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Research Article

E-Bayesian Estimation of Hierarchical Poisson-Gamma Model on the Basis of Restricted and Unrestricted Parameter Spaces

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In this study, we use the idea of the hierarchical model (HM) to estimate an unknown parameter of the hierarchical Poisson-Gamma model using the E-Bayesian (E-B) theory. We propose the idea of hierarchical probability function instead of the traditional hierarchical prior density function. We aim to infer E-B estimates with respect to the conjugate Gamma prior distribution along with the E-posterior risks on the basis of different symmetric and asymmetric loss functions (LFs) under restricted and unrestricted parameter spaces using uniform hyperprior. Whereas, E-B estimators are compared with maximum likelihood estimators (MLEs) using mean squared error (MSE). Monte Carlo simulations are prosecuted to study the efficiency of E-B estimators empirically. It is shown that the LFs under a restricted parameter space dominate to estimate the parameter of the hierarchical Poisson-Gamma model. It is also found that the E-B estimators are more precise than MLEs, and Stein's LF has the least E-PR. Moreover, the application of outcomes to a real-life example has been made for analysis, comparison, and motivation.

1. Introduction

The data, model, prior, and LF are the four essential components of Bayesian analysis. According to Mood et al. [1], the Bayes estimator of δ, using the LF $L(\delta;\delta^*)$ and prior distribution $h(\delta)$, is the estimator that minimizes the posterior risk (PR) with respect to the posterior distribution $(\delta \mid \mathbf{Y})$. The information about restricted and unrestricted parameter spaces is useful in an effort to estimate a parameter using the Bayesian theory. Zhang et al. [2] suggested that the LFs under unrestricted space $(-\infty,+\infty)$ penalize evenly for both underestimation and overestimation. Whereas, the LFs under restricted space $(0,+\infty)$ penalize evenly for gross underestimation and gross overestimation. The most ordinarily used symmetric LF is squared error LF which applies equal weightage to underestimation and overestimation. Parsian and Nematollahi [3] figured out a better asymmetric LF known as Stein's LF.

In some situations, when available information has various levels of the observational units, then the hierarchical modelling is preferred. The HM has not only a major interest in processing computational schemes but also supports in inferring the multiparameter problems. According to Grover and Vriens [4], a statistical model scripted with multiple levels is known as Bayesian HM. It uses hyperparameters and hyperpriors. The HM uses the

Bayesian method to estimate the unknown parameter(s) of posterior distribution. Hierarchical Bayesian (HB) models are truly the synthesis of two parts: (i) a model scripted in hierarchical form that is (ii) computed using the Bayesian theory. Garthwaite et al. [5] stated that the Bayesian inference considers the unknown parameters as variables, thus the parameters may have the probability distributions. The HMs make use of this variability. According to Lynch [6], HMs make use of the resilience given by the Bayesian theory; i.e., it assumes the parameters of interest as variables that may have the probability distributions. The models in which parameters follow some kind of hierarchical patterns are known as HMs.

Lindley and Smith [7] introduced the concept of hierarchical prior distribution firstly in 1972. The E-B and HB methods have been used by many authors in the literature. Han [8] proposed E-B and HB methods to estimate failure rate using exponential distribution. Iqbal and Yousuf Shad [9] derived E-Bayesian estimates of the hierarchical normal and inverse gamma model under different LFs. Han [10] evaluated E-B and HB estimates of binomial distribution under three distinct hyperpriors. Athirakrishnan and Abdul-Sathar [11] computed E-B and HB estimations to estimate the parameter of inverse Rayleigh distribution. Basheer et al. [12] derived E-B and HB estimates of inverse Weibull distribution. Reyad et al. [13] analyzed E-B and HB methods to estimate the parameter of inverse Weibull distribution on the basis of dual generalized order statistics. Yaghoobzadeh Shahrastani [14] computed E-B and HB estimates of Gompertz distribution under the type-II censoring scheme. Han [15] derived the E-B and HB estimations of the shape parameter of Pareto distribution. For more details about E-Bayesian estimation (E-BE), we may refer to [16–19]. However, those authors used hierarchical prior density function. Whereas, we exploit the idea of hierarchical probability function and propose E-BE to estimate the unknown parameter of hierarchical Poisson-Gamma model using various LFs under restricted and unrestricted parameter spaces.

The rest of the study is demonstrated as follows: the hierarchical Poisson-Gamma model is stated in Section 2. In Section 3, the E-BEs are presented. Monte Carlo simulation study and the real data example are analyzed in Sections 4 and 5, respectively. Finally, the conclusions are given in Section 6.

2. Hierarchical Poisson-Gamma Model

Suppose $\mathbf{Y}=\{Y_i, \text{ and } (i=1,\ldots,m)\}$ represents the set of values drawn from the following hierarchical Poisson-Gamma model ([20–22]):

$$\left.\begin{array}{ll} \mathrm{Y} \mid \delta & \sim \text{Poisson}(\delta) \\ \delta & \sim \Gamma(\gamma,\varphi) \end{array}\right\}, \tag{1}$$

where Poisson (δ) is the Poisson distribution having an unknown parameter $\delta>0$, and $\Gamma\ (\gamma,\varphi)$ is the Gamma distribution with unknown shape and rate parameters $\gamma(>0)$ and $\varphi(>0)$, respectively. The $\Gamma\ (\gamma,\varphi)$ is the prior distribution, and γ and φ are the hyper parameters. The probability mass function of Poisson distribution given that parameter δ is stated as follows:

$$\mathrm{f}(\mathrm{y}\mid\delta)=\frac{\delta^{\mathrm{y}}\exp(-\delta)}{\mathrm{y}!}. \tag{2}$$

The pdf of conjugate prior distribution is:

$$\mathrm{g}(\delta;\gamma,\varphi)=\frac{\varphi^{\gamma}}{\Gamma(\gamma)}\delta^{\gamma-1}\exp[-\varphi\delta]. \tag{3}$$

The likelihood function of the following equation is as under:

$$\mathrm{L}(\delta\mid\mathbf{Y})=\frac{\delta^{\sum_{i=1}^{m}y_i}\exp(-m\delta)}{\prod_{i=1}^{m}(y_i!)}. \tag{4}$$

Now, the log-likelihood function is given by:

$$\ell=\log[\mathrm{L}(\delta\mid\mathbf{Y})]=\left[\sum_{i=1}^{m}y_i\right]\log\delta-m\delta-\sum_{i=1}^{m}\log[y_i!], \tag{5}$$

setting $\partial\ell/\partial\delta=0$, we get the MLE of δ and is given by:

$$\delta^{*}_{\mathrm{ML}}=\frac{1}{m}\sum_{i=1}^{m}y_i. \tag{6}$$

Now, the posterior distribution of δ can be evaluated:

$$\begin{aligned}\pi(\delta\mid\mathbf{Y})&=\frac{\delta^{\left(\gamma+\sum_{i=1}^{m}y_i-1\right)}\exp[-(\varphi+m)\delta]}{\int_0^{\infty}\delta^{\left(\gamma+\sum_{i=1}^{m}y_i-1\right)}\exp[-(\varphi+m)\delta]d\delta}\\&=\frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\delta^{\gamma_o-1}\exp[-\varphi_o\delta].\end{aligned} \tag{7}$$

The posterior distribution of δ follows the Gamma distribution having $\gamma_o=(\gamma+\sum_{i=1}^{m}y_i)$ and $\varphi_o=(\varphi+m)$.

The marginal probability distribution of $\mathbf{Y}$ can be derived as follows:

$$\begin{aligned}g_Y(y)&=\int_0^{\infty}f(y\mid\delta)g(\delta;\gamma,\varphi)d\delta\\&=\frac{\varphi_o^{\gamma_o}}{y!\Gamma(\gamma_o)}\int_0^{\infty}\delta^{(\gamma_o+y)-1}\exp[-(\varphi_o+1)\delta]d\delta\\&=\frac{\varphi_o^{\gamma_o}}{y!\Gamma(\gamma_o)}\frac{\Gamma(\gamma_o+y)}{(\varphi_o+1)^{\gamma_o+y}}\\&=\frac{\Gamma(\gamma_o+y)}{y!\Gamma(\gamma_o)}\left(\frac{\varphi_o}{1+\varphi_o}\right)^{\gamma_o}\left(1-\frac{\varphi_o}{1+\varphi_o}\right)^{y}.\end{aligned} \tag{8}$$

If γ is a positive whole number, then $g_Y(y)$ follows the negative binomial distribution, given in the following equation, with $E(Y) = \gamma_o/\varphi_o$ and $\mathrm{Var}(Y) = \gamma_o/\varphi_o + \gamma_o/\varphi_o^2$.

$$g_Y(y) = \binom{\gamma_o + y - 1}{\gamma_o - 1}\left(\frac{\varphi_o}{1+\varphi_o}\right)^{\gamma_o}\left(1 - \frac{\varphi_o}{1+\varphi_o}\right)^{y},$$
$$\Rightarrow Y \sim \mathrm{Nb}\left(\gamma_o, \frac{\varphi_o}{1+\varphi_o}\right), \tag{9}$$

where $\gamma_o = (\gamma + \sum_{i=1}^{m} y_i)$ & $\varphi_o = (\varphi + m)$.

