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The NBRULC Reliability Class: Mathematical Theory and Goodness-of-Fit Testing with Applications to Asymmetric Censored and Uncensored Data

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Abstract: The majority of approaches proposed in the past few decades to solve life test problems have differed markedly from those used for closely related, yet broader, issues. Due to the complexity of data that are generated each day in many practical domains, as a result of the development of scales for rating the success or failure of reliability, a new domain of reliability has been created. This domain is referred to as life classes, where specific probability distributions are presented. In this study, it is shown that the use of the quality-of-fit technique to solve problems involving life testing makes sense, and produces simpler processes that are roughly equivalent or superior to those used in traditional procedures. They may also behave better in limited samples. This work investigates a novel quality-of-fit test statistic; it is based on an exponential transform and is compared to the best renewal used Laplace test in increasing convex ordering (NBRULC). Evidence for approach normality is provided. The calculated variables include powers, Pitman asymptotic effectiveness, and critical points. Methods on how to handle censored data were also studied. Our experiments have real-world applications in the fields of medicine and engineering.

Keywords: NBRULC reliability class; goodness-of-fit approach; aging; life testing; right censored data; computer simulation; comparative study; COVID-19; statistics and numerical data

MSC: 62C07; 62G99; 62N01; 62N02; 62N05

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1. Introduction

Reliability can be compared to quality over time. Since workmanship and manufacturing are quality factors, a product would be deemed as low quality if it malfunctions or breaks right after purchase. Poor reliability, on the other hand, is when product components fail before one anticipates them to. Therefore, there is a distinction between quality and reliability that has to do with time, more precisely, the product's lifetime. Reliability is officially defined as the capacity of a thing to carry out an essential task under predetermined circumstances for a predetermined amount of time. The likelihood of an item performing a necessary function under specified conditions for a specific amount of time is another

definition that addresses the probabilistic nature of assessing reliability. As a result, reliability is a measure of technical uncertainty, and estimating reliability requires the application of statistics, specifically the probability theory. Testing completed items or components under settings that mimic real-world use until failure occurs is the typical method for gauging system reliability. Although it may seem that having more data would increase one's trust in the reliability level, this practice is frequently costly and time-consuming. Therefore, it is particularly important to consider the lifespan of products before all test units experience a malfunction. Aging concepts explain how a part or a system can experience improvements or deterioration as it ages. According to the aging characteristics, many types of life distributions are categorized or characterized in the literature. The exponential distribution almost always falls into one of these classifications, which is a significant feature in such classifications. Any reliability analysis must take stochastic aging into account, and numerous test statistics have been established in the literature, comparing exponential growth to various aging options. Our objective is to provide an overview of these advancements. Researchers in statistics and reliability analysis have looked at testing exponentiality problems using different age classes of life distributions from a variety of angles; for more information, see Bryson and Siddiqui [1], Bhattacharyya et al. [2], Barlow and Proschan [3], Klefsjo [4], Khan et al. [5], Kumazawa [6], Mahmoud et al. [7,8], Majumder and Mitra [9], Bhattacharyya et al. [10], Navarro [11], Abu-Youssef et al. [12], Ghosh and Mitra [13], Navarro and Pellerey [14], El-Morshedy et al. [15], EL-Sagheer et al. [16] Gadallah et al. [17], Mansour [18], Ghosh and Mitra [19], Majumder and Mitra [20], Lai and Xie [21], Etman et al. [22], Ghosh and Majumder [23], Bakr and Al-Babtain [24], and Ghosh and Mitra [25] and Alqifar et al. [26]. The goodness-of-fit test indicates whether the data in one's sample correspond to the data one would anticipate seeing in the actual population. More specifically, it is used to determine if sample data from a population with a normal distribution or a Weibull distribution fit the distribution. Numerous researchers have created tests for exponentiality against particular types of life distributions using the goodness-of-fit approach. To provide examples, Mahmoud and Abdul Alim [27], Kayid et al. [28], Bera et al. [29], Mahmoud et al. [30], Abu-Youssef et al. [31], Bakr et al. [32], Abu-Youssef and El-Toony [33], Abu-Youssef and Gerges [34], and Etman et al. [35]. We created a brand-new class of life distribution models to account for the efficacy and power of the test utilized in this study. The goal of creating a systematic method for studying events and processes occurring in the world was driven by the essential requirements of modern science and technology. It follows that the need for such an approach in the investigation of technological products and system reliability is quite natural. There are real-life instances where system components gradually degrade over time, eventually reaching the end of the manufacturer's warranty period; they then need to be renewed through the replacement of spare parts. In this case, renewal is intended to enhance the system's functionality, but it cannot surpass the superior state it had at age t. For example, after several hours of flight, the aviation administration may wish to replace a portion of an airplane engine. The airlines contend that this replacement is, at best, unneeded, and may potentially be detrimental to the aircraft. Airlines will examine if an aviation engine, after hours of renewal, is as good as a new engine by using operational data to support their claims.

The structure of this essay can be summarized as follows: The NBRU, NBRUL, and NBRULC classes of life distributions are defined in the remaining portions of this section. In Section 2, a goodness-of-fit test based on an exponentiality test is discussed regarding the NBRULC class. For several popular alternatives, the Pitman asymptotic is presented in Section 3. Section 4 includes simulations of the key areas, power estimates for the Monte Carlo null distribution, and a suggested test for right-censored data. Section 5 tabulates a few crucial numbers and deals with right-censored data. Finally, we discuss a few examples to show the utility of the statistical test in Section 6.

Preliminary:
A random variable X is said to be

(i) *'New better than renewal used'*, denoted by $X \in NBRU$, if

$$\overline{W}_F(x+t) \leq \overline{F}(x)\overline{W}_F(t); \quad x,t, \geq 0.$$

(ii) *'New better (worse) than renewal used' in the Laplace transform order,* denoted by $X \in NBRUL$, if

$$\int_0^\infty e^{-sx}\overline{W}_F(x+t)dx \leq \overline{W}_F(t)\int_0^\infty e^{-sx}\overline{F}(x)dx; \quad x,t,s \geq 0,$$

or

$$\int_0^\infty \int_{x+t}^\infty e^{-sx}\overline{F}(u)dudx \leq \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(u)dudx,$$

where $\overline{W}_F(x+t) = \frac{1}{\mu}\int_{x+t}^\infty \overline{F}(u)du$.

(iii) *'New better than renewal used' in the Laplace transform in increasing convex order*, denoted by $X \in NBRULC$, if

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{W}_F(x+u)dxdu \leq (\geq) \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{W}_F(u)dxdu; \;\; x,t,s \geq 0,$$

or

$$\int_0^\infty \int_{x+t}^\infty e^{-sx}\overline{W}_F(u)dxdu \leq (\geq) \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{W}_F(u)dxdu,$$

this could be rewritten as

$$\int_0^\infty e^{-sx}\Gamma(x+t)dx \leq \int_0^\infty e^{-sx}\overline{F}(x)\Gamma(t)dx,$$

where $\Gamma(x+t) = \int_{x+t}^\infty \overline{W}_F(u)du$. It is clear that

- $IFR \subset IFRA \subset NBU \subset NBUE \subset NBRU \subset NBRUL \subset NBRULC$.
- $IFR \subset IFRA \subset NBU \subset NBRU \subset NBRUL \subset NBRULC$.
- $IFR \subset IFRA \subset NBU \subset NBUE \subset HNBUE$.

2. Testing against NBRULC Alternatives

We discuss the potential that $H_0 : F$ is exponential in this section as opposed to the related hypothesis $H_1 : F$, which states that it is not exponential but NBRULC. Using the subsequent theorem is necessary for creating our test statistic.

Theorem 1. *Let X be an NBRULC random variable with distribution function F, then*

$$\zeta(s) - 1 \geq \mu[s^2 - s] + \mu_{(2)}\zeta(s)[\frac{1}{2}s^2 - \frac{1}{2}s^3] + (s^3 - s^2)[\mu\zeta(s) + \zeta(1)\zeta(s) - \zeta(s)] + s^2\zeta(1) - s^2, \tag{1}$$

for $s \geq 0$ and $s \neq 1$, where

$$\zeta(s) = Ee^{-sX} = \int_0^\infty e^{-sx}dF(x).$$

Proof. Since F is NBRULC, then

$$\int_0^\infty e^{-sx}\Gamma(x+t)dx \leq \int_0^\infty e^{-sx}\overline{F}(x)\Gamma(t)dx; \quad x,t \geq 0.$$

Consider the following integral

$$\int_0^\infty \int_0^\infty e^{-t}e^{-sx}\Gamma(x+t)dxdt \leq \int_0^\infty e^{-t}\Gamma(t)\int_0^\infty e^{-sx}\overline{F}(x)dxdt. \tag{2}$$

Setting

$$I_1 = \int_0^\infty \int_0^\infty e^{-t} e^{-sx} \Gamma(x+t) dx dt,$$

hence

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty e^{-v} e^{-s(u-v)} \Gamma(u) du dv \\ &= \int_0^\infty \int_0^v e^{-u} e^{-s(v-u)} \Gamma(v) du dv \\ &= \frac{1}{1-s}[\int_0^\infty e^{-sv} \Gamma(v) dv - \int_0^\infty e^{-v} \Gamma(v) dv]; s \neq 1. \end{aligned}$$

Note that,

$$\begin{aligned} \int_0^\infty e^{-sv} \Gamma(v) dv &= \frac{1}{2} E \int_0^\infty e^{-sv} [V-v]^2 I(V > v) dv \\ &= \frac{1}{2} E \int_0^V e^{-sv} [V^2 - 2vV + v^2] dv \\ &= \frac{\mu_{(2)}}{2s} - \frac{\mu}{s^2} - \frac{1}{s^3}\zeta(s) + \frac{1}{s^3}. \end{aligned}$$

Therefore,

$$I_1 = \frac{1}{1-s}\{\frac{\mu_{(2)}}{2s} - \frac{\mu}{s^2} - \frac{1}{s^3}\zeta(s) + \frac{1}{s^3} - \frac{\mu_{(2)}}{2} + \mu + \zeta(1) - 1\}. \quad (3)$$

Setting

$$\begin{aligned} I_2 &= \int_0^\infty e^{-t} \Gamma(t) \int_0^\infty e^{-sx} \overline{F}(x) dx dt \\ &= E \int_0^\infty e^{-t} \Gamma(t) \int_0^\infty e^{-sx} I(X > x) dx dt. \\ &= E \int_0^\infty e^{-t} \Gamma(t) \int_0^X e^{-sx} dx dt = \frac{1}{s}(1 - \zeta(s)) \int_0^\infty e^{-t} \Gamma(t) dt. \end{aligned}$$

Therefore,

$$I_2 = (\frac{1}{s} - \frac{1}{s}\zeta(s))(\frac{\mu_{(2)}}{2} - \mu - \zeta(1) + 1). \quad (4)$$

