x_t = X_t(w)$ is the observation of a sequence of random vectors $(X_t)_{t \in T}$ and T can be a subset of $\mathbb{Z}$

Panel/Longitudinal Data

- Panel/Longitudinal data is a time-series data taken from multiple individuals. It's a data obtained when we observe multiple entities over multiple points in time.

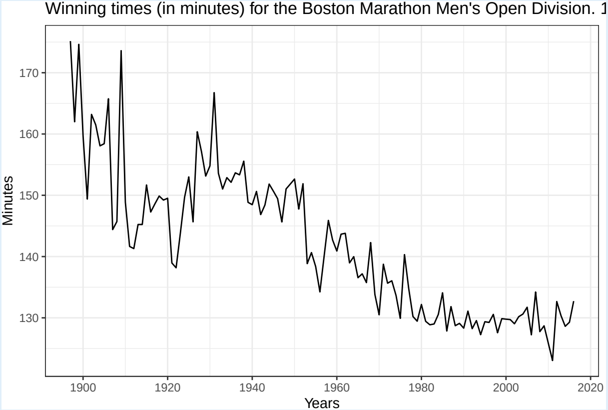
- Formally a panel data will be sequence of vectors $(x_t^1, \ldots, x_t^n)_{t \in T}$ where each $i = 1, \ldots, n$ $(x_t^i)_{t \in T}$ is a time-series data.

Examples, Visualizations

Example: CAC France: 1991 to 1990

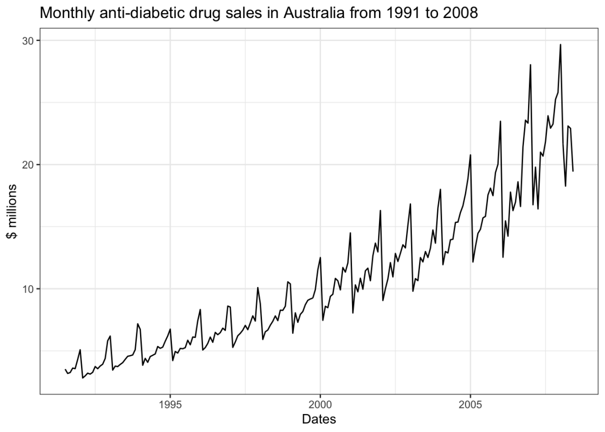
Source: `EuStockMarkets` from `datasets` `R` package (French index, stock market).

Example: Winning times for the Boston Marathon.



Source: `marathon` data from `fpp2` package.

Example: Anti-diabetic drug sales.



Source: `a10` data from `fpp2` package.

Visualizing Time Series

1. Import your data file into `R`
   - `R`: `read_csv` , from `readr` library.

2. Converting the column containing the date-times to a Date-time Object.

3. Using specific commands for visualization:
   - `R`: `ggplot` from `ggplot2` library, `autoplot` from `forecast` library

With R, using ggplot

- Import data

```
> library(readr)
> marathon <- read_csv("marathon.csv",
+  col_types = cols(year = col_datetime(format = "%Y")))
```

- Convert to Date-time object

```
> marathon$year=as.Date(marathon$year,format = "%Y")
```

- Visualization

```
> p<-ggplot(data=marathon,aes(x=year,y=marathon))+
+   geom_line()+theme_bw()+
+   scale_x_date(breaks = date_breaks("10 year"),
+                labels = date_format("%Y")) +
+   labs(x="Year",y="Minutes",
+        title="Winning times (in minutes) \n
+               for the Boston Marathon
+               Men's Open Division. ",
+        subtitle="1897-2016.")
> p
```

With R using ggplot

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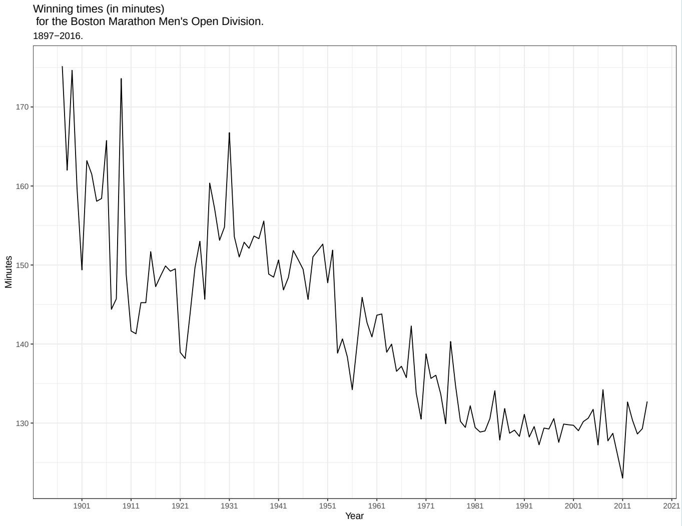

