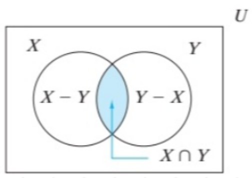
# < CH6. Counting Methods and The Pigeonhole Principle >

## ※ 6.1 Basic Principles

- **Multiplication Principle**
  - **Def**: If an activity can be constructed in $t$ successive steps and step $t$ can be done in $n_t$ ways, then the number of different possible activities is $\prod_{i=1}^{t} n_i$
    - → Given $t$ events, the number of ways is $\prod_{i=1}^{t} n_i = n_1 \cdot n_2 \cdots n_t$
  - We multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

- **Addition Principle**
  - Def: If sets $\{X_1, X_2, \cdots, X_t\}$ is a pairwise disjoint family $(i \neq j,\ X_i \cap X_j = \emptyset)$, the number of possible elements that can be selected from $X_1$ or $X_2$ or, ⋯, $X_t$ is $\sum_{i=1}^{t} n_i$ $(X_1 \cup X_2 \cup \cdots \cup X_t)$
    - → we add the numbers of elements in each subset when the elements being counted can be decomposed into pairwise disjoint subsets.

- **Inclusion - Exclusion Principle**
  - Def: Generalize the Addition Principle by giving a formular to compute the number of elements in a union without requiring the sets to be pairwise disjoint.
    - → If $X$ and $Y$ are finites sets, then
      $|X \cup Y| = |X| + |Y| - |X \cap Y|$




Pf) $|X| = |X-Y| + |X \cap Y|$ ⋯ ①

$|Y| = |Y-X| + |X \cap Y|$ ⋯ ②

$|X \cup Y| = |X-Y| + |Y-X| + |X \cap Y|$ ⋯ ③

① − ③ = $|X| - |X \cup Y| = |X \cap Y| - |Y-X| - |X \cap Y|$

$|X| - |X \cup Y| = -|Y| + |X \cap Y|$

$\therefore |X \cup Y| = |X| + |Y| - |X \cap Y|$

## ※ 6.2 Permutations and Combinations

- **Permutations**
  - Def: A permutation of $n$ is distinct elements $x_1, x_2, \cdots, x_n$ is an ordering of the $n$ element $x_1, x_2, \cdots, x_n$
  - → Arranging $r$ elements from $n$ different elements in order without duplication.
  - Formular: There are $n!$ permutations of $n$ elements.
    - ▫ $n! = n(n-1)(n-2)\cdots 2 \cdot 1$
  - $r$-permutation
    - Def: Consider an ordering of $r$ elements selected from $n$ available elements.
      - → A permutation of selecting $r$ elements from $n$ different elements
    - formular: $_nP_r = P(n,r) = \frac{n!}{(n-r)!}$
      $= n(n-1)(n-2)\cdots(n-r+1)$

      Pf) $n(n-1)(n-2)\cdots(n-r+1)$

      $= \frac{[n(n-1)(n-2)\cdots(n-r+1)](n-r)(n-r-1)\cdots 2 \cdot 1}{(n-r)(n-r-1)\cdots 2 \cdot 1}$

      $= \frac{n!}{(n-r)!}$

- **Combinations**
  - Def: Selection of objects without regard to order
    - → since the order doesn't matter, the number of times overlaps by $r$.
  - formular: $_nC_r = C(n,r) = \frac{P(n,r)}{r!}$
    $= \frac{n!}{(n-r)!\,r!}$
  - Catalan numbers: $\frac{C(2n,n)}{n+1}$
    - ▫ $C_0=1,\ C_1=1,\ C_2=2,\ C_3=5, \cdots$

    Pf) The number of shortest paths through grid from $(0,0)$ to $(n,n)$ : $C(2n,n)$ ⋯①

    The number of path beyond $y=x$ is equal to from $(0,0)$ to $(n+1, n-1)$ : $C(2n, n+1)$ ⋯②

    Thus ① − ② = $C(2n,n)/(n+1)$