Mathematical Methods - III

Week 2

Jayampathy Ratnayake

Faculty of Science
Department of Mathematics
Room 209(B)
jratnaya@indiana.edu

September 23, 2016

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Remark

Reimann, with other mathematician, studied the Theory of Integration and derived many techniques to find it, some of which we now learn in the high school.
His definition of Integral is based on the two types of approximations mentioned above.

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4. Find the Riemann sum

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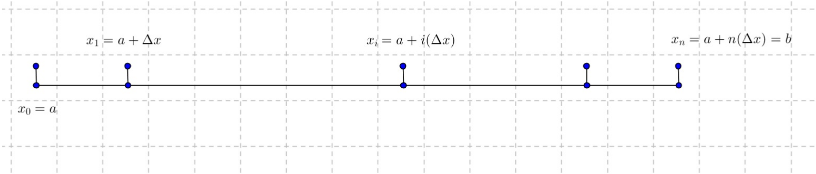

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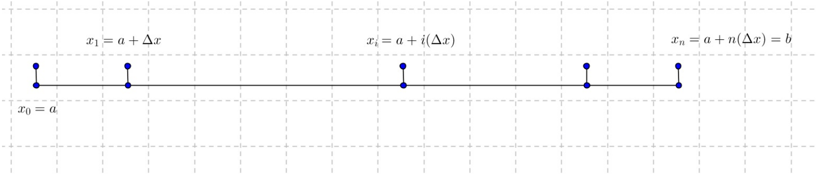

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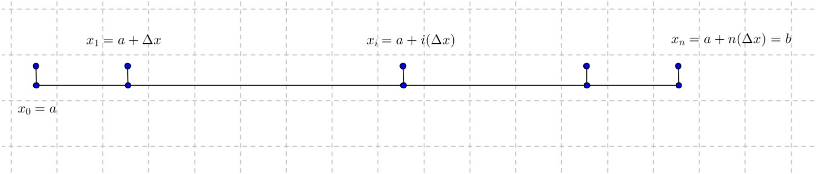

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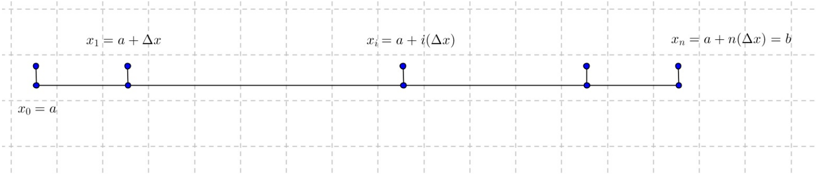

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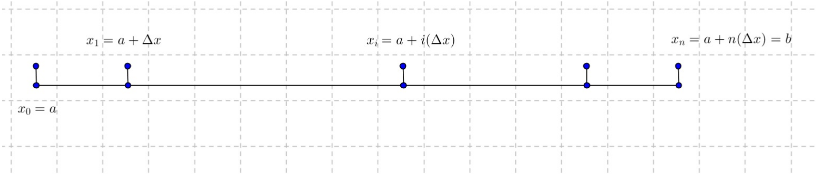

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We understand this limit as the "Area Under the Curve".

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4. Different ways of taking finite approximations gives rise to various numerical methods for integration.

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- *in each sub-interval i the function is approximated by the horizontal line passing through the point $(\zeta_i, f(\zeta_i))$.*

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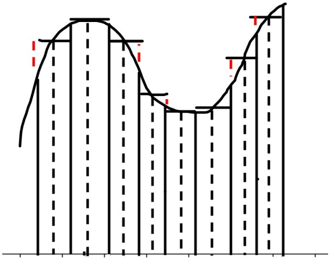

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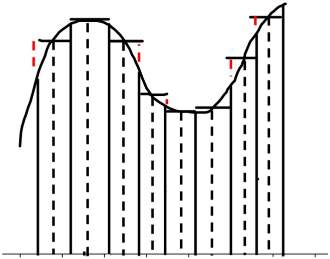

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Where

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Trapezoidal Rule

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1. approximate the function by a straight line connecting $(0, f(0))$ and $(2, f(2))$. What is the approximation for the area with this approximation for the function.

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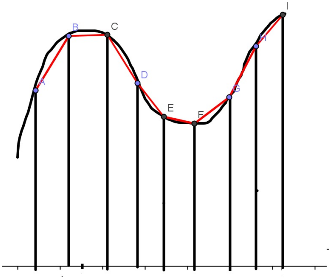

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