Substituting (3) and (4) into (2), we have

$$\zeta(s) - 1 \geq \mu[s^2 - s] + \mu_{(2)}\zeta(s)[\frac{1}{2}s^2 - \frac{1}{2}s^3] + (s^3 - s^2)[\mu\zeta(s) + \zeta(1)\zeta(s) - \zeta(s)] + s^2\zeta(1) - s^2.$$

Thus, the proof is complete. Let us put forward the starting point from exponentiality as follows

$$\begin{aligned} \delta(s) &= [\tfrac{1}{2}s^3 - \tfrac{1}{2}s^2]\mu_{(2)}\zeta(s) + [s^2 - s^3]\zeta(1)\zeta(s) + [s^2 - s^3]\mu\zeta(s) \\ &\quad + [s^3 - s^2 + 1]\zeta(s) + [s - s^2]\mu - s^2\zeta(1) + s^2 - 1. \end{aligned} \quad (5)$$

Be aware that while $\delta(s) = 0$ for H_0, $\delta(s) > 0$ for H_1. The empirical estimate of $\delta(s)$, given $X_1, X_2, ..., X_n$ as a sample at random from distribution F, can be found as

$$\begin{aligned} \hat{\delta}(s) &= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \{[\frac{1}{2}s^3 - \frac{1}{2}s^2]X_i^2 e^{-sX_j} + [s^2 - s^3]e^{-X_i}e^{-sX_j} + [s^2 - s^3]X_i e^{-sX_j} \\ &\quad + [s^3 - s^2 + 1]e^{-sX_i} + [s - s^2]X_i - s^2 e^{-X_i} + s^2 - 1\}. \end{aligned}$$

Let $\Delta(s) = \frac{\delta(s)}{\mu^3}$, which is approximated by the formula $\hat{\Delta}(s) = \frac{\hat{\delta}(s)}{\overline{X}^3}$, $\overline{X}$ is the sample mean to make the test invariant. Then,

$$\begin{aligned}\hat{\Delta}(s) &= \frac{1}{n^2\overline{X}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\{[\tfrac{1}{2}s^3-\tfrac{1}{2}s^2]X_i^2e^{-sX_j}+[s^2-s^3]e^{-X_i}e^{-sX_j}+[s^2-s^3]X_ie^{-sX_j}\\ &\quad+[s^3-s^2+1]e^{-sX_i}+[s-s^2]X_i-s^2e^{-X_i}+s^2-1\}.\end{aligned} \tag{6}$$

It should be noted that $\hat{\Delta}(s)$ is an impartial estimator of $\delta(s)$. Set

$$\begin{aligned}\phi(X_i,X_j) &= [\tfrac{1}{2}s^3-\tfrac{1}{2}s^2]X_i^2e^{-sX_j}+[s^2-s^3]e^{-X_i}e^{-sX_j}+[s^2-s^3]X_ie^{-sX_j}\\ &\quad+[s^3-s^2+1]e^{-sX_i}+[s-s^2]X_i-s^2e^{-X_i}+s^2-1,\end{aligned} \tag{7}$$

and define the symmetric kernel

$$\psi_s(X_i,X_j)=\frac{1}{2!}\sum\phi_s(X_i,X_j),$$

where the total of all configurations of X_i, X_j. Then, $\hat{\Delta}(s)$ in (6) is comparable to the U_n-statistic, given by

$$U_n=\frac{1}{\binom{n}{2}}\sum_{i<j}^{n}\psi_s(X_i,X_j). \tag{8}$$

The following theorem can be used to prove that $\hat{\Delta}(s)$ is asymptotically normal. □

Theorem 2.

(a) *As $n\to\infty$, $\sqrt{n}(\hat{\Delta}(s)-\Delta(s))$ is asymptotically normal, having a mean of 0 and variance $\sigma^2(s)$, where $\sigma^2(s)$ is determined by*

$$\begin{aligned}\sigma^2(s) &= Var\{[\tfrac{1}{2}s^3-\tfrac{1}{2}s^2]X^2\zeta(s)+[s^2-s^3]e^{-X}\zeta(s)+[s^2-s^3]X\zeta(s)\\ &\quad+[s^3-s^2+1]e^{-sX}+[s-s^2]X-s^2e^{-X}+[\tfrac{1}{2}s^3-\tfrac{1}{2}s^2]\mu_{(2)}e^{-sX}\\ &\quad+[s^2-s^3]e^{-sX}\zeta(1)+[s^2-s^3]\mu e^{-sX}+[s^3-s^2+1]\zeta(s)\\ &\quad+[s-s^2]\mu-s^2\zeta(1)+2s^2-2\}.\end{aligned} \tag{9}$$

(b) *Under H_0, the variance $\sigma_0^2(s)$ is*

$$\sigma_0^2(s)=\frac{s^6(-1+s)^2(8+3s)(7+s)}{12(1+s)^3(1+2s)(2+s)}. \tag{10}$$

Proof. We may determine the mean and variance using the conventional U-statistics theory, see Lee [36], and by conducting the following calculations

$$\sigma^2=Var\{\eta(X)\},$$

where

$$\eta(X)=\eta_1(X)+\eta_2(X),$$

$$\begin{aligned}\eta_1(X) &= E(\phi(X_1,X_2)\mid X_1)=[\frac{1}{2}s^3-\frac{1}{2}s^2]X^2\int_0^\infty e^{-sx}dF(x)+[s^2-s^3]e^{-X}\int_0^\infty e^{-sx}dF(x)\\ &\quad+[s^2-s^3]X\int_0^\infty e^{-sx}dF(x)+[s^3-s^2+1]e^{-sX}+[s-s^2]X-s^2e^{-X}+s^2-1,\end{aligned}$$

and

$$\begin{aligned}\eta_2(X) &= E(\phi(X_1,X_2)\mid X_2)=[\frac{1}{2}s^3-\frac{1}{2}s^2]e^{-sX}\int_0^\infty x^2dF(x)+[s^2-s^3]e^{-sX}\int_0^\infty e^{-x}dF(x)\\ &\quad+[s^2-s^3]e^{-sX}\int_0^\infty xdF(x)+[s^3-s^2+1]\int_0^\infty e^{-sx}dF(x)+[s-s^2]\int_0^\infty xdF(x)\\ &\quad-s^2\int_0^\infty e^{-x}dF(x)+s^2-1.\end{aligned}$$

Therefore,

$$\begin{aligned}\eta(X) &= [\frac{1}{2}s^3 - \frac{1}{2}s^2]X^2\int_0^\infty e^{-sx}dF(x) + [s^2 - s^3]e^{-X}\int_0^\infty e^{-sx}dF(x)\\ &+[s^2 - s^3]X\int_0^\infty e^{-sx}dF(x) + [s^3 - s^2 + 1]e^{-sX} + [s - s^2]X - s^2e^{-X}\\ &+[\frac{1}{2}s^3 - \frac{1}{2}s^2]e^{-sX}\int_0^\infty x^2dF(x) + [s^2 - s^3]e^{-sX}\int_0^\infty e^{-x}dF(x)\\ &+[s^2 - s^3]e^{-sX}\int_0^\infty xdF(x) + [s^3 - s^2 + 1]\int_0^\infty e^{-sx}dF(x)\\ &+[s - s^2]\int_0^\infty xdF(x) - s^2\int_0^\infty e^{-x}dF(x) + 2s^2 - 2.\end{aligned}$$

Upon using (7), we have

$$\begin{aligned}\sigma_0^2(s) &= Var\{[\frac{s^3 - s^2 + 2}{2}]e^{-sX} - [\frac{2s^3}{s+1}]e^{-X} + [\frac{s^3 - s^2}{2(s+1)}]X^2\\ &+[\frac{s^2 - 2s^3 + s}{s+1}]X + [\frac{3s^3 + s^2 - 2s - 2}{2(s+1)}]\}.\end{aligned}$$

From (9) and after some computations, (10) can be reported. □

3. Pitman's Asymptotic Efficiency of $\hat{\Delta}(s)$

Pitman's asymptotic efficiency is a term used to compare the effectiveness of two statistical tests based on the sizes of their sample populations. It is referred to as the percentage of the minimum sample sizes that must be used for each test to obtain a specific degree of significance and power, at which point, the alternative hypothesis begins to approach the null hypothesis. A test with a greater Pitman's efficiency needs fewer observations than a test with a lower efficiency to obtain the same accuracy. A comparison of several tests for mean, variance, or proportion equality between two populations, for instance, can be made using the asymptotic Pitman efficiency. Based on the bare minimum sample size necessary, the Pitman efficiency can assist the method in selecting the optimal test for a specific problem. The following are some possible restrictions or presumptions of the asymptotic Pitman efficiency: Its asymptotic foundation makes it susceptible to becoming unreliable for samples of small or medium sizes. The rate of convergence of the test statistics' finite distributions is influenced by the selection of alternative hypotheses and the level of significance; this rate of convergence also has an impact. Other factors, such as robustness, simplicity, or interpretability, which could influence the test selection, are not taken into consideration. It is conceivable that this does not accurately reflect how tests for other or higher levels are performed. Additionally, it does not account for the cost of errors or the loss function. In this section, we evaluate the effectiveness of Pitman's asymptotic efficiency (PAE) method for the linear failure rate (LFR), Weibull, and Makeham distributions, using the probability models listed below.

(i) The Weibull distribution: $\overline{F}_1(x) = e^{-x^\theta};\ x \geq 0,\ \theta \geq 1.$

(ii) The LFR distribution: $\overline{F}_2(x) = e^{-x-\frac{\theta}{2}x^2};\ x \geq 0,\ \theta \geq 0.$

(iii) The Makeham distribution: $\overline{F}_3(x) = e^{-x-\theta\left(x+e^{-x}-1\right)};\ x \geq 0,\ \theta \geq 0.$

Be mindful that for $\theta = 0$, $\overline{F}_1(u)$ and $\overline{F}_3(u)$ minimize exponential distributions, whereas for $\theta = 1$, $\overline{F}_2(u)$ does the same. The PAE can be reported as

$$PAE(\Delta(s)) = \frac{1}{\sigma_0(s)}\left|\frac{d}{d\theta}\delta_\theta(s)\right|_{\theta\to\theta_\circ}, \tag{11}$$

where

$$\begin{aligned}\delta_\theta(s) &= [\frac{1}{2}s^3 - \frac{1}{2}s^2]\mu_{(2\theta)}\zeta_\theta(s) + [s^2 - s^3]\zeta_\theta(1)\zeta_\theta(s) + [s^2 - s^3]\mu_\theta\zeta_\theta(s) \\ &\quad + [s^3 - s^2 + 1]\zeta_\theta(s) + [s - s^2]\mu_\theta - s^2\zeta_\theta(1) + s^2 - 1,\end{aligned}$$

$$\zeta_\theta(s) = \int_0^\infty e^{-sx}dF_\theta(x), \mu_{(2\theta)} = 2\int_0^\infty x\overline{F}_\theta(x)dx, \mu_\theta = \int_0^\infty \overline{F}_\theta(x)dx.$$