2.1. The Loss Function (LF), Bayesian Estimator (BE), and Posterior Risk (PR). Suppose a random variable Y has a pdf $f(y; \delta)$, for an estimator δ^* of parameter δ, the LFs, BEs, and PRs of δ are given in Table 1. The E-B estimators are computed using distinct LFs on the basis of restricted as well as unrestricted parameter spaces.

The LFs according to the unrestricted parameter space $(-\infty, +\infty)$ are: squared error LF (SELF), quadratic LF (QLF), and weighted squared error LF (WSELF).

The LFs according to the restricted parameter space $(0, +\infty)$ are: Stein's LF (STLF), power-power LF (PPLF), precautionary LF (PLF), degroot LF (DLF), and squared log error LF (SLELF).

3. E-Bayesian Estimation (E-BE)

3.1. Statements of E-BE and E-PR. Han [23] proposed that, in a conjugate prior distribution of δ given in the following equation, γ and φ should be designated to assure that $g(\delta; \gamma, \varphi)$ is a nonincreasing function of δ. The derivative of $g(\delta; \gamma, \varphi)$ w.r.t. δ is as follows:

$$\frac{d}{d\delta}[g(\delta; \gamma, \varphi)] = \frac{\varphi^{\gamma}}{\Gamma(\gamma)}\delta^{\gamma-2}\exp(-\varphi\delta)(\gamma - \varphi\delta - 1). \tag{10}$$

Note that $\gamma > 0$, $\varphi > 0$, and $\delta > 0$, if follows $0 < \gamma < 1$, $\varphi > 0$ due to $d[g(\delta; \gamma, \varphi)]/d\delta < 0$, and consequently, $g(\delta; \gamma, \varphi)$ is a nonincreasing function of δ.

The definitions of E-BE and E-PR as are stated as follows:

Definition 1. The E-BE of δ_{BE}^* is defined as:

$$\delta_{EBE}^* = \iint_S \delta_{BE}^* h(\gamma, \varphi) d\gamma\, d\varphi, \tag{11}$$

where γ and φ are the hyper parameters, $h(\gamma, \varphi)$ is the hyperprior, S be the range of hyper parameters, and δ_{BE}^* is the Bayesian estimation of δ.

Definition 2. The E-PR of δ_{EBE}^* is defined as:

$$E - PR(\delta_{EBE}^*) = \iint_S PR(\delta_{BE}^*) h(\gamma, \varphi) d\gamma\, d\varphi. \tag{12}$$

where $PR(\delta_{BE}^*)$ is the PR of the Bayesian estimation of δ.

The following hyperprior of γ and φ is assumed to derive the E-B estimators [24]:

$$h(\gamma, \varphi) = \frac{1}{k}$$
$$0 < \gamma < 1\ \&\ 0 < \varphi < k. \tag{13}$$

3.2. E-B and E-PR Estimations under the Assumed LFs

Theorem 1. *Suppose $\mathbf{Y} = \{y_i$, and $(i = 1, \ldots, m)\}$ be the set of values drawn from (1), then, we have the undermentioned estimators using SELF:*

(i) The BE of δ is as follows:

$$\delta_{BSE}^* = \frac{(\gamma + \sum_{i=1}^{m} y_i)}{(\varphi + m)}. \tag{14}$$

(ii) Using hyperprior (9), the E-B estimator of δ is:

$$\delta_{EBSE}^* = \frac{1}{k}\left(\frac{1}{2} + \sum_{i=1}^{m} y_i\right)\log\left(1 + \frac{k}{m}\right). \tag{15}$$

(iii) The PR and E-PR are given by:

$$PR(\delta_{BSE}^*) = \frac{(\gamma + \sum_{i=1}^{m} y_i)}{(\varphi + m)^2},$$
$$E - PR(\delta_{EBSE}^*) = \frac{(1/2 + \sum_{i=1}^{m} y_i)}{n(k + m)}. \tag{16}$$

Proof

(i) The BE of δ (i.e., δ_{BSE}^*) is:

$$\delta_{BSE}^* = E_{\delta|y}(\delta) = \frac{\gamma_o}{\varphi_o} = \frac{(\gamma + \sum_{i=1}^{m} y_i)}{(\varphi + m)}. \tag{17}$$

(ii) The E-B estimator of δ (i.e., δ_{EBSE}^*) is:

$$\delta_{EBSE}^* = \iint_S \delta_{BSE}^* h(\gamma, \varphi) d\gamma\, d\varphi = \int_0^k \int_0^1 \frac{(\gamma + \sum_{i=1}^{m} y_i)}{(\varphi + m)} \frac{1}{k} d\gamma\, d\varphi = \frac{1}{k}\left(\frac{1}{2} + \sum_{i=1}^{m} y_i\right)\log\left(1 + \frac{k}{m}\right). \tag{18}$$

Table 1: LF, BE, and PR.

LF	BE	PR
$\text{SELF} = (\delta-\delta^*)^2$	$\delta^*_{BSE} = E_{\delta\vert y}(\delta)$	$PR(\delta^*_{BSE}) = Var_{\delta\vert y}(\delta)$
$\text{STLF} = \delta^*/\delta - \log(\delta^*/\delta) - 1$	$\delta^*_{BST} = [E_{\delta\vert y}(\delta^{-1})]^{-1}$	$PR(\delta^*_{BST}) = [E_{\delta\vert y}(\log(\delta))] + \log[E_{\delta\vert y}(\delta^{-1})]$
$\text{QLF} = (\delta-\delta^*)^2/\delta^2$	$\delta^*_{BQ} = [E_{\delta\vert y}(\delta^{-1})]/[E_{\delta\vert y}(\delta^{-2})]$	$PR(\delta^*_{BQ}) = 1 - [E_{\delta\vert y}(\delta^{-1})]^2/[E_{\delta\vert y}(\delta^{-2})]$
$\text{PPLF} = (\delta-\delta^*)^2/\delta\delta^*$	$\delta^*_{BPP} = \sqrt{E_{\delta\vert y}(\delta)/E_{\delta\vert y}(\delta^{-1})}$	$PR(\delta^*_{BPP}) = 2[\sqrt{E_{\delta\vert y}(\delta^{-1})E_{\delta\vert y}(\delta)} - 1]$
$\text{PLF} = (\delta-\delta^*)^2/\delta^*$	$\delta^*_{BP} = \sqrt{E_{\delta\vert y}(\delta^2)}$	$PR(\delta^*_{BP}) = 2[\sqrt{E_{\delta\vert y}(\delta^2)} - E_{\delta\vert y}(\delta)]$
$\text{DLF} = (\delta-\delta^*)^2/(\delta^*)^2$	$\delta^*_{BD} = [E_{\delta\vert y}(\delta^2)]/[E_{\delta\vert y}(\delta)]$	$PR(\delta^*_{BD}) = 1 - [E_{\delta\vert y}(\delta)]^2/[E_{\delta\vert y}(\delta^2)]$
$\text{SLELF} = (\log(\delta) - \log(\delta^*))^2$	$\delta^*_{BSLE} = \exp[E_{\delta\vert y}(\log\delta)]$	$PR(\delta^*_{BSLE}) = Var_{\delta\vert y}(\log\delta)]$
$\text{WSELF} = (\delta-\delta^*)^2/\delta$	$\delta^*_{BW} = [E_{\delta\vert y}(\delta^{-1})]^{-1}$	$PR(\delta^*_{BW}) = E_{\delta\vert y}(\delta) - [E_{\delta\vert y}(\delta^{-1})]^{-1}$

(iii) The PR and E-PR under SELF are calculated:

$$\begin{aligned} PR(\delta^*_{BSE}) &= Var_{\delta|y}(\delta) \\ &= \frac{\gamma_o}{\varphi_o^2} \\ &= \frac{(\gamma+\sum_{i=1}^m y_i)}{(\varphi+m)^2}, \end{aligned} \tag{19}$$

$$E - PR(\delta^*_{EBSE}) = \iint_S PR(\delta^*_{BSE}) h(\gamma,\varphi) d\gamma\, d\varphi = \int_0^k\int_0^1 \frac{(\gamma+\sum_{i=1}^m y_i)}{(\varphi+m)^2}\frac{1}{k} d\gamma\, d\varphi = \frac{(1/2+\sum_{i=1}^m y_i)}{n(k+m)}.$$

□

Theorem 2. *Suppose $\mathbf{Y}=\{y_i$, and $(i=1,\ldots,m)\}$ denotes the sample values from (1), then we have the succeeding results under STLF:*

(i) The BE of δ is:

$$\delta^*_{BST} = \frac{(\gamma+\sum_{i=1}^m y_i - 1)}{(\phi+m)}. \tag{20}$$

(ii) Using hyperprior (9), the E-B estimator of δ is:

$$\delta^*_{EBST} = \frac{1}{k}\left(-\frac{1}{2}+\sum_{i=1}^m y_i\right)\log\left(1+\frac{k}{m}\right). \tag{21}$$

(iii) The PR and E-PR are given by:

$$PR(\delta^*_{BST}) = \text{PolyGamma}\left(0,\gamma+\sum_{i=1}^m y_i\right) - \log\left(\gamma+\sum_{i=1}^m y_i - 1\right), \tag{22}$$

and

$$E - PR(\delta^*_{EBST}) = 1 + \left(1-\sum_{i=1}^m y_i\right)\log\left(\frac{\sum_{i=1}^m y_i}{\sum_{i=1}^m y_i - 1}\right). \tag{23}$$

