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Bivariate Discrete Odd Generalized Exponential Generator of Distributions for Count Data: Copula Technique, Mathematical Theory, and Applications

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Abstract: In this article, a new family of bivariate discrete distributions is proposed based on the copula concept, in the so-called bivariate discrete odd generalized exponential-G family. Some distributional properties, including the joint probability mass function, joint survival function, joint failure rate function, median correlation coefficient, and conditional expectation, are derived. After proposing the general class, one special model of the new bivariate family is discussed in detail. The maximum likelihood approach is utilized to estimate the family parameters. A detailed simulation study is carried out to examine the bias and mean square error of maximum likelihood estimators. Finally, the importance of the new bivariate family is explained by means of two distinctive real data sets in various fields.

Keywords: copula technique; bivariate discrete distributions; failure analysis; conditional expectation; simulation; maximum likelihood estimators; statistics and numerical data

MSC: 60E05; 62E10

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1. Introduction

Probability distributions are statistical functions that describe all possible values of a random variable and their likelihood, within a specific range. These are a result of the data generating process of an occurrence or its probability density function (PDF). Probability distributions are useful in modeling our environment to acquire estimates of the likelihood that a specific event will occur or to determine the variability of occurrence. There are many types of probability distributions, including binomial distribution, uniform distribution, Poisson distribution, normal distribution, lognormal distribution, beta distribution, exponential, and bivariate distribution. In our study, we highlight the exponential distribution (E) as a basic probability model for the proposed generator. The E model is a fundamental and widely recognized probability distribution in research. For lifetime phenomenon and reliability studies, the E distribution is rarely utilized due to the constant failure rate (FR); however, its memoryless property proves useful in queuing theory. Different extensions or generalizations of the E distribution have been introduced in the literature, with the generalized exponential (GE) garnering increased attention. The cumulative distribution function (CDF) of the GE distribution can be formulated as expression

$$F_{\mathrm{GE}}(x;\xi,\nu)=\left(1-e^{-\xi x}\right)^{\nu};x\geq 0, \tag{1}$$

where $\xi > 0$ is the scale parameter and $\nu > 0$ is the shape parameter. Based on the technique of Alzaatreh et al. [1] for generating families of distributions, Tahir et al. [2] proposed the new odd generator of distributions called the odd generalized exponential-G (OGE-G), which exhibits flexible FR shapes such as increasing, decreasing, bathtub or upside-down bathtub. The CDF of the OGE-G family is given by the following formula

$$F_{\mathrm{OGE\text{-}G}}(x;\xi,\nu,\mathbf{\Theta})=\left(1-e^{-\xi\Psi(x;\mathbf{\Theta})}\right)^{\nu};x\geq 0, \tag{2}$$

where $\Psi(x;\mathbf{\Theta})=G(x;\mathbf{\Theta})/(1-G(x;\mathbf{\Theta}))$, $G(x;\mathbf{\Theta})$ is a non-negative function of any model that depends on positive parameter vector $\mathbf{\Theta} > 0$. For more detail around the odd ratio, $\Psi(x;\mathbf{\Theta})$ (see Tahir et al., [2]). Due to the flexibility of Alzaatreh et al. [1] approach, several authors utilized this technique to generate new models for analyzing different types of data sets in various fields, for instance, see Silva et al. [3], Alizadeh et al. [4], Korkmaz et al. [5], Djibrila [6], Reyad et al. [7], Alizadeh et al. [8], among others.

The above probability models can be applied to discuss and analyze univariate data. In some applied sciences, bivariate data can be generated from different fields. Bivariate data refer to a data set that contains exactly two variables. In statistics, bivariate data represents data where each value of one variable is paired with a corresponding value of the other variable. Typically, it would be of interest to investigate the possible association between the two variables. A probability distribution involving two random variables is called a bivariate probability distribution. The bivariate (BV) distributions have been introduced, developed, and discussed by many authors and have wide applications in various fields like engineering, weather, sports, drought, among others. More detail is given in Balakrishnan and Lai [9]. The construction of the BV discrete and continuous distributions are mainly via (I) the copulas (II) compounding (III) marginals (IV) reduction and (V) conditioning, for instance, Johnson and Tenenbein [10], Quesada and Rodrguez [11], Fang et al. [12], Durante [13], Kundu and Gupta [14], Sarabia et al. [15], Roozegar and Jafari [16], Eliwa et al. [17], among others.

In several cases, the BV lifetimes need to be recorded on a discrete scale rather than on a continuous one. Due to the previous reasons, discretizing continuous BV distributions has received much attention in the statistical literature, for instance, Lee and Cha [18], Kundu and Nekoukhou [19], El-Morshedy et al. [20], Nekoukhou et al. [21], De Oliveira and Achcar [22], among others. Herein, we focus on the copulas approach. A copula function for bivariate discrete probability distributions is a function that connects the joint probability mass function of two discrete variables to their marginal probability mass functions. A copula function can be obtained by discretizing a continuous copula function, which is a function that links a multivariate distribution function to its one-dimensional marginal distribution functions. Alternatively, a copula function can be derived from a specific bivariate discrete distribution, such as the bivariate geometric, binomial, Poisson, or negative binomial distributions. A copula function for bivariate discrete probability distributions can be used to model the dependence structure between the variables and to simulate bivariate discrete data with given marginals and correlation; for more details, see, Kobus and Kurek, [23], Najarzadegan et al., [24], and Yamaguchi and Maruo, [25]. One of the important constructions of the bivariate distribution in survival and lifetime models is frailty models. So, both copulas and frailty are important in lifetime models (see Emura et al., [26]). The copula function can be formulated as follows

$$C(F_1(x_1),F_2(x_2))=G(x_1,x_2)\text{ for all }x_1,x_2=0,1,\ldots, \tag{3}$$

where $G(x_1, x_2)$ represents the CDF of a given BV continuous model on $(0, \infty)^2$, whereas $F_1(x_1)$ and $F_2(x_2)$ are the marginals. There are some known parametric copulas such as the following: the normal copulas, Farlie-Gumbel-Morgenstern (FGM) copulas, Marshall and Olkin (MO) copulas, Cuadras and Augé (CA) copulas (see Cuadras and Augé, [27]), among others. The CA copulas family was proposed as an extension of the BV distributions, defined as follows

$$C_\theta(x_1, x_2) = \min\{x_1, x_2\} \max\{x_1, x_2\}^{1-\theta}, \tag{4}$$

where $0 \leq \theta \leq 1$ and $0 \leq x_1, x_2 \leq 1$. The parameter θ measures the degree of dependence and plays the role of the parameter of upper-tail dependence of $C_\theta(x_1, x_2)$. The CA copulas family is utilized in a variety of modeling's for exchangeable random vectors because it is symmetric in its marginals. The aim of this paper is to propose a new discrete BV family of distributions based on the CA copulas approach, in the so-called BV discrete odd generalized exponential-G (BDsOGE-G) family, where the marginals have the OGE-G families.

The article is organized as follows: In Section 2, the BDsOGE-G family of distributions is introduced. Some mathematical and statistical properties are derived in Section 3. In Section 4, the BDsOGE-Weibull distribution is discussed in detail. The parameters of the new bivariate family are estimated via the maximum likelihood technique in Section 5. In Section 6, a simulation study is performed to discuss the performance of the maximum likelihood estimators. Two distinctive data sets are analyzed to discuss the flexibility of the proposed family in Section 7. Finally, some concluding remarks and future work are listed in Section 8.

2. The BDOGE-G Class

Recall that, in Equation (2), the CDF and probability mass function (PMF) of the discrete OGE-G (DsOGE-G) family can be formulated as follows

$$F_{\text{DsOGE-G}}(x; p, \nu, \boldsymbol{\Theta}) = \left(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\right)^\nu;\ x = 0, 1, 2, 3, \ldots \tag{5}$$

and

$$f_{\text{DsOGE-G}}(x; p, \nu, \boldsymbol{\Theta}) = \left(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\right)^\nu - \left(1 - p^{\Psi(x;\boldsymbol{\Theta})}\right)^\nu; x = 0, 1, 2, 3, \ldots, \tag{6}$$

respectively, where $p = e^{-\xi}$, $0 < p < 1$, $\nu > 0$ and $\boldsymbol{\Theta}$ is the vector of parameters $(1 \times k)$. Equation (6) can be writen as follows

$$f_{\text{DsOGE-G}}(x; p, \nu, \boldsymbol{\Theta}) = \sum_{k=1}^{\nu} (-1)^{k+1} \tbinom{\nu}{k} \left[p^{k\Psi(x;\boldsymbol{\Theta})} - p^{k\Psi(x+1;\boldsymbol{\Theta})}\right]. \tag{7}$$

Utilizing the CA copulas approach with $\gamma = 1 - \theta$, the joint PMF of the BDsOGE-G can be expressed as

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} P_1(x_1, x_2) & \text{if } x_1 < x_2 \\ P_2(x_1, x_2) & \text{if } x_2 < x_1 \\ P_3(x) & \text{if } x_1 = x_2 = x, \end{cases} \tag{8}$$

where

$$\begin{aligned} P_1(x_1, x_2) &= \left[\left(1 - p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^\nu - \left(1 - p^{\Psi(x_1;\boldsymbol{\Theta})}\right)^\nu\right]\left[\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right] \\ &= f_{\text{DsOGE-G}}(x_1; p, \nu, \boldsymbol{\Theta}) f_{\text{DsOGE-G}}(x_2; p, \nu\gamma, \boldsymbol{\Theta}), \end{aligned}$$

$$\begin{aligned} P_2(x_1,x_2) &= \left[\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}-\left(1-p^{\Psi(x_1;\mathbf{\Theta})}\right)^{\nu\gamma}\right]\left[\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}\right] \\ &= f_{\text{DsOGE-G}}(x_1;p,\nu\gamma,\mathbf{\Theta})f_{\text{DsOGE-G}}(x_2;p,\nu,\mathbf{\Theta}), \end{aligned}$$

and

$$\begin{aligned} P_3(x) &= \left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left[\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\right]-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu} \\ &\quad\times\left[\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu\gamma}\right] \\ &= \left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}f_{\text{DsOGE-G}}(x;p,\nu,\mathbf{\Theta})-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}f_{\text{DsOGE-G}}(x;p,\nu\gamma,\mathbf{\Theta}). \end{aligned}$$

The joint CDF of the BDsOGE-G family can be proposed as

$$F_{X_1,X_2}(x_1,x_2)=\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left(1-p^{\Psi(z+1;\mathbf{\Theta})}\right)^{\nu(1-\gamma)};\ x_1,x_2=0,1,2,3\ldots,$$

where $z=\min\{x_1,x_2\}$. The marginal CDF of the BDsOGE-G family is

$$F_{\text{DsOGE-G}}(x;p,\nu,\mathbf{\Theta})=\left(1-p^{\Psi(x_i+1;\mathbf{\Theta})}\right)^{\nu};\ i=1,2. \tag{9}$$

If $(X_1,X_2)\sim$ BDsOGE-G$(p,\nu,\gamma,\mathbf{\Theta})$, then X_1 and X_2 are positive quadrant dependent (PQD) where

$$\Pr(X_1\le x_1,X_2\le x_2)\ge\Pr(X_1\le x_1)\Pr(X_2\le x_2),\ \text{for all } x_1 \text{ and } x_2.$$

The joint RF is given by

$$S_{X_1,X_2}(x_1,x_2)=\begin{cases} S_1(x_1,x_2) & \text{if } x_1<x_2 \\ S_2(x_1,x_2) & \text{if } x_2<x_1 \\ S_3(x) & \text{if } x_1=x_2=x, \end{cases} \tag{10}$$

where

$$S_1(x_1,x_2)=1-\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu}\left(1-\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu\gamma}\right),$$

$$S_2(x_1,x_2)=1-\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}\left(1-\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}\right),$$

and

$$S_3(x)=1-2\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu}+\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu(1+\gamma)}.$$

The joint HRF function can be expressed by using $h_{X_1,X_2}(x_1,x_2)=f_{X_1,X_2}(x_1,x_2)/S_{X_1,X_2}(x_1-1,x_2-1)$ whereas the joint reversed HRF function can be proposed via $v_{X_1,X_2}(x_1,x_2)=f_{X_1,X_2}(x_1,x_2)/F_{X_1,X_2}(x_1,x_2)$. If the bivariate vector X have the BDsOGE-G model, then the distribution for each $W=\max\{X_1,X_2\}$ and $V=\min\{X_1,X_2\}$ can be written as follows

$$F_W\ (z)=F_{\text{DsOGE-G}}\ (z;p,\nu_1+\nu_2+\nu_3,\mathbf{\Theta}) \tag{11}$$

and

$$F_V\ (z)=F_{\text{DsOGE-G}}\ (z;p,\nu_1+\nu_3,\mathbf{\Theta})+F_{\text{DsOGE-G}}\ (z;p,\nu_2+\nu_3,\mathbf{\Theta})-F_W\ (z), \tag{12}$$

respectively.