With R, using autoplot

- Import data

```
> library(readr)
> marathon <- read_csv("marathon.csv",
+  col_types = cols(year = col_datetime(format = "%Y")))
```

- Convert to Date-time object

```
> z=ts(marathon$marathon,start=c(1897,1),end=c(2016,1),frequency = 1)
> z
Time Series:
Start = 1897
End = 2016
Frequency = 1
  [1] 175.1667 162.0000 174.6333 159.7333
  ... Truncated output
```

- Visualization

```
> autoplot(z)+ggtitle("Winning times (in minutes)\n
+                     for the Boston Marathon Men's Open Division.
+                     \n 1897-2016.")+
+   theme_bw()+xlab("Years")+ylab("Minutes")
```

With R using autoplot

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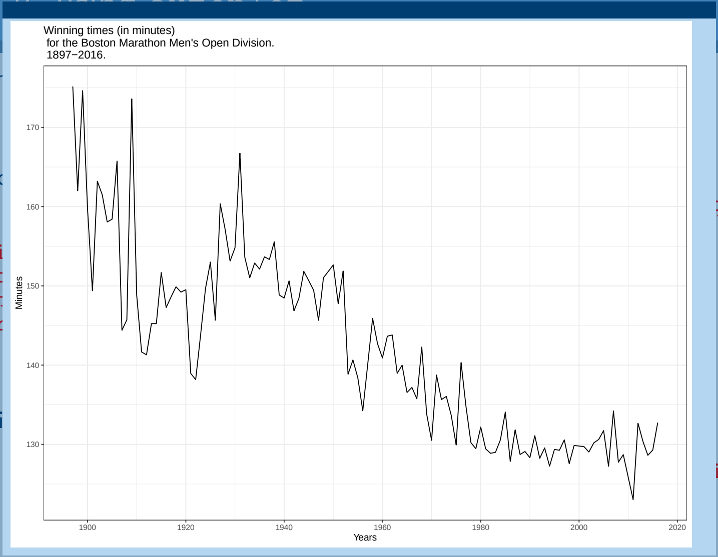


WDI data with `R`

Exercise 1 (Project)

Use `WDI` package provides convenient access to over 40 databases hosted by the World Bank, including the World Development Indicators (WDI), International Debt Statistics, Doing Business, Human Capital Index, and Sub-national poverty indicators.

Explore WDI package and draw some time series extracted from the package.

Stationary time series, White Noise

Stochastic process/Time Series

- $\mathbf{X} = (X_t)_{t \in \mathbb{Z}}$ is called a stochastic process or time series, if for all $t \in \mathbb{Z}$, X_t is a random variable:

$$\begin{array}{rcrcl} \mathbf{X} & : & \Omega \times T & \rightarrow & \mathbb{R} \\ & & (w, t) & \mapsto & X_t(w) \end{array}$$

- We're interested in the Probability distribution of $\mathbf{X}$: $\forall\, n \in \mathbb{N}^*$, $\forall\, (t_1, \ldots, t_n) \in \mathbb{Z}^n$

$$F_{\mathbf{X}}^{t_1, \ldots, t_n}(x_1, \ldots, x_n) = \mathbb{P}\left(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_b\right), \ \ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

- $\mathbf{X} = (X_t)_{t \in \mathbb{Z}}$ is a Gaussian stochastic process if $\forall\, n \in \mathbb{N}$, $\forall\, (t_1, \ldots, t_n) \in \mathbb{Z}^n$, $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian random vector.

