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ORIGINAL ARTICLE

Statistical inference based on lower record values for the inverse Weibull distribution under the modified loss function

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Abstract The main objective of this article is to develop a linear, exponential loss function (LXLF) to the dependability value and variables of the Inverse Weibull Distribution (IWD) according to Lowest Record Values. By weighting the LXLF to construct current function loss, the modified linear, exponential loss function (WLXLF) was named. Then use WLXLF to estimate the parameters and reliability functions of the (IWD). After that, we evaluated how well the prediction made in this article performed against the Bayesian estimator using the symmetric and asymmetric loss functions and maximum likelihood estimation (MLE). The credibility range for the values of the parameters and durability function has been created using the parametric bootstrap approach, and the outcomes have been contrasted via a Monte Carlo simulation. An actual data set from a realistic situation was utilized to explain the concept. The computer simulation results demonstrated that the suggested method, performed as expected, was the best for estimating the scale parameter and reliability function. The proposed technique performed well in estimating the shape parameter based on real data and had an acceptable performance for estimating scale parameters and reliability functions.

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1. Introduction

The size model is defined by the sequence of values and the incidence of the measurements, so the recorded values can be considered sequence statistics. The Recording is valuable in various real-world activities, including weather, economy, sports, and reliability. The highest recorded value and the lowest recorded value are two different recorded value kinds. The value of a measurement is more than the value of all previous measurements; it is referred to as lower record values. Records and the corresponding statistics have been studied by several authors, for example, [1–5]. Moreover, several studies discussed the estimate of the parameters and reliability of probability distributions of Failure based on record values from them Generalized Inverted Exponential [6], Gompertz [7], Extended Burr XII [8–9], inverse Weibull [10–12] and Weibull [13–14].

The IWD is frequently employed for data modeling in practical domains: the dynamic components of diesel, reliability, and biological studies. It may be modeled using many features of failure, like germanium transistors, automotive radiators, infant mortality, and motors. And marketing life. It can also determine the ideal time to do routine upkeep [15]; for more details, see [16–17].

The *p.d.f* and the *c.d.f* of the IWD as follows:

$$f(y;\rho,\sigma)=\begin{cases}\rho\sigma y^{-\sigma-1}\exp\left[-\rho y^{-\sigma}\right], & y\geq 0,\ \rho,\sigma>0,\\ 0 & \text{o.w.}\end{cases} \tag{1}$$

$$F(y;\rho,\sigma)=\exp\left[-\rho y^{-\sigma}\right],\qquad y\geq 0,\rho,\sigma>0 \tag{2}$$

where ρ is the shape parameter and σ is the scale parameter.

also, the $R(t)$ at time t can be obtained from equation (2). as follows:

$$R(t;\rho,\sigma)=1-\exp\left[-\rho t^{-\sigma}\right],\qquad t\geq 0,\rho,\sigma>0 \tag{3}$$

Let $Y_{L(1)}Y_{L(2)}\cdots Y_{L(k)}$ represent the first k lower record values from the IW distributions.

The *p.d.f* of $Y_{L(k)}$ 'is

$$f_{(k)}(y)=\frac{1}{(k-1)!}\{-\ln F(y)\}^{k-1}f(y),$$

$$=\frac{\sigma\rho^{k}}{\Gamma(k)}y^{-k\sigma-1}\exp\left[-\rho y^{-\sigma}\right]. \tag{5}$$

2. Point estimation (MLE)

In this part, we will discuss the MLE the estimator of parameters, and dependability work. When data under study is decreased record numbers from the IWD with *Pdf* is given [1]. Depending on the first k lower record values the joint function of density of $\underset{-}{y}\equiv y_{L(1)},y_{L(2)},\cdots\cdots y_{L(k)}$ is given by [18]

$$f_{1,2,\ldots\ldots k}\left(\underset{-}{y}\right)=f\left(y_{L(k)}\right)\prod_{i=1}^{k-1}\frac{f\left(y_{L(i)}\right)}{F\left(y_{L(i)}\right)},-\infty\leq\underset{-}{y}<\infty \tag{6}$$

Then the likelihood function $L\left(\rho,\sigma)|\underset{-}{y}\right)$ of the first K lower record values is

$$L\left(\rho,\sigma)|\underset{-}{y}\right)=(\rho\sigma)^{k}\mathscr{D}\exp\left[-\rho\mathscr{k}_{k}\right],$$

$$\mathscr{D}=\prod_{i=1}^{k-1}y_{L(i)}^{-\sigma-1}\quad,\quad \mathscr{k}_{k}=y_{L(k)}^{-\sigma}, \tag{7}$$

To locate the estimator of the MLE, we take the natural logarithm of equation (6) to convert it to

$$\mathscr{L}=\log L=k\log(\rho\sigma)-(\sigma+1)\sum_{i=1}^{k-1}\log y_{L(i)}-\rho\mathscr{k}_{k}. \tag{8}$$

when σ is identified: MLE estimate of scale limitation ρ denoted $\widehat{\rho}_{ML}$ can be obtained from (8) as

$$\frac{\partial\mathscr{L}}{\partial\rho}=\frac{k}{\widehat{\rho}}-y_{L(k)}^{-\sigma}=0,$$

Therefore, the MLE estimator of ρ is

$$\widehat{\rho}_{ML}=\frac{k}{y_{L(k)}^{-\sigma}}. \tag{9}$$

when σ and ρ are unknown: the MLE estimator of σ and ρ denoted by $\widehat{\sigma}_{ML}$ and $\widehat{\rho}_{ML}$ respectively can be obtained from (8) as

$$\frac{\partial\mathscr{L}}{\partial\sigma}=\frac{k}{\widehat{\sigma}}-\sum_{i=1}^{k-1}\log y_{L(i)}+\rho y_{L(k)}^{-\widehat{\sigma}}\log y_{L(k)}=0,$$

$$\frac{\partial\mathscr{L}}{\partial\rho}=\frac{k}{\widehat{\rho}}-y_{L(k)}^{-\sigma}=0,$$

By solving the above two equations, we get

$$\widehat{\sigma}_{ML}=\frac{k}{\sum_{i=1}^{k-1}\log y_{L(i)}-k\log y_{L(k)}}. \tag{10}$$

and

$$\widehat{\rho}_{ML}=\frac{K}{y_{L(k)}^{-\widehat{\sigma}_{ML}}}. \tag{11}$$

Since the MLE estimators have the invariance property, we can find the reliability function estimator by substituting $\widehat{\rho}_{ML}$ and $\widehat{\sigma}_{ML}$ for ρ and σ in equation (3) to be $R(t)$ as follows:

$$\widehat{R}(t;\rho,\sigma)=1-\exp\left[-\widehat{\rho}_{ML}t^{-\widehat{\sigma}_{ML}}\right],\qquad t\geq 0 \tag{12}$$

3. Loss function

The Loss function determines how much a limit's predicted or assessed value differs from the actual value. The squared error loss function was already widely used in the procedure, but it is recommended when the damage incurred by overstatement and undervaluation is equally important. However, utilizing a symmetric gradient descent is not always suitable. An overestimation is more important than an understated, or vice versa. In this part of the research, symmetric and asymmetric loss functions will be used addition to the proposed loss function.