3. Distributional Properties

3.1. Median Correlation Coefficient

In statistics and probability theory, the median is a type of summary statistics used to give important information about a certain data point or population. It is the value separating the higher half from the lower half of an ordered data sample, a population, or a probability distribution. For a bivariate version, the median correlation coefficient, say L_{X_1,X_2}, can be expressed as a form $L_{X_1,X_2} = 4F_{X_1,X_2}(L_{X_1}, L_{X_2}) - 1$, where L_{X_1} and L_{X_2} denote the median of X_1 and X_2, respectively. If $X_1 \sim$ DsOGE-G $(\boldsymbol{\Theta}, \nu_1 + \nu_3)$ and $X_2 \sim$ DsOGE-G $(\boldsymbol{\Theta}, \nu_2 + \nu_3)$, then

$$L_{X_1,X_2} = \begin{cases} 4F_{\text{DsOGE-G}}\left(L_{X_2}; p, \nu_2, \boldsymbol{\Theta}\right) F_{\text{DsOGE-G}}\left(L_{X_1}; p, \nu_1 + \nu_3, \boldsymbol{\Theta}\right) - 1 & \text{if } x_1 \leq x_2 \\ 4F_{\text{DsOGE-G}}\left(L_{X_1}; p, \nu_1, \boldsymbol{\Theta}\right) F_{\text{DsOGE-G}}\left(L_{X_2}; p, \nu_2 + \nu_3, \boldsymbol{\Theta}\right) - 1 & \text{if } x_1 > x_2, \end{cases} \tag{13}$$

where $L_{X_i}; i = 1, 2$ is the quantile function for the marginals.

3.2. The Conditional CDF of X_1 Given $X_2 = x_2$ ($X_2 \leq x_2$)

A conditional distribution is a probability distribution for a sub-population. In other words, it shows the probability that a randomly selected item in a sub-population has a characteristic of interest. Conditional distributions have many applications in statistics and machine learning. For example, they can be used in regressing structured response variables. In addition, conditional distributions can be embedded into a Hilbert space, which is potentially useful in applications where conditional distributions are the key quantities of interest. Assuming X follows the BDsOGE-G family, the conditional CDF of $(X_1 \mid X_2 = x_2)$, say $F_{X_1|X_2=x_2}(x_1)$, is given by

$$F_{X_1|X_2=x_2}(x_1 \mid x_2) = \begin{cases} F_1(x_1 \mid x_2) & \text{if } 0 \leq x_1 < x_2 \\ F_2(x_1 \mid x_2) & \text{if } 0 \leq x_2 < x_1 \\ F_3(x_1 \mid x_2) & \text{if } 0 \leq x_1 = x_2 = x, \end{cases} \tag{14}$$

where

$$F_1(x_1 \mid x_2) = \frac{\left(1 - p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu} \left[\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right]}{\left[\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu} - \left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu}\right]},$$

$$F_2(x_1 \mid x_2) = \left(1 - p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu\gamma},$$

and

$$F_3(x_1 \mid x_2) = \left(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\right)^{\nu\gamma},$$

where $F_{X_1|X_2=x_2}(x_1 \mid x_2) = \frac{P(X_1 \leq x_1, X_2 = x_2)}{P(X_2 = x_2)}$. Similarly, the conditional CDF of $X_1 \mid X_2 \leq x_2$, say $F_{X_1|X_2 \leq x_2}(x_1 \mid x_2)$, is given by

$$F_{X_1|X_2 \leq x_2}(x_1 \mid x_2) = \begin{cases} \left(1 - p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu} \left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{-\nu(1-\gamma)} & \text{if } 0 \leq x_1 < x_2, \\ \left(1 - p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} & \text{if } 0 \leq x_2 < x_1, \\ \left(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} & \text{if } 0 \leq x_1 = x_2 = x. \end{cases} \tag{15}$$

3.3. The Conditional Expectation of X_1 Given $X_2 = x_2$

In probability theory, the conditional expectation of a random variable is its expected value given that a certain set of "conditions" is known to occur. The conditional expectation is the expected value of a random variable, computed with respect to a conditional probability distribution. To derive the conditional expectation of X_1, given $X_2 = x_2$, the PMF of

X_1 given $X_2 = x_2$ should be calculated first. Assuming X follow the BDsOGE-G family, the PMF of X_1 given $X_2 = x_2$ can be expressed as

$$f_{X_1|X_2=x_2}(x_1 \mid x_2) = \begin{cases} f_1(x_1 \mid x_2) & \text{if } \; 0 \le x_1 < x_2 \\ f_2(x_1 \mid x_2) & \text{if } \; 0 \le x_2 < x_1 \\ f_3(x_1 \mid x_2) & \text{if } \; 0 \le x_1 = x_2 = x, \end{cases} \tag{16}$$

where

$$f_1(x_1 \mid x_2) = \frac{f_{\text{DsOGE-G}}(x_1; p, \nu, \boldsymbol{\Theta}) f_{\text{DsOGE-G}}(x_2; p, \nu\gamma, \boldsymbol{\Theta})}{f_{\text{DsOGE-G}}(x_2; p, \nu, \boldsymbol{\Theta})},$$

$$f_2(x_1 \mid x_2) = f_{\text{DsOGE-G}}(x_1; p, \nu\gamma, \boldsymbol{\Theta}),$$

and

$$f_3(x_1 \mid x_2) = \frac{\left(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} f_{\text{DsOGE-G}}(x; p, \nu, \boldsymbol{\Theta}) - \left(1 - p^{\Psi(x;\boldsymbol{\Theta})}\right)^{\nu} f_{\text{DsOGE-G}}(x; p, \nu\gamma, \boldsymbol{\Theta})}{f_{\text{DsOGE-G}}(x; p, \nu, \boldsymbol{\Theta})}.$$

Thus, the conditional expectation of $X_1 \mid X_2 = x_2$, say $E(X_1 \mid X_2 = x_2)$, is given by

$$\begin{aligned} \mathbf{E}(X_1 \mid X_2 = x_2) &= \sum_{i=0}^{\nu\gamma} \sum_{x_1=x_2+1}^{\infty} (-1)^i x_1 \binom{\nu\gamma}{i} \left[p^{i\Psi(x_1+1;\boldsymbol{\Theta})} - p^{i\Psi(x_1;\boldsymbol{\Theta})}\right] + x_2\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} \\ &+ \frac{\sum_{i=0}^{\nu} \sum_{x_1=0}^{x_2-1} (-1)^i x_1 \binom{\nu}{i} \left[p^{i\Psi(x_1+1;\boldsymbol{\Theta})} - p^{i\Psi(x_1;\boldsymbol{\Theta})}\right] \left[\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right]}{\left[(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})})^{\nu} - (1 - p^{\Psi(x_2;\boldsymbol{\Theta})})^{\nu}\right]} \\ &- \frac{x_2\left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu} \left[\left(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1 - p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right]}{\left[(1 - p^{\Psi(x_2+1;\boldsymbol{\Theta})})^{\nu} - (1 - p^{\Psi(x_2;\boldsymbol{\Theta})})^{\nu}\right]}, \end{aligned}$$

where $\mathrm{E}(X_1 \mid X_2 = x_2) = \sum_{x_1=0}^{\infty} x_1 f_{X_1|X_2=x_2}(x_1 \mid x_2)$. The accurate estimation of conditional expectations is an important problem arising in different branches of science and engineering as well as finance, economics and various business applications (see Casella and Berger, [28]; Pfeiffer, [29]; and Emura et al., [26]).

4. The BDsOGE-Weibull (BDsOGEW) Distribution

Considering the CDF of Weibull distribution with one positive shape parameter a, the joint PMF of the BDsOGEW model is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} P_1(x_1, x_2) & \text{if } \; x_1 < x_2 \\ P_2(x_1, x_2) & \text{if } \; x_2 < x_1 \\ P_3(x) & \text{if } \; x_1 = x_2 = x, \end{cases} \tag{17}$$

where

$$P_1(x_1, x_2) = \left[\left(1 - p^{(e^{(x_1+1)^a} - 1)}\right)^{\nu} - \left(1 - p^{(e^{x_1{}^a} - 1)}\right)^{\nu}\right] \left[\left(1 - p^{(e^{(x_2+1)^a} - 1)}\right)^{\nu\gamma} - \left(1 - p^{(e^{x_2{}^a} - 1)}\right)^{\nu\gamma}\right],$$

$$P_2(x_1, x_2) = \left[\left(1 - p^{(e^{(x_1+1)^a} - 1)}\right)^{\nu\gamma} - \left(1 - p^{(e^{x_1{}^a} - 1)}\right)^{\nu\gamma}\right] \left[\left(1 - p^{(e^{(x_2+1)^a} - 1)}\right)^{\nu} - \left(1 - p^{(e^{x_2{}^a} - 1)}\right)^{\nu}\right]$$

and

$$P_3(x) = \left(1-p^{\left(e^{(x+1)^a}-1\right)}\right)^{\nu\gamma}\left[\left(1-p^{\left(e^{(x+1)^a}-1\right)}\right)^{\nu}-\left(1-p^{\left(e^{x^a}-1\right)}\right)^{\nu}\right]-\left(1-p^{\left(e^{x^a}-1\right)}\right)^{\nu}\times \left[\left(1-p^{\left(e^{(x+1)^a}-1\right)}\right)^{\nu\gamma}-\left(1-p^{\left(e^{x^a}-1\right)}\right)^{\nu\gamma}\right].$$

Figure 1 shows the joint PMF of the BDsOGEW model for various values of the parameters.

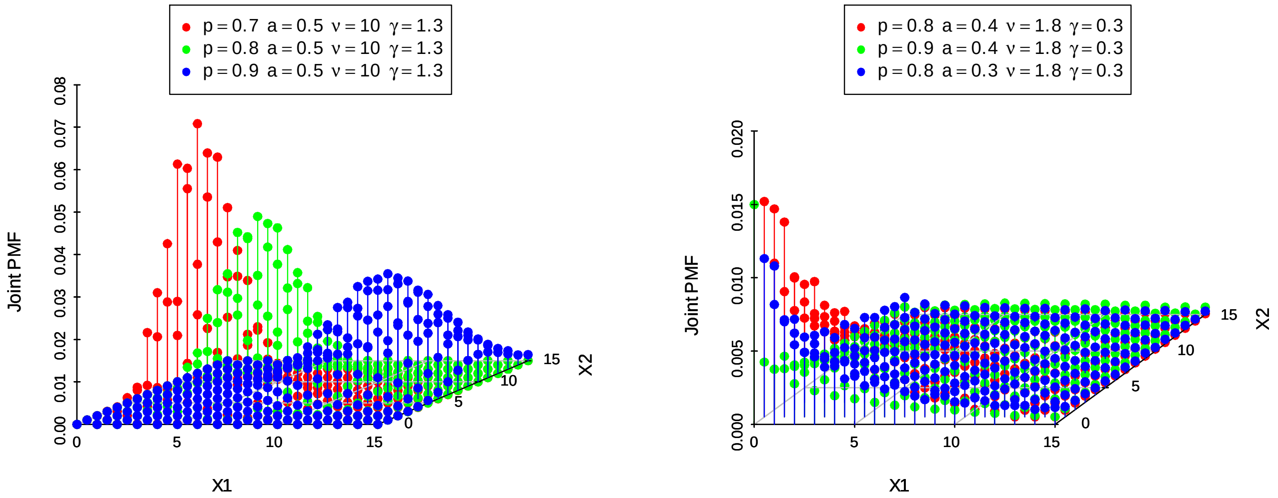

Figure 1. The joint PMF of the BDsOGEW model.