Hence,

$$\begin{aligned}\frac{d}{d\theta}\delta_\theta(s) &= [\frac{1}{2}s^3 - \frac{1}{2}s^2][\mu_{(2\theta)}\zeta_\theta^{\backprime}(s) + \mu_{(2\theta)}^{\backprime}\zeta_\theta(s)] + [s^2 - s^3][\zeta_\theta(1)\zeta_\theta^{\backprime}(s) + \zeta_\theta^{\backprime}(1)\zeta_\theta(s)] \\ &\quad + [s^2 - s^3][\mu_\theta\zeta_\theta^{\backprime}(s) + \mu_\theta^{\backprime}\zeta_\theta(s)] + [s^3 - s^2 + 1]\zeta_\theta^{\backprime}(s) + [s - s^2]\mu_\theta^{\backprime} - s^2\zeta_\theta^{\backprime}(1),\end{aligned}$$

where

$$\mu_\theta^{\backprime} = \int_0^\infty \overline{F}_\theta^{\backprime}(x)dx, \mu_{(2\theta)}^{\backprime} = 2\int_0^\infty x\overline{F}_\theta^{\backprime}(x)dx, \zeta_\theta^{\backprime}(s) = -\int_0^\infty e^{-sx}d\overline{F}_\theta^{\backprime}(x).$$

Using the PAE definition in (11), we have

$$PAE(\delta) = \frac{1}{\sigma_0}\left|\frac{[\frac{1}{2}s^3 - \frac{1}{2}s^2][\mu_{(2\theta)}\zeta_\theta^{\backprime}(s) + \mu_{(2\theta)}^{\backprime}\zeta_\theta(s)] + [s^2 - s^3][\zeta_\theta(1)\zeta_\theta^{\backprime}(s) + \zeta_\theta^{\backprime}(1)\zeta_\theta(s)]}{+[s^2 - s^3][\mu_\theta\zeta_\theta^{\backprime}(s) + \mu_\theta^{\backprime}\zeta_\theta(s)] + [s^3 - s^2 + 1]\zeta_\theta^{\backprime}(s) + [s - s^2]\mu_\theta^{\backprime} - s^2\zeta_\theta^{\backprime}(1)}\right|_{\theta\to\theta_0}.$$

For a few different values of s, Table 1 compares our test $\hat{\Delta}(s)$ to those of Mugdadi and Ahmad [37] ($\delta_{(3)}$), Kango [38] (K^*), and Abdel-Aziz [39] ($\hat{\Delta}_{RN}$).

Table 1 demonstrates that statistic $\hat{\Delta}(s)$ is more effective than statistics $\delta_{(3)}$, K^* and $\hat{\Delta}_{RN}$ for the three alternative families.

Table 1. Some competitive tests are contrasted with the PAE test.

Distribution	$\delta_{(3)}$	K^*	$\hat{\Delta}_{RN}$	**Our Test $\hat{\Delta}(s)$**		
				$s = 0.09$	$s = 0.9$	$s = 2$
Weibull	0.170	0.132	0.223	0.597	0.851	1.023
LFR	0.408	0.433	0.535	0.851	0.974	0.996
Makeham	0.039	0.144	0.184	0.148	0.213	0.249

4. The Monte Carlo Method

The Monte Carlo simulation makes use of random sampling to predict the potential outcomes of an unknown event. It was developed in World War II by Stanislaw Ulam and John von Neumann. Using an expected range of values rather than using predetermined input values, a Monte Carlo simulation forecasts a succession of outcomes. It creates random samples using a probability distribution and then computes the outcomes for each sample. This procedure is repeated numerous times to produce a distribution of potential conclusions that may be statistically examined. The steps below can be used to evaluate the likelihood of an event occurring using a Monte Carlo simulation: choose the random variables that will affect the occurrence whose likelihood one wants to calculate.; identify each random variable's probability distribution and create random samples using it; analyze the event for each sample and determine its frequency; to calculate the percentage, divide the total number of samples by the number of iterations, and then multiply the result by 100; repeat this method numerous times, then determine the mean and standard deviations of the percentages. One will receive an assessment of the event likelihood and the level of uncertainty surrounding it as a result.

4.1. Critical Points

A critical value (C.V) in statistics refers to a specific point in a distribution of a test statistic; it is conducted to determine whether the null hypothesis may be rejected. This group is referred to as the rejection region or the critical zone. One critical value typically exists for one-sided tests, while two critical values often exist for two-sided tests. The statistical literature does not provide a clear consensus on whether the critical value or probability value method is preferable. Depending on the circumstance and preference, each has benefits and drawbacks. Using 10,000 samples, n = 5(5)100, this part mimics the crucial points of the null Monte Carlo distribution. For 90%, 95%, and 99%, the upper percentile of $\hat{\Delta}\ (0.09)$ was identified. As seen in Table 2 and Figure 1, the critical values rose with higher confidence levels and fell with higher sample sizes.

Table 2. The statistic's critical values of $\hat{\Delta}(0.09)$.

n	90%	95%	99%
5	0.0001268	0.0001343	0.0001457
10	0.0001041	0.0001100	0.0001194
15	0.0000951	0.0001003	0.0001085
20	0.0000906	0.0000955	0.0001034
25	0.0000868	0.0000916	0.0000994
30	0.0000839	0.0000887	0.0000952
35	0.0000817	0.0000865	0.0000942
40	0.0000795	0.0000843	0.0000916
45	0.0000775	0.0000822	0.0000890
50	0.0000763	0.0000810	0.0000887
55	0.0000745	0.0000799	0.0000868
60	0.0000734	0.0000785	0.0000861
65	0.0000713	0.0000769	0.0000848
70	0.0000706	0.0000765	0.0000839
75	0.0000693	0.0000748	0.0000824
80	0.0000685	0.0000738	0.0000818
85	0.0000667	0.0000725	0.0000802
90	0.0000662	0.0000722	0.0000805
95	0.0000655	0.0000711	0.0000793
100	0.0000647	0.0000703	0.0000781

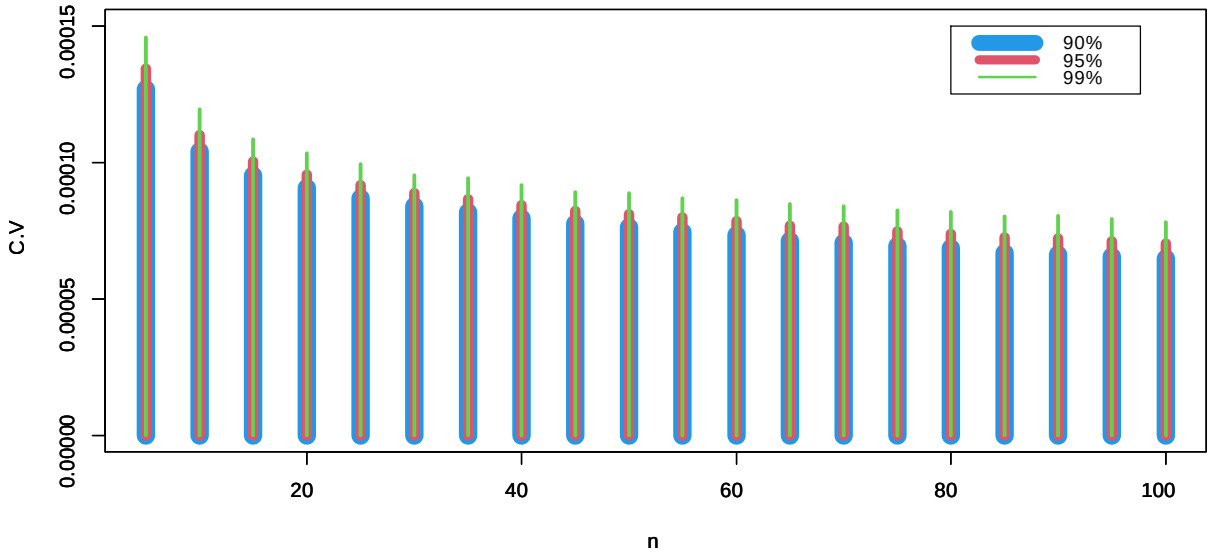

Figure 1. Relationship between the sample size, the level of confidence, and the critical values.

4.2. The Power Estimates of the Test

The calculations used to establish the minimal sample size for a study are referred to as power estimates in statistics. The probability that the null hypothesis will be correctly rejected, assuming it is true, is known as strength. It is based on four basic factors: power, significance level, sample size, and the number of observations or research participants. The proposed test's potency was evaluated at the $(1-\alpha)\%$ confidence level, $\alpha = 0.05$, using the 10,000 samples provided in Table 3 as a basis. Assume that θ has proper values for the Weibull distribution (WD), and gamma distribution (GD) at $n = 10, 20$, and 30, respectively. Table 3 demonstrates that the $\Delta_n(0.09)$ test we employed has adequate power for all other alternatives.

Table 3. Estimates of the power of $\Delta_n(0.09)$.

n	θ	Weibull	Gamma
10	2	0.9650	0.5671
	3	0.9997	0.8950
	4	1.0000	0.9568
20	2	0.9931	0.7116
	3	1.0000	0.9320
	4	1.0000	0.9847
30	2	0.9994	0.7498
	3	1.0000	0.9560
	4	1.0000	0.9922

5. Testing for Censored Data

It is recommended to use a test statistic to contrast H_0 and H_1 with data that have been randomly right-censored. In a life-testing model or a clinical study where patients may be lost (censored) before the completion of a trial, such censored data are frequently the only types of information that are accessible. This hypothetical experiment can be formalized as follows. Assume that n items are checked, with $X_1, X_2, \ldots, X_n$ representing the actual lifespan of each item. We allow $X_1, X_2, \ldots, X_n$ to be independently and identically distributed (i.i.d.) under the assumption of a continuous life distribution, F. Assume that $Y_1, Y_2, \ldots, Y_n$ are (i.i.d.) according to a continuous life distribution, G. Additionally, assume that X and Y are independent variables. The pairs are visible in the randomly right-censored model (Z_j, δ_j), $j = 1, \ldots, n$, where $Z_j = \min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1, & \text{if} \quad Z_j = X_j \text{ (j-th observation is uncensored)} \\ 0, & \text{if} \quad Z_j = Y_j \text{ (j-th observation is censored).} \end{cases} \tag{1}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ denote the ordered Z and $\delta_{(j)}$ is δ_j corresponding to $Z_{(j)}$. The product limit estimator was suggested by Kaplan and Meier [40] using the censored data (Z_j, δ_j), $j = 1, \ldots, n$.

$$\overline{F}_n(X) = \prod_{[j:Z(j)\leq X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}};\ X \in [0, Z_{(n)}].$$