3.1. Squared error loss function (SLF)

The SLF is represented as

$$L_{SLF}(\varrho,\widehat{\varrho})=(\varrho-\widehat{\varrho})^{2} \tag{13}$$

From this, $\widehat{\varrho}$ is denoted as any ϱ estimate. The ϱ Bayesian estimator under SLF as [19]

$$\widehat{\varrho}_{BSLF} = E\left(\varrho|\underset{-}{y}\right). \tag{14}$$

3.2. Linex loss function (LXLF)

When underestimation is much more expensive than overestimation, its Linex loss function (LXLF) has been recommended, which gives as follows

$$\widehat{\varrho}_{BLXLF} \propto \exp[\delta\nabla] - \delta\nabla - 1, \nabla = (\widehat{\varrho} - \varrho), \delta \neq 0, \tag{15}$$

The sign of the delta indicates the perspective of asymmetry, and the level of asymmetry is represented by its size. Under the identical distinction $\widehat{\varrho} - \varrho$, the greater the magnitude of δ, the greater the expense. LXLF is nearly symmetric and vearer to SE when $|\delta|$ is narrow [20].

The ϱ Bayesian estimator under LXlF is

$$\widehat{\varrho}_{BLXFL} = -\frac{1}{\delta} lnE_{\varrho}\left(e^{-\delta\varrho}|\underset{-}{y}\right) \tag{16}$$

Provided that $E_{\varrho}\left(e^{-\delta\varrho}|\underset{-}{y}\right)$ is specified and exists.

3.3. Modified Linex loss function

The planned layout of WLXF based on the weighting component LXLF and WLXF is defined as

$$L_{WLXF}(\widehat{\varrho} - \varrho) = W(\varrho)[\exp[\delta\nabla] - \delta\nabla - 1]; \delta \neq 0. \tag{17}$$

Here W (ϱ) represents proposed weight and given the following formula

$$W(\varrho) = \exp[-c\varrho] \quad ; c \neq 0. \tag{18}$$

According to the proposed loss function in equation (17), calculate Bayes estimators for the parameter ϱ by using the risk function $R = (\widehat{\varrho} - \varrho)$ given by the following equation

$$R(\widehat{\varrho} - \varrho) = E(L_{WLXF}(\widehat{\varrho} - \varrho))$$

$$= \int_{\forall\varrho} W(\varrho)[\exp[\delta(\widehat{\varrho} - \varrho)] - \delta(\widehat{\varrho} - \varrho) - 1]g\left(\varrho|\underset{-}{y}\right)d\varrho$$

$$= \int_{\forall\varrho} \exp[-c\varrho]\Big[\exp[\delta(\widehat{\varrho} - \varrho)]g\left(\varrho|\underset{-}{y}\right)d\varrho - \exp[-c\varrho]\delta(\widehat{\varrho} - \varrho)$$
$$- \exp[-c\varrho]g\left(\varrho|\underset{-}{y}\right)d\varrho$$

$$= \begin{cases} \exp[c\widehat{\varrho}]\int_{\forall\varrho} \exp[-\varrho(\delta+c)]g\left(\varrho|\underset{-}{y}\right)d\varrho \\ -\delta\widehat{\varrho}\int_{\forall\varrho} \exp[-c\varrho]g\left(\varrho|\underset{-}{y}\right)d\varrho \\ +\delta\int_{\forall\varrho} \exp[-c\varrho]g\left(\varrho|\underset{-}{y}\right)d\varrho \\ -\int_{\forall\varrho} \exp[-c\varrho]g(\varrho|\exp[-\varrho(\delta+c)])d\varrho \end{cases}$$

$$= \exp[c\widehat{\varrho}]E_{\varrho}\left(\exp[-\varrho(\delta+c)]|\underset{-}{y}\right) - \delta\widehat{\varrho}E_{\varrho}\left(\exp[-c\varrho]|\underset{-}{y}\right)$$
$$+ \delta E_{\varrho}\left(\varrho\exp[-c\varrho]|\underset{-}{y}\right) - E_{\varrho}\left(\exp[-c\varrho]|\underset{-}{y}\right)$$

$$\frac{\partial R(\widehat{\varrho} - \varrho)}{\partial\widehat{\varrho}} = \delta\exp[\delta\widehat{\varrho}]E_{\varrho}\left(\exp[-\varrho(\delta+c)]|\underset{-}{y}\right)$$
$$- \delta\left(\varrho\exp[-c\varrho]|\underset{-}{y}\right)$$

Setting.$\frac{\partial R\left(\widehat{\varrho}-\varrho\right)}{\partial\widehat{\varrho}} = 0$

The Bayesian estimator for the parameter ϱ Based on the WLXLF is given by

$$\widehat{\varrho}_{WLXLF} = \frac{1}{\delta}\log\left[\frac{E_{\varrho}\left(\exp[-c\varrho]|\underset{-}{y}\right)}{E_{\varrho}\left(\exp[-\varrho(\delta+c)]|\underset{-}{y}\right)}\right]; \delta \neq 0. \tag{19}$$

Provided that $\left(\exp[-c\varrho]|\underset{-}{y}\right)$ and $E_{\varrho}\left(\exp[-\varrho(\delta+c)]|\underset{-}{y}\right)$ are specified and exists.

4. Bayes estimation

In this part of the article, we will derive a Bayes estimate for the parameters and $R(t)$ of IWD in 2 cases, the first case when σ is identified and ρ unknown, and the next case after both σ and ρ are unknown. They believe gamma (Θ, φ) be a conjugate prior distribution for ρ as follows

$$g(\rho) = \frac{\Theta^{\varphi}}{\Gamma(\varphi)}\rho^{\varphi-1}\exp[-\Theta\rho], \rho > 0, \Theta, \varphi > 0, \tag{20}$$

Via Bayes theorem, by merging the $L\left(\rho,\sigma)|\underset{-}{y}\right)$ in (7) with the $g(\rho)$ in (20), $\pi\left(\rho|\underset{-}{y}\right)$ of ρ can be obtained as

$$\pi\left(\rho|\underset{-}{y}\right) = \frac{L\left(\rho,\sigma)|\underset{-}{y}\right)g(\rho)}{\int_0^{\infty} L\left(\rho,\sigma)|\underset{-}{y}\right)g(\rho)}\rho^{\varphi-1}\exp[-\Theta\rho], \rho > 0, \Theta, \varphi > 0,$$

$$= \frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp[-\rho\gamma], \rho > 0. \tag{21}$$

where $\gamma = \Theta + y_{L(k)}^{-\sigma}$

4.1. Bayes estimator of One-Parameter case

In this subsection of the article, we will derive a Bayes estimate for a scaling factor ρ, and reliability function $R(t)$ for IWD, using three diverse loss functions which include, SLF, LXLF, and the proposed loss function WLXLF.