It is worth noting that the joint probability mass function (PMF) can exhibit either a unimodal or a decreasing surface shape. This flexibility allows it to be used for modeling and analyzing both asymmetric and symmetric data. The corresponding joint CDF and joint RF to Equation (17) can be formulated as

$$F_{X_1,X_2}(x_1,x_2) = \left(1-p^{\left(e^{x_1{}^a}-1\right)}\right)^{\nu\gamma}\left(1-p^{\left(e^{x_2{}^a}-1\right)}\right)^{\nu\gamma}\left(1-p^{\left(e^{z^a}-1\right)}\right)^{\nu(1-\gamma)}$$

and

$$S_{X_1,X_2}(x_1,x_2) = 1-\left(1-p^{\left(e^{x_1{}^a}-1\right)}\right)^{\nu}-\left(1-p^{\left(e^{x_2{}^a}-1\right)}\right)^{\nu}+\left(1-p^{\left(e^{x_1{}^a}-1\right)}\right)^{\nu\gamma} \times\left(1-p^{\left(e^{x_2{}^a}-1\right)}\right)^{\nu\gamma}\left(1-p^{\left(e^{z^a}-1\right)}\right)^{\nu(1-\gamma)},$$

respectively. Figures 2 and 3 illustrate the joint RF and joint HRF of the proposed bivariate model based on different values of its parameters.

It is noted that the joint HRF can be either a unimodal, constant, or decreasing surface. Thus, the joint HRF of this model can be used to analyze several types of data in various fields. Figure 4 lists different plots of the conditional PMF of $X_{1,}$ given that $x_2 = 5, 15, 25$ under various specific schemes Figure 4a: $a = 0.4$, $p = 0.8$, $\gamma = 1.3$, $v = 10$ and Figure 4b: $a = 0.4$, $p = 0.8$, $\gamma = 0.3$, $v = 1.8$. Whereas the attitude of the conditional PMF of $X_{2,}$ given that $x_1 = 5, 15, 25$ can be proposed through schemes Figure 5a: $a = 0.4$, $p = 0.8$, $\gamma = 1.3$, $v = 10$ and Figure 5b: $a = 0.4$, $p = 0.8$, $\gamma = 0.3$, $v = 1.8$ in Figure 5.

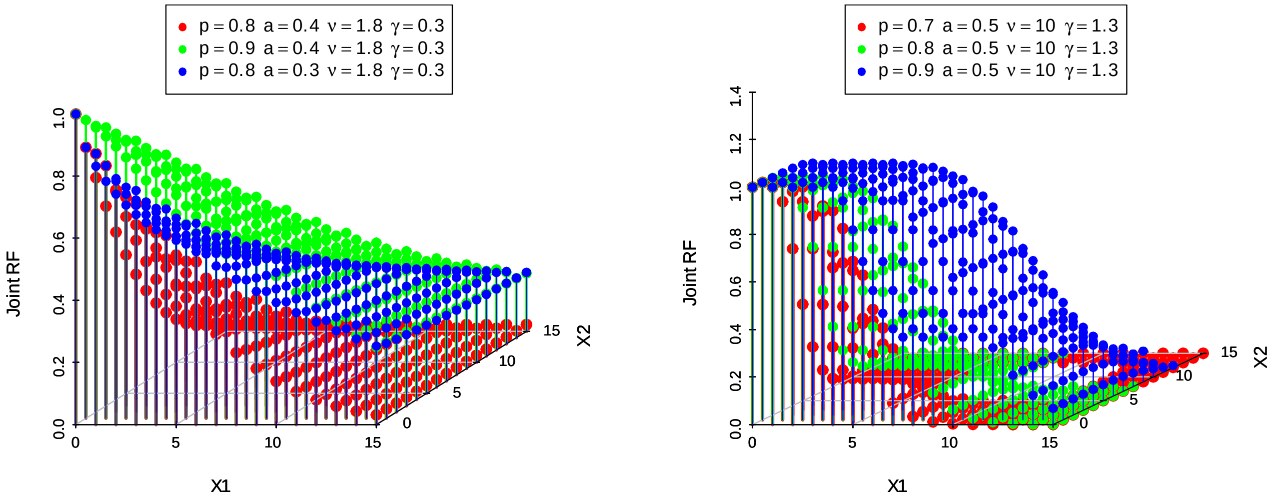

Figure 2. The joint RF of the BDsOGEW model.

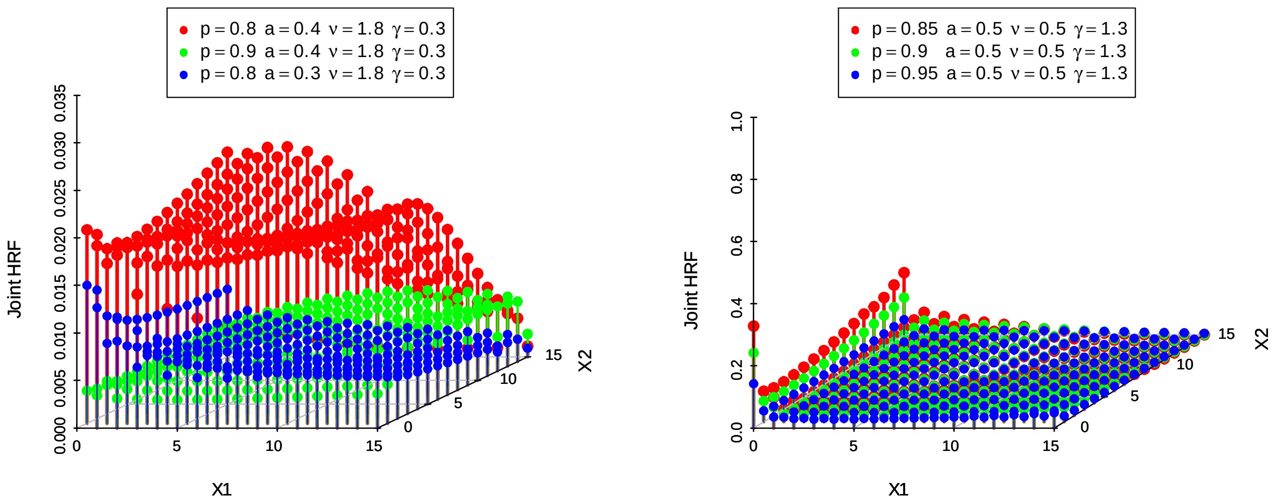

Figure 3. The joint HRF of the BDsOGEW model.

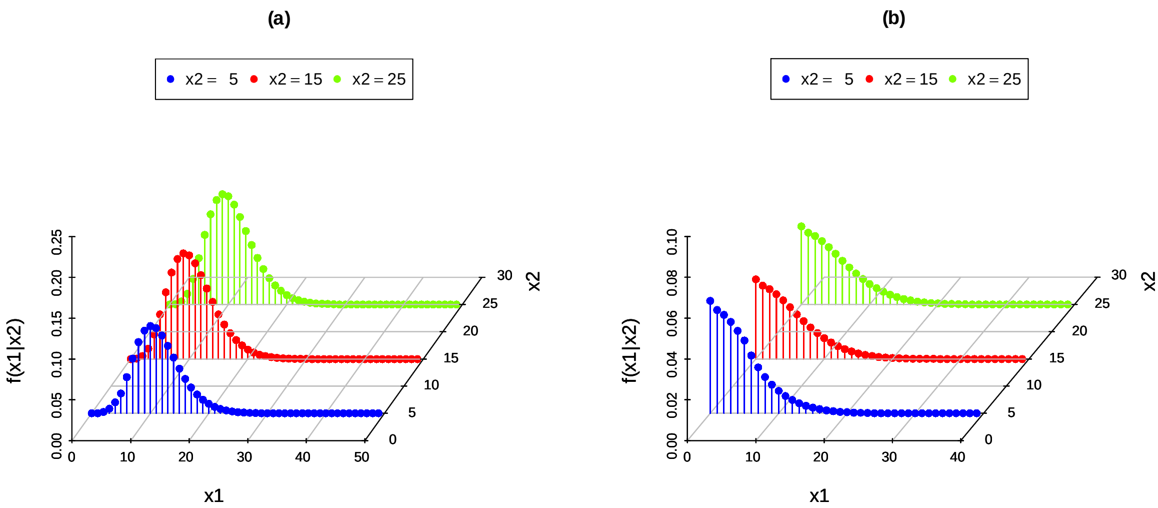

Figure 4. Some scatter plots of the conditional PMF for X_1 given x_2.

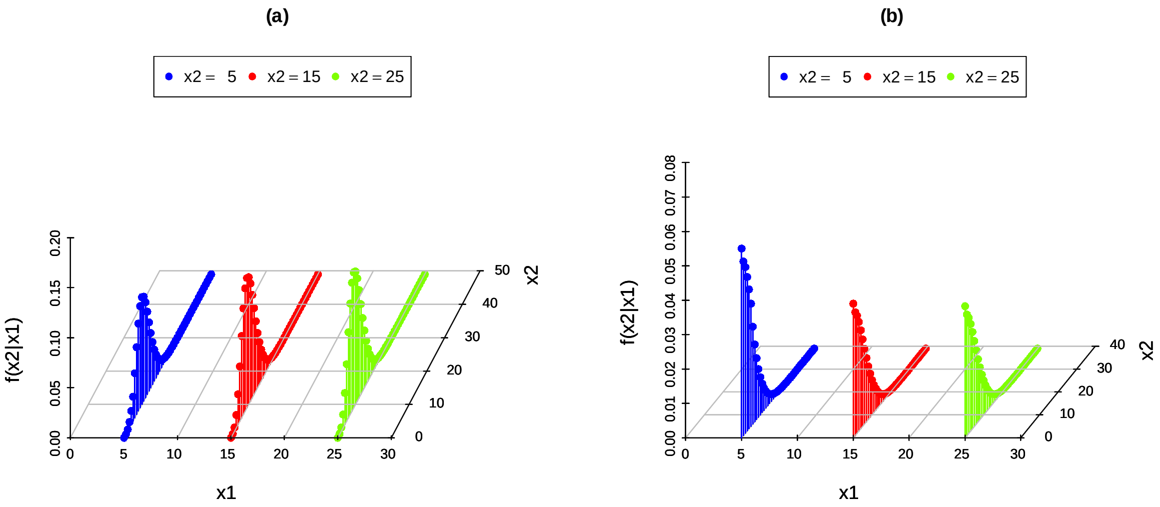

Figure 5. Some scatter plots of the conditional PMF for X_2 given x_1.