We now propose the following test statistic to compare $H_0 : \hat{\phi}_c = 0$ to $H_1 : \hat{\phi}_c > 0$, using the randomized right censored data

$$\begin{aligned} \hat{\phi}_c \quad = \quad & [\frac{1}{2}s^3 - \frac{1}{2}s^2]\mu_{(2)}\zeta(s) + [s^2 - s^3]\zeta(1)\zeta(s) + [s^2 - s^3]\mu\zeta(s) \\ & + [s^3 - s^2 + 1]\zeta(s) + [s - s^2]\mu - s^2\zeta(1) + s^2 - 1, \end{aligned}$$

where $\zeta(s) = \int_0^\infty e^{-sx} dF_n(x)$. It is possible to rewrite $\hat{\phi}_c$ for computational purposes.

$$\begin{aligned}\hat{\phi}_c &= [\frac{1}{2}s^3 - \frac{1}{2}s^2]\Phi\eta + [s^2 - s^3]\tau\eta + [s^2 - s^3]\Omega\eta \\ &\quad + [s^3 - s^2 + 1]\eta + [s - s^2]\Omega - s^2\tau + s^2 - 1,\end{aligned}$$

where

$$\begin{aligned}\Omega &= \sum_{k=1}^{n}[\prod_{m=1}^{k-1} C_m^{\delta(m)}\left(Z_{(k)} - Z_{(k-1)}\right)], \\ \Phi &= 2\sum_{i=1}^{n}[\prod_{v=1}^{i-1} Z_{(i)} C_v^{\delta(v)}\left(Z_{(i)} - Z_{(i-1)}\right)], \\ \eta &= \sum_{j=1}^{n} e^{-sZ_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}], \\ \tau &= \sum_{j=1}^{n} e^{-Z_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}],\end{aligned}$$

and

$$dF_n(Z_j) = \overline{F}_n(Z_{j-1}) - \overline{F}_n(Z_j), c_k = [n-k][n-k+1]^{-1}.$$

The test invariance is achieved by letting

$$\hat{\Delta}_c = \frac{\hat{\phi}_c}{\overline{Z}^3}, \text{ where } \overline{Z} = \sum_{i=1}^{n} \frac{Z_{(i)}}{n}. \tag{12}$$

In Table 4 and Figure 2, the crucial percentages of the $\hat{\Delta}_c$ tests for the samples taken $n = 10(10)100$ are displayed. The null Monte Carlo distribution's crucial values were found using the common exponential distribution, using the Mathematica 12 program at $s = 0.9$, and with 10,000 replications. The critical values increased as the confidence level increased and decreased as sample numbers increased, respectively, as shown in Figure 2 and Table 4.

Table 4. The superior percentages of $\hat{\Delta}_c$ with 10,000 replications at $s = 0.9$.

n	90%	95%	99%
10	0.0018839	0.0069971	0.0171796
20	0.0018178	0.0066184	0.0158503
30	0.0017509	0.0061387	0.0137694
40	0.0016926	0.0059580	0.0133870
50	0.0015313	0.0057832	0.0123716
60	0.0014402	0.0055886	0.0119811
70	0.0012656	0.0052264	0.0112588
80	0.0012082	0.0045355	0.0101906
90	0.0011823	0.0044339	0.0098655
100	0.0010760	0.0040471	0.0090138

Estimates of the Test Power $\Delta_c(s)$

At a significance level of $\alpha = 0.05$, we assessed the test power using the occasion parameter values of θ based on 10,000 samples, at $n = 10, 20$ and 30 for the gamma, LFR, and Weibull distributions. Table 5 exemplifies how the power estimations for this test $\Delta_c(0.9)$ for all other options were suitable.

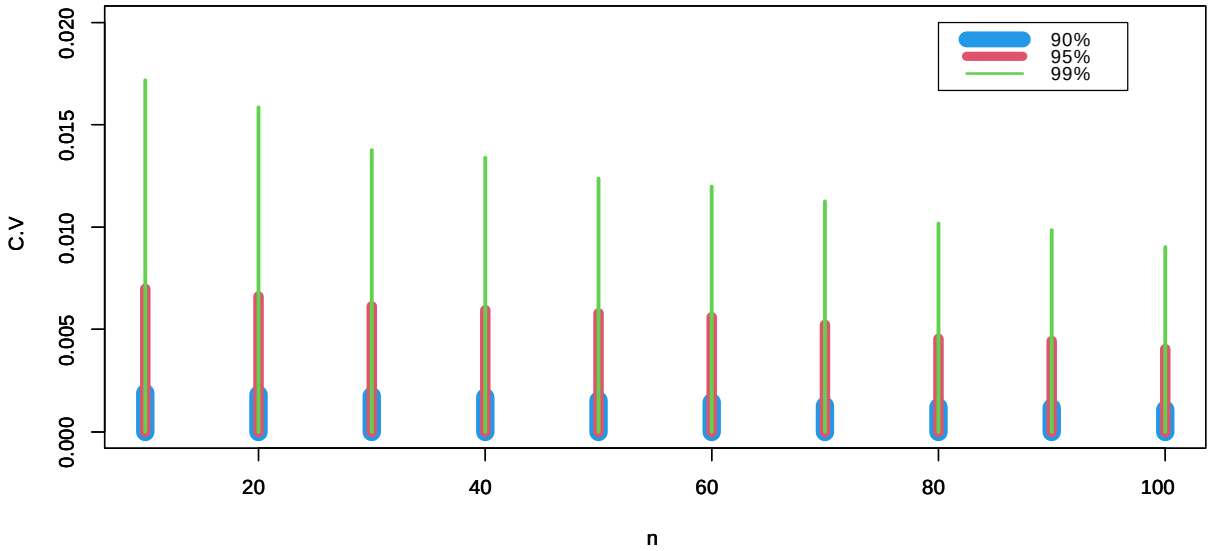

Figure 2. Relationship between the sample size, the level of confidence, and the critical values.

Table 5. Estimates of the power of $\Delta_c(0.9)$.

n	θ	Weibull	LFR	Gamma
10	2	0.6162	0.4496	0.3541
	3	0.6219	0.4843	0.3954
	4	0.6222	0.5035	0.4173
20	2	0.8289	0.6308	0.4049
	3	0.8328	0.6792	0.4647
	4	0.8351	0.7203	0.5278
30	2	0.9148	0.7376	0.5497
	3	0.9193	0.7994	0.6949
	4	0.9193	0.8224	0.8167

6. Censored and Uncensored Observations in Applications to Real Data

Data that are controlled (censored) exist, suggesting that some details are omitted or are not comprehensive. Data that have not been filtered are completely known, indicating that they contain all pertinent information. Controlled data can be difficult to analyze statistically since they call for unique approaches and presumptions to deal with missing or insufficient data. Data that are unsupervised (uncensored) do not suffer from these problems, making them easier to analyze. In this Section, the data sets have been discussed and analyzed based on a significance level $\alpha = 0.05$.

6.1. Non-Censored Data

6.1.1. Dataset I: Methylmercury Poisoning

The listed times of death within the week were recorded (using single dosage levels) in a Florida State University experiment to study the effects of methylmercury poisoning on the life spans of fish (see Kochar [41]).

6.000	6.143	7.286	8.714	9.429
9.857	10.143	11.571	11.714	11.714

The data representation plots are presented in Figure 3; it was found that there are no extreme observations and the nucleation intensity is asymmetric (left-bimodal-skewed). It was discovered that the critical value, which can be found in Table 2, is exceeded by $\hat{\Delta} = 0.0001122$. We then accept H_1, which says that the data collection has $NBRULC$ properties rather than exponential growth.

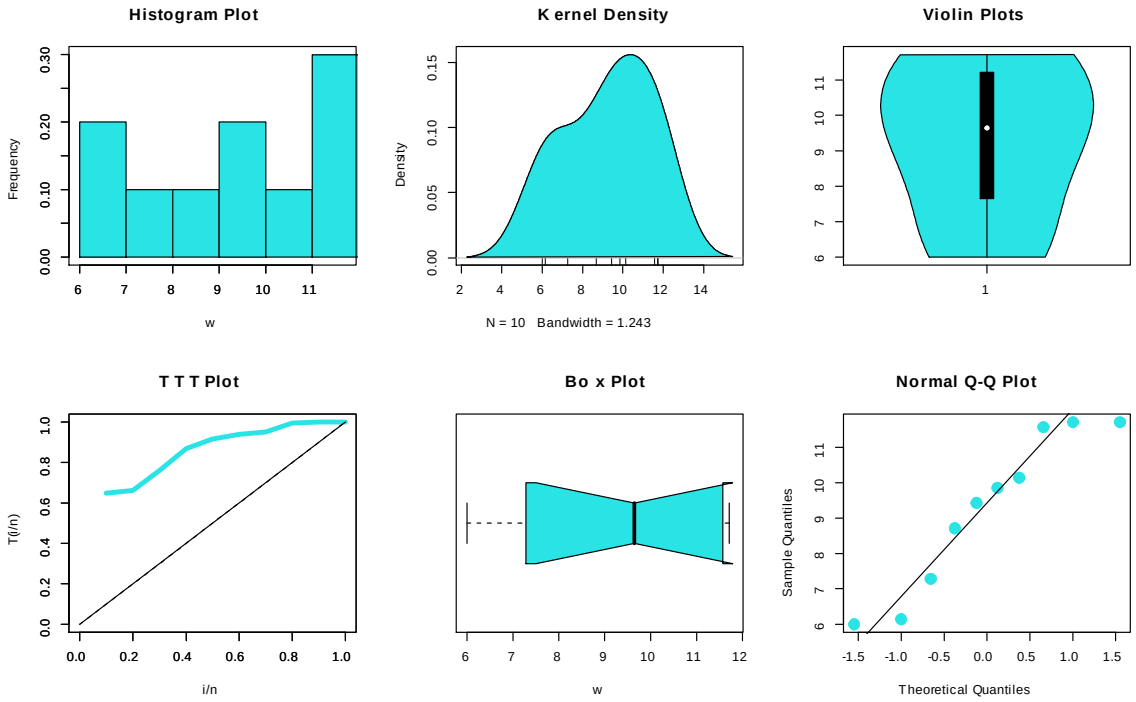

Figure 3. Non-parametric plots for dataset I.

6.1.2. Dataset II: Endurance of Ball Bearings

Pavur et al. [42] took into consideration the information below. The findings of a life test investigation on 23 ball bearings are shown in the following, indicating the number of revolutions (in ten million) until failure.

1.788	2.892	3.300	4.152	4.212	4.560	4.848	5.184
5.196	5.412	5.556	6.78	6.864	6.864	6.988	8.412
9.312	9.864	10.512	10.584	12.792	12.804	17.340	

The data visualization plots are presented in Figure 4. It was reported that there are some extreme observations and the nucleation intensity is asymmetric (bimodal-right-skewed). The dataset's exponential features are demonstrated by the null hypotheses, which are disproved. When $\hat{\Delta} = 0.0001074$, it is found to exceed the pertinent critical value in Table 2.