Mean, autocovariance functions

- The **mean** function, μ_X, is a function from $\mathbb{Z}$ to $\mathbb{R}$ such that $\forall\, t \in \mathbb{Z}$,
$$\mu_X(t) = \mathbb{E}(X_t)$$
where $\mathbb{E}(X_t)$ is the expectation of X_t

- The **autocovariance** function (ACF) is defined as the second moment product, i.e., it's a function γ_X from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{R}$ such that
$$\gamma_X(s,t) = \mathrm{cov}(X_t, X_s) = \mathbb{E}\left[(X_t - \mu_X(t))\,(X_s - \mu_X(s))\right]$$

Mean, autocovariance functions

- The **mean** function, μ_X, is a function from $\mathbb{Z}$ to $\mathbb{R}$ such that $\forall\, t \in \mathbb{Z}$,

$$\mu_X(t) = \mathbb{E}(X_t)$$

  where $\mathbb{E}(X_t)$ is the expectation of X_t

- The **autocorrelation** function is defined as a function, $\rho_X(s, t)$, from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{R}$ such that

$$\rho_X(s, t) = \frac{\gamma_X(s, t)}{\sqrt{\gamma_X(s, s)\gamma_X(t, t)}} \in [-1, 1]$$

Stationary processes, strictly stationary

Let $\mathbf{X} = (X_t)_{t \in \mathbb{Z}}$ be a stochastic process,

A stochastic process is called **strictly stationary** if $\forall\ n \in \mathbb{N}^*$,
$\forall\ n-$uplets $(t_1, \ldots, t_n)$, and $\forall\ h \in \mathbb{Z}$,

the random vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1+h}, \ldots, X_{t_n+h})$ have the same probability distribution, i.e,

$$F_{\mathbf{X}}^{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{\mathbf{X}}^{t_1+h,\ldots,t_n+h}(x_1, \ldots, x_n)\ \ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$\Longleftrightarrow$ *The distribution of a number of random variables of the stochastic process is the same as we shift them along the time index.*

Stationary processes, weakly stationary

Let $\mathbf{X} = (X_t)_{t \in \mathbb{Z}}$ be a stochastic process,

A stochastic process is called **weakly stationary** or **stationary** if the following conditions are satisfied

i. $\forall\, t \in \mathbb{Z},\ \mathbb{E}\left[|X_t|^2\right] < \infty$

ii. $\forall\, t,\ h \in \mathbb{Z},\ \mu_X(t) = \mu_X(t+h)$

ii. $\forall\, t,\ s \in \mathbb{Z},\ \gamma_X(t,s) = \gamma_X(t-s,0)$

Examples

- *Example 1:* Let Y be a random variable, and define a stochastic process $(X_t)_{t\in\mathbb{Z}}$, by $X_t = Y$ for all t and $(X_t)_{t\in\mathbb{Z}}$ are mutually independent. Then $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary.

- *Example 2:* Let Y be a random variable with uniform distribution on $[0, 2\pi]$, and define a stochastic process $(X_t)_{t\in\mathbb{Z}}$, by $X_t = \cos(t + Y)$.

- *Example 3:* Let $(X_t)_{t\in\mathbb{Z}}$ be sequence of independent variables such that of t is even $X_t \sim \mathrm{Exp}(1)$ and if t is odd $X_t \sim \mathcal{N}(1, 1)$.

Exercise 2

Prove that the processes in Examples 2 & 3 are a stationary, but they are not a strictly stationary.

Example: "Checking" stationary graphically

Anti-diabetic drug sales

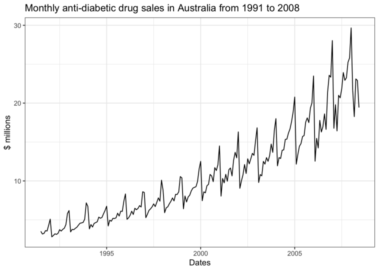


fig a: Time series

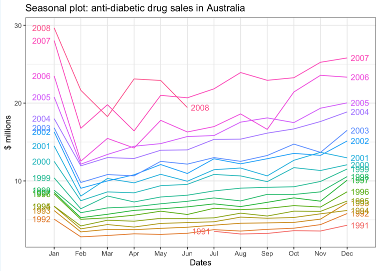


fig b: Yearly graph

Is this this time series stationary?

Theorem

Theorem 1
If $\mathbf{X}$ is strictly stationary and has finite second moment, then $\mathbf{X}$ is stationary.