Proof

(i) The BE of δ (i.e., δ^*_{BST})) is:

$$\begin{aligned} \delta^*_{BST} &= E_{\delta|x}\left(\delta^{-1}\right)\Big]^{-1} \\ &= \frac{\gamma_o - 1}{\varphi_o} \\ &= \frac{(\gamma+\sum_{i=1}^m y_i - 1)}{(\varphi+m)}. \end{aligned} \tag{24}$$

(ii) The E-B estimator of δ (i.e., δ^*_{EBST}) is:

$$\delta^*_{EBST} = \iint_S \delta^*_{BST} h(\gamma,\varphi) d\gamma\, d\varphi = \int_0^k\int_0^1 \frac{(\gamma+\sum_{i=1}^m y_i - 1)}{(\varphi+m)}\frac{1}{k} d\gamma\, d\varphi = \frac{1}{k}\left(-\frac{1}{2}+\sum_{i=1}^m y_i\right)\log\left(1+\frac{k}{m}\right). \tag{25}$$

(iii) To derive the PR, we require the following calculation:

$$E_{\delta|y}(\log\delta) = \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty (\log\delta)\delta^{\gamma_o-1}\exp[-\varphi_o\delta]d\delta = \text{PolyGamma}(0,\gamma_o) - \log(\varphi_o), \tag{26}$$

where $\Gamma'(\gamma_o)/\Gamma(\gamma_o) = \text{PolyGamma}(0, \gamma_o)$ that may be solved using the PolyGamma function in Mathematica. The details are given in the appendix.

Now, the PR and E-PR are calculated:

$$\begin{aligned}
\text{PR}(\delta^*_{BST}) &= \left[E_{\delta|x}(\log(\delta))\right] + \log\left[E_{\delta|x}(\delta^{-1})\right]\\
&= \left(\text{PolyGamma}\left(0,\gamma+\sum_{i=1}^m y_i\right) - \log(\varphi+m)\right) + \log\left(\frac{\varphi+m}{\gamma+\sum_{i=1}^m y_i - 1}\right)\\
&= \text{PolyGamma}\left(0,\gamma+\sum_{i=1}^m y_i\right) - \log\left(\gamma+\sum_{i=1}^m y_i - 1\right),\\
E-\text{PR}(\delta^*_{EBST}) &= \iint_S \text{PR}(\delta^*_{BST})h(\gamma,\varphi)d\gamma\, d\varphi\\
&= \int_0^k\int_0^1\left(\text{PolyGamma}\left(0,\gamma+\sum_{i=1}^m y_i\right) - \log\left(\gamma+\sum_{i=1}^m y_i - 1\right)\right)\frac{1}{k}d\gamma\, d\varphi\\
&= \log\left(\sum_{i=1}^m y_i\right) - \left(\log\left(\sum_{i=1}^m y_i\right) - 1 - \left(1-\sum_{i=1}^m y_i\right)\log\left(\frac{\sum_{i=1}^m y_i}{\sum_{i=1}^m y_i - 1}\right)\right)\\
&= 1 + \left(1-\sum_{i=1}^m y_i\right)\log\left(\frac{\sum_{i=1}^m y_i}{\sum_{i=1}^m y_i - 1}\right),
\end{aligned} \tag{27}$$

where $\text{PR}(\delta^*_{BST})$ and $E-\text{PR}(\delta^*_{EBST})$ are derived in closed form using Mathematica. □

Theorem 3. *Suppose* $\boldsymbol{Y} = \{y_i$, *and* $(i = 1, \ldots, m)\}$ *represents the sample drawn from (1), then, we have the undermentioned results under QLF:*

(i) The BE of δ *is:*

$$\delta^*_{BQ} = \frac{(\gamma+\sum_{i=1}^m y_i - 2)}{(\varphi+m)}. \tag{28}$$

(ii) Using hyperprior (9), the E-B estimator of δ *is:*

$$\delta^*_{EBQ} = \frac{1}{k}\left(-\frac{3}{2}+\sum_{i=1}^m y_i\right)log\left(1+\frac{k}{m}\right). \tag{29}$$

(iii) The PR and E-PRs are given by:

$$\begin{aligned}
\text{PR}(\delta^*_{BQ}) &= \frac{1}{(\gamma+\sum_{i=1}^m y_i - 1)},\\
E-\text{PR}(\delta^*_{EBQ}) &= \log\left(\frac{\sum_{i=1}^m y_i}{\sum_{i=1}^m y_i - 1}\right).
\end{aligned} \tag{30}$$

Proof

(i) The BE of δ (i.e., δ^*_{BQ}) is:

$$\delta^*_{BQ} = \frac{E_{\delta|y}(\delta^{-1})}{E_{\delta|y}(\delta^{-2})} = \frac{\gamma_o - 2}{\varphi_o} = \frac{(\gamma+\sum_{i=1}^m y_i - 2)}{(\varphi+m)}. \tag{31}$$

(ii) The E-B estimator of δ (i.e., δ^*_{EBQ}) is:

$$\delta_{EBQ}^{*} = \iint_{S} \delta_{BQ}^{*} h(\gamma,\varphi) d\gamma\, d\varphi = \int_{0}^{k}\int_{0}^{1} \frac{\left(\gamma + \sum_{i=1}^{m} y_i - 2\right)}{(\varphi + m)} \frac{1}{k} d\gamma\, d\varphi = \frac{1}{k}\left(\frac{-3}{2} + \sum_{i=1}^{m} y_i\right)\log\left(1 + \frac{k}{m}\right). \quad (32)$$

(iii) The PR and E-PR are calculated:

$$\begin{aligned} \mathrm{PR}\left(\delta_{BQ}^{*}\right) &= 1 - \frac{\left[E_{\delta|y}\left(\delta^{-1}\right)\right]^{2}}{E_{\delta|y}\left(\delta^{-2}\right)} \\ &= \frac{1}{\gamma_o - 1} \\ &= \frac{1}{\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}, \end{aligned} \quad (33)$$

$$\begin{aligned} E - \mathrm{PR}\left(\delta_{EBQ}^{*}\right) &= \iint_{S} \mathrm{PR}\left(\delta_{BQ}^{*}\right) h(\gamma,\varphi) d\gamma\, d\varphi \\ &= \int_{0}^{k}\int_{0}^{1} \frac{1}{\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)} \frac{1}{k} d\gamma\, d\varphi \\ &= \log\left(\frac{\sum_{i=1}^{m} y_i}{\sum_{i=1}^{m} y_i - 1}\right). \end{aligned}$$

□

Theorem 4. *Suppose* $\mathbf{Y} = \{y_i,$ *and* $(i = 1, \ldots, m)\}$ *be the set of sample drawn from (1), then, we obtain the succeeding estimators using PPLF:*

(i) The BE of δ *is:*

$$\delta_{BPP}^{*} = \frac{\sqrt{\left(\gamma + \sum_{i=1}^{m} y_i\right)\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}}{(\varphi + n)}. \quad (34)$$

(ii) Using hyperprior (9), the E-B estimator of δ *is:*

$$\delta_{\mathrm{EBPP}}^{*} = \frac{1}{k}\log\left(1 + \frac{k}{m}\right)\int_{0}^{1} \sqrt{\left(\gamma + \sum_{i=1}^{m} y_i\right)\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)} d\gamma. \quad (35)$$

(iii) The PR and E-PR are given by:

$$\mathrm{PR}\left(\delta_{BPP}^{*}\right) = 2\left[\sqrt{\frac{\left(\gamma + \sum_{i=1}^{m} y_i\right)}{\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}} - 1\right], \quad (36)$$

$$E - \mathrm{PR}\left(\delta_{EBPP}^{*}\right) = 2\left[\int_{0}^{1} \sqrt{\frac{\left(\gamma + \sum_{i=1}^{m} y_i\right)}{\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}} d\gamma - 1\right]. \quad (37)$$

Proof

(i) The BE of δ (i.e., δ_{BPP}^{*}) is:

$$\begin{aligned} \delta_{BPP}^{*} &= \sqrt{\frac{E_{\delta|y}(\delta)}{E_{\delta|y}\left(\delta^{-1}\right)}} \\ &= \sqrt{\frac{\gamma_o\left(\gamma_o - 1\right)}{\varphi_o^{2}}} \\ &= \frac{\sqrt{\left(\gamma + \sum_{i=1}^{m} y_i\right)\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}}{(\varphi + m)}. \end{aligned} \quad (38)$$

(ii) The E-B estimator of δ (i.e., δ_{EBPP}^{*}) is:

$$\begin{aligned} \delta_{EBPP}^{*} &= \iint_{S} \delta_{BPP}^{*} h(\gamma,\varphi) d\gamma\, d\varphi = \int_{0}^{k}\int_{0}^{1} \frac{\sqrt{\left(\gamma + \sum_{i=1}^{m} y_i\right)\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}}{k(\varphi + m)} d\gamma\, d\varphi \\ &= \frac{1}{k}\log\left(1 + \frac{k}{m}\right)\int_{0}^{1} \sqrt{\left(\gamma + \sum_{i=1}^{m} y_i\right)\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)} d\gamma, \end{aligned} \quad (39)$$

δ^*_{EBPP} cannot be derived in closed form and numerically integrated using R.