4.1.1. Estimates under SLF

Through equation (21) we can find a Bayesian estimation of ρ and R (t) depending on SLF, respectively, as follows

$$\widehat{\rho}_{BSLF} = E\left(\rho|\underset{-}{y}\right) = \frac{\varphi + k}{\gamma}. \tag{22}$$

and

$$\widehat{R}(t)_{BSLF} = E\left(1 - \exp\left[-\rho t^{-\sigma}\right]|\underset{-}{y}\right) = 1 - \frac{\gamma + k}{\gamma + t^{-\sigma}}, \quad t > 0. \tag{23}$$

4.1.2. Estimates under LXLF

Through equation (21) we can find a Bayesian estimation of ρ and R (t) depending on LXLF, respectively, as follows

$$\widehat{\rho}_{BLXLF} = -\frac{1}{\delta}\log\left[E\left[\exp\left[\left(-\delta\rho|\underset{-}{y}\right)\right]\right]\right]$$

$$= \frac{k+\varphi}{\delta}\log\left(1 + \frac{\delta}{\gamma}\right) \tag{23}$$

$$\widehat{R}(t)_{BLXLF} = -\frac{1}{\delta}\log\left[E\left[\exp\left[\left(-\delta\left(1 - exp[-\rho\, t^{-\sigma}]|\underset{-}{y}\right)\right]\right]\right]\right],$$

$$= \frac{-1}{\delta}\log\left[\exp[-\delta] + \exp[-\delta]\,\gamma^{k+\varphi}\right]\sum_{i=1}^{\infty}\frac{\delta^i}{i!}\left(\gamma + it^{-\sigma}\right)^{-k-\varphi.} \tag{24}$$

4.1.3. Estimates under WLXLF

Through equation (21) we can find a Bayesian estimation of ρ and R (t) depending on SLF, respectively, as follows

$$\widehat{\rho}_{WBLXLF} = \frac{1}{\delta}\log\left[\frac{E_\rho\left(\exp[-c\rho]|\underset{-}{y}\right)}{E_\rho\left(\exp[-\rho(\delta + c)]|\underset{-}{y}\right)}\right] = \frac{1}{\delta}\ln\left(\frac{\mathbb{T}_1}{\mathbb{T}_2}\right). \tag{24}$$

where

$$\mathbb{T}_1 = \begin{cases} E_\rho\left(\exp[-c\rho]|\underset{-}{y}\right) \\ \int_0^\infty \left(\exp[-c\rho]|\underset{-}{y}\right)\pi\left(\rho|\underset{-}{y}\right)d\rho \\ \int_0^\infty \exp[-c\rho]\frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp[-\rho\gamma]\,d\rho \\ \int_0^\infty \frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp\left[-\rho(\gamma+c]\,d\rho\right. \left[\frac{\gamma}{\gamma+c}\right]^{k+\varphi}, \end{cases}$$

and

$$\mathbb{T}_2 = \begin{cases} E_\rho\left(\exp[-\rho(\delta+c)]|\underset{-}{y}\right) \\ \int_0^\infty \exp[-\rho(\delta+c)]\pi\left(\rho|\underset{-}{y}\right)d\rho \\ \int_0^\infty \exp[-\rho(\delta+c)]\frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp[-\rho\gamma]\,d\rho \\ \int_0^\infty \frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp\left[-\rho(\delta+\gamma+c]\,d\rho\right. \quad \left[\frac{\gamma}{\gamma+c+\delta}\right]^{k+\varphi}, \end{cases}$$

Therefore the $\widehat{\rho}_{WBLXLF}$ is

$$\widehat{\rho}_{WBLXLF} = \frac{1}{\delta}\log\frac{\mathbb{T}_1}{\mathbb{T}_2}$$

$$= \frac{k+\varphi}{\delta}\log\left[1 + \frac{\delta}{\gamma + c}\right]. \tag{25}$$

As for the reliability operating inside the proposed loss meaning, it can be obtained as follows

$$\widehat{R}(t)_{BLXLF} = \frac{1}{\delta}\log\left[\frac{E_\rho\left(\exp[-cR(t)]|\underset{-}{y}\right)}{E_\rho\left(\exp[-R(t)(\delta+c)]|\underset{-}{y}\right)}\right] = \frac{1}{\delta}\log\left[\frac{\mathbb{T}_3}{\mathbb{T}_4}\right]. \tag{26}$$

where

$$\mathbb{T}_3 = \begin{cases} E\left[\exp\left[\left(-c\left(1 - exp[-\rho\, t^{-\sigma}]|\underset{-}{y}\right)\right]\right]\right. \\ \int_0^\infty \exp\left[\left(-c\left(1 - exp[-\rho\, t^{-\sigma}]|\underset{-}{y}\right)\right]\pi\left(\rho|\underset{-}{y}\right)d\rho\right. \\ \int_0^\infty \exp[-c(1 - exp[-\rho\, t^{-\sigma}]] \\ \frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\exp[-\rho\gamma]\,d\rho \\ \exp[-c] + \exp[-c]\,\gamma^{k+\varphi}\sum_{i=1}^{\infty}\frac{c^i}{i!}\left(\gamma + it^{-\sigma}\right)^{-k-\varphi}, \end{cases}$$

and

$$\mathbb{T}_4 = \begin{cases} E\left[\exp\left[\left(-(c+\delta)\left(1 - exp[-\rho\, t^{-\sigma}]|\underset{-}{y}\right)\right]\right]\right. \\ \int_0^\infty \exp\left[\left(-c\left(1 - exp[-\rho\, t^{-\sigma}]|\underset{-}{y}\right)\right]\pi\left(\rho|\underset{-}{y}\right)d\rho\right. \\ \int_0^\infty \exp\left[-(c+\delta)(1 - exp[-\rho\, t^{-\sigma}]]\frac{1}{\Gamma(k+\varphi)}\gamma^{k+\varphi}\rho^{k+\varphi-1}\right. \\ \exp[-\rho\gamma]\,d\rho \\ \exp[-(c+\delta)] + \exp[-(c+\delta)] \\ \gamma^{k+\varphi}\sum_{i=1}^{\infty}\frac{(c+\delta)^i}{i!}\left(\gamma + it^{-\sigma}\right)^{-k-\varphi}, \end{cases}.$$