However, if $\mathbf{X}$ is a weakly stationary Gaussian stochastic process, then $\mathbf{X}$ is strongly stationary.

ACF of a Stationary processes

$X = (X_t)_{t \in \mathbb{Z}}$ a stationary processes

- The function mean is constant: $\mu = \mathbb{E}(X_t)$, $\forall\, t \in \mathbb{Z}$

- The autocovariance function (ACF) is

$$\gamma_X(h) = \gamma_X(h, 0) = \gamma_X(t + h, t), \ \ \forall\, t,\ h \in \mathbb{Z}$$

- The autocorrelation function is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} \in \ [-1, 1]$$

ACF of a Stationary processes, Properties

The ACF γ of a stationary process satisfies the following

i. $\gamma(0) \geq 0$

ii. $|\gamma(h)| \leq \gamma(0)$

iii. $\gamma(-h) = \gamma(h)$

iv. γ is a positive semidefinite function, i.e.,
$\forall\, n,\ \forall\, (t_1, \ldots, t_n) \in \mathbb{Z}^n,\ \forall\, (v_1, \ldots, v_n) \in \mathbb{R}^n$:

$$\sum_{j=1}^{n} \sum_{k=1}^{n} v_j v_k \gamma(t_j - t_k) \geq 0$$

Furthermore, any function γ that satisfies (iii) and (iv) is an ACF of some stationary time series.
(iv.) $\Rightarrow$ (i) and (ii).

Exercise

Exercise 3, Proof of ii. and iv.

1. Let $(t_1, \ldots, t_n) \in \mathbb{Z}^n$ and $V = [X_{t_1} - \mathbb{E}(X_{t_1}), \ldots, X_{t_n} - \mathbb{E}(X_{t_n})]$
   1.1 Prove that $\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} v_j v_k \gamma(t_j - t_k) = \mathrm{Var}(v^T V)$
   1.2 Deduce then iv.
2. Prove ii. using Cauchy-Schwarz inequality.

ACF of Gaussian process

Theorem 2

In the class of stationary, zero mean, Gaussian processes there is a one-to-one correspondence between the family of the finite dimensional distributions and autocovariance function

$\iff$ *Gaussian processes are entirely determined by its autocovariance and mean functions*

Example: Periodic processes

- Let A and B two independent random vectors, $\mathbb{E}(A) = \mathbb{E}(B) = 0$ and $\mathrm{Var}(A) = \mathrm{Var}(B) = \sigma^2$, let $\theta \in [-\pi, \pi]$
- The following
$$X_t = A\cos(\theta t) + B\sin(\theta t)$$

Exercise 4

Prove that (X_t) is a stationary process and $\gamma_X(h) = \sigma^2 \cos(\theta h)$.

White Noise

Let $\mathbf{Z} = (Z_t)_{t\in\mathbb{Z}}$ be a (weak)-stationary stochastic process.
$(Z_t)_{t\in\mathbb{Z}}$ is a

- *weak white noise* if for every $(s,t) \in \mathbb{Z}\times\mathbb{Z}$, $\gamma_Z(s,t) = 0$ if $s \neq t$. Since $\mathbf{Z}$ is stationary then $\gamma_Z(t) = \sigma^2 \mathbf{1}_{t=0}$ et the mean function is constant and equal $\mu_Z \in \mathbb{R}$. We denote then $\mathbf{Z} \sim \mathrm{WN}(\mu_Z, \sigma^2)$

- *strong white noise* if $\mathbf{Z} = (Z_t)_{t\in\mathbb{Z}}$ is a sequence of independent variables.

- If $\mathbf{Z}$ is a Gaussian process and a WN, we denote $\mathbf{Z} \sim \mathrm{GWN}(\mu_Z, \sigma^2)$

Exercise

Exercise 5

Let X be a random variable with Gaussian distribution $\mathcal{N}(0,1)$, and let

$$Y = X\mathbf{1}_{U=1} - X\mathbf{1}_{U=0}$$

where U is a random variable with Bernoulli distribution with parameter $1/2$, independent from X.

1. Show that X and Y have the same probability distribution.
2. Show that $\mathrm{Cov}(X, Y) = 0$ and X and Y are not independent.
3. Can you come up with an example of a weak white noise that cannot be considered a strong white noise?