(iii) The PR and E-PR are calculated:

$$\begin{aligned}
\mathrm{PR}\left(\delta^*_{BPP}\right) &= 2\left[\sqrt{E_{\delta|y}\left(\delta^{-1}\right)E_{\delta|y}(\delta)} - 1\right] = 2\left[\sqrt{\frac{\gamma_o}{\gamma_o - 1}} - 1\right] \\
&= 2\left[\sqrt{\frac{\left(\gamma + \sum_{i=1}^m y_i\right)}{\left(\gamma + \sum_{i=1}^m y_i - 1\right)}} - 1\right], \\
E - PR\left(\delta^*_{EBPP}\right) &= \iint_S \mathrm{PR}\left(\delta^*_{BPP}\right) h(\gamma,\varphi) d\gamma\, d\varphi \\
&= \int_0^k \int_0^1 2\left[\sqrt{\frac{\left(\gamma + \sum_{i=1}^m y_i\right)}{\left(\gamma + \sum_{i=1}^m y_i - 1\right)}} - 1\right]\frac{1}{k} d\gamma\, d\varphi.
\end{aligned} \tag{40}$$

$E - \mathrm{PR}\left(\delta^*_{EBPP}\right)$ can be numerically integrated using R software. □

Theorem 5. *Suppose* $\mathbf{Y} = \{y_i$, *and* $(i = 1, \ldots, m)\}$ *be the sample values drawn from (1), then, we get the following estimators using PLF:*

(i) The BE of δ *is:*

$$\delta^*_{BP} = \frac{\sqrt{\left(\gamma + \sum_{i=1}^m y_i\right)\left(\gamma + \sum_{i=1}^m y_i + 1\right)}}{(\varphi + n)}. \tag{41}$$

(ii) Using hyperprior (9), the E-B estimator of δ *is:*

$$\delta^*_{EBP} = \frac{1}{k}\log\left(1 + \frac{k}{m}\right)\int_0^1 \sqrt{\left(\gamma + \sum_{i=1}^m y_i\right)\left(\gamma + \sum_{i=1}^m y_i + 1\right)} d\gamma. \tag{42}$$

(iii) The PR and E-PR are given by

:

$$\mathrm{PR}\left(\delta^*_{BP}\right) = \frac{2\left[\sqrt{\left(\gamma + \sum_{i=1}^m y_i\right)\left(\gamma + \sum_{i=1}^m y_i + 1\right)} - \left(\gamma + \sum_{i=1}^m y_i\right)\right]}{(\varphi + n)}, \tag{43}$$

$$E - \mathrm{PR}\left(\delta^*_{EBP}\right) = \frac{2}{k}\log\left(1 + \frac{k}{m}\right)\left[\int_0^1 \sqrt{\left(\gamma + \sum_{i=1}^m y_i\right)\left(\gamma + \sum_{i=1}^m y_i + 1\right)} d\gamma - \left(\frac{1}{2} + \sum_{i=1}^m y_i\right)\right], \tag{44}$$

Proof

(i) The BE of δ (i.e., δ^*_{BP}) is:

$$\delta_{BP}^* = \sqrt{E_{\delta|y}(\delta^2)} = \sqrt{\frac{\gamma_o(\gamma_o+1)}{\varphi_o^2}} = \frac{\sqrt{(\gamma+\sum_{i=1}^m y_i)(\gamma+\sum_{i=1}^m y_i+1)}}{(\varphi+m)}. \tag{45}$$

(ii) The E-B estimator of δ (i.e., δ_{EBP}^*) is:

$$\begin{aligned}\delta_{EBP}^* &= \iint_S \delta_{BP}^* h(\gamma,\varphi)d\gamma\, d\varphi \\ &= \int_0^k \int_0^1 \frac{\sqrt{(\gamma+\sum_{i=1}^m y_i)(\gamma+\sum_{i=1}^m y_i+1)}}{(\varphi+n)} \frac{1}{k} d\gamma\, d\varphi \\ &= \frac{1}{k}\log\left(1+\frac{k}{m}\right)\int_0^1 \sqrt{\left(\gamma+\sum_{i=1}^m y_i\right)\left(\gamma+\sum_{i=1}^m y_i+1\right)}d\gamma,\end{aligned} \tag{46}$$

δ_{EBP}^* cannot be derived analytically in closed form and integrated using R.

(iii) The PR and E-PR are calculated:

$$\begin{aligned}\mathrm{PR}(\delta_{BP}^*) &= 2\left[\sqrt{E_{\delta|y}(\delta^2)} - E_{\delta|y}(\delta)\right] = 2\left[\sqrt{\frac{\gamma_o(\gamma_o+1)}{\varphi_o^2}} - \frac{\gamma_o}{\varphi_o}\right] \\ &= \frac{2\left[\sqrt{(\gamma+\sum_{i=1}^m y_i)(\gamma+\sum_{i=1}^m y_i+1)} - (\gamma+\sum_{i=1}^m y_i)\right]}{(\varphi+n)}, \\ E-\mathrm{PR}(\delta_{EBP}^*) &= \iint_S \mathrm{PR}(\delta_{BP}^*)h(\gamma,\varphi)d\gamma\, d\varphi \\ &= \int_0^k\int_0^1 \frac{2\left[\sqrt{(\gamma+\sum_{i=1}^m y_i)(\gamma+\sum_{i=1}^m y_i+1)} - (\gamma+\sum_{i=1}^m y_i)\right]}{k(\varphi+n)} d\gamma\, d\varphi \\ &= \frac{2}{k}\log\left(1+\frac{k}{m}\right)\left[\int_0^1\sqrt{\left(\gamma+\sum_{i=1}^m y_i\right)\left(\gamma+\sum_{i=1}^m y_i+1\right)}d\gamma - \left(\frac{1}{2}+\sum_{i=1}^m y_i\right)\right].\end{aligned} \tag{47}$$

$E-\mathrm{PR}(\delta_{EBP}^*)$ cannot be derived in closed form and integrated using R. □

Theorem 6. *Suppose* $\mathbf{Y}=\{y_i$, *and* $(i=1,\ldots,m)\}$ *denotes the sample drawn from (1), then, we have the undermentioned results under DLF:*

(i) The BE of δ *is:*

$$\delta_{BD}^* = \frac{(\gamma+\sum_{i=1}^m y_i+1)}{(\varphi+m)}. \tag{48}$$

(ii) Using hyperprior (9), the E-B estimator of δ *is:*

$$\delta_{EBD}^* = \frac{1}{k}\left(\frac{3}{2}+\sum_{i=1}^m y_i\right)\log\left(1+\frac{k}{m}\right). \tag{49}$$

(iii) The PR and E-PR are given by:

$$\begin{aligned}\mathrm{PR}(\delta_{BD}^*) &= \frac{1}{(\gamma+\sum_{i=1}^m y_i+1)}, \\ E-\mathrm{PR}(\delta_{EBD}^*) &= \log\left(\frac{\sum_{i=1}^m y_i+2}{\sum_{i=1}^m y_i+1}\right).\end{aligned} \tag{50}$$

Proof

(i) The BE of δ (i.e., δ^*_{BD}) is:

$$\delta^*_{BD} = \frac{E_{\delta|y}(\delta^2)}{E_{\delta|y}(\delta)} = \frac{\gamma_o + 1}{\varphi_o} = \frac{(\gamma + \sum_{i=1}^m y_i + 1)}{(\varphi + m)}. \quad (51)$$

(ii) The E-B estimator of δ (i.e., δ^*_{EBD}) is:

$$\delta^*_{EBD} = \iint_S \delta^*_{BD} h(\gamma,\varphi) d\gamma\, d\varphi = \int_0^k \int_0^1 \frac{(\gamma + \sum_{i=1}^m y_i + 1)}{(\varphi + m)} \frac{1}{k} d\gamma\, d\varphi = \frac{1}{k}\left(\frac{3}{2} + \sum_{i=1}^m y_i\right)\log\left(1 + \frac{k}{m}\right). \quad (52)$$

(iii) The PR and E-PR are calculated:

$$\mathrm{PR}(\delta^*_{BD}) = 1 - \frac{[E_{\delta|y}(\delta)]^2}{E_{\delta|y}(\delta^2)} = \frac{1}{\gamma_o + 1} = \frac{1}{(\gamma + \sum_{i=1}^m y_i + 1)},$$
$$E - \mathrm{PR}(\delta^*_{EBD}) = \iint_S \mathrm{PR}(\delta^*_{BD}) h(\gamma,\varphi) d\gamma\, d\varphi = \int_0^k \int_0^1 \frac{1}{(\gamma + \sum_{i=1}^m y_i + 1)} \frac{1}{k} d\gamma\, d\varphi = \log\left(\frac{\sum_{i=1}^m y_i + 2}{\sum_{i=1}^m y_i + 1}\right). \quad (53)$$

□

Theorem 7. *Suppose* $\mathbf{Y} = \{y_i$, *and* $(i = 1, \ldots, m)\}$ *denotes the set of values drawn from (1), then, we derive the succeeding results under SLELF:*