Therefore the $\widehat{R}(t)_{WBLXLF}$ is

$$\widehat{R}(t)_{WBLXLF} = \frac{1}{\delta}\log\left[\frac{\mathbb{T}_3}{\mathbb{T}_4}\right]$$

$$= \frac{1}{\delta}\log\left[\frac{\exp[-c] + \exp[-c]\,\gamma^{k+\varphi}\sum_{i=1}^{\infty}\frac{c^i}{i!}\left(\gamma + it^{-\sigma}\right)^{-k-\varphi}}{\exp[-(c+\delta)] + \exp[-(c+\delta)]\,\gamma^{k+\varphi}\sum_{i=1}^{\infty}\frac{(c+\delta)^i}{i!}\left(\gamma + it^{-\sigma}\right)^{-k-\varphi}}\right]. \tag{27}$$

4.2. Bayes estimator of two-Parameters case

In this part of the article, we will discuss Bayes estimations when both parameters are unknown. It was done using the family of previous joint dispersion. Thus, discrete ranges were assigned the shape variable and continuous ones the scale variable.

Postulate that shape limitation σ is limited a bounded various value $\sigma_1, \sigma_2 \cdots \sigma_m$ using historical probabilities $\tau_1, \tau_2 \cdots \tau_m$ such $0 \le \tau_j \le 1, \sum_{j=1}\tau_j = 1$ and $p(\sigma = \sigma_j) = \tau_j$. Moreover, suppose that conditional on to $\sigma = \sigma_j, j = 1, 2, \cdots \rho$ has a natural gamma distribution (φ_j, Θ_j) prior, with the density function [12]

$$g(\rho|\sigma=\sigma_j)=\frac{\Theta_j^{\varphi_j}}{\Gamma(\varphi_j)}\rho^{\varphi_j-1}\exp\left[-\Theta_j\rho\right],\ \ \rho>0,\ \ \Theta_j,\varphi_j>0\,, \tag{28}$$

Combining the $L\left(\rho,\sigma)|\underset{-}{y}\right)$ in equation (7) and the $g(\rho|\sigma=\sigma_j)$ in equation (28), The unconditional posterior is obtained of $(\rho|\sigma=\sigma_j)$ following is

$$g^*(\rho|\sigma=\sigma_j)=\frac{L\left(\rho,\sigma)|\underset{-}{y}\right)g(\rho|\sigma=\sigma_j)}{\int_0^\infty L\left(\rho,\sigma)|\underset{-}{y}\right)g(\rho|\sigma=\sigma_j)\,d\rho}\,,$$

$$=\frac{1}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j-1}\exp\left[-\rho\gamma_j\right], \tag{29}$$

where

$\gamma_j=\Theta_j+y_{L(k)}^{-\sigma_j}$

On application separate style Bayesian theorem, marginal probability circulation of σ It may be written by way of follows

$$\mathbb{H}\left(\sigma_j|\underset{-}{y}\right)=g_j=\begin{cases}p\left(\sigma=\sigma_j|\underset{-}{y}\right),\\ \mathbb{Q}\int_\rho g(\rho|\sigma=\sigma_j)\,p(\sigma)L\left(\rho,\sigma)|\underset{-}{y}\right)d\rho,\\ \frac{\Theta_j^{\varphi_j}\sigma_j^k\,\mathbb{U}_j\tau_j\Gamma(\varphi_j+k)}{\mathbb{Q}\Gamma(\varphi_j)\gamma_j^{k+\varphi_j}}\end{cases} \tag{30}$$

where

$$\mathbb{Q}=\sum_{j=1}^m\frac{\Theta_j^{\varphi_j}\sigma_j^k\,\mathbb{U}_j\tau_j\Gamma(\varphi_j+k)}{\Gamma(\varphi_j)\gamma_j^{k+\varphi_j}}\,,$$

$$\mathbb{U}_j=\prod_{i=1}^k y_{L(i)}^{-\sigma_j-1}.$$

4.2.1. Estimates by using SLF

On applying SLF in equation (13) the Bayesian estimation of ρ, σ and R (t), respectively, as

$$\widehat{\rho}_{BSLF}=\begin{cases}\int_\rho\sum_{j=1}^m g_j\,\rho g^*(\rho|\sigma=\sigma_j)d\rho,\\ \int_\rho\sum_{j=1}^m\frac{g_j}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j}\exp\left[-\rho\gamma_j\right]d\rho\\ \sum_{j=1}^m\frac{g_j\,\rho(k+\varphi_j)}{\gamma_j}.\end{cases} \tag{30}$$

$$\widehat{\sigma}_{BSLF}=\sum_{j=1}^m g_j\sigma_{j.} \tag{31}$$

and

$$\widehat{R}(t)_{BSLF}=\begin{cases}\int_\rho\sum_{j=1}^m g_j\,R(t)g^*(\rho|\sigma=\sigma_j)d\rho,\\ \int_\rho\sum_{j=1}^m g_j(1-\exp\left[-\rho t^{-\sigma_j}\right])g^*(\rho|\sigma=\sigma_j)d\rho\\ 1-\sum_{j=1}^m g_j\left(\frac{\gamma_j}{\gamma_j+t^{-\sigma_j}}\right)^{k+\varphi_j}.\end{cases} \tag{32}$$

4.2.2. Estimates by using LXLF

On applying LXLF in equation (16) the Bayesian estimation of ρ, σ and R (t), respectively, as

$$\widehat{\rho}_{BLXLF}=\begin{cases}-\frac{1}{\delta}\log\left[\int_\rho\sum_{j=1}^m g_j\exp\left[-\delta\rho\right]g^*(\rho|\sigma=\sigma_j)d\rho\right],\\ -\frac{1}{\delta}\log\left[\sum_{j=1}^m g_j\int_0^\infty\frac{1}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j-1}\right.\\ \left.\exp\left[-\rho(\gamma_j+\delta)\right]d\rho\right]-\frac{1}{\delta}\log\left[\sum_{j=1}^m\frac{g_j\,\gamma_j^{\varphi_j+k}}{(\delta+\gamma_j)^{\varphi_j+k}}\right],\end{cases} \tag{33}$$