Learning outcomes

- A time series is sequence of random variables $\mathbf{X} = (X_t)_{t\in\mathbb{Z}}$ indexed by the time. Studying a time series means determining the joint distribution $(X_{t_1}, \ldots, X_{t_n})$, $\forall n \in \mathbb{N}^*$, $\forall\, t_1, \ldots, t_n \in \mathbb{Z}$.
- For a given time series $\mathbf{X} = (X_t)_{t\in\mathbb{Z}}$, we consider the following functions:
  - The mean function
  $$\mu(t) = \mathbb{E}(X_t)$$
  - The Autocovariance function (ACF)
  $$\gamma(t, s) = \mathrm{cov}(X_t, X_s)$$

Learning outcomes

- Two stationary definitions:
  - Strong: $\forall n$, $\forall (t_1, \ldots, t_n)$, $\forall h$, $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1+h}, \ldots, X_{t_n+h})$ have the same probability distribution.
  - Weak: $\mu(t)$ is constant and $\gamma(t,s) = \gamma(t-s)$
- Two white Noises: (X_t) a stationary time series
  - Strict: (X_t) is a sequence of independent variables
  - Weak: $\forall t \neq s$, $\gamma(t,s) = 0 \iff \gamma(t) = \sigma^2 \mathbb{1}_{\{t=0\}}$

Exercises

Exercise 6

Let (Z_t) be a strong white noise such that $\mathbb{E}(Z_t) = 0$ and $\text{Var}(Z_t) = \sigma^2$ and let

$$X_t = Z_t + \theta Z_{t-1}, \text{ where } \theta \in \mathbb{R}^*$$

1. Derive the ACF of (X_t)
2. Deduce that (X_t) is a stationary time series
3. Is (X_t) strictly stationary?
4. Write an `R` code simulating (X_t)

Exercises

Exercise 7

Let (Y_t) be a time series and we consider

$$X_t = \begin{cases} Y_t & \text{if } t \text{ even} \\ a + Y_t & \text{if } t \text{ odd} \end{cases}$$

where $a > 0$.

1. Derive the ACF of (X_t)
2. Is (X_t) stationary?

Solutions

- *Example 1:* Let Y be a random variable, and define a stochastic process $(X_t)_{t\in\mathbb{Z}}$, by $X_t = Y$ for all t and $(X_t)_{t\in\mathbb{Z}}$ are mutually independent. Then $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary.

Let $(t_1, \dots, t_n) \in \mathbb{Z}^n$; $h \in \mathbb{Z}$; $(x_1, \dots, x_n) \in \mathbb{R}^n$

$$\mathbb{P}(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n) = \prod_{k=1}^{n} \mathbb{P}(X_{t_k} \le x_k) = \prod_{k=1}^{n} \mathbb{P}(Y \le x_k).$$

$$\mathbb{P}(X_{t_1+h} \le x_1, \dots, X_{t_n+h} \le x) = \prod_{k=1}^{n} \mathbb{P}(X_{t_k+h} \le x) = \prod_{k=1}^{n} \mathbb{P}(Y \le x_k)$$

$$\Rightarrow \mathbb{P}(X_{t_1} \le x_1, \dots X_{t_n} \le x_n) =$$

$$\mathbb{P}(X_{t_1+h} \le x_1, \dots, X_{t_n+h} \le x).$$

$\Rightarrow (X_t)_{t\in\mathbb{Z}}$ is strictly stationary

Exercise 3, Proof of ii. and iv.

1. Let $(t_1, \dots, t_n) \in \mathbb{Z}^n$ and $V = [X_{t_1} - \mathbb{E}(X_{t_1}), \dots, X_{t_n} - \mathbb{E}(X_{t_n})]$
   1.1 Prove that $\sum_{j=1}^{n}\sum_{k=1}^{n} v_j v_k \gamma(t_j - t_k) = \mathrm{Var}(v^T V)$
   1.2 Deduce then iv.
2. Prove ii. using Cauchy-Schwarz inequality.