5. Point and Interval Estimations

5.1. Maximum Likelihood Estimation (MLE)

In statistics, MLE is a method of estimating the parameters of an assumed probability distribution, given some observed data. This is achieved by maximizing a likelihood function so that, under the assumed statistical model, the observed data is most probable. MLE has applications in many fields such as physics, engineering, economics, finance, and biology. In this section, the approach of MLE is used to estimate the unknown parameters p, ν, γ and $\boldsymbol{\Theta}$ of the BDsOGE-G family. Suppose that a sample size n, of the form $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})\}$ from BDsOGE-G family. Under the following notations: $I_1 = \{x_{1j} < x_{2j}\}$, $I_2 = \{x_{2j} < x_{1j}\}$, $I_0 = \{x_{1j} = x_{2j} = x_j\}$, $I = I_1 \cup I_2 \cup I_0$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_0| = n_0$ and $n = n_1 + n_2 + n_0$, the likelihood function can be expressed as

$$L(p, \nu, \gamma, \boldsymbol{\Theta}) = \prod_{j \in I_1} P_1(x_{1j}, x_{2j}) \prod_{j \in I_2} P_2(x_{1j}, x_{2j}) \prod_{j \in I_3} P_3(x_j). \tag{18}$$

The log-likelihood function becomes

$$\begin{aligned} l(p, \nu, \gamma, \boldsymbol{\Theta}) \quad = \quad & \sum_{j \in I_1} \ln\big(g_1(x_{1j}; \nu)\big) + \sum_{j \in I_1} \ln\big(g_1(x_{2j}; \nu\gamma)\big) \\ & + \sum_{j \in I_2} \ln\big(g_1(x_{1j}; \nu\gamma)\big) + \sum_{j \in I_2} \ln\big(g_1(x_{2j}; \nu)\big) \\ & + \sum_{j \in I_0} \ln\Big(\Big(1 - p^{\Psi(x_j+1;\boldsymbol{\Theta})}\Big)^{\nu\gamma} g_1(x_j; \nu) - \Big(1 - p^{\Psi(x_j;\boldsymbol{\Theta})}\Big)^{\nu} g_1(x_j; \nu\gamma)\Big), \end{aligned} \tag{19}$$

where

$$g_1(x; \nu) = \Big(1 - p^{\Psi(x+1;\boldsymbol{\Theta})}\Big)^{\nu} - \Big(1 - p^{\Psi(x;\boldsymbol{\Theta})}\Big)^{\nu}.$$

The MLEs of the parameters p, ν, γ and $\boldsymbol{\Theta}$ can be obtained by computing the first partial derivatives of Equation (19) with respect to p, ν, γ and $\boldsymbol{\Theta}$, respectively, and setting the resulting equations equal to zero. The first partial derivatives are

$$
\begin{aligned}
\frac{\partial l(p,\nu,\gamma,\boldsymbol{\Theta})}{\partial p} &= \sum_{j\in I_1}\frac{g_2(x_{1j},\nu)-g_2(x_{1j}+1,\nu)}{g_1(x_{1j};\nu)}+\sum_{j\in I_1}\frac{g_2(x_{2j},\nu\gamma)-g_2(x_{2j}+1,\nu\gamma)}{g_1(x_{2j};\nu\gamma)} \\
&+\sum_{j\in I_2}\frac{g_2(x_{1j},\nu\gamma)-g_2(x_{1j}+1,\nu\gamma)}{g_1(x_{1j};\nu\gamma)}+\sum_{j\in I_2}\frac{g_2(x_{2j},\nu)-g_2(x_{2j}+1,\nu)}{g_1(x_{2j};\nu)} \\
&+\sum_{j\in I_0}\frac{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}(g_2(x_j,\nu)-g_2(x_j+1,\nu))-g_2(x_j+1,\nu\gamma)g_1(x_j;\nu)}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)} \\
&-\sum_{j\in I_0}\frac{\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}(g_2(x_j,\nu\gamma)-g_2(x_j+1,\nu\gamma))-g_2(x_j,\nu)g_1(x_j;\nu\gamma)}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(p,\nu,\gamma,\boldsymbol{\Theta})}{\partial \nu} &= \sum_{j\in I_1}\frac{g_3(x_{1j},\nu)-g_3(x_{1j}+1,\nu)}{g_1(x_{1j};\nu)}+\sum_{j\in I_1}\frac{\gamma(g_3(x_{2j},\nu\gamma)-g_3(x_{2j}+1,\nu\gamma))}{g_1(x_{2j};\nu\gamma)} \\
&+\sum_{j\in I_2}\frac{\gamma(g_3(x_{1j},\nu\gamma)-g_3(x_{1j}+1,\nu\gamma))}{g_1(x_{1j};\nu\gamma)}+\sum_{j\in I_2}\frac{g_3(x_{2j},\nu)-g_3(x_{2j}+1,\nu)}{g_1(x_{2j};\nu)} \\
&+\sum_{j\in I_0}\frac{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}(g_3(x_j,\nu)-g_3(x_j+1,\nu))-\gamma g_3(x_j+1,\nu\gamma)g_1(x_j;\nu)}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)} \\
&-\sum_{j\in I_0}\frac{\gamma\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}(g_3(x_j,\nu\gamma)-g_3(x_j+1,\nu\gamma))-g_3(x_j,\nu)g_1(x_j;\nu\gamma)}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(p,\nu,\gamma,\boldsymbol{\Theta})}{\partial \gamma} &= \sum_{j\in I_1}\frac{\nu(g_3(x_{2j},\nu\gamma)-g_3(x_{2j}+1,\nu\gamma))}{g_1(x_{2j};\nu\gamma)}+\sum_{j\in I_2}\frac{\nu(g_3(x_{1j},\nu\gamma)-g_3(x_{1j}+1,\nu\gamma))}{g_1(x_{1j};\nu\gamma)} \\
&+\sum_{j\in I_0}\frac{-\nu g_3(x_j+1,\nu\gamma)g_1(x_j;\nu)-\nu\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}(g_3(x_j,\nu\gamma)-g_3(x_j+1,\nu\gamma))}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial l(p,\nu,\gamma,\boldsymbol{\Theta})}{\partial \boldsymbol{\Theta}_k} &= \sum_{j\in I_1}\frac{g_4(x_{1j},\nu)-g_4(x_{1j}+1,\nu)}{g_1(x_{1j};\nu)}+\sum_{j\in I_1}\frac{g_4(x_{2j},\nu\gamma)-g_4(x_{2j}+1,\nu\gamma)}{g_1(x_{2j};\nu\gamma)} \\
&+\sum_{j\in I_2}\frac{g_4(x_{1j},\nu\gamma)-g_4(x_{1j}+1,\nu\gamma)}{g_1(x_{1j};\nu\gamma)}+\sum_{j\in I_2}\frac{g_4(x_{2j},\nu)-g_4(x_{2j}+1,\nu)}{g_1(x_{2j};\nu)} \\
&+\sum_{j\in I_0}\frac{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}(g_4(x_j,\nu)-g_4(x_j+1,\nu))-g_4(x_j+1,\nu\gamma)g_1(x_j;\nu)}{\left(1-p^{\Psi(x_j+1;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)} \\
&-\sum_{j\in I_0}\frac{\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}(g_4(x_j,\nu\gamma)-g_4(x_j+1,\nu\gamma))-g_4(x_j,\nu)g_1(x_j;\nu\gamma)}{\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu\gamma}g_1(x_j;\nu)-\left(1-p^{\Psi(x_j;\boldsymbol{\Theta})}\right)^{\nu}g_1(x_j;\nu\gamma)}.
\end{aligned}
$$

where

$$
\begin{aligned}
g_2(x;\nu) &= \nu p^{\Psi(x;\mathbf{\Theta})-1}\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu-1}, \\
g_3(x;\nu) &= \ln\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}, \\
g_4(x;\nu) &= \nu \dot{H}(x;\mathbf{\Theta}) \ln p \; p^{\Psi(x;\mathbf{\Theta})}\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu-1},
\end{aligned}
$$

where $\dot{H}(x;\mathbf{\Theta}) = \frac{\partial}{\partial \mathbf{\Theta}_k}\Psi(x;\mathbf{\Theta})$ and $\mathbf{\Theta}_k$ is a vector of the parameters of the baseline model ($k = 1,2,3,\ldots$). The MLEs of the parameters p, ν, γ and $\mathbf{\Theta}$ can be obtained by solving the above system of non-linear equations. The solution of these equations is not easy to solve. Thus, the "NMaximize" function in the Mathematica software was utilized. The "NMaximize" is a built-in function that attempts to find a global maximum of a function or an expression numerically see Appendix A.

5.2. Asymptotic Confidence Intervals

Asymptotic confidence intervals (ACIs) are confidence intervals (CIs) that are based on the normal approximation of the sampling distribution of a statistic. They are also called large-sample CIs because they are only valid when the sample size is large enough. The asymptotic normal distribution of the MLE is the most widely utilized approach for establishing confidence bounds for the model parameters. With respect to the Fisher information matrix (FIM), denoted as $I(\Upsilon)$, which consists of the negative second derivatives of the $l(p,\nu,\gamma,\mathbf{\Theta}_k)$ evaluated at $\widehat{\mathbf{Y}} = (\widehat{p},\widehat{\nu},\widehat{\gamma},\widehat{\mathbf{\Theta}_k})$, the asymptotic variance-covariance matrix (AVCM) of the MLE of the parameters, assuming that the AVCM of the parameter vector can be formulated as

$$
I(\widehat{\mathbf{Y}}) = -E\begin{bmatrix}
\frac{\partial^2 l(\mathbf{Y})}{\partial p^2} & & & \\
\frac{\partial^2 l(\mathbf{Y})}{\partial \nu \partial p} & \frac{\partial^2 l(\mathbf{Y})}{\partial \nu^2} & & \\
\frac{\partial^2 l(\mathbf{Y})}{\partial \gamma \partial p} & \frac{\partial^2 l(\mathbf{Y})}{\partial \gamma \partial \nu} & \frac{\partial^2 l(\mathbf{Y})}{\partial \gamma^2} & \\
\frac{\partial^2 l(\mathbf{Y})}{\partial \mathbf{\Theta}_k \partial p} & \frac{\partial^2 l(\mathbf{Y})}{\partial \mathbf{\Theta}_k \partial \nu} & \frac{\partial^2 l(\mathbf{Y})}{\partial \mathbf{\Theta}_k \partial \gamma} & \frac{\partial^2 l(\mathbf{Y})}{\partial \mathbf{\Theta}_k^2}
\end{bmatrix}, \tag{20}
$$

where $Var(\widehat{\mathbf{Y}}) = I^{-1}(\widehat{\mathbf{Y}})$. Based on the asymptotic normality of the MLE, a $100(1-\alpha)\%$ confidence interval for parameter Y can be constructed as follows: $\widehat{p} \pm Z_{\alpha/2}\sqrt{Var(p)}$, $\widehat{\nu} \pm Z_{\alpha/2}\sqrt{Var(\nu)}$, $\widehat{\gamma} \pm Z_{\alpha/2}\sqrt{Var(\gamma)}$, and $\widehat{\mathbf{\Theta}_k} \pm Z_{\alpha/2}\sqrt{Var(\mathbf{\Theta}_k)}$. The second derivatives of the likelihood function can be calculated with Maple software.