$$\widehat{\sigma}_{BLXLF}=\begin{cases}-\frac{1}{\delta}\log\left[\int_\rho\sum_{j=1}^m g_j\exp\left[-\delta\sigma_j\right]g^*(\rho|\sigma=\sigma_j)d\rho\right]\\ -\frac{1}{\delta}\log\left[\sum_{j=1}^m g_j\int_0^\infty\exp\left[-\delta\sigma_j\right]\right.\\ \left.\frac{1}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j-1}\exp\left[-\rho\gamma_j\right]d\rho\right]\\ -\frac{1}{\delta}\log\left[\sum_{i=1}^m g_j\exp\left[-\delta\sigma_j\right]\right],\end{cases} \tag{34}$$

and

$$\widehat{R}(t)_{BLXLF}=\begin{cases}-\frac{1}{\delta}\log\left[\int_\rho\sum_{j=1}^m g_j\exp\left[-\delta R(t)\right]g^*(\rho|\sigma=\sigma_j)d\rho\right],\\ -\frac{1}{\delta}\log\left[\sum_{j=1}^m g_j\int_0^\infty\frac{1}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j-1}\right.\\ \left.\exp\left[-\delta(R(t)+\gamma_j)\right]d\rho\right]\\ -\frac{1}{\delta}\log\left[\sum_{i=1}^m g_j(\exp\left[-\delta\right]+\exp\left[-\delta\right]\right.\\ \left.\left.\exp\left[-\delta\gamma_j^{k+\varphi_j}\right]\sum_{i=1}^\infty\frac{\delta^i}{i!\left[\gamma_j+it^{-\sigma_j}\right]^{k+\varphi_j}}\right)\right].\end{cases} \tag{35}$$

4.2.3. Estimates by using WLXLF

On applying WLXLF in equation (19) the Bayesian estimation of ρ, σ and R (t), respectively, as

$$\widehat{\rho}_{WBLXLF}=\frac{1}{\delta}\log\left[\frac{E_\rho\left(\exp\left[-c\rho\right]|\underset{-}{y}\right)}{E_\rho\left(\exp\left[-\rho(\delta+c)\right]|\underset{-}{y}\right)}\right]$$

$$=\frac{1}{\delta}\ln\left(\frac{\mathbb{T}_5}{\mathbb{T}_6}\right) \tag{36}$$

where

$$\mathbb{T}_5=\begin{cases}E_\rho\left(\exp\left[-c\rho\right]|\underset{-}{y}\right)\\ \int_\rho\sum_{j=1}^m g_j\exp\left[-c\rho\right]g^*(\rho|\sigma=\sigma_j)d\rho\\ \sum_{j=1}^m g_j\int_0^\infty\frac{1}{\Gamma(k+\varphi_j)}\gamma_j^{k+\varphi_j}\rho^{k+\varphi_j-1}\exp\left[-\rho(c+\gamma_j)\right]d\rho\\ \sum_{j=1}^m\frac{g_j\,\gamma_j^{\varphi_j+k}}{(c+\gamma_j)^{\varphi_j+k}},\end{cases}$$

and

$$\mathbb{T}_6 = \begin{cases} E_\rho\left(\exp\left[-(c+\delta)\rho\right] | \underset{-}{y}\right) \\ \int_\rho \sum_{j=1}^m g_j \exp\left[-(c+\delta)\rho\right] g^*\left(\rho|\sigma=\sigma_j\right) d\rho \\ \sum_{j=1}^m g_j \int_0^\infty \frac{1}{\Gamma\left(k+\varphi_j\right)} \gamma_j^{k+\varphi_j} \rho^{k+\varphi_j-1} \exp\left[-\rho\left((c+\delta)+\gamma_j\right)\right] d\rho \\ \sum_{j=1}^m \frac{g_j\, \gamma_j^{\varphi_j+k}}{\left(c+\delta+\gamma_j\right)^{\varphi_j+k}}, \end{cases}$$

Therefore the $\widehat{\rho}_{WBLXLF}$ is

$$\widehat{\rho}_{WBLXLF} = \frac{1}{\delta} \log \frac{\mathbb{T}_5}{\mathbb{T}_6}$$

$$= \frac{1}{\delta} \log \left[\frac{\sum_{j=1}^m \frac{g_j\, \gamma_j^{\varphi_j+k}}{\left(c+\gamma_j\right)^{\varphi_j+k}}}{\sum_{j=1}^m \frac{g_j\, \gamma_j^{\varphi_j+k}}{\left(c+\delta+\gamma_j\right)^{\varphi_j+k}}} \right]. \tag{37}$$

$$\widehat{\sigma}_{WBLXLF} = \frac{1}{\delta} \log \left[\frac{E_\rho\left(\exp\left[-c\sigma_j\right] | \underset{-}{y}\right)}{E_\rho\left(\exp\left[-\sigma_j(\delta+c)\right] | \underset{-}{y}\right)} \right]$$

$$= \frac{1}{\delta} \ln \left(\frac{\mathbb{T}_7}{\mathbb{T}_8} \right). \tag{38}$$

where

$$\mathbb{T}_7 = \begin{cases} \int_\rho \sum_{j=1}^m g_j \exp\left[-c\sigma_j\right] g^*\left(\rho|\sigma=\sigma_j\right) d\rho \\ \sum_{j=1}^m g_j \int_0^\infty \exp\left[-c\sigma_j\right] \frac{1}{\Gamma\left(k+\varphi_j\right)} \gamma_j^{k+\varphi_j} \rho^{k+\varphi_j-1} \exp\left[-\rho\gamma_j\right] d\rho \\ \sum_{i=1}^m g_j \exp\left[-c\sigma_j\right], \end{cases}$$

$$\mathbb{T}_8 = \begin{cases} \int_\rho \sum_{j=1}^m g_j \exp\left[-\left(c+\delta\right)\sigma_j\right] g^*\left(\rho|\sigma=\sigma_j\right) d\rho \ \sum_{j=1}^m g_j \\ \int_0^\infty \exp\left[-(c+\delta)\sigma_j\right] \frac{1}{\Gamma\left(k+\varphi_j\right)} \gamma_j^{k+\varphi_j} \rho^{k+\varphi_j-1} \exp\left[-\rho\gamma_j\right] d\rho \\ \sum_{i=1}^m g_j \exp\left[-(c+\delta)\,\sigma_j\right], \end{cases}$$