(i) The BE of δ is:

$$\delta^*_{BSLE} = \frac{1}{(\varphi + m)} \exp\left(\mathrm{PolyGamma}\left(0, \gamma + \sum_{i=1}^m y_i\right)\right). \quad (54)$$

(ii) Using hyperprior (9), the E-B estimator of δ is

:

$$\delta^*_{\mathrm{EBSLE}} = \frac{1}{k}\log\left(1 + \frac{k}{m}\right)\int_0^1 \exp\left(\mathrm{PolyGamma}\left(0, \gamma + \sum_{i=1}^m y_i\right)\right) d\gamma. \quad (55)$$

(iii) The PR and E-PR are given by:

$$\mathrm{PR}(\delta^*_{BSLE}) = \mathrm{PolyGamma}\left(1, \gamma + \sum_{i=1}^m y_i\right), \quad (56)$$

$$E - \mathrm{PR}(\delta^*_{\mathrm{EBSLE}}) = \int_0^1 \left(\mathrm{PolyGamma}\left(1, \gamma + \sum_{i=1}^m y_i\right)\right) d\gamma. \quad (57)$$

Proof

(i) The BE of δ (i.e., δ^*_{BSLE}) is:

$$\delta^*_{\mathrm{BSLE}} = \exp\left[E_{\delta|y}(\log\delta)\right] = \frac{1}{(\varphi+m)}\exp\left(\mathrm{PolyGamma}\left(0, \gamma + \sum_{i=1}^{m} y_i\right)\right). \quad (58)$$

(ii) The E-B estimator of δ (i.e., δ^*_{EBSLE}) is:

$$\begin{aligned}
\delta^*_{EBSLE} &= \iint_S \delta^*_{BSLE} h(\gamma,\varphi) d\gamma\, d\varphi \\
&= \int_0^k \int_0^1 \frac{1}{(\varphi+m)} \exp\left(\mathrm{PolyGamma}\left(0, \gamma + \sum_{i=1}^{m} y_i\right)\right)\frac{1}{k} d\gamma\, d\varphi \\
&= \frac{1}{k}\log\left(1+\frac{k}{m}\right)\int_0^1 exp\left(\mathrm{PolyGamma}\left(0,\gamma+\sum_{i=1}^{m} y_i\right)\right)d\gamma,
\end{aligned} \quad (59)$$

δ^*_{EBSLE} cannot be derived analytically in closed form and integrated using *R*.

(iii) To solve PR, we need the following calculation which is given in the appendix:

$$\begin{aligned}
E_{\delta|y}(\log\delta)^2 &= \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty (\log\delta)^2 \delta^{\gamma_o - 1}\exp\left[-\varphi_o\delta\right]d\delta \\
&= \left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)} - \log\varphi_o\right)^2 + \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right) \\
&\therefore\ Var_{\delta|y}(\log\delta) \\
&= E_{\delta|y}(\log\delta)^2 - \left[E_{\delta|y}(\log\delta)\right]^2 \\
&= \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right) \\
&= \mathrm{PolyGamma}(1,\gamma_o),
\end{aligned} \quad (60)$$

where $\partial/\partial\gamma_o(\Gamma'(\gamma_o)/\Gamma(\gamma_o)) = \mathrm{PolyGamma}(1, \gamma_o)$ that may be solved using the PolyGamma function in Mathematica.

$$\begin{aligned}
\mathrm{PR}(\delta^*_{BSLE}) &= Var_{\delta|y}(\log\delta) \\
&= \mathrm{PolyGamma}\left(1, \gamma + \sum_{i=1}^{m} y_i\right).
\end{aligned} \quad (61)$$

$\mathrm{PR}(\delta^*_{BSLE})$ can be derived analytically in closed form using Mathematica.

Now, the PR and E-PR are evaluated:

$$\begin{aligned}
E-\mathrm{PR}(\delta^*_{EBSLE}) &= \iint_S \mathrm{PR}(\delta^*_{BSLE}) h(\gamma,\varphi)d\gamma\, d\varphi \\
&= \int_0^k\int_0^1 \mathrm{PolyGamma}\left(1,\gamma+\sum_{i=1}^{m} y_i\right)\frac{1}{k}d\gamma\, d\varphi,
\end{aligned} \quad (62)$$

$E-\mathrm{PR}(\delta^*_{EBSLE})$ cannot be derived in closed form and integrated using *R*. □

Theorem 8. *Suppose $Y = \{y_i$, and $(i = 1, \ldots, m)\}$ represents the sample values from (1), then, we get the following estimators under WSELF:*

(i) The BE of δ is:

$$\delta^*_{BW} = \frac{\left(\gamma + \sum_{i=1}^{m} y_i - 1\right)}{(\varphi+m)}. \quad (63)$$

(ii) Using hyperprior (9), the E-B estimator of δ is:

$$\delta^*_{\mathrm{EBW}} = \frac{1}{k}\left(-\frac{1}{2} + \sum_{i=1}^{m} y_i\right)\log\left(1+\frac{k}{m}\right). \quad (64)$$

Table 2: E-B and E-PR estimates of SELF.

n	k	MLE ($\delta = 0.5$)	δ^*_{EB} ($\delta = 0.5$)	E-PR (δ^*_{EB}) ($\delta = 0.5$)	MLE ($\delta = 2$)	δ^*_{EB} ($\delta = 2$)	E-PR (δ^*_{EB}) ($\delta = 2$)
20	1	0.498900 (0.025022)	0.511223 (0.023951)	0.024947	1.999065 (0.101092)	1.975089 (0.096878)	0.096384
	5		0.467619 (0.020983)	0.020956		1.806628 (0.117931)	0.080963
30	1	0.498696 (0.016829)	0.506960 (0.016332)	0.016624	1.997787 (0.067249)	1.981607 (0.065408)	0.064982
	5		0.476661 (0.014940)	0.014724		1.863176 (0.076244)	0.057556
50	1	0.499456 (0.010070)	0.504428 (0.009891)	0.009989	1.999846 (0.040407)	1.990012 (0.039713)	0.039408
	5		0.485563 (0.009356)	0.009263		1.915588 (0.043832)	0.036543
80	1	0.499570 (0.006279)	0.006245 (0.006209)	0.006245	1.999597 (0.025359)	1.993414 (0.025089)	0.024763
	5		0.490642 (0.005996)	0.005951		1.945661 (0.026813)	0.023598
100	1	0.499750 (0.004943)	0.502243 (0.004899)	0.004997	2.000241 (0.019706)	1.995281 (0.019534)	0.019854
	5		0.492537 (0.004762)	0.004807		1.956721 (0.020637)	0.019097
200	1	0.500041 (0.002507)	0.501288 (0.002497)	0.002500	2.001203 (0.010101)	1.998710 (0.010051)	0.009969
	5		0.496362 (0.002459)	0.002451		1.979066 (0.010291)	0.009774
500	1	0.499876 (0.001006)	0.500376 (0.001004)	0.000999	1.999934 (0.004068)	1.998936 (0.004061)	0.003993
	5		0.498388 (0.000998)	0.000992		1.990995 (0.004109)	0.003962

Table 3: E-B and E-PR estimates of STLF.

n	k	MLE ($\delta = 0.5$)	δ^*_{EB} ($\delta = 0.5$)	E-PR (δ^*_{EB}) ($\delta = 0.5$)	MLE ($\delta = 2$)	δ^*_{EB} ($\delta = 2$)	E-PR (δ^*_{EB}) ($\delta = 2$)
20	1	0.498900 (0.025022)	0.462433 (0.025236)	0.046132	1.999065 (0.101092)	1.926299 (0.101690)	0.012955
	5		0.422991 (0.025864)	0.046132		1.761999 (0.137182)	0.012954
30	1	0.498696 (0.016829)	0.47417 (0.016951)	0.037040	1.997787 (0.067249)	1.948817 (0.067689)	0.008537
	5		0.445831 (0.017329)	0.037040		1.832346 (0.085632)	0.008537
50	1	0.499456 (0.010070)	0.484626 (0.010108)	0.021217	1.999846 (0.040407)	1.970209 (0.040501)	0.005069
	5		0.466501 (0.010269)	0.021217		1.896526 (0.047413)	0.005069
80	1	0.499570 (0.006279)	0.490262 (0.006296)	0.012958	1.999597 (0.025359)	1.980992 (0.025407)	0.003152
	5		0.478517 (0.006369)	0.012958		1.933535 (0.028278)	0.003152
100	1	0.499750 (0.004943)	0.492293 (0.004953)	0.010284	2.000241 (0.019706)	1.985331 (0.019726)	0.002516
	5		0.482779 (0.005003)	0.010284		1.946963 (0.021577)	0.002516
200	1	0.500041 (0.002507)	0.496301 (0.002509)	0.005068	2.001203 (0.010101)	1.993723 (0.010088)	0.001253
	5		0.491423 (0.002520)	0.005068		1.974128 (0.010522)	0.001253
500	1	0.499876 (0.001006)	0.498378 (0.001006)	0.002011	1.999934 (0.004068)	1.996938 (0.004067)	0.000501
	5		0.496398 (0.001009)	0.002011		1.989005 (0.0041482)	0.000501

Table 4: E-B and E-PR estimates of QLF.