① $$\sum_{j=1}^{k}\sum_{k=1}^{k} v_j v_k \gamma(t_j - t_k) = \sum_{j=1}^{k}\sum_{k=1}^{n} v_j v_k \mathbb{E}\left((X_{t_j} - \mathbb{E}(X_{t_j}))(X_{t_k} - \mathbb{E}(X_{t_k}))\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{E}\left(v_j (X_{t_k} - \mathbb{E}(X_{t_k}))(X_{t_k} - \mathbb{E}(X_{t_k})) v_k\right).$$

$$= \mathbb{E}\left((vV)^T (vV)\right) = \mathrm{Var}(v^T V)$$

since $\mathrm{Var}(v^T V) \geq 0 \Rightarrow \forall v_1, \dots, v_n \quad \sum_{j=1}^{n}\sum_{k=1} v_j v_k \gamma(t_k - t_j) \geq 0$

② $\gamma(h) = \mathrm{cov}(X_h, X_0)$

$$|\gamma(h)| \leq \underbrace{V(X_h)}_{\gamma(0)}{}^{1/2} \underbrace{V(X_0)}_{\gamma(0)}{}^{1/2} \Rightarrow |\gamma(h)| \leq \gamma(0)$$

- Let A and B two independent random vectors, $\mathbb{E}(A) = \mathbb{E}(B) = 0$ and $\mathrm{Var}(A) = \mathrm{Var}(B) = \sigma^2$, let $\theta \in [-\pi, \pi]$
- The following

$$X_t = A\cos(\theta t) + B\sin(\theta t)$$

Exercise 4

Prove that (X_t) is a stationary process and $\gamma_X(h) = \sigma^2 \cos(\theta h)$.

$\mathbb{E}(X_t^2) < \infty$ (1) ; $\mathbb{E}(X_t) = \mathbb{E}(A)\cos\theta t + \mathbb{E}(B)\sin\theta t = 0 = \mathbb{E}(X_{t+h})$ (2)

$\forall t;\ \forall h$

$\gamma(t,s) = \mathrm{cov}(X_t, X_s) = \mathbb{E}(X_t X_s)$

$= \mathbb{E}(A^2)\cos\theta t\cos\theta s + \mathbb{E}(B^2)\sin\theta t\sin\theta s$

$= \sigma^2\cos(\theta(t-s))$ (3) then: (1) + (2) + (3) (X_t) is stati...

$\gamma(t,s) = \gamma(t-s) = \sigma^2\cos\theta(t-s)$

$\Rightarrow \gamma(h) = \sigma^2\cos(\theta h)$

Exercise 5

Let X be a random variable with Gaussian distribution $\mathcal{N}(0,1)$, and let

$$Y = X\mathbf{1}_{U=1} - X\mathbf{1}_{U=0}$$

where U is a random variable with Bernoulli distribution with parameter $1/2$, independent from X.

1. Show that X and Y have the same probability distribution.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y \leq y \mid U=1)\,\mathbb{P}(U=1) + \mathbb{P}(Y \leq y \mid U=0)\,\mathbb{P}(U=0)$$

$$= \mathbb{P}(X \leq y \mid U=1)\,\tfrac{1}{2} + \mathbb{P}(X \leq y \mid U=0)\,\tfrac{1}{2}$$

$$= \mathbb{P}(X \leq y) = F_X(y)$$

$\Rightarrow$ X and Y have the same probability distrib.

$\operatorname{cov}(X, Y) = \mathbb{E}(XY)$ because $\mathbb{E}(X) = \mathbb{E}(Y) = 0$

$= \mathbb{E}\left(X\left(X\mathbb{1}_{U=1} - X\mathbb{1}_{U=0}\right)\right) = \mathbb{E}(X^2)\,\mathbb{E}(\mathbb{1}_{U=1}) - \mathbb{E}(X^2)\,\mathbb{E}(\mathbb{1}_{U=0})$

$= \frac{1}{2}\left(\mathbb{E}(X^2) - \mathbb{E}(X^2)\right) = 0 \Rightarrow \operatorname{cov}(X, Y) = 0$