Therefore the $\widehat{\sigma}_{WBLXLF}$ is

$$\widehat{\sigma}_{WBLXLF} = \frac{1}{\delta} \log \frac{\mathbb{T}_7}{\mathbb{T}_8}$$

$$= \frac{1}{\delta} \log \left[\frac{\sum_{i=1}^m g_j \exp\left[-c\sigma_j\right]}{\sum_{i=1}^m g_j \exp\left[-(c+\delta)\,\sigma_j\right]} \right]. \tag{39}$$

and

$$\widehat{R}(t)_{WBLXLF} = \frac{1}{\delta} \log \left[\frac{E_\rho\left(\exp\left[-cR(t)\right] | \underset{-}{y}\right)}{E_\rho\left(\exp\left[-R(t)(\delta+c)\right] | \underset{-}{y}\right)} \right],$$

$$= \frac{1}{\delta} \ln \left(\frac{\mathbb{T}_9}{\mathbb{T}_{10}} \right) \tag{40}$$

$$\mathbb{T}_9 = \begin{cases} \int_\rho \sum_{j=1}^m g_j \exp\left[-cR(t)\right] g^*\left(\rho|\sigma=\sigma_j\right) d\rho \\ \sum_{j=1}^m g_j \int_0^\infty \frac{1}{\Gamma\left(k+\varphi_j\right)} \gamma_j^{k+\varphi_j} \rho^{k+\varphi_j-1} \exp\left[-c\left(R(t)+\gamma_j\right)\right] d\rho \\ \sum_{i=1}^m g_j \left(\exp\left[-c\right] + \exp\left[-c\right] \exp\left[-c\gamma_j^{k+\varphi_j}\right] \sum_{i=1}^\infty \frac{c^i}{i!\left[\gamma_j + it^{-\sigma_j}\right]^{k+\varphi_j}} \right) \end{cases}$$

and

$$\mathbb{T}_{10} = \begin{cases} \int_\rho \sum_{j=1}^m g_j \exp\left[(-c+\delta)R(t)\right] g^*\left(\rho|\sigma=\sigma_j\right) d\rho, \\ \sum_{j=1}^m g_j \int_0^\infty \frac{1}{\Gamma\left(k+\varphi_j\right)} \gamma_j^{k+\varphi_j} \rho^{k+\varphi_j-1} \\ \exp\left[-(-c+\delta)\left(R(t)+\gamma_j\right)\right] d\rho \\ \sum_{i=1}^m g_j (\exp\left[-(-c+\delta)\right] + \exp\left[-(-c+\delta)\right] \\ \exp\left[-(c+\delta)\gamma_j^{k+\varphi_j}\right] \sum_{i=1}^\infty \frac{(-c+\delta)^i}{i!\left[\gamma_j + it^{-\sigma_j}\right]^{k+\varphi_j}} \Big) \end{cases}$$

Therefore the $\widehat{R}(t)_{WBLXLF}$ is

$$\widehat{R}(t)_{WBLXLF} = \frac{1}{\delta} \ln \left(\frac{\mathbb{T}_9}{\mathbb{T}_{10}} \right).$$

$$= \frac{1}{\delta} \log \left[\frac{\sum_{i=1}^m g_j \left(\exp\left[-c\right] + \exp\left[-c\right] \exp\left[-c\gamma_j^{k+\varphi_j}\right] \sum_{i=1}^\infty \frac{c^i}{i!\left[\gamma_j + it^{-\sigma_j}\right]^{k+\varphi_j}} \right)}{\sum_{i=1}^m g_j \left(\exp\left[-(c+\delta)\right] + \exp\left[-(-c+\delta)\right] \exp\left[-(-c+\delta)\gamma_j^{k+\varphi_j}\right] \sum_{i=1}^\infty \frac{(-c+\delta)^i}{i!\left[\gamma_j + it^{-\sigma_j}\right]^{k+\varphi_j}} \right),}. \right] \tag{41}$$

5. Bootstrap technique

It is one of the statistical sampling techniques related to simulation methods by [21] This method aims to obtain unbiased or less biased estimates from among a set of biased estimators, by replacing several randomly drawn samples with replacement from the observed data. These drawn samples are called Bootstrap The percentage bootstrap algorithm is described in a particular way the percentile bootstrap as a method:

Algorithm A: Algorithm of pareto bootstrapping

Stage 1: Based on the original data set $Y_1, Y_2, Y_3, \cdots Y_k$ draw U independent bootstrap samples $Y_1^*, Y_2^*, Y_3^*, \cdots Y_U^*$

with replacement, each of size k
Stage 2: Collect the series of bootstrap MLE
$\rho_i^* and \sigma_i^* i = 1, 2, 3, \cdots U.$
Stage 3: Compute the mean of all values in $\widehat{\rho}^*$ and $\widehat{\sigma}^*$
Stage 4: Arrange the value of $\underset{-}{\rho^*}$ and $\underset{-}{\sigma^*}$ in ascending order corresponding based on the $\left(\widehat{\rho}_{(1)}^*, \widehat{\rho}_{(2)}^*, \cdots, \widehat{\rho}_{(U)}^*\right)$ and $\left(\widehat{\sigma}_{(1)}^*, \sigma_{(2)}^*, \cdots, \widehat{\sigma}_{(U)}^*\right)$
Stage 5: The estimated 100(1-ζ)% boot-ρ CIs for ρ and σ is respectively defined, by $\widehat{\rho}^{*[\alpha U/2]}, \cdots \widehat{\rho}^{*[(1-\alpha/2)U/2]}$ and $\widehat{\sigma}^{*[\alpha U/2]}, \cdots \widehat{\sigma}^{*[(1-\alpha/2)U/2]}$

6. The simulation

In this part of the article, the simulation will be used to evaluate the recital of the planned method by comparing it with the MLE estimator and the Bayes estimator using two loss functions, including SLF and LXLF, calculating the scale variable (ρ) and $R(t)$ of IW distribution when shape parameter is known.

The simulation was carried out according to the following steps:

1. To observe the impact of IWD factors on projections, three distinct values of the scale parameter were chosen $\rho = 0.5$. and 0.75 with known shape parameter $\sigma = 1.5$ and 2.5
2. The parameter values of LXLF's (δ) were selected to be (δ = -1.5 and 1.5) which represents under and upper estimates, respectively, the values of WLXLF constant (c) were selected to be (c = 2), and The values of gamma prior distribution(φ, Θ) were selected to be $(\varphi = 1$ and $\Theta = 2)$.
3. Select the sample size $(k = 3, 4, 5, 6$ and $7)$, produced information with reduced record values IW distribution given by (1).
4. Calculate guesses of ρ and $R(t)$ with $t = 4$ using the estimations under the study.
5. Repetition of steps 1 through 3 5,000 times. Then determined the MSE for every estimate. (say ϱ) using

$$MSE = \frac{1}{5000}\sum_{i=1}^{5000}(\widehat{\varrho} - \varrho)^2.$$

where ϱ can be ρ or $R(t)$ and $\widehat{\varrho}$ is the estimate at the i$^{\text{th}}$ run.