6.1.3. Dataset III: Leukaemia

Check out the information from Abouammoh et al. [43]. These numbers indicate a group of 43 leukemia patients from a Saudi Arabian Ministry of Health hospital, with order values in years, as follows:

0.315	0.496	0.699	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.718	3.751	3.858	3.986	4.049	4.244
4.323	4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074
5.203	5.274	5.384							

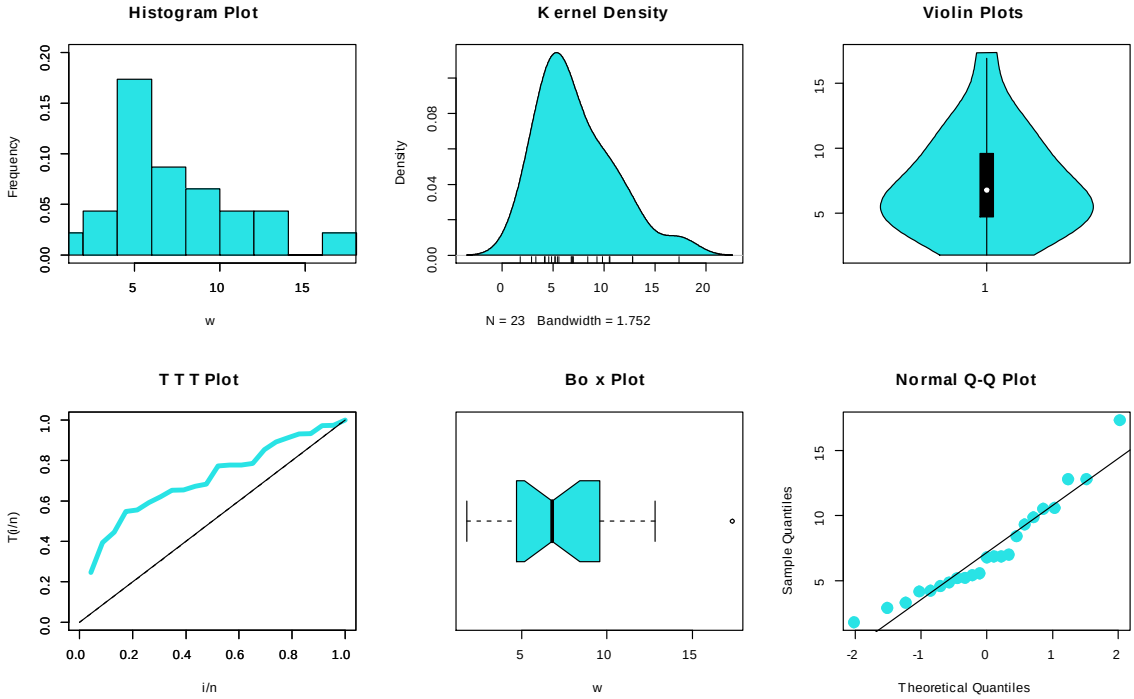

Figure 4. Non-parametric plots for dataset II.

The data representation plots are sketched in Figure 5; there are no extreme observations and the nucleation intensity is asymmetric (left-skewed). It was discovered that the critical value, which can be found in Table 2, is exceeded by $\hat{\Delta} = 0.0001222$. We then accept H_1, which states that the data collection has $NBRULC$ properties rather than exponential growth.

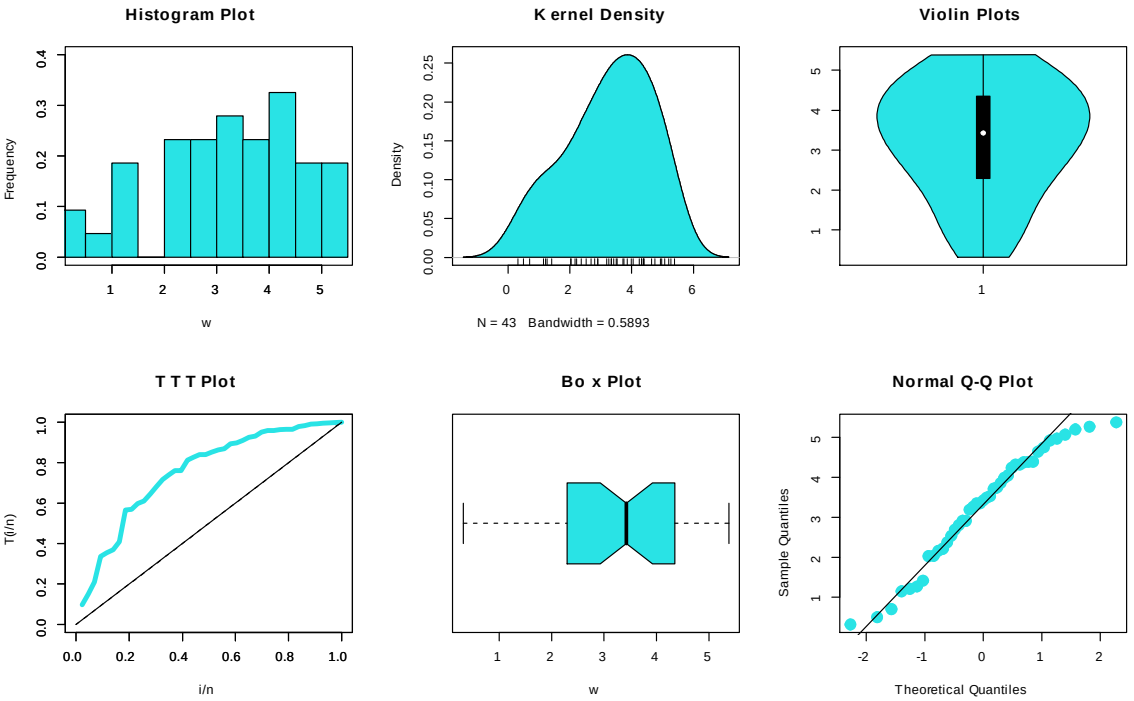

Figure 5. Non-parametric plots for dataset III.

6.1.4. Dataset IV: Leukaemia

Take a look at the data below, which show the survival periods (in years) following the diagnoses of 43 patients with specific kinds of leukemia (see Kotz and Johnson [44]). The data can be listed as follows:

0.019	0.129	0.159	0.203	0.485	0.636	0.748	0.781	0.869	1.175
1.206	1.219	1.219	1.282	1.356	1.362	1.458	1.564	1.586	1.592
1.781	1.923	1.959	2.134	2.413	2.466	2.548	2.652	2.951	3.038
3.6	3.655	3.754	4.203	4.690	4.888	5.143	5.167	5.603	5.633
6.192	6.655	6.874							

The data visualization plots are presented in Figure 6; it is clear that there are no extreme observations and the nucleation intensity is asymmetric-bimodal (right-skewed). The dataset's exponential features are demonstrated by the null hypotheses, which are disproved. When $\hat{\Delta} = 0.0001048$, is found, it exceeds the relevant critical value in Table 2.

6.1.5. Dataset V: Carbon Fibers

Two sets of data that were introduced by Badar and Priest [45] and used by Kundu and Gupta [46] are discussed in this segment. The strength of individual carbon fibers evaluated under tension at gauge lengths of 20 mm is measured in set A using GPA. For set B, individual carbon fibers were tested under stress at gauge lengths of 10 mm; the strength was evaluated in GPA. Size 63 applies to sets A and B. The data visualization plots are listed in Figures 7 and 8.

Dataset A:

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944
1.958	1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098
2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274	2.301
2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633
2.642	2.648	2.684	2.697	2.726	2.770	2.773	2.800	2.809
2.818	2.821	2.848	2.880	2.954	3.012	3.067	3.084	3.090

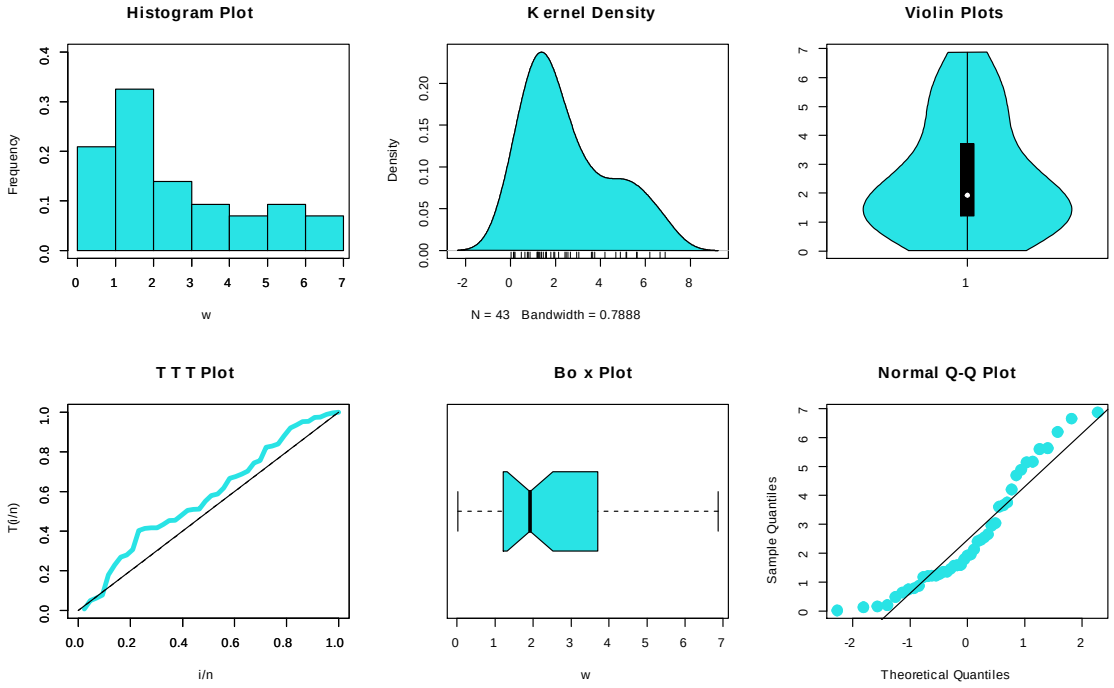

Figure 6. Non-parametric plots for dataset IV.

In such a case, the estimated value, $\hat{\Delta} = 0.0001036$, exceeds the crucial value shown in Table 2 by a significant amount. Given the $\alpha = 0.05$ significant level, it is true because these types of data satisfy the $NBRULC$ characteristic.

Dataset B:

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020

In such a case, the estimated value, $\hat{\Delta} = 0.0001131$, exceeds the crucial value shown in Table 2 by a significant amount. Given the $\alpha = 0.05$ significant level, it is true because these types of data satisfy the $NBRULC$ characteristic.