Assume that X and Y are indep then $\forall x \geq 0$

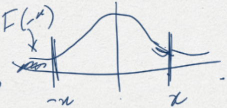
$$\mathbb{P}(X \leq x, Y \leq x) = \mathbb{P}(X \leq x)\, \mathbb{P}(Y \leq x)$$

but $\mathbb{P}(X \leq x, Y \leq x) =$

$$\mathbb{P}(X \leq x \mid U = 1) \times \mathbb{P}(U = 1) + \mathbb{P}(X \leq x, -X \leq x \mid U = 0)\, \mathbb{P}(U = 0)$$

$$= F_X(x) \times \frac{1}{2} + \left(F_X(x) - F(-x)\right) \times \frac{1}{2} = F_X(x) - \frac{1}{2} F_X(-x)$$

$$\mathbb{P}(X \leq x)\, \mathbb{P}(Y \leq x) = F_X(x)^2$$



if X and Y are indep $\Rightarrow$ $\forall x \geq 0$

$$F_X(x) - \frac{1}{2} F_X(-x) = \overline{F_X(x)}^2 \qquad F(x) = 1 - F(-x)$$

$$F_X(x) - \frac{1}{2}\left(1 - F(x)\right) = F_X(x)^2 \quad \forall x \geq 0$$

$$\frac{3}{2} F_X(x) - \frac{1}{2} = F_X(x)^2$$

$$\Leftrightarrow \quad F_X(x)^2 - \frac{3}{2} F_X(x) + \frac{1}{2} = 0$$

```
> pnorm(1)^2-1.5*pnorm(1)+.5
[1] -0.05415614
>
```

3. Can you come up with an example of a weak white noise that cannot be considered a strong white noise?

$X \sim \mathcal{N}(0,1)$. $U \sim \text{Bernoulli}(1/2)$

$X_0 = X$

$X_1 = X_0 \mathbb{1}_{U=1} - X_0 \mathbb{1}_{U=0}$

$X_t = X_{t-1} \mathbb{1}_{\{U=1\}} - X_{t-1} \mathbb{1}_{\{U=0\}}$.

1/ $\forall t$ $X_t \sim \mathcal{N}(0,1)$

2/ X_t and U are indep.

3/ $\text{cov}(X_t, X_{t+1}) = 0$ and $\forall h$ $\text{cov}(X_t, X_{t+h}) = 0$

but X_t and X_{t+1} are not indep.

$\Rightarrow$ (X_t) is a weak WN and not a strong WN.

- *Example 2:* Let Y be a random variable with uniform distribution on $[0, 2\pi]$, and define a stochastic process $(X_t)_{t\in\mathbb{Z}}$, by $X_t = \cos(t + Y)$.

(X_t) is stationary:

$$\mathbb{E}(X_t) = \mathbb{E}(\cos(t+Y)) = \frac{1}{2\pi}\int_0^{2\pi} \cos(t+y)\,dy$$

$$= \frac{1}{2\pi}\left[\sin(t+y)\right]_0^{2\pi} = \frac{1}{2\pi}\left(\sin(t+2\pi) - \sin t\right) = 0$$

$\Rightarrow \mathbb{E}(X_t)$ is constant $\forall t$

$$\gamma(t,s) = \mathbb{E}(X_t X_s) = \mathbb{E}\left(\cos(t+Y)\cos(s+Y)\right)$$

$$= \frac{1}{2}\mathbb{E}\left(\cos(t-s)\cos(t+s+2Y)\right)$$

$$= \frac{1}{4\pi}\cos(t-s)\int_0^{2\pi} \cos(t+s+2y)\,dy$$

$$= -\frac{1}{4\pi}\cos(t-s)\left[-\sin(t+s+2y)\right]_0^{2\pi} = 0$$

$\Rightarrow \gamma(t,s) = 0 \quad \forall t,s$

$\Rightarrow (X_t)$ is stationary

- *Example 3:* Let $(X_t)_{t\in\mathbb{Z}}$ be sequence of independent variables such that of t is even $X_t \sim \text{Exp}(1)$ and if t is odd $X_t \sim \mathcal{N}(1,1)$.

t even $\mathbb{E}(X_t) = 1$; t odd $\mathbb{E}(X_t) = 1$

$\gamma(t,s) = \mathbb{E}(X_t X_s) - \mathbb{E}(X_t)\mathbb{E}(X_s) = \underbrace{\mathbb{E}(X_t X_s)} - 1$.

or X_t and X_s are indep when $t \neq s$.

$\gamma(t,s) = 0$

$\Rightarrow (X_t)$ is stationary.

(X_t) isn't strongly stationary. since

$X_{2t} \sim \mathcal{E}(1)$ and $X_{2t+1} \sim \mathcal{N}(1,1)$

don't have the same prob. distribut°