n	K	MLE ($\delta = 0.5$)	δ^*_{EB} ($\delta = 0.5$)	E-PR (δ^*_{EB}) ($\delta = 0.5$)	MLE ($\delta = 2$)	δ^*_{EB} ($\delta = 2$)	E-PR (δ^*_{EB}) ($\delta = 2$)
20	1	0.498900 (0.025022)	0.413643 (0.031282)	0.098125	1.999065 (0.101092)	1.877509 (0.111262)	0.026025
	5		0.378362 (0.034729)	0.098125		1.717371 (0.160417)	0.026025
30	1	0.498696 (0.016829)	0.441381 (0.019720)	0.075119	1.997787 (0.067249)	1.916027 (0.072121)	0.017124
	5		0.415001 (0.021620)	0.075119		1.801516 (0.096920)	0.017124
50	1	0.499456 (0.010070)	0.464823 (0.011109)	0.042753	1.999846 (0.040407)	1.950406 (0.042073)	0.010156
	5		0.447439 (0.011910)	0.042753		1.877464 (0.051721)	0.010156
80	1	0.499570 (0.006279)	0.477839 (0.006693)	0.026031	1.999597 (0.025359)	1.968569 (0.026033)	0.006311
	5		0.466392 (0.007038)	0.026031		1.92141 (0.030036)	0.006311
100	1	0.499750 (0.004943)	0.482342 (0.005206)	0.020641	2.000241 (0.019706)	1.975380 (0.020117)	0.005037
	5		0.473021 (0.005435)	0.020641		1.937205 (0.022707)	0.005037
200	1	0.500041 (0.002507)	0.491313 (0.002571)	0.010154	2.001203 (0.010101)	1.988735 (0.010176)	0.002508
	5		0.486484 (0.002629)	0.010154		1.969189 (0.010802)	0.002508
500	1	0.499876 (0.001006)	0.496380 (0.001016)	0.004025	1.999934 (0.004068)	1.994940 (0.004085)	0.001002
	5		0.494408 (0.001027)	0.004025		1.987015 (0.004196)	0.001002

Table 5: E-B and E-PR estimates of PPLF.

n	k	($\delta = 0.5$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})	($\delta = 2$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})
20	1	0.498900 (0.025022)	0.488486 (0.023974)	0.117204	1.999065 (0.101092)	1.953046 (0.096816)	0.025802
	5		0.446822 (0.022776)	0.117204		1.786465 (0.124757)	0.025802
30	1	0.498696 (0.016829)	0.490418 (0.015900)	0.073375	1.997787 (0.067249)	1.964395 (0.064544)	0.017047
	5		0.461108 (0.015488)	0.073375		1.846993 (0.079350)	0.017047
50	1	0.499456 (0.010070)	0.493587 (0.009752)	0.042327	1.999846 (0.040407)	1.977618 (0.038715)	0.010139
	5		0.475128 (0.009617)	0.042327		1.903658 (0.044691)	0.010139
80	1	0.499570 (0.006279)	0.496562 (0.006069)	0.025835	1.999597 (0.025359)	1.987256 (0.024155)	0.006300
	5		0.484666 (0.006006)	0.025835		1.939649 (0.026499)	0.006300
100	1	0.499750 (0.004943)	0.497176 (0.004898)	0.020536	2.000241 (0.019706)	1.989730 (0.019350)	0.005032
	5		0.487568 (0.020536)	0.020536		1.951276 (0.020882)	0.005032
200	1	0.500041 (0.002507)	0.49834 (0.002462)	0.010134	2.001203 (0.010101)	1.994546 (0.009964)	0.002508
	5		0.493499 (0.002453)	0.010134		1.974943 (0.010368)	0.002508
500	1	0.499876 (0.001006)	0.499312 (0.000966)	0.004021	1.999934 (0.004068)	1.997382 (0.003919)	0.001002
	5		0.497328 (0.000965)	0.004021		1.989448 (0.003992)	0.001002

Table 6: E-B and E-PR estimates of PLF.

n	K	($\delta = 0.5$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})	($\delta = 2$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})
20	1	0.498900 (0.025022)	0.537357 (0.025009)	0.147573	1.999065 (0.101092)	2.001840 (0.094614)	0.048486
	5		0.491524 (0.024075)	0.143515		1.831097 (0.107688)	0.044350
30	1	0.498696 (0.016829)	0.523229 (0.016745)	0.092243	1.997787 (0.067249)	1.997186 (0.063284)	0.032653
	5		0.491958 (0.015837)	0.090316		1.877824 (0.070866)	0.030702
50	1	0.499456 (0.010070)	0.513395 (0.010989)	0.079604	1.999846 (0.040407)	1.997421 (0.038221)	0.019753
	5		0.494194 (0.010903)	0.078871		1.922720 (0.041381)	0.019014
80	1	0.499570 (0.0062795)	0.508985 (0.009138)	0.042345	1.999597 (0.025359)	1.999678 (0.023992)	0.012403
	5		0.496792 (0.007781)	0.042049		1.951774 (0.025182)	0.012106
100	1	0.499750 (0.004943)	0.507128 (0.005941)	0.029901	2.000241 (0.019706)	1.999680 (0.019245)	0.009938
	5		0.497327 (0.005710)	0.029709		1.961035 (0.020027)	0.009746
200	1	0.500041 (0.002507)	0.503385 (0.003471)	0.014975	2.001203 0.010101	1.999534 0.009935	0.004984
	5		0.498438 (0.003413)	0.014926		1.979882 (0.010145)	0.004935
500	1	0.499876 (0.001006)	0.501310 (0.000986)	0.004996	1.999934 (0.004068)	1.999380 (0.003912)	0.001998
	5		0.499318 (0.000974)	0.004988		1.991438 (0.003954)	0.001990

Table 7: E-B and E-PR estimates of DLF.

n	K	($\delta = 0.5$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})	($\delta = 2$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})
20	1	0.498900 (0.025022)	0.562360 (0.027654)	0.094461	1.999065 (0.101092)	2.026387 (0.095292)	0.024658
	5		0.514395 (0.020091)	0.094461		1.853551 (0.100594)	0.024658
30	1	0.498696 (0.016829)	0.539898 (0.017380)	0.064411	1.997787 (0.067249)	2.013649 (0.063458)	0.016545
	5		0.507631 (0.014016)	0.064411		1.893303 (0.067319)	0.016545
50	1	0.499456 (0.010070)	0.523395 (0.010254)	0.039282	1.999846 (0.040407)	2.007347 (0.038267)	0.00996
	5		0.503821 (0.009009)	0.039282		1.932275 (0.039995)	0.00996
80	1	0.499570 (0.006279)	0.515235 (0.006288)	0.024689	1.999597 (0.025359)	2.005899 (0.024027)	0.00623
	5		0.502892 (0.005778)	0.024689		1.957845 (0.024633)	0.006230
100	1	0.499750 (0.004943)	0.512128 (0.005037)	0.019809	2.000241 (0.019706)	2.004661 (0.019267)	0.004988
	5		0.502230 (0.004708)	0.019809		1.965920 (0.019670)	0.004988
200	1	0.500041 (0.002507)	0.505885 (0.002494)	0.009956	2.001203 (0.010101)	2.002029 (0.009939)	0.002497
	5		0.500913 (0.002412)	0.009956		1.982353 (0.010052)	0.002497
500	1	0.499876 (0.001006)	0.502310 (0.000971)	0.003993	1.999934 (0.004068)	2.000379 (0.003912)	0.001000
	5		0.500314 (0.000958)	0.003993		1.992433 (0.003938)	0.001000

Table 8: E-B and E-PR estimates of SLELF.

n	K	($\delta = 0.5$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})	($\delta = 2$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})
20	1	0.498900 (0.025022)	0.487058 (0.023966)	0.113247	1.999065 (0.101092)	1.950746 (0.098678)	0.025680
	5		0.445516 (0.022881)	0.113247		1.784362 (0.127033)	0.025680
30	1	0.498696 (0.016829)	0.490664 (0.016364)	0.072096	1.997787 (0.067249)	1.965235 (0.066276)	0.016976
	5		0.461339 (0.015884)	0.072096		1.847783 (0.080693)	0.016976
50	1	0.499456 (0.010070)	0.494561 (0.009901)	0.041808	1.999846 (0.040407)	1.980119 (0.040009)	0.010104
	5		0.476065 (0.009719)	0.041808		1.906065 (0.045529)	0.010104
80	1	0.499570 (0.006279)	0.496486 (0.006214)	0.025686	1.999597 (0.025359)	1.987206 (0.025209)	0.006291
	5		0.484592 (0.006145)	0.025686		1.939601 (0.027508)	0.006291
100	1	0.499750 (0.004943)	0.497276 (0.004901)	0.020425	2.000241 (0.019706)	1.990308 (0.019605)	0.005024
	5		0.487666 (0.004858)	0.020425		1.951844 (0.021083)	0.005024
200	1	0.500041 (0.002507)	0.498796 (0.002497)	0.010102	2.001203 (0.010101)	1.996217 (0.010063)	0.002505
	5		0.493894 (0.002484)	0.010102		1.976598 (0.010400)	0.002505
500	1	0.499876 (0.001006)	0.499377 (0.001004)	0.004017	1.999934 (0.004068)	1.997937 (0.004064)	0.001001
	5		0.497394 (0.001002)	0.004017		1.99 (0.004127)	0.001001