6.1.6. Dataset VI: COVID-19

According to EL-Sagheer et al. [47], this information indicates the fatality rate for COVID-19 in the Netherlands from 31 March to 30 April 2020. The information is as follows:

14.918	10.656	12.274	10.289	10.832	7.099	5.928	13.211
7.968	7.584	5.555	6.027	4.097	3.611	4.960	7.498
6.940	5.307	5.048	2.857	2.254	5.431	4.462	3.883
3.461	3.647	1.974	1.273	1.416	4.235		

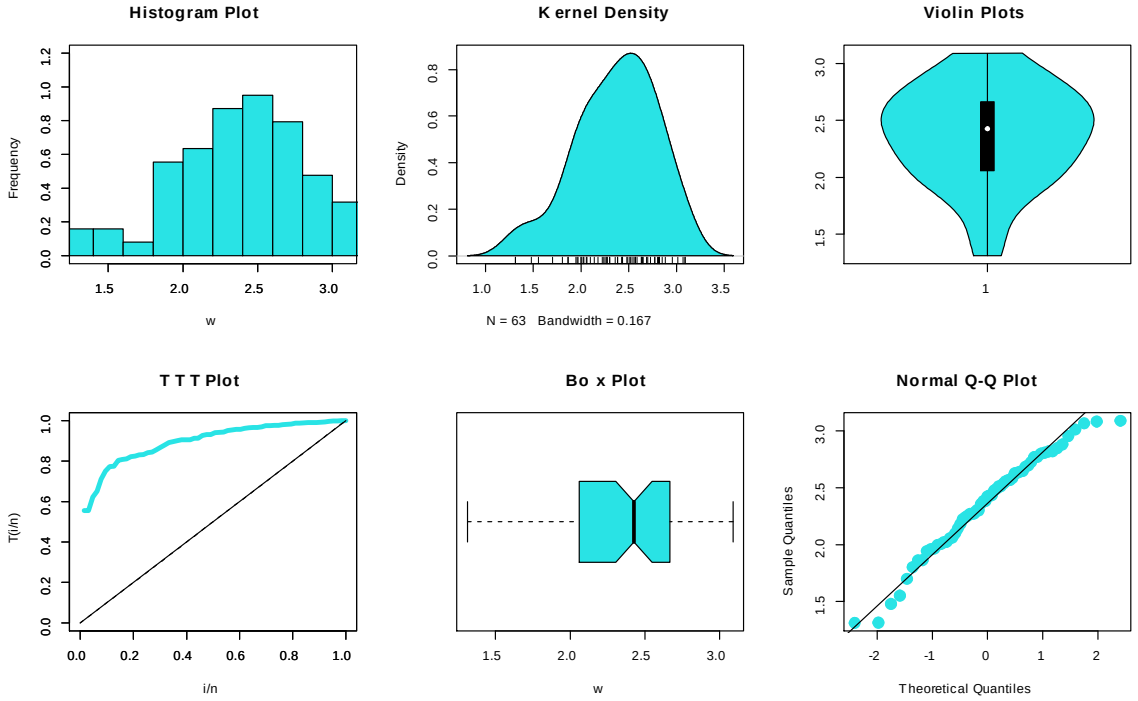

Figure 7. Non-parametric plots for set A of dataset V.

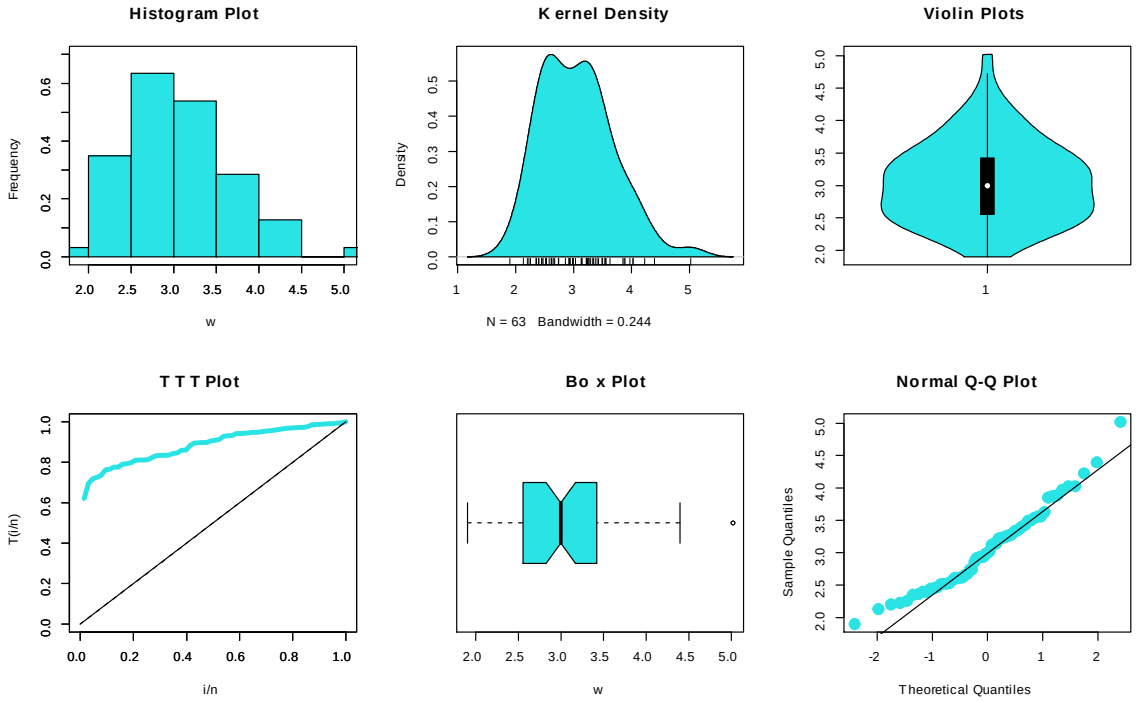

Figure 8. Non-parametric plots for set B of dataset V.

The data visualization plots are presented in Figure 9. It was discovered that the critical value, which can be found in Table 2, is exceeded by $\hat{\Delta} = 0.0001081$. We then accept H_1, which states that the data collection has $NBRULC$ properties rather than exponential growth.

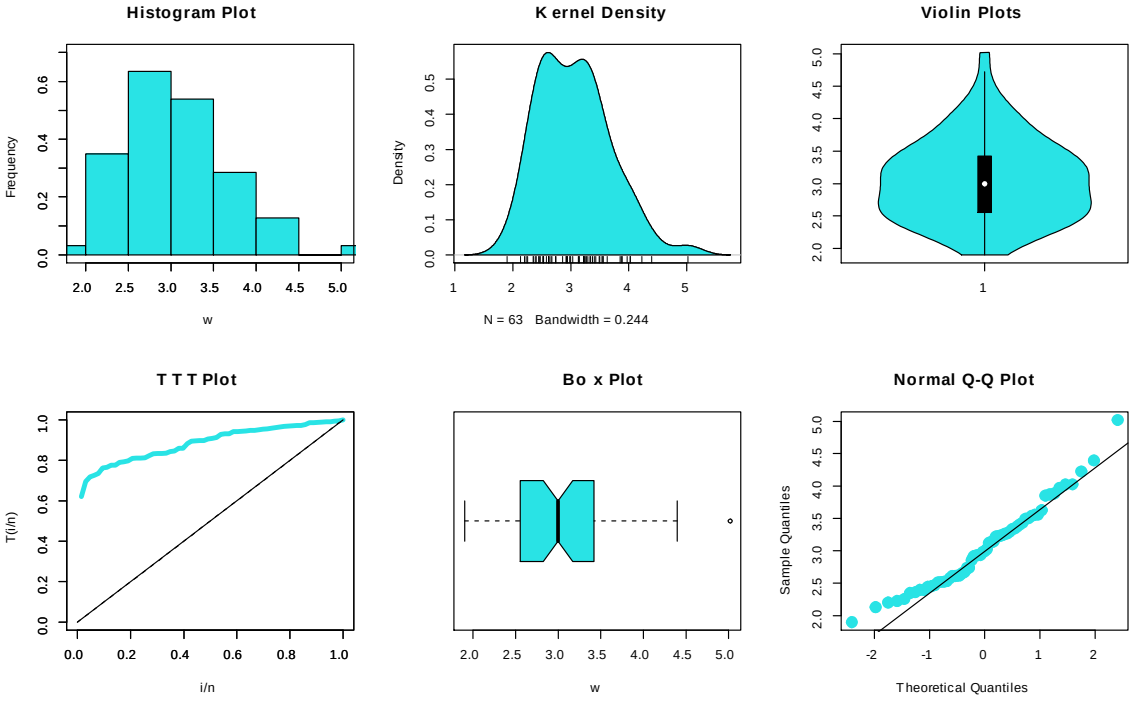

Figure 9. Non-parametric plots for dataset VI.

6.1.7. Dataset VII: COVID-19

According to Almongy et al. [48], this information indicates the fatality rate for COVID-19 in Italy from 27 February to 27 April 2020. The information is as follows:

4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908	6.503
18.474	11.010	17.337	16.561	13.226	15.137	8.697	15.787	13.333
11.822	14.242	11.273	14.330	16.046	11.950	10.282	11.775	10.138
9.037	12.396	10.644	8.646	8.905	8.906	7.407	7.445	7.214
6.194	4.640	5.452	5.073	4.416	4.859	4.408	4.639	3.148
4.040	4.253	4.011	3.564	3.827	3.134	2.780	2.881	3.341
2.686	2.814	2.508	2.450	1.518				

The data representation plots are listed in Figure 10. The dataset's exponential features are demonstrated by the null hypotheses, which are disproved. When $\hat{\Delta} = 0.0001027$, it is determined to exceed the critical value specified in Table 2.

6.1.8. Dataset VIII: Wind Speed

We used the following dataset, which was originally obtained from NCDC. These data represent the wind speed measured in knots for the first sample (set A) for 23 days and the second sample for 25 days (see, Ghazal et al. [49]).

Dataset A:

8.6	3.8	5.4	4.4	2.2	3.8	4.5	6.3
3.4	4.1	3.8	8.6	13.0	11.3	12.4	12.4
5.0	3.4	3.8	5.3	3.6	5.8	4.2	

The data visualization plots are presented in Figure 11 (set A); there are no extreme observations and the nucleation intensity is asymmetric-bimodal (right-skewed).

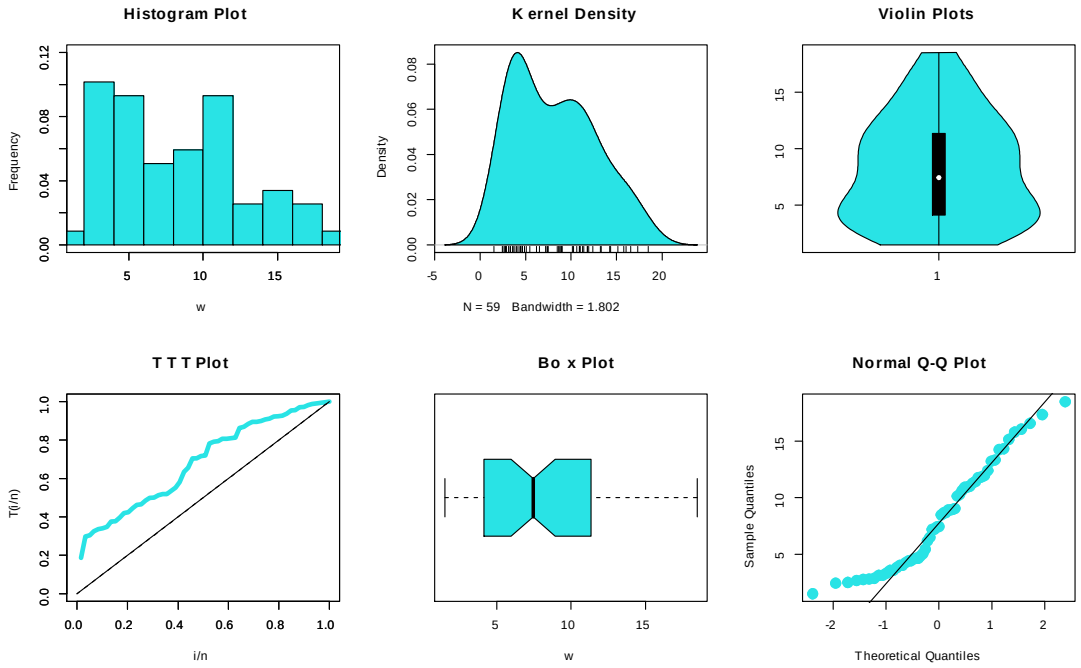

Figure 10. Non-parametric plots for dataset VII.