6. Simulation: Estimators Performance

Simulation for bivariate discrete data in statistics is a technique to generate random data that follows a joint probability distribution of two discrete variables. There are different methods to simulate bivariate discrete data, depending on the type and structure of the distribution. One common method is to use copulas, which are functions that link univariate marginal distributions to a multivariate distribution. Copulas can capture the dependence structure between the variables and allow for flexible modeling of different types of correlations. Another method is to use discretization of continuous bivariate distributions. This involves rounding or truncating the continuous values to discrete values, and adjusting the probabilities accordingly. A third method is to use specific bivariate discrete distributions. Depending on the type of estimator, different metrics may be used to measure the performance of the estimator. These measures may include bias, mean squared error (MSE), confidence intervals (CIs), and coverage probability (CP) of the 95% CIs. In this segment, the second approach has been used. The performance of the MLE is tested under different schemes for the BDsOGEW parameter as follows:

- Scheme I: ($\forall$ p = 0.1, ν = 0.3, γ = 0.5, a = 0.2 | n_1 = 20, n_2 = 50, n_3 = 150, n_4 = 300, n_5 = 500, n_6 = 700);
- Scheme II: ($\forall$ p = 0.2, ν = 0.5, γ = 0.3, a = 0.5 | n_1 = 20, n_2 = 50, n_3 = 150, n_4 = 300, n_5 = 500, n_6 = 700);
- Scheme III: ($\forall$ p = 0.5, ν = 0.3, γ = 0.7, a = 0.9 | n_1 = 20, n_2 = 50, n_3 = 150, n_4 = 300, n_5 = 500, n_6 = 700)

The numerical assessments are performed depending on the bias, MSE, CP, and 95% CIs "lower bound (LB) and upper bound (UB)" using software *R* package. In general, an estimator is considered good if it has low bias and low variance. The empirical results are reported in Tables 1–3.

Table 1. Simulation results for scheme I.

	n_1 = 20				n_2 = 50			
	p	ν	γ	a	p	ν	γ	a
Bias	0.10125	0.21458	0.18256	0.25203	0.05236	0.18256	0.12469	0.19636
MSE	0.09856	0.18256	0.15246	0.21552	0.04254	0.15635	0.09850	0.16504
CP	0.88719	0.92335	0.93419	0.83963	0.88436	0.92144	0.93223	0.83745
95% CI_{LB}	0.06523	0.24714	0.42302	0.14025	0.07136	0.25748	0.44120	0.15699
95% CI_{UB}	0.14569	0.35125	0.59368	0.27145	0.13825	0.34283	0.56774	0.26632
	n_3 = 150				n_4 = 300			
	p	ν	γ	a	p	ν	γ	a
Bias	0.01365	0.12145	0.08254	0.14231	0.00858	0.07156	0.0236	0.11258
MSE	0.01258	0.10395	0.06632	0.12569	0.00652	0.05236	0.01428	0.09254
CP	0.88253	0.92074	0.93064	0.83164	0.88162	0.91886	0.92846	0.82854
95% CI_{LB}	0.07512	0.26314	0.46256	0.16369	0.08124	0.27142	0.47428	0.17402
95% CI_{UB}	0.12415	0.34472	0.54239	0.24012	0.11932	0.33748	0.53201	0.23148
	n_5 = 500				n_6 = 700			
	p	ν	γ	a	p	ν	γ	a
Bias	0.00103	0.01254	0.00125	0.06317	0.00003	0.00235	0.00024	0.00301
MSE	0.00084	0.00825	0.00052	0.01426	0.00001	0.00082	0.00011	0.00082
CP	0.87949	0.91278	0.92676	0.81746	0.87552	0.91075	0.92019	0.81298
95% CI_{LB}	0.08412	0.28301	0.48241	0.18012	0.09174	0.28102	0.49102	0.19125
95% CI_{UB}	0.11363	0.32823	0.53294	0.21989	0.10857	0.31025	0.52138	0.21011

According to the simulation results/performance as $n \to +\infty$, the bias and MSE decrease; and consequently, an unbiased estimator was achieved for large samples under consistency condition. Thus, the maximum likelihood approach can be used effectively to estimate the BDsOGEW parameters.

Table 2. Simulation results for scheme II.

	n_1 = 20				n_2 = 50			
	p	ν	γ	a	p	ν	γ	a
Bias	0.12369	0.23529	0.13085	0.16625	0.10238	0.20136	0.09820	0.14823
MSE	0.10925	0.22968	0.12821	0.15530	0.09569	0.18014	0.07742	0.13224
CP	0.91203	0.82734	0.94537	0.85016	0.91027	0.82256	0.94521	0.84771
95% CI_{LB}	0.15365	0.40123	0.22303	0.46369	0.16325	0.42013	0.24120	0.47102
95% CI_{UB}	0.26636	0.59325	0.38204	0.54215	0.24932	0.57012	0.36025	0.53325

Table 2. *Cont.*

	$n_3 = 150$				$n_4 = 300$			
	p	ν	γ	a	p	ν	γ	a
Bias	0.08256	0.16328	0.04230	0.12012	0.02137	0.11205	0.00803	0.08825
MSE	0.06636	0.14200	0.03047	0.09825	0.01452	0.09852	0.00625	0.06328
CP	0.90734	0.82019	0.94338	0.84309	0.90193	0.81578	0.94219	0.84290
95% CI_{LB}	0.17714	0.43025	0.25202	0.48158	0.18230	0.45201	0.28323	0.48623
95% CI_{UB}	0.23852	0.55236	0.34525	0.52745	0.22825	0.54414	0.32256	0.51445
	$n_5 = 500$				$n_6 = 700$			
	p	ν	γ	a	p	ν	γ	a
Bias	0.00615	0.02167	0.00012	0.00215	0.00023	0.00273	0.00003	0.00052
MSE	0.00321	0.00825	0.00008	0.00102	0.00004	0.00019	0.00001	0.00008
CP	0.89723	0.81426	0.94139	0.84311	0.89227	0.81337	0.93772	0.84382
95% CI_{LB}	0.18936	0.46236	0.29201	0.49012	0.19102	0.47525	0.29025	0.49523
95% CI_{UB}	0.21985	0.53926	0.30941	0.51125	0.20824	0.53101	0.30125	0.50585

Table 3. Simulation results for scheme III.

	$n_1 = 20$				$n_2 = 50$			
	p	ν	γ	a	p	ν	γ	a
Bias	0.09858	0.24223	0.19302	0.16014	0.07145	0.19073	0.16325	0.14085
MSE	0.07145	0.21014	0.16328	0.14025	0.05241	0.17452	0.14145	0.13748
CP	0.92747	0.81457	0.97194	0.95712	0.91265	0.82602	0.97188	0.95639
95% CI_{LB}	0.46568	0.19365	0.58636	0.75145	0.47188	0.22325	0.61258	0.79569
95% CI_{UB}	0.54258	0.41256	0.82414	1.15369	0.53521	0.38525	0.79201	1.10254
	$n_3 = 150$				$n_4 = 300$			
	p	ν	γ	a	p	ν	γ	a
Bias	0.02035	0.14625	0.13205	0.12638	0.00413	0.12025	0.11015	0.11365
MSE	0.01996	0.13236	0.11859	0.12025	0.00251	0.11301	0.09852	0.09852
CP	0.91746	0.82746	0.96984	0.95612	0.92957	0.88230	0.96901	0.95598
95% CI_{LB}	0.48194	0.24189	0.64302	0.82638	0.48831	0.25885	0.66358	0.86677
95% CI_{UB}	0.52825	0.34125	0.76012	0.98510	0.51254	0.33236	0.73258	0.93748
	$n_5 = 500$				$n_6 = 700$			
	p	ν	γ	a	p	ν	γ	a
Bias	0.00074	0.08236	0.08858	0.09014	0.00002	0.00714	0.00752	0.01173
MSE	0.00066	0.06326	0.07968	0.06852	0.00001	0.00513	0.00524	0.00721
CP	0.90166	0.88109	0.96836	0.95583	0.90654	0.88112	0.96554	0.95566
95% CI_{LB}	0.49134	0.27256	0.68225	0.87480	0.49014	0.28541	0.69221	0.88369
95% CI_{UB}	0.51015	0.32748	0.71529	0.93254	0.50895	0.31526	0.70418	0.91596

7. Data Analysis

In this section, we clear the experimental importance of the BDsOGE-G family by using three applications. In each data set, we compare the fits of the proposed BDsOGEW distribution with well-known bivariate models. The tested distributions are compared using some criteria namely, the negative maximized log-likelihood $(-l)$, Akaike information criterion (AIC), and Hannan-Quinn information criterion (HQIC). The AIC is an estimator of prediction error and thereby relative quality of statistical models for a given set of data. It estimates the quality of each model, relative to each of the other models. AIC provides a means for model selection. The HQIC is a criterion for model selection. It is an alternative to AIC and Bayesian information criterion.

7.1. Data Set I: Nasal Drainage Severity Score

This data set represents the efficacy of steam inhalation in the treatment of common cold symptoms "0 = no symptoms; 1 = mild symptoms; 2 = moderate symptoms; 3 = severe symptoms" (see Davis, [30]). Figure 6 shows scatter and violin plots of the Nasal drainage severity score data.

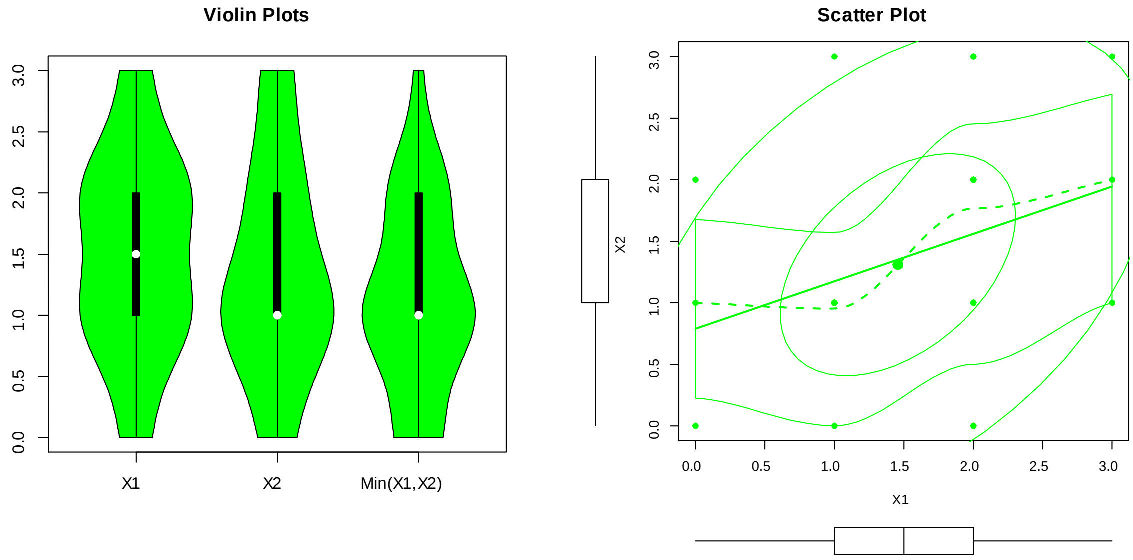

Figure 6. Scatter and violin plots of data set I.

Before analyzing the bivariate data, we first fit the marginals X_1 and X_2 separately and $\min(X_1, X_2)$. The MLEs of the parameters ν, p and α of the corresponding univariate distribution for X_1, X_2 and $\min(X_1, X_2)$ are $(1.915, 0.809, 0.911)$, $(5.073, 0.516, 0.601)$ and $(1.591, 0.769, 0.923)$, respectively. The $-l$ values are 78.855, 78.031 and 76.781, respectively. Moreover, the p-values ranged from 0.658 to 0.759. Now, we compare the BDsOGEW distribution with some competitive distributions like bivariate discrete exponential (BDsE), bivariate Poisson with four parameters (BPo-4P), independent bivariate Poisson (IBPo), bivariate discrete inverse exponential (BDsIE), bivariate discrete inverse Rayleigh (BDsIR), and bivariate discrete inverse Weibull (BDsIW) distributions. Table 4 lists the MLEs and some goodness-of-fit measures (GOFM).

Table 4. The MLEs and GOFM for data set I.