Table 9: E-B and E-PR estimates of WSELF.

n	K	($\delta = 0.5$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})	($\delta = 2$) MLE	δ^*_{EB}	E-PR (δ^*_{EB})
20	1	0.498900 (0.025022)	0.462433 (0.025236)	0.048790	1.999065 (0.101092)	1.926299 (0.101690)	0.048790
	5		0.422991 (0.025864)	0.044629		1.761999 (0.137182)	0.044628
30	1	0.498696 (0.016829)	0.474170 (0.016951)	0.032789	1.997787 (0.067249)	1.948817 (0.067689)	0.032789
	5		0.445831 (0.017329)	0.030830		1.832346 (0.085632)	0.03083
50	1	0.499456 (0.010070)	0.484626 (0.010108)	0.019802	1.999846 (0.040407)	1.970209 (0.040501)	0.019803
	5		0.466501 (0.010269)	0.019062		1.896526 (0.047413)	0.019062
80	1	0.499570 (0.006279)	0.490262 (0.006296)	0.012422	1.999597 (0.025359)	1.980992 (0.025407)	0.012422
	5		0.478517 (0.006369)	0.012125		1.933535 (0.028278)	0.012125
100	1	0.499750 (0.004943)	0.492293 (0.004953)	0.009950	2.000241 (0.019706)	1.985331 (0.019726)	0.009950
	5		0.482779 (0.005003)	0.009758		1.946963 (0.021577)	0.009758
200	1	0.500041 (0.002507)	0.496301 (0.002509)	0.004988	2.001203 (0.010101)	1.993723 (0.010088)	0.004987
	5		0.491423 (0.002520)	0.004938		1.974128 (0.010522)	0.004938
500	1	0.499876 (0.001006)	0.498378 (0.001006)	0.001998	1.999934 (0.004068)	1.996938 (0.004067)	0.001998
	5		0.496398 (0.001009)	0.001990		1.989005 (0.004148)	0.001990

(iii) The PR and E-PR are given by:

$$\mathrm{PR}\left(\delta^*_{BW}\right) = \frac{1}{(\varphi + m)},$$
$$E - \mathrm{PR}\left(\delta^*_{BW}\right) = \frac{1}{k}\log\left(1 + \frac{k}{m}\right). \tag{65}$$

Proof

(i) The BE of δ (i.e., δ^*_{BW}) is:

$$\begin{aligned}\delta^*_{BW} &= \left[E_{\delta|x}\left(\delta^{-1}\right)\right]^{-1} \\ &= \frac{\gamma_o - 1}{\varphi_o} \\ &= \frac{\left(\gamma + \sum_{i=1}^m y_i - 1\right)}{(\varphi + m)}.\end{aligned} \tag{66}$$

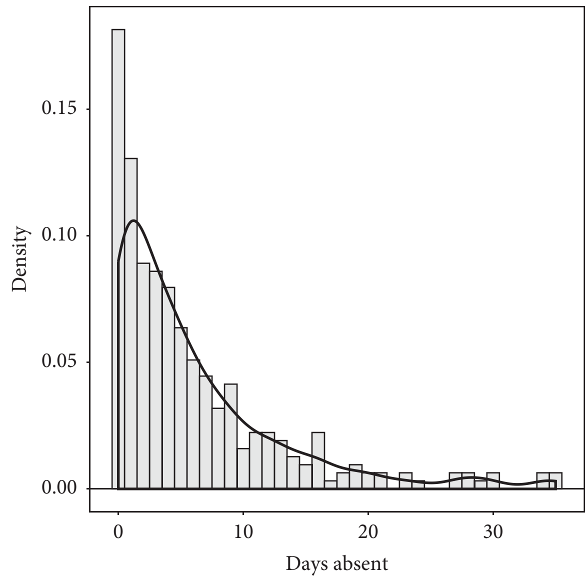


Figure 1: Histogram and density using the real data.

Table 10: MLE, AIC, and BIC values of the real data.

Distribution	Parameter	MLE	Std. error	Log-L	AIC	BIC
Poisson	δ	5.9554	0.1377	−1550.5090	3103.0180	3106.7680
Negative binomial	γ	0.7999	0.0758	−896.4724	1796.9450	1804.4440
	δ	5.9558	0.4003			

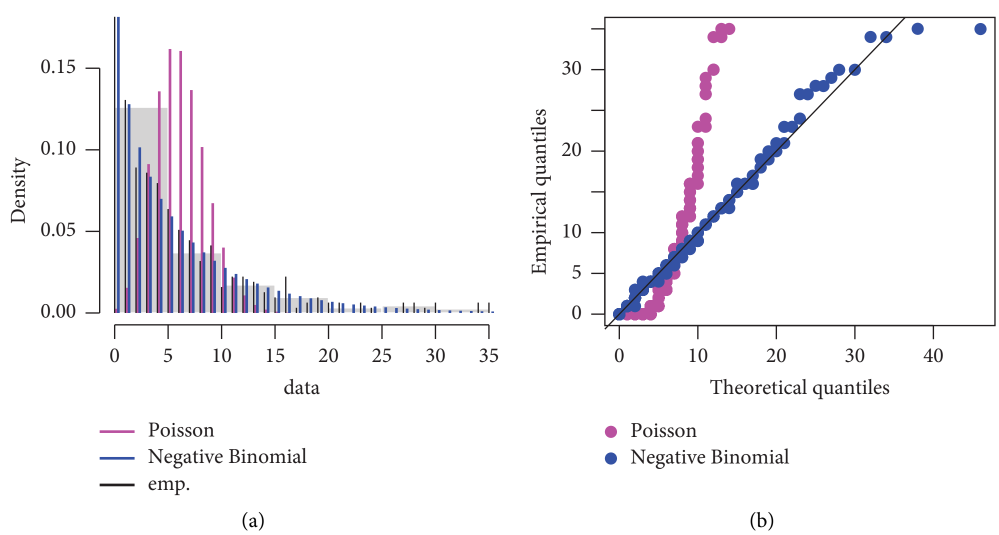
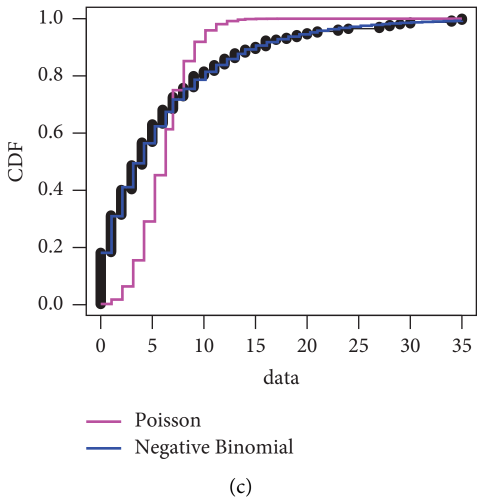


Figure 2: Continued.



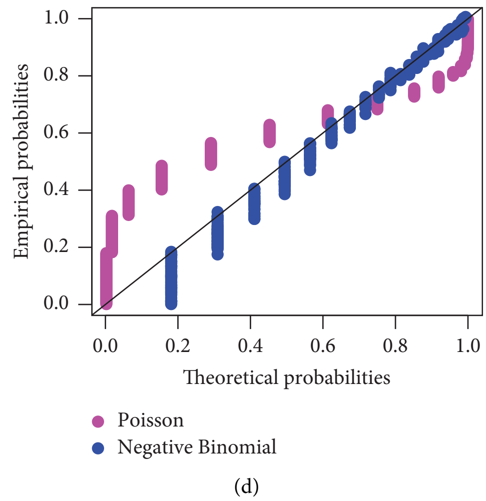


FIGURE 2: Density, Q-Q, CDF, and P-P plots of the real data. (a) Histogram and theoretical densities. (b) Q-Q plot. (c) Empirical and theoretical CDFs. (d) P-P plot.

TABLE 11: E-B and E-PR estimates of real life data.