In such a case, the estimated value, $\hat{\Delta} = 0.0001084$, exceeds the crucial value shown in Table 2 by a significant amount. Given the $\alpha = 0.05$ significant level, it is true because these types of data satisfy the $NBRULC$ characteristic.

Dataset B:

2.4	2.9	3.3	3.4	3.5	3.7	3.8	3.9	4.0
4.1	4.2	4.5	4.6	4.8	5.1	5.3	5.5	6.0
6.2	6.5	7.8	8.2	8.4	9.4	10.9		

The data visualization plots are presented in Figure 12 (set B); there are some extreme observations and the nucleation intensity is asymmetric-multimodal (right-skewed). In such a case, the estimated value, $\hat{\Delta} = 0.0001183$, exceeds the crucial value shown in Table 2 by a significant amount. Considering the significance level of $\alpha = 0.05$, it can be concluded that these types of data satisfy the $NBRULC$ characteristic.

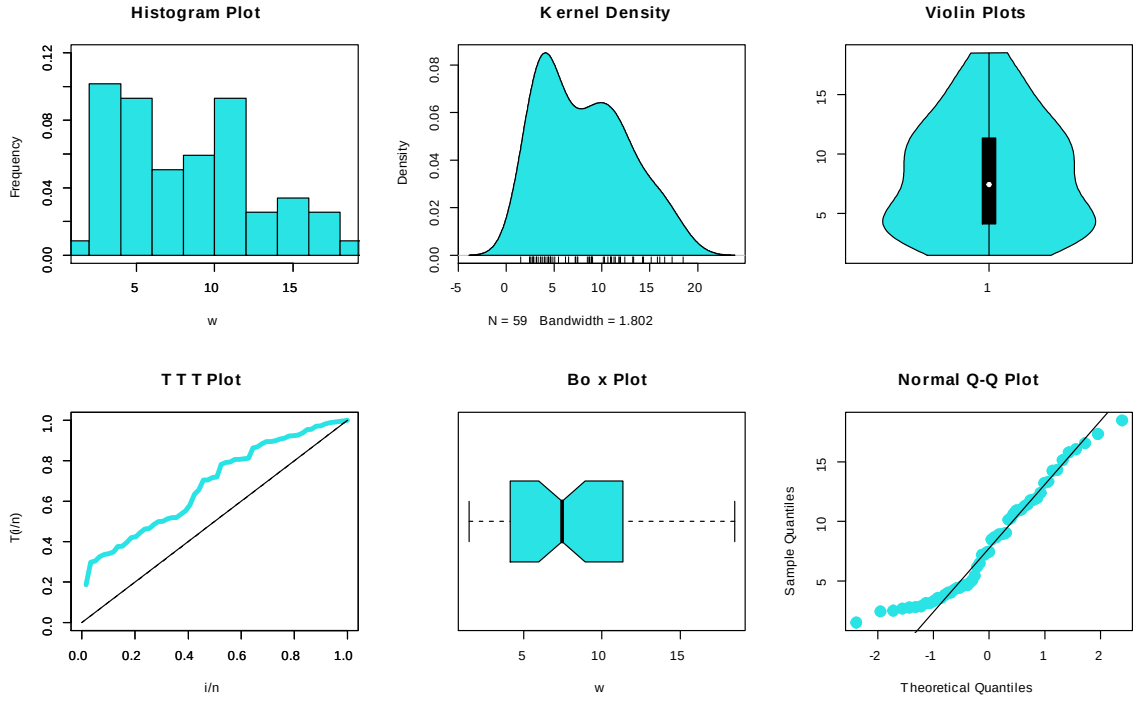

Figure 11. Non-parametric plots for set A of dataset VIII.

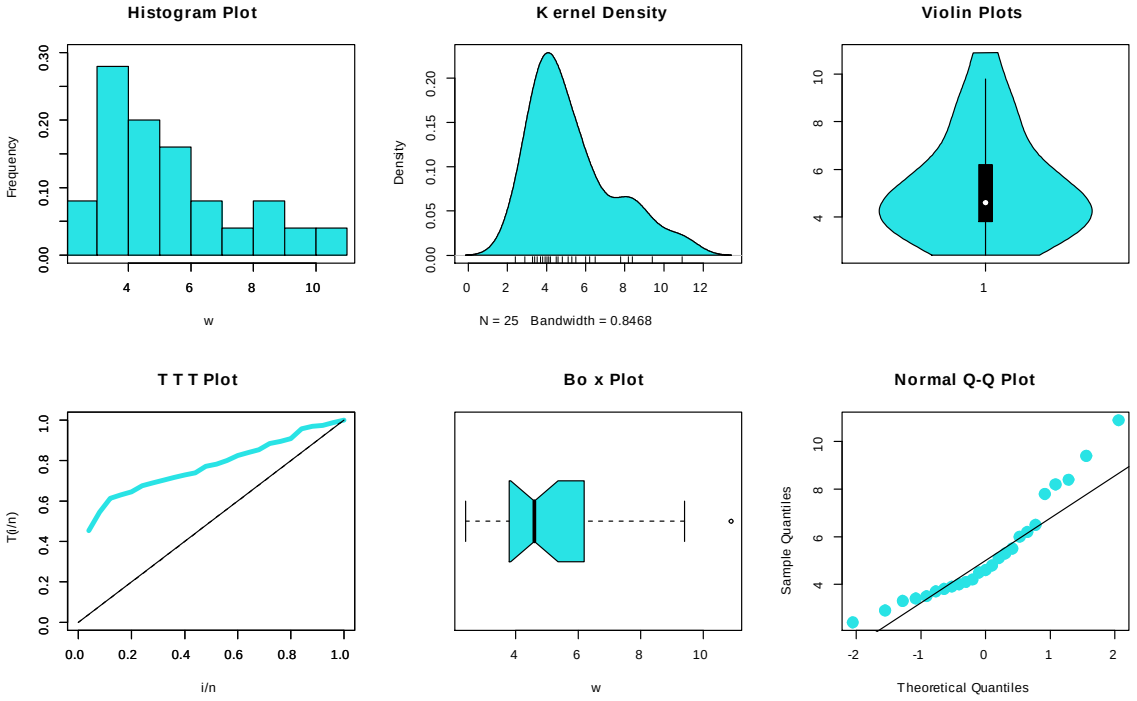

Figure 12. Non-parametric plots for set B of dataset VIII.

6.2. Censored Data

6.2.1. Dataset IX: Lung Cancer

The following data represent the number of weeks that 61 individuals, who received cyclophosphamide for incurable lung cancer, survived and were considered in the studies by Lagakos and Williams [50] and Lee and Wolfe [51]. For the patients whose treatments were stopped due to a deteriorating condition, there are 28 censored observations and 33 uncensored observations.

- Censored observations:

0.14	0.14	0.29	0.43	0.57	0.57	1.86	3.00	3.00	3.29
3.29	6.00	6.00	6.14	8.71	10.57	11.86	15.57	16.57	17.29
18.71	21.29	23.86	26.00	27.57	32.14	33.14	47.29		

- Uncensored observations:

0.43	2.86	3.14	3.14	3.43	3.43	3.71	3.86	6.14	6.86	9.00
9.43	10.71	10.86	11.14	13.00	14.43	15.71	18.43	18.57	20.71	29.14
29.71	40.57	48.57	49.43	53.86	61.86	66.57	68.71	68.96	72.86	72.86

When considering all available survival statistics, both censored and uncensored, we discover that Table 4's essential value exceeds our conclusion, which is $\Delta_c(0.9) = -0.00099279$. We can, therefore, clearly see the exponential nature of the data.

6.2.2. Dataset XI: Melanoma

Consider the information from Susarla and Van Ryzin [52]. These numbers represent the survival rates of 46 melanoma patients. Among them, 35 (non-censored material) have complete lifetime information. The following observations have been redacted, in order, as follows:

13	14	19	19	20	21	23	23	25	26
26	27	27	31	32	34	34	37	38	38
40	46	50	53	54	57	58	59	60	65
65	66	70	85	90	98	102	103	110	118
124	130	136	138	141	234				

The following is the hierarchy of the censored observations:

16	21	44	50	55	67	73	76	80	81
86	93	100	108	114	120	124	125	129	130
132	134	140	147	148	151	152	152	158	181
190	193	194	213	215					

If we take into account both censored and uncensored survival statistics, the essential value in Table 4 exceeds our outcome, which is $\Delta_c(0.9) = -2.65634 \times 10^{-23}$. The exponential properties of the data are, therefore, evident to us.

7. Conclusions

An exponential comparison of different life distributions has garnered much attention. Based on the goodness-of-fit technique, we introduced a new test statistic in this study to evaluate the exponential versus non-exponential NBRULC class of life distributions. For the uncensored and censored data, Tables 1 and 3 calculated and tabulated the critical values for this test. The efficiency of the Pitman asymptotic efficiency was discussed, and power estimates were simulated for families of the most common age distribution in reliability. The Monte Carlo critical points for the null distribution were simulated. Moreover, the power estimations of this test were computed for a few typical alternative distributions. Several key values were listed along with the power estimates for the censored and uncensored

data for this test; moreover, the issue of properly suppressed data was also discussed. To illustrate the usefulness of the recommended test in the reliability analyses of censored and uncensored data, this paper covered several applications.