Model	MLEs	$-l$	AIC	HQIC
BDsE	$\widehat{v}_1 = 0.846, \widehat{v}_2 = 0.792, \widehat{v}_3 = 0.693$	88.002	182.004	183.349
BPo-4P	$\widehat{\lambda}_1 = 0.262, \widehat{\gamma}_1 = 0.165, \widehat{\lambda}_2 = 0.405, \widehat{\gamma}_2 = 2.971$	77.664	163.328	165.121
IBPo	$\widehat{\lambda}_1 = 1.499, \widehat{\lambda}_2 = 1.367$	92.478	188.956	189.853
BDsIE	$\widehat{v}_1 = 0.501, \widehat{v}_2 = 0.622, \widehat{v}_3 = 0.383$	92.482	190.964	192.309
BDsIR	$\widehat{v}_1 = 0.262, \widehat{v}_2 = 0.405, \widehat{v}_3 = 0.363$	78.659	163.318	164.663
BDsIW	$\widehat{v}_1 = 0.192, \widehat{v}_2 = 0.337, \widehat{v}_3 = 0.360, \widehat{\zeta} = 2.453$	76.513	161.026	162.815
BDsOGEW	$\widehat{\gamma} = 0.574, \widehat{v} = 2.457, \widehat{p} = 0.722, \widehat{a} = 0.780$	71.162	150.324	152.117

It is observed that the BDsOGEW model is the best among all tested models because it has the smallest values among $-l$, AIC, and HQIC when compared to the other competitive models. The 95% CIs for the BDsOGEW parameters $\gamma, v,\ p,\ a$ can be listed as $[0.378, 0.770]$, $[2.165, 2.748]$, $[0.624, 0.820]$, and $[0.594, 0.965]$, respectively. Figure 7 shows the estimated joint PMF for the BDsOGEW distribution and the other competitive models, which support the results of Table 4.

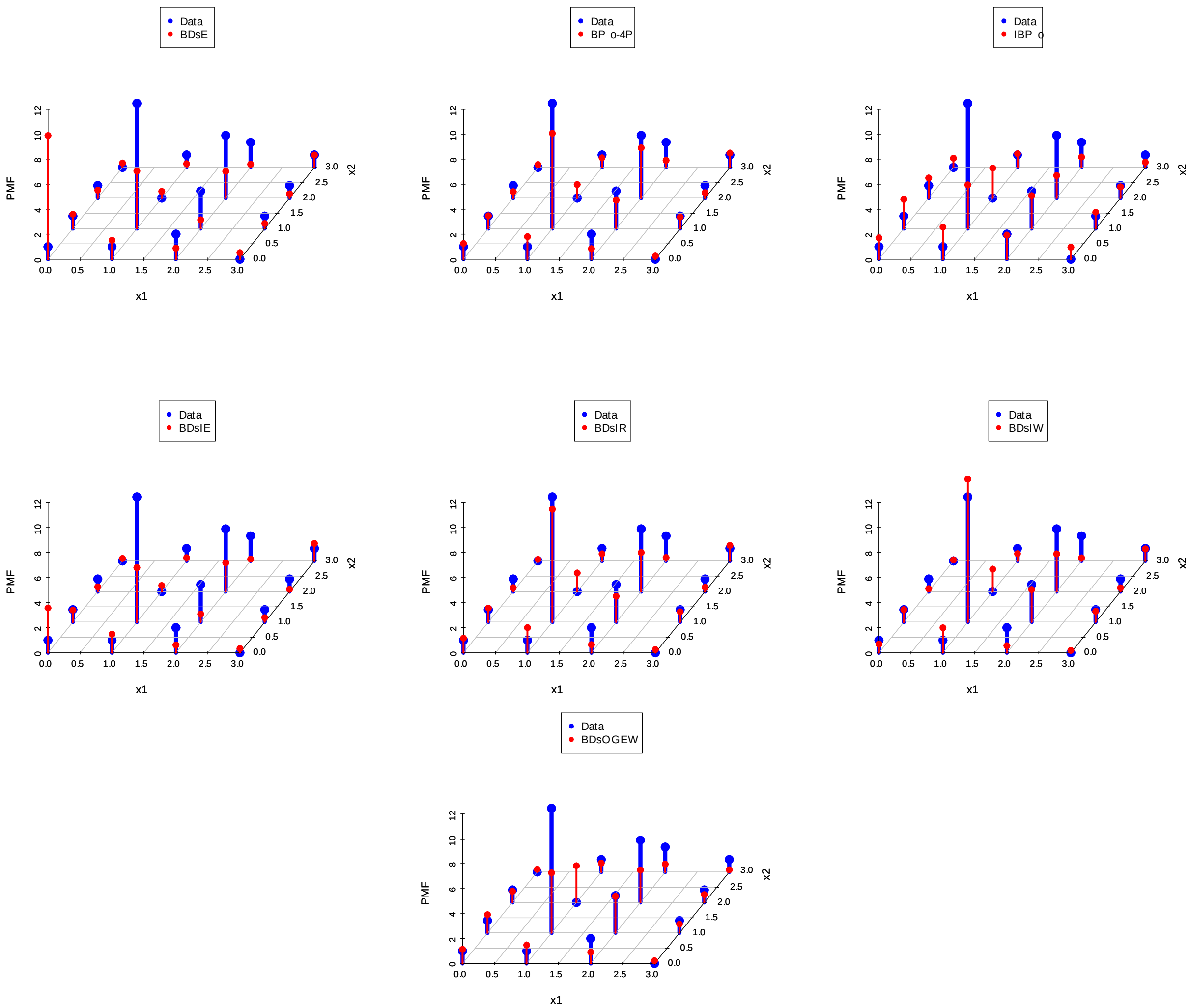

Figure 7. The estimated joint PMF based on data set I.

Figures 8–10 show the profile l for each parameter as well as contour plots. It has been found that the maximum likelihood estimates are unique.

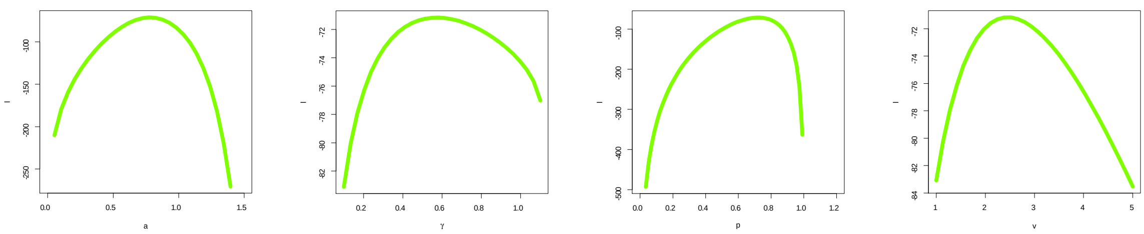

Figure 8. The profile l for parameters a, γ, p and v for data set I.

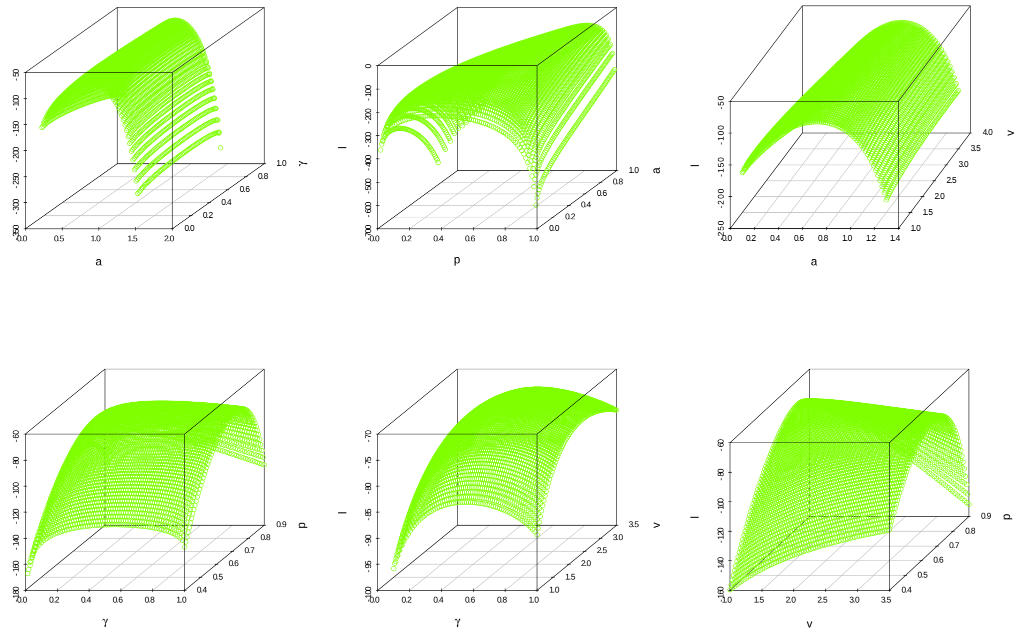

Figure 9. A 3-dimensional surface profile likelihood of a, γ, p and v based on data set I.

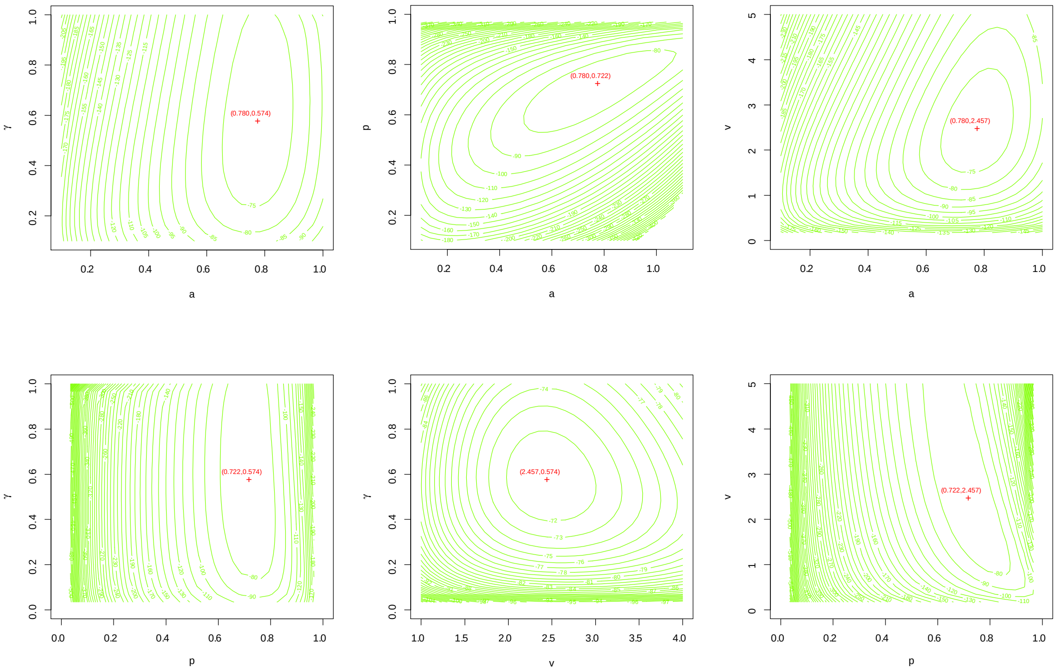

Figure 10. Contour diagrams of the model estimators based on data set I.

7.2. Data Set II: Football Score

The data set consists of a football match score in talian football matches from 1996 to 2011, between "ACF Fiorentina" (X_1) and "Juventus" (X_2). The data source is "http://www.worldfootball.net/competition/ita-serie-a/ (accessed on 16 March 2023)". Figure 11 shows scatter and violin plots of the football score data.