Table 1 shows the effectiveness of the suggested approach (WLXLF) for approximating scale restriction is the best estimator compared to MLE, BSLF, and BLXF for all cases of (ρ, σ) and most sample sizes K. The results also show that the MSE's increase with the increase in the value of the scale parameter.

Table 2 demonstrates that the effectiveness of the proposed method (WLXLF) for estimating the reliability function is the best estimator compared to MLE, BSLF, and BLXF for all cases of (ρ, σ) and sample sizes K. Results also indicate that MSE's increases with increase in value of the scale parameter.

7. Application example

For the exam, the loss function developed in the section, parameters and reliability function of the IWD will be estimated on an actual data set provided by Nelson [22], Which represents At a voltage of 34 KV, the number of seconds it takes for an insulating fluid between poles to break down. These 19 breakdown times are

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Then the real data set form IWD

1.042	0.241	5.263	1.282	0.125	0.031	0.136	0.154	0.121	0.029
0.031	0.316	0.206	0.36	0.214	0.763	0.083	0.027	0.014	

It is possible to see the following seven lower record values:

1.042	0.241	0.125	0.031	0.029	0.027	0.014

We can get the values of (φ_j, Θ_j) by calculating Expected values of $R(t)$

Table 1 MSEs of the scale variable estimations.

(ρ, σ)	k	$\widehat{\rho}_{ML}$	$\widehat{\rho}_{\mathrm{BSLF}}$	$\widehat{\rho}_{\mathrm{BLXLF}}$		$\widehat{\rho}_{\mathrm{WBLXLF}}$	
				$\delta = -1.5$	$\delta = 1.5$	$\delta = -1.5$	$\delta = 1.5$
(0.5,1.5)	3	0.644077	0.079441	0.183369	0.041309	0.023155	0.027693
	4	0.246205	0.067226	0.127974	0.042256	0.022299	0.023298
	5	0.135980	0.054890	0.091457	0.037120	0.021511	0.020788
	6	0.096778	0.047046	0.072168	0.033889	0.020563	0.019060
	7	0.067739	0.039668	0.057238	0.029220	0.018286	0.016950
(0.5, 2.5)	3	0.650930	0.074958	0.172669	0.044549	0.022841	0.026795
	4	0.231211	0.064596	0.121794	0.038779	0.022142	0.023235
	5	0.158145	0.060382	0.101727	0.038276	0.022417	0.020835
	6	0.092151	0.045610	0.069387	0.034066	0.020601	0.018798
	7	0.067107	0.039186	0.055911	0.030763	0.019274	0.017442
(0.75,1.5)	3	1.177369	0.079817	0.217278	0.053688	0.071655	0.109612
	4	0.524547	0.080406	0.179517	0.051886	0.058191	0.086256
	5	0.312928	0.074221	0.144215	0.052020	0.049052	0.072635
	6	0.219060	0.069750	0.121033	0.048442	0.045197	0.060554
	7	0.152843	0.060880	0.097432	0.045502	0.039982	0.053043
(0.75, 2.5)	3	1.334376	0.080508	0.221773	0.054333	0.070497	0.110166
	4	0.523175	0.078569	0.174514	0.052708	0.057876	0.086834
	5	0.294445	0.071038	0.135206	0.051863	0.050390	0.071714
	6	0.236184	0.072345	0.127120	0.048953	0.044956	0.061372
	7	0.163933	0.063220	0.101779	0.047342	0.040835	0.051611

Table 2 MSEs of estimates of Reliability function.

(ρ, σ)	k	$\widehat{R}(t)_{ML}$	$\widehat{R}(t)_{\mathrm{BSLF}}$	$\widehat{R}(t)_{\mathrm{BLXLF}}$		$\widehat{R}(t)_{\mathrm{WBLXLF}}$	
				$\delta = -1.5$	$\delta = 1.5$	$\delta = -1.5$	$\delta = 1.5$
(0.5,1.5)	3	0.005453	0.000975	0.001044	0.000842	0.000799	0.000655
	4	0.002703	0.000834	0.000882	0.000801	0.000710	0.000651
	5	0.001623	0.000690	0.000722	0.000663	0.000604	0.000559
	6	0.001182	0.000596	0.000619	0.000581	0.000532	0.000502
	7	0.000864	0.000507	0.000526	0.000487	0.000458	0.000428
(0.5, 2.5)	3	0.000503	0.000069	0.000070	0.000072	0.000065	0.000067
	4	0.000205	0.000060	0.000061	0.000056	0.000057	0.000053
	5	0.000143	0.000056	0.000057	0.000052	0.000054	0.000049
	6	0.000084	0.000042	0.000043	0.000044	0.000041	0.000042
	7	0.000062	0.000036	0.000037	0.000038	0.000035	0.000037
(0.75,1.5)	3	0.009103	0.000958	0.001017	0.000907	0.000827	0.000763
	4	0.005138	0.000959	0.001011	0.000871	0.000834	0.000740
	5	0.003311	0.000886	0.000929	0.000849	0.000780	0.000735
	6	0.002428	0.000837	0.000872	0.000778	0.000750	0.000681
	7	0.001766	0.000737	0.000764	0.000712	0.000667	0.000632
(0.75, 2.5)	3	0.001017	0.000073	0.000075	0.000072	0.000069	0.000067
	4	0.000447	0.000072	0.000073	0.000070	0.000068	0.000065
	5	0.000259	0.000065	0.000066	0.000067	0.000062	0.000063
	6	0.000210	0.000066	0.000067	0.000061	0.000063	0.000058
	7	0.000148	0.000058	0.000058	0.000059	0.000056	0.000056

$$E\left[\left(R(t)|\sigma = \sigma_j\right)\right], = \int_{\rho} \left(1 - \exp\left[-\rho t^{-\sigma_j}\right]\right) \frac{\Theta_j^{\varphi_j}}{\Gamma\left(\varphi_j\right)} \rho^{\varphi_j - 1} \times \exp\left[-\Theta_j \rho\right] d\rho = 1 - \left[1 + \frac{t^{-\sigma_j}}{\Theta_j}\right]^{-\varphi_j}, \ t > 0 \tag{42}$$

Assuming Prior knowledge of the distribution allows one to recognize two values $(R(t_1, t_1))$ and $(R(t_1, t_2))$, so values of $\left(\varphi_j, \Theta_j\right)$ may be got If there is no previous knowledge, a stochastic strategy may be utilized to calculate the two values of R(t) directly from (42) following formal

$$R\left(t_i = Y_i = \frac{k - i + 0.625}{k + 0.25}\right. \tag{43}$$

By applying the nonparametric method to the $R(t)$, we selected $t_1 = 0.027$ and $t_2 = 0.241$ in equation (43), we get $R(t_1) = 0.78)$ and $R(t_2) = 0.36)$, we assume that σ_j takes the values: (0.5 (0.07) 1.13) with 0.1 probability for each, equally likely. And with hyper-parameter value $\left(\varphi_j, \Theta_j\right)$ at an apiece value of σ_j are derived by applying Newton-Raphson to the following solutions [23].