LF	K	δ^*_{EB}	E-PR (δ^*_{EB})
SELF	1	5.947541	0.01891113
	5	5.910076	0.01867400
STLF	1	5.944361	0.00026743
	5	5.906916	0.00026743
QLF	1	5.941181	0.00053490
	5	5.903756	0.00053490
PPLF	1	5.945951	0.00053483
	5	5.908496	0.00053483
PLF	1	5.949130	0.00317923
	5	5.911655	0.00315920
DLF	1	5.950720	0.00053433
	5	5.913235	0.00053433
SLELF	1	5.945951	0.00053476
	5	5.908496	0.00053476
WSELF	1	5.944361	0.00317965
	5	5.906916	0.00315962

(ii) The E-B estimator of δ (i.e., δ^*_{EBW}) is:

$$\begin{aligned}\delta^*_{EBW} &= \iint_S \delta^*_{BW} h(\gamma,\varphi)d\gamma\, d\varphi\\ &= \int_0^k\int_0^1 \frac{\left(\gamma+\sum_{i=1}^m y_i - 1\right)}{(\varphi+m)}\frac{1}{k}d\gamma\, d\varphi \\ &= \frac{1}{k}\left(-\frac{1}{2}+\sum_{i=1}^m y_i\right)\log\left(1+\frac{k}{m}\right).\end{aligned} \tag{67}$$

(iii) The PR and E-PR are calculated:

$$\begin{aligned}\mathrm{PR}\left(\delta^*_{BW}\right) &= E_{\delta|x}(\delta) - \left[E_{\delta|x}\left(\delta^{-1}\right)\right]^{-1}\Big]\\ &= \frac{1}{\varphi_o}\\ &= \frac{1}{(\varphi+m)},\\ E-\mathrm{PR}\left(\delta^*_{EBW}\right) &= \iint_S \mathrm{PR}\left(\delta^*_{BW}\right)h(\gamma,\varphi)d\gamma\, d\varphi\\ &= \int_0^k\int_0^1 \frac{1}{(\varphi+m)}\frac{1}{k}d\gamma\, d\varphi\\ &= \frac{1}{k}\log\left(1+\frac{k}{m}\right).\end{aligned} \tag{68}$$

□

4. Monte Carlo Simulation

In this section, a Monte Carlo simulation study has been conducted to incur the estimates of E-B, E-PR, and MSE under SELF, STLF, QLF, PPLF, PLF, DLF, SLELF, and WSELF. It has been studied to compare the general behavior of proposed estimators using the following steps:

(i) Select the sample size $n = 20$, 30, 50, 80, 100, 200, or 500

(ii) Select the parameters $\delta = 0.5$ or 2 and $k = 1$ or 5

(iii) Generate the sample values $\mathbf{Y} = \{y_i, (i = 1, \ldots, m)\}$, for sample size n, from the hierarchical Poisson-Gamma model (1)

(iv) The estimates of δ^*_{ML} are calculated from equation (4) using different values of parameters (δ). Moreover, the respective MSEs of δ^*_{ML} are calculated

(v) Under all the assumed LFs, the E-B and E-PR estimates are calculated from equations (10)–(25), respectively, using different values of parameter (δ). The MSEs (the values in ()) of respective E-B estimates and MLEs are also calculated

(vi) Simulate the steps (iii)–(v) 10,000 times for a given sample size n. The average of E-B, E-PR, and MSE estimates are then calculated.

The outcomes are given in Tables 2–9.

5. Real Data Illustration

In this section, a real data example, associated with the attendance of 314 students from two high schools located in city, is analyzed to elaborate the proposed methods [25]. The variable "daysabs," i.e., days absent, is the response variable under study. Suppose the variable "daysabs" is represented by Y, then its mean and variance are: $\overline{Y} = 5.9554$ and $s^2 = 49.5188$. Figure 1 shows the histogram and density of Y. It is observed that the Poisson distribution is not an appropriate distribution when the variance is much higher than mean. The Poisson and negative binomial distributions are fitted by the maximum likelihood method. It is obvious from Table 10 that the negative binomial distribution has a better fit to the aforementioned data. Figure 2 depicts the density, CDF, Q-Q, and P-P plots of Poisson and negative binomial distributions, respectively, and it can be clearly seen that the negative binomial distribution fits the real data very well. Table 11 provides the E-B and E-PR estimates of real data under the assumed LFs.

6. Conclusions

In this study, we have derived the E-B estimates of the hierarchical Poisson-Gamma model under various LFs using restricted and unrestricted parameter spaces by considering uniform hyperprior. The performance of E-B estimates have been compared on the basis of E-PR. The E-B estimates are also compared empirically with MLEs on the basis of MSEs.

After assessing the results, both E-PRs and MSEs tend to decrease as the sample size increases. It is worth mentioning that the E-B estimates of STLF dominate, under restricted parameter space, and incurs precise estimates as far as E-PR is concerned. The LFs under unrestricted parameter space (i.e., SELF and WSELF) are not appropriate, because these LFs inflict an equal penalty on underestimation as well as overestimation. For $\delta < 1$, it can be seen clearly that SELF underestimates the E-PR which does not support the outcomes of real data application. A real data example is analyzed that supported the numerical simulation results under restricted parameter space. It has been observed that the asymmetric LFs performed better than the symmetric LFs. It is pertinent to mention that the performance of E-BE is efficient as compared to the classical estimation in terms of minimum MSE. If we consider E-PR as an evaluation standard, then, it has been found that the STLF is an appropriate LF and E-BE provides satisfactory outputs to estimate the parameter of the hierarchical Poisson-Gamma model. In the future, this work may be extended using a random censoring scheme.

Appendix

A. Additional Calculations

The additional calculations are stated as follows:

$$\begin{aligned} E_{\delta|y}(\log\delta) &= \int_0^\infty (log\delta)\pi(\delta \mid \mathbf{Y})d\delta \\ &= \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty (\log\delta)\delta^{\gamma_o - 1}\exp[-\varphi_o\delta]d\delta. \end{aligned} \tag{A.1}$$

Let us suppose, $z = \varphi_o\delta \Rightarrow \delta = z/\varphi_o \Rightarrow d\delta = dz/\varphi_o$

$$\begin{aligned}
\therefore E_{\delta|y}(\log\delta) &= \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty \left(\log\frac{z}{\varphi_o}\right)\left(\frac{z}{\varphi_o}\right)^{\gamma_o-1} \exp(-z)\frac{dz}{\varphi_o} \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty [\log(z) - \log(\varphi_o)] z^{\gamma_o-1}\exp(-z)dz \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty \left[z^{\gamma_o-1}\log(z)\right] exp(-z)dz - \frac{\log(\varphi_o)}{\Gamma(\gamma_o)}\int_0^\infty z^{\gamma_o-1}\exp(-z)dz \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty \left[\frac{\partial}{\partial\gamma_o} z^{\gamma_o-1}\right]\exp(-z)dz - \frac{\log(\varphi_o)}{\Gamma(\gamma_o)}\Gamma(\gamma_o) \\
&= \frac{1}{\Gamma(\gamma_o)}\frac{\partial}{\partial\gamma_o}\int_0^\infty z^{\gamma_o-1}\exp(-z)dz - \log(\varphi_o) \\
&= \frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)} - \log(\varphi_o) = \text{PolyGamma}(0,\gamma_o) - \log(\varphi_o),
\end{aligned} \tag{A.2}$$

where $\Gamma'(\gamma_o)/\Gamma(\gamma_o) = \text{PolyGamma}(0, \gamma_o)$ that may be solved using PolyGamma function in Mathematica.

Now,

$$\begin{aligned}
E_{\delta|y}(\log\delta)^2 &= \int_0^\infty (\log\delta)^2\pi(\delta \mid \mathbf{Y})d\delta \\
&= \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty (\log\delta)^2\delta^{\gamma_o-1}\exp[-\varphi_o\delta]d\delta.
\end{aligned} \tag{A.3}$$

Let us suppose, $\delta = z/\varphi_o \Rightarrow d\delta = dz/\varphi_o$

$$\begin{aligned}
\therefore E_{\delta|y}(\log\delta)^2 &= \frac{\varphi_o^{\gamma_o}}{\Gamma(\gamma_o)}\int_0^\infty \left(\log\frac{z}{\varphi_o}\right)^2\left(\frac{z}{\varphi_o}\right)^{\gamma_o-1}\exp(-z)\frac{dz}{\varphi_o} \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty [logz - \log\varphi_o]^2 z^{\gamma_o-1}\exp(-z)dz \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty \left[(\log\varphi_o)^2\right] z^{\gamma_o-1}\exp(-z)dz \\
&= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty (logz)^2 z^{\gamma_o-1}\exp(-z)dz - \frac{2(\log\varphi_o)}{\Gamma(\gamma_o)}\int_0^\infty (logz) z^{\gamma_o-1}\exp(-z)dz \\
&\quad + \frac{(\log\varphi_o)^2}{\Gamma(\gamma_o)}\int_0^\infty z^{\gamma_o-1}\exp(-z)dz \\
&= I_c - 2(\log\varphi_o)\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)} + (\log\varphi_o)^2,
\end{aligned} \tag{A.4}$$

where I_c involves complex integration that can be solved using Mathematica:

$$
\begin{aligned}
I_c &= \frac{1}{\Gamma(\gamma_o)}\int_0^\infty (logz)^2 z^{\gamma_o - 1}\exp(-z)dz \\
&= \left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right)^2 + \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right), \\
\Rightarrow E_{\delta|y}(\log\delta)^2 &= \left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right)^2 + \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right) - 2(\log\varphi_o)\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)} + (\log\varphi_o)^2 \\
&= \left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)} - \log\varphi_o\right)^2 + \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right), \\
\therefore Var_{\delta|y}(\log\delta) &= E_{\delta|y}(\log\delta)^2 - \left[E_{\delta|y}(\log\delta)\right]^2 \\
&= \frac{\partial}{\partial\gamma_o}\left(\frac{\Gamma'(\gamma_o)}{\Gamma(\gamma_o)}\right) = \text{PolyGamma}(1,\gamma_o),
\end{aligned}
\tag{A.5}
$$

where $\partial/\partial\gamma_o(\Gamma'(\gamma_o)/\Gamma(\gamma_o)) = \text{PolyGamma}(1,\gamma_o)$ that may be solved using PolyGamma function in Mathematica.

Data Availability

The data that support the findings of this study are available from the corresponding author on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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