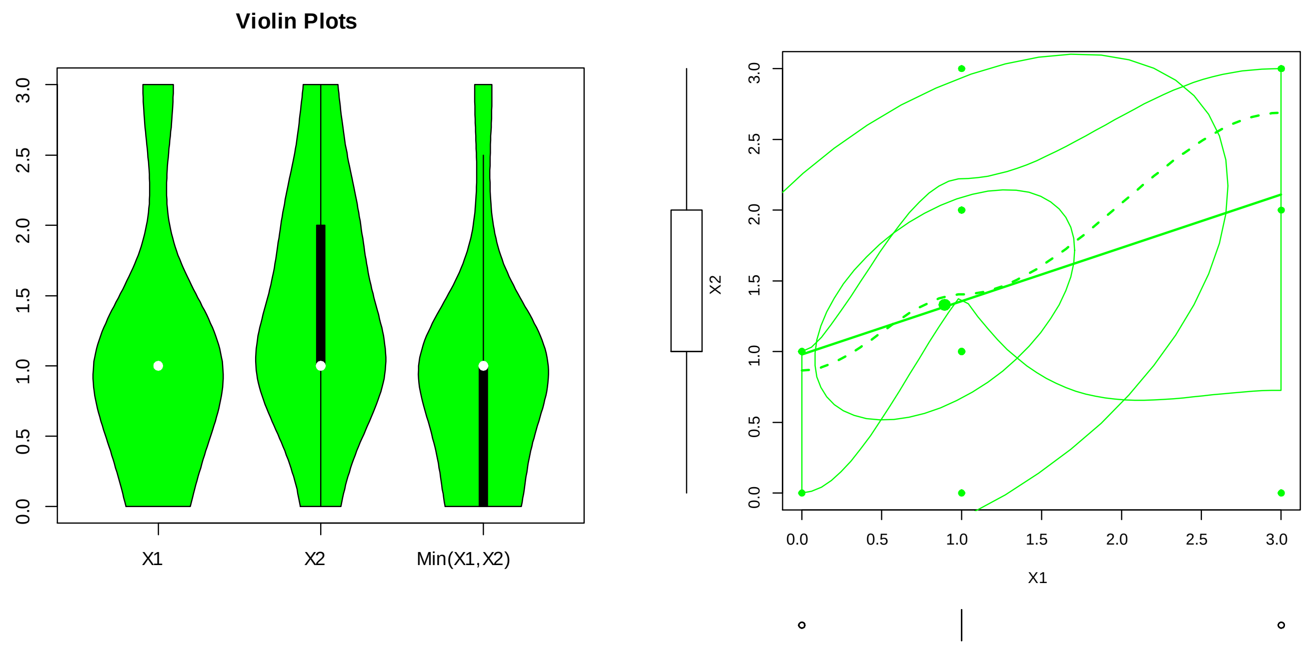

Figure 11. Scatter and violin plots of data set II.

We first fit the marginals X_1 and X_2 separately and $\min(X_1, X_2)$. The MLEs of the parameters ν, p and α of the corresponding univariate distribution for X_1, X_2 and $\min(X_1, X_2)$ are (29.269, 0.165, 0.357), (5.083, 0.534, 0.631) and (28.051, 0.152, 0.384), respectively. The $-l$ values are 65.898, 67.856 and 64.723, respectively. Moreover, the p-values ranged from 0.786 to 0.801. Now, we compare the BDsOGEW distribution with some competitive distributions like BDsE, bivariate discrete Rayleigh (BDsR), bivariate discrete Weibull (BDsW), bivariate Poisson with minimum operator ($\text{BPo}_{\min}$), bivariate Poisson with three parameters (BPo-3P), inverse bivariate Poisson (IBPo), BDsIE, and BDsIR distributions. Table 5 lists the MLEs and some GOFM.

Table 5. The MLEs and GOFM for data set II.

Model	MLEs	$-l$	AIC	HQIC
BDsE	$\widehat{v}_1 = 0.625, \widehat{v}_2 = 0.812, \widehat{v}_3 = 0.713$	75.421	156.842	157.929
BDsR	$\widehat{v}_1 = 0.790, \widehat{v}_2 = 0.872, \widehat{v}_3 = 0.905$	63.931	133.862	134.949
BDsW	$\widehat{v}_1 = 1.36, \widehat{v}_2 = 2.10, \widehat{v}_3 = 2.27, \widehat{\zeta} = 2.125$	63.911	135.822	137.271
$\text{BPo}_{\min}$	$\widehat{v}_1 = 1.36, \widehat{v}_2 = 2.10, \widehat{v}_3 = 2.27$	64.228	134.456	135.543
BPo-3P	$\widehat{\gamma}_1 = 1.08, \widehat{\gamma}_2 = 1.38, \widehat{\gamma}_3 = 0.70$	64.932	135.864	136.951
IBPo	$\widehat{\lambda}_1 = 1.08, \widehat{\lambda}_2 = 1.38$	67.623	139.246	139.971
BDsIE	$\widehat{v}_1 = 0.669, \widehat{v}_2 = 0.388, \widehat{v}_3 = 0.514$	78.541	163.082	164.169
BDsIR	$\widehat{v}_1 = 0.493, \widehat{v}_2 = 0.212, \widehat{v}_3 = 0.561$	64.102	134.204	135.291
BDsOGEW	$\widehat{\gamma} = 0.759, \widehat{v} = 23.807, \widehat{p} = 0.213, \widehat{a} = 0.399$	62.589	133.178	134.627

It is noted that the BDsOGEW distribution is the best among all tested models. The 95% CIs for the BDsOGEW parameters γ, v, p, a can be reported as [0.620, 0.897], [23.529, 24.084], [0.0611, 0.3648], and [0.253, 0.544], respectively. Figure 12 shows the estimated joint PMF for the BDsOGEW distribution and the other competitive distributions, which prove the results of Table 5.

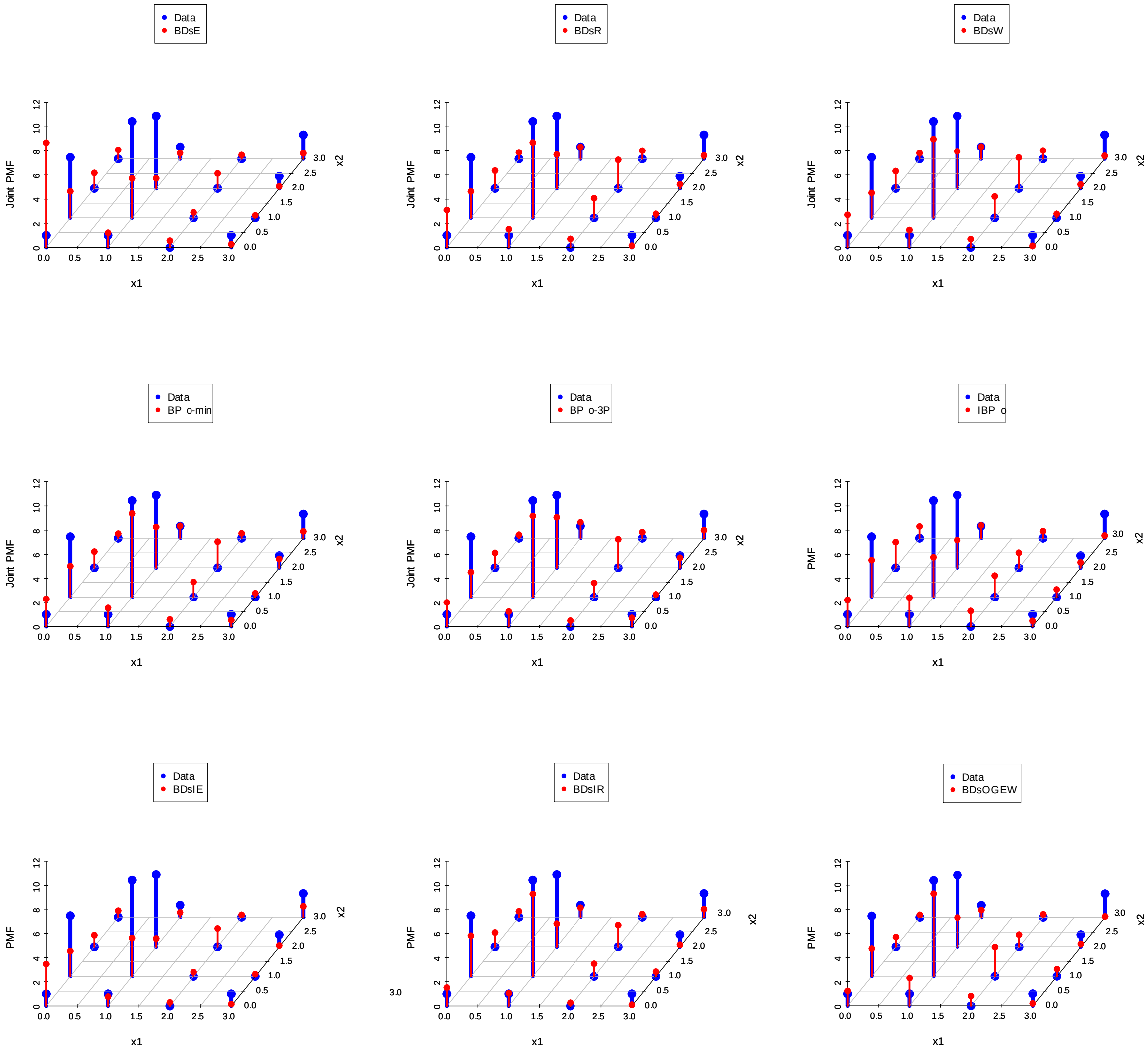

Figure 12. The estimated joint PMF based on data set II.

Figures 13–15 prove that the maximum likelihood estimates are unique.

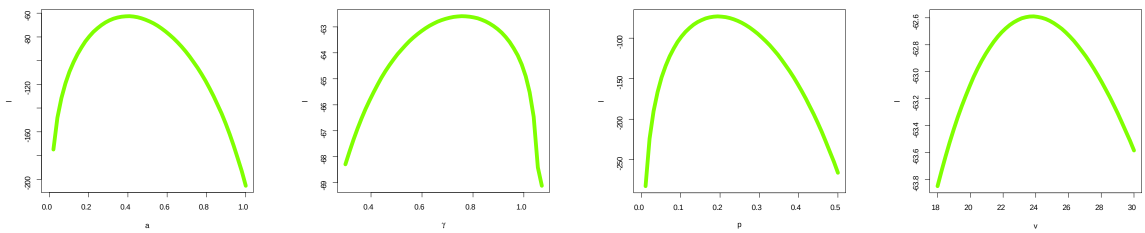

Figure 13. The profile l of a, γ, p and v according to data set II.

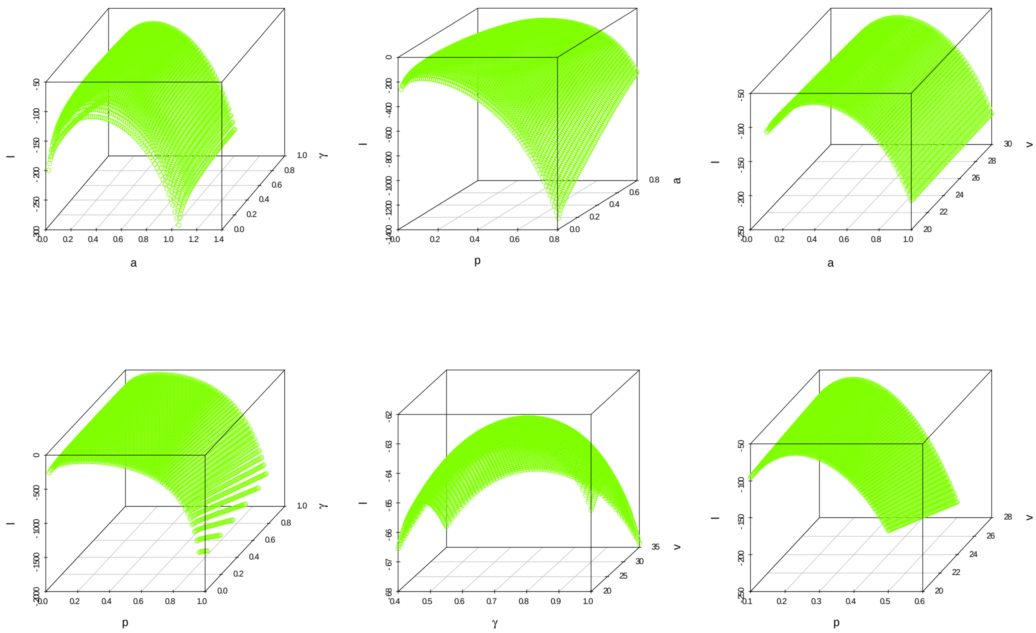

Figure 14. A 3-dimensional surface profile likelihood of a, γ, p and v based on data set II.