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References

1. Bryson, M.C.; Siddiqui, M.M. Some criteria for aging. *J. Am. Stat. Assoc.* **1969**, *64*, 1472–1483. [CrossRef]
2. Bhattacharyya, D.; Khan, R.A.; Mitra, M. A test of exponentiality against DMTTF alternatives via L-statistics. *Stat. Probab. Lett.* **2020**, *165*, 108853. [CrossRef]
3. Barlow, R.E.; Proschan, F. *Statistical Theory of Reliability and Life Testing*; To Begin With: Silver Spring, MD, USA, 1981.
4. Klefsjo, B. The HNBUE and HNWUE classes of life distributions. *Nav. Res. Logist.* **1982**, *29*, 331–344. [CrossRef]
5. Khan, R.A.; Bhattacharyya, D.; Mitra, M. Exact and asymptotic tests of exponentiality against nonmonotonic mean time to failure type alternatives. *Stat. Pap.* **2021**, *62*, 3015–3045. [CrossRef]
6. Kumazawa, Y. A class of tests statistics for testing whether new is better than used. *Commun. Stat. Theory Methods* **1983**, *12*, 311–321. [CrossRef]
7. Mahmoud, M.A.W.; EL-Sagheer, R.M.; Etman, W.B.H. Testing exponentiality against new better than renewal used in Laplace transform order. *J. Stat. Appl. Probab.* **2016**, *5*, 279–285. [CrossRef]
8. Mahmoud, M.A.W.; EL-Sagheer, R.M.; Etman, W.B.H. Moments inequalities for NBRUL distributions with hypotheses testing applications. *Austrian J. Stat.* **2018**, *47*, 95–104.
9. Majumder, P.; Mitra, M. A test for detecting Laplace order dominance and related Bahadur efficiency issues. *Stat. Pap.* **2019**, *60*, 1921–1937. [CrossRef]
10. Bhattacharyya, D.; Khan, R.A.; Mitra, M. Tests for Laplace order dominance with applications to insurance data. *Insur. Math. Econ.* **2021**, *99*, 163–173. [CrossRef]
11. Navarro, J. Preservation of DMRL and IMRL aging classes under the formation of order statistics and coherent systems. *Stat. Probab. Lett.* **2018**, *137*, 264–268. [CrossRef]
12. Abu-Youssef, S.E.; Hassan, E.M.A.; Gerges, S.T. A new nonparametric class of life distributions based on ordering moment generating approach. *J. Stat. Appl. Probab. Lett.* **2020**, *7*, 151–162.
13. Ghosh, S.; Mitra, M. A new test for exponentiality against HNBUE alternatives. *Commun. Stat. Theory Methods* **2020** *49*, 27–43. [CrossRef]
14. Navarro, J.; Pellerey, F. Preservation of ILR and IFR aging classes in sums of dependent random variables. *Appl. Stoch. Model. Bus. Ind.* **2022**, *38*, 240–261. [CrossRef]
15. El-Morshedy, M.; Al-Bossly, A.; EL-Sagheer, R.M.; Almohaimeed, B.; Etman, W.B.H.; Eliwa, M.S. A moment inequality for the NBRULC class: Statistical properties with applications to model asymmetric data. *Symmetry* **2022**, *14*, 2353. [CrossRef]
16. EL-Sagheer, R.M.; Abu-Youssef, S.E.; Sadek, A.; Omar, K.M.; Etman, W.B.H. Characterrization and testing NBRUL class of life distributions based on Laplace transform technique. *J. Stat. Appl. Probab.* **2022**, *11*, 1–14.
17. Gadallah, A.M.; Mohammed, B.I.; Al-Babtain, A.A.; Khosa, S.K.; Kilai, M.; Yusuf, M.; Bakr, M.E. Modeling various survival distributions using a nonparametric hypothesis testing based on Laplace transform approach with some real applications. *Comput. Math. Methods Med.* **2022**, *2022*, 5075716. [CrossRef]
18. Mansour, M.M.M. Assessing treatment methods via testing exponential property for clinical data. *J. Stat. Appl. Probab.* **2022**, *11*, 109–113.

19. Ghosh, S.; Mitra, M. A weighted integral approach to testing against HNBUE alternatives. *Stat. Probab. Lett.* **2017**, *129*, 58–64. [CrossRef]
20. Majumder, P.; Mitra, M. A test of exponentiality against alternatives. *J. Nonparametr. Stat.* **2019**, *31*, 794–812. [CrossRef]
21. Lai, C.D.; Xie, M. *Stochastic Ageing and Dependence for Reliability*; Springer: New York, NY, USA, 2006.
22. Etman, W.B.H.; EL-Sagheer, R.M.; Abu-Youssef, S.E.; Sadek, A. On some characterizations to NBRULC class with hypotheses testing application. *Appl. Math. Inf. Sci.* **2022**, *16*, 139–148.
23. Ghosh, S.; Majumder, P. A moment inequality for decreasing mean time to failure distributions with hypothesis testing application. *J. Stat. Comput. Simul.* **2022**, *92*, 2875–2890. [CrossRef]
24. Bakr, M.E.; Al-Babtain, A.A. Non-parametric hypothesis testing for unknown aged class of life distribution using real medical data. *Axioms* **2023**, *12*, 369. [CrossRef]
25. Ghosh, S.; Mitra, M. On the exact distribution of generalized Hollander-Proschan type statistics. *Commun. Stat. Simul. Comput.* **2022**, *51*, 5051–5067. [CrossRef]
26. Alqifar, H.; Eliwa, M.S.; Etman, W.B.H.; El-Morshedy, M.; Al-Essa, L.A.; EL-Sagheer, R.M. Reliability class testing and hypothesis specification: NBRULC-$t_{\circ}$ characterizations with applications for medical and engineering data modeling. *Axioms* **2023**, *12*, 414. [CrossRef]
27. Mahmoud, M.A.W.; Abdul Alim, N.A. A goodness of fit approach to NBURFR and NBARFR classes. *Econ. Qual. Control* **2006** *21*, 59–75. [CrossRef]
28. Kayid, M.; Diab, L.S.; Alzughaibi, A. Testing NBU (2) class of life distribution based on goodness of fit approach. *J. King Saud-Univ.-Sci.* **2010**, *22*, 241–245. [CrossRef]
29. Bera, S.; Bhattacharyya, D.; Khan, R.A.; Mitra, M. Test for harmonic mean residual life function: A goodness of fit approach. *Math. Comput. Simul.* **2023**, *203*, 58–70. [CrossRef]
30. Mahmoud, M.A.W.; Diab, L.S.; Radi, D.M. Testing exponentiality against RNBUL class of life distribution based on goodness of fit. *J. Stat. Appl. Probab.* **2019**, *8*, 57–66. [CrossRef]
31. Abu-Youssef, S.E.; Ali, N.S.A.; Bakr, M.E. Used better than aged in mgf ordering class of life distribution with application of hypothesis testing. *J. Stat. Appl. Probab. Lett.* **2020**, *7*, 23–32.
32. Bakr, M.E.; Nagy, M.; Al-Babtain, A.A. Non-parametric hypothesis testing to model some cancers based on goodness of fit. *AIMS Math.* **2022**, *7*, 13733–13745. [CrossRef]
33. Abu-Youssef, S.E.; El-Toony, A.A. A new class of life distribution based on Laplace transform and It's applications. *Inf. Sci. Lett.* **2022**, *11*, 355–362.
34. Abu-Youssef, S.E.; Gerges, S.T. Based on the goodness of fit approach, a new test statistics for testing NBUC$_{mgf}$ class of life distributions. *Pak. J. Stat.* **2022**, *38*, 129–144.
35. Etman, W.B.H.; El-Morshedy, M.; Eliwa, M.S.; Almohaimeed, A.; EL-Sagheer, R.M. A new reliability class-test statistic for life distributions under convolution, mixture and homogeneous shock model: Characterizations and applications in engineering and medical fields. *Axioms* **2023**, *12*, 331. [CrossRef]
36. Lee, A.J. U-Statistics-Theorem and Practice. In *Statistics Textbooks Monographs*; Marcel Dekker: New York, NY, USA, 1990; p. 110.
37. Mugdadi, A.R.; Ahmad, I.A. Moment inequalities derived from comparing life with its equilibrium form. *J. Stat. Plan. Inference* **2005**, *134*, 303–317. [CrossRef]
38. Kango, A.I. Testing for new is better than used. *Commun. Stat. Theory Methods* **1993**, *12*, 311–321.
39. Abdel Aziz, A.A. On testing exponentiality against RNBRUE alternatives. *Appl. Math. Sci.* **2007**, *35*, 1725–1736.
40. Kaplan, E.L.; Meier, P. Nonparametric estimation from incomplete observation. *J. Am. Stat. Assoc.* **1958**, *53*, 457–481. [CrossRef]
41. Kocha S.C. Testing exponenttality against monotone failure rate average. *Commun. Stat. Theory Methods* **1985**, *14*, 381–392. [CrossRef]
42. Pavur, R.J.; Edgeman, R.L.; Scott, R.C. Quadratic statistics for the goodness-of-fit test of the inverse Gaussian distribution. *IEEE Trans. Reliab.* **1992**, *41*, 118–123. [CrossRef]
43. Abouammoh, A.M.; Abdulghani, S.A.; Qamber, I.S. On partial orderings and testing of new better than renewal used classes. *Reliab. Eng. Syst. Saf.* **1994**, *43*, 37–41. [CrossRef]
44. Kotz, S.; Johnson, N.L. *Encyclopedia of Statistical Sciences*; Wiley: New York, NY, USA, 1983.
45. Bader, M.G.; Priest, A.M. Statistical aspects of fibre and bundle strength in hybrid composites. In *Progress in Science and Engineering of Composites*; Hayashi, T., Kawata, K., Umekawa, S., Eds.; ICCM-IV: Tokyo, Japan, 1982; pp. 1129–1136.
46. Kundu, D.; Gupta, R.D. Estimation of P[Y<X] for Weibull Distributions. *IEEE Trans. Reliab.* **2006**, *55*, 270–280.
47. EL-Sagheer, R.M.; Eliwa, M.S.; Alqahtani, K.M.; EL-Morshedy, M. Asymmetric randomly censored mortality distribution: Bayesian framework and parametric bootstrap with application to COVID-19 data. *J. Math.* **2022**, *2022*, 8300753. [CrossRef]
48. Almongy, H.M.; Almetwally, E.M.; Aljohani, H.M.; Alghamdi, A.S.; Hafez, E.H. A new extended rayleigh distribution with applications of COVID-19 data. *Results Phys.* **2021**, *23*, 104012. [CrossRef] [PubMed]
49. Ghazal, M.G.M.; EL-Sagheer, R.M.; Mahmoud, M.A.W.; Al-Babtain, A.A.; Bakr, M.E.; Hasaballah, M.M. Inference for two populations 3-Burr-XII in presence of joint progressive censoring. *Hindawi Complex.* **2022**, *2022*, 6171714. [CrossRef]
50. Lagakos, S.W.; Williams, J.S. Models for censored survival analysis: A cone class of variable-sum models. *Biometrika* **1978**, *65*, 181–189. [CrossRef]

51. Lee, S.Y.; Wolfe, R.A. A simple test for independent censoring under the proportional hazards model. *Biometrics* **1998**, *54*, 1176–1182. [CrossRef]
52. Susarla, V.; Vanryzin, J. Empirical Bayes estimations of a survival function right censored observation. *Ann. Stat.* **1978**, *6*, 710–755. [CrossRef]