$$1 - \left[1 + \frac{0.027^{-\sigma_j}}{\Theta_j}\right]^{-\varphi_j} = 0.78,$$

$$1 - \left[1 + \frac{0.241^{-\sigma_j}}{\Theta_j}\right]^{-\varphi_j} = 0.36.$$

Table 3 displayed values of hyper-parameters and posterior probabilities calculated for every value in σ_j. While Table 4 demonstrates the calculated estimators of ρ, σ, and $R(t)$ using (.)MLE, (.) Bootstrap estimator depending on the algorithm A, and the Bayes estimators ((.)BSLF, (.)BLXLF, (.) WBLXLF). based on algorithmic analysis of real data with lower record values. The range of trust levels for ρ, σ, and $R(t)$ were presented in Table 5. Noted that the MLE estimator for the complete real data, was found using Newton-Raphson

Table 3 Prior information, Hyperparameter values and the posterior probabilities.

j	τ_j	σ_j	φ_j	Θ_j	$\mathbb{U}_j$	g_j
1	0.1	0.50	432480	$1.68819*10^{6}$	9682.26	0.844412
2	0.1	0.57	20.6385	102.9460	2678.8	0.117361
3	0.1	0.64	3.13273	16.2374	741.145	0.0345205
4	0.1	0.71	1.75420	9.48002	205.053	0.00345098
5	0.1	0.78	1.24185	7.01631	56.7322	0.000240537
6	0.1	0.85	0.97292	5.75890	15.6961	0.0000141621
7	0.1	0.92	0.80637	5.00908	4.34266	$7.59127*10^{-7}$
8	0.1	0.99	0.692582	4.52112	1.20149	$3.83784*10^{-8}$
9	0.1	1.06	0.609579	4.18641	0.332417	$1.86468*10^{-9}$
10	0.1	1.13	0.546141	3.94956	0.09197	$8.80217*10^{-11}$

Table 4 The MLE, Bootstrap and estimator Bayesian estimators of ρ, σ, and.$R(t)$

	$\widehat{\rho}$	$\lvert\widehat{\rho}-\rho\rvert$	$\widehat{\sigma}$	$\lvert\widehat{\sigma}-\sigma\rvert$	$\widehat{R}(t)$	$\left\lvert\widehat{R}(t)-R(t)\right\rvert$
$(.)_{ML}$	0.6074	0.001	0.5237	0.294	0.5989	0.072
$(.)_{\text{Bootstrap}}$	0.7321	0.125	0.6382	0.229	0.6231	0.048
$(.)_{\text{BSLF}}$	0.5138	0.093	0.2603	0.031	0.3403	0.187
$(.)_{\text{BLXLF}}(z=-2)$	0.5152	0.092	0.2612	0.032	0.9544	0.428
$(.)_{\text{BLXLF}}(z=2)$	0.5127	0.094	0.2596	0.030	0.8503	0.323
$(.)_{\text{WBLXLF}}(z=-2)$	0.5098	0.097	0.2612	0.032	0.7748	0.248
$(.)_{\text{WBLXLF}}(z=2)$	0.5082	0.099	0.2564	0.027	0.7757	0.249

Table 5 Two –side 90% and95% confidence intervals of ρ, σ, and $R(t)$ by Bootstrap estimator.

	90% P Interval	Length	95% P Interval	Length
ρ	[0.0277, 1.7653]	1.7376	[0.0147, 1.8762]	1.8615
σ	[0.5432, 1.6652]	1.1220	[0.5232, 1.6784]	1.1552
$R(t)$	[0.0626, 0.8715]	0.8089	[0.0481, 0.9450]	0.8969

Method $\widehat{\sigma}=0.2293$ and $\widehat{\rho}=0.6067$. The results showed the proposed method was well than other methods in estimating shape parameters.

8. Discussion

The research's findings show that the suggested technique, which uses the WLXLF and the parametric bootstrap method, does a good job at predicting the Inverse IWD's parameters and validity functional based on Lesser Record Levels [24–25]. SLF, LXLF, and the newly suggested loss function WLXLF are the three loss functions employed. Initially it's critical to emphasize the benefits of the approach that has been established. The estimate of parameters and reliability functions in the IWD is handled particularly by the LXLF and WLXLF loss functions. Many actual disciplines, including the dynamic components of diesel, reliability, and biological investigations, employ the IWD to simulate data. It may be used to simulate a variety of failure traits, including those related to motors, infant mortality, automobile radiators, and germanium transistors. Insights may be gained by contrasting the performance of the proposed method with other techniques like MLE and the Bayesian estimator employing symmetric and asymmetric loss functions [26–27]. The findings imply that the suggested technique predicts the magnitude parameter and accuracy function better than existing conventional methods. This shows that the parameterized bootstrap approach and the WLXLF effectively represent the data's fundamental properties.

SLF, Modified Linex loss function, and LXLF is used to calculate how much the projected or evaluated value of a limit deviates from the actual value. The suggested technique, however, still has potential for real-world applications when considering the overall performance and the importance of the scale value and dependability function in reliability evaluation.

An essential component of the research is the construction of intervals of trust for the predicted parameters and reliability function using the parametric bootstrap approach. This method permits trustworthy inference and offers a measure of uncertainty related to the estimations. Further evidence for the suggested strategy's legitimacy and efficacy comes from comparing the findings obtained through Monte Carlo simulation. The study not only offers theoretical insights, but also offers a useful illustration based on facts. The results of the simulation research are supported by this real data analysis, which shows how well the suggested technique performs in predicting the dimension parameter and provides reasonable predictions for the scale parameter and reliability function.

9. Conclusion

This article developed LXLF to calculate parameters and dependability performance of IWD based on Lower Record Rates. By weighting LXLF to construct a new loss purpose named modified the loss is linear logarithmic WLXLF. Then use WLXLF to derive parameters and reliability functions of IWD. The shape and scale parameters are unknown when the shape parameter is known. In addition, we compared the proposed method WLXLF with the Bayesian estimator under the symmetric and asymmetric loss functions and MLE. The results showed that the proposed method had the best performance in estimating the scale parameter and the reliability function when the shape parameter was known. At the same time, the results showed that the proposed method had the best performance for estimating the shape parameter compared to other methods and acceptable performance for estimating the reliability function and scale parameter when both the shape and scale parameters are unknown. Finally, A range of trust has been built for the parameters and reliability function using a parametric bootstrap approach.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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