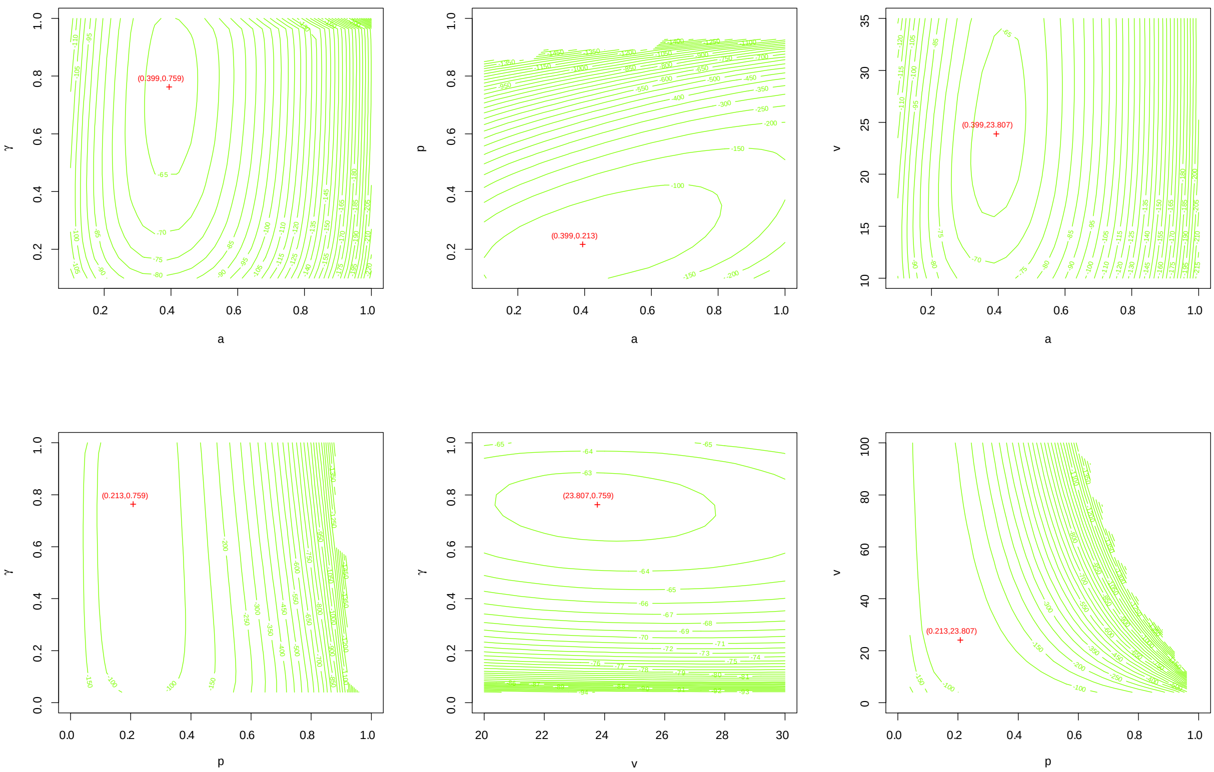

Figure 15. Contour diagrams of the model estimators based on data set II.

8. Concluding Remarks and Future Work

In this paper, a flexible bivariate odd discrete generator has been introduced based on the copula concept. Some of its statistical properties have been investigated. After reporting the general class, one special model of the new bivariate family has been studied in-detail. It was found that the new joint probability mass function can be utilized to model asymmetric as well as symmetric data surfaces. Moreover, the hazard rate function of the newly created distribution can be used to discuss various types of failures, including increasing-, decreasing-, bathtub-, and unimodal-shaped surface. The bivariate model parameters have been estimated utilizing the maximum likelihood technique. A simulation has been performed to test the performance of the maximum likelihood estimators based on different sample sizes, and it was found that the maximum likelihood approach could be used to discuss the real data herein. Finally, two distinctive real data sets, "*Nasal drainage severity score" and "Football score*", have been analyzed, and it was found that the proposed bivariate family has worked quite well in modeling the data. As a future work, the bivariate quantile residual life and multivariate extension of the proposed model with its applications will be discussed.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

- The PMF of the BDsOGE-G

  Utilizing Equations (3)–(6), the join PMF of the BDsOGE-G can be expressed as

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= C(F_{X_1}(x_1+1), F_{X_2}(x_2+1)) - C(F_{X_1}(x_1+1), F_{X_2}(x_2)) \\ &\quad - C(F_{X_1}(x_1), F_{X_2}(x_2+1)) + C(F_{X_1}(x_1), F_{X_2}(x_2)) \\ &= \begin{cases} P_1(x_1,x_2) & \text{if } \; x_1 < x_2 \\ P_2(x_1,x_2) & \text{if } \; x_2 < x_1 \\ P_3(x) & \text{if } \; x_1 = x_2 = x, \end{cases} \end{aligned}$$

where

$$\begin{aligned} P_1(x_1,x_2) &= \left(1-p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1-p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma} \\ &\quad -\left(1-p^{\Psi(x_1;\boldsymbol{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} + \left(1-p^{\Psi(x_1;\boldsymbol{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma} \\ &= \left(1-p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu}\left[\left(1-p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1-p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right] - \\ &\quad \left(1-p^{\Psi(x_1;\boldsymbol{\Theta})}\right)^{\nu}\left[\left(1-p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1-p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right] \\ &= \left[\left(1-p^{\Psi(x_1+1;\boldsymbol{\Theta})}\right)^{\nu} - \left(1-p^{\Psi(x_1;\boldsymbol{\Theta})}\right)^{\nu}\right]\left[\left(1-p^{\Psi(x_2+1;\boldsymbol{\Theta})}\right)^{\nu\gamma} - \left(1-p^{\Psi(x_2;\boldsymbol{\Theta})}\right)^{\nu\gamma}\right] \\ &= f_{\text{DsOGE-G}}(x_1;p,\nu,\boldsymbol{\Theta}) f_{\text{DsOGE-G}}(x_2;p,\nu,\gamma,\boldsymbol{\Theta}), \end{aligned}$$

$$
\begin{aligned}
P_2(x_1,x_2) &= \left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}-\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}\\
&\quad -\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}+\left(1-p^{\Psi(x_1;\mathbf{\Theta})}\right)^{\nu\gamma}\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}\\
&= \left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}[\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}]-\\
&\quad \left(1-p^{\Psi(x_1;\mathbf{\Theta})}\right)^{\nu\gamma}[\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}]\\
&= \left[\left(1-p^{\Psi(x_1+1;\mathbf{\Theta})}\right)^{\nu\gamma}-\left(1-p^{\Psi(x_1;\mathbf{\Theta})}\right)^{\nu\gamma}\right]\left[\left(1-p^{\Psi(x_2+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x_2;\mathbf{\Theta})}\right)^{\nu}\right]\\
&= f_{\text{DsOGE-G}}(x_1;p,\nu,\gamma,\mathbf{\Theta})f_{\text{DsOGE-G}}(x_2;p,\nu,\mathbf{\Theta})
\end{aligned}
$$

and

$$
\begin{aligned}
P_3(x) &= \left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\\
&\quad -\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}+\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu\gamma}\\
&= \left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}\left[\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu}-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\right]-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}\\
&\quad \times\left[\left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu\gamma}\right]\\
&= \left(1-p^{\Psi(x+1;\mathbf{\Theta})}\right)^{\nu\gamma}f_{\text{DsOGE-G}}(x;p,\nu,\mathbf{\Theta})-\left(1-p^{\Psi(x;\mathbf{\Theta})}\right)^{\nu}f_{\text{DsOGE-G}}(x;p,\nu,\gamma,\mathbf{\Theta}).
\end{aligned}
$$

- Estimation code for real data

$$
\begin{aligned}
Lok[a\_,p\_v\_,\gamma\_] &:= \textstyle\sum_{i=1}^{n1}\text{Log}\left[\left(1-p^{\left(-1+e^{(x1[[i]]+1)^a}\right)}\right)^{v}-\left(1-p^{\left(-1+e^{(x1[[i]])^a}\right)}\right)^{v}\right]\\
&+\textstyle\sum_{i=1}^{n1}\text{Log}\left[\left(1-p^{\left(-1+e^{(x2[[i]]+1)^a}\right)}\right)^{v*\gamma}-\left(1-p^{\left(-1+e^{(x2[[i]])^a}\right)}\right)^{v*\gamma}\right]\\
&+\textstyle\sum_{i=1}^{n2}\text{Log}\left[\left(1-p^{\left(-1+e^{(Y1[[i]]+1)^a}\right)}\right)^{v*\gamma}-\left(1-p^{\left(-1+e^{(Y1[[i]])^a}\right)}\right)^{v*\gamma}\right]\\
&+\textstyle\sum_{i=1}^{n2}\text{Log}\left[\left(1-p^{\left(-1+e^{(Y2[[i]]+1)^a}\right)}\right)^{v}-\left(1-p^{\left(-1+e^{(Y2[[i]])^a}\right)}\right)^{v}\right]\\
&+\textstyle\sum_{i=1}^{n3}\text{Log}\left[\begin{array}{c}\left(1-p^{\left(-1+e^{(z[[i]]+1)^a}\right)}\right)^{v*\gamma}*\left(\left(1-p^{\left(-1+e^{(z[[i]]+1)^a}\right)}\right)^{v}-\left(1-p^{\left(-1+e^{(z[[i]])^a}\right)}\right)^{v}\right)\\-\left(1-p^{\left(-1+e^{(z[[i]])^a}\right)}\right)^{v}*\left(\left(1-p^{\left(-1+e^{(z[[i]]+1)^a}\right)}\right)^{v*\gamma}-\left(1-p^{\left(-1+e^{(z[[i]])^a}\right)}\right)^{v*\gamma}\right)\end{array}\right]
\end{aligned}
$$

```
parameter = 4;
large = {{2, 0}, {3, 2}, {1, 0}, {2, 1}, {2, 1}, {3, 1}, {2, 1}, {2, 0}};
small = {{0, 2}, {2, 3}, {1, 3}, {2, 3}, {0, 1}};
equal = {{1, 1}, {0, 0}, {1, 1}, {1, 1}, {2, 2}, {1, 1}, {2, 2}, {1, 1}, {2, 2}, {1, 1}, {2, 2}, {1, 1}, {2, 2},
{1, 1}, {1, 1}, {1, 1}, {3, 3}};
Unprotect[Power]; Power[0 | 0., 0 | 0.] = 1; Protect[Power];
Y1 = large[[All, 1]];
Y2 = large[[All, 2]];
n1 = Length[x1]
x1 = small[[All, 1]];
x2 = small[[All, 2]];
n2 = Length[Y1]
z = equal[[All, 1]];
n3 = Length[z]
```

NMaximize$[\{Lok[a_,p_v_,\gamma_],0<p<1,0<a<\infty,0<v<\infty,0<\gamma<1\},\{a,p,v,\gamma\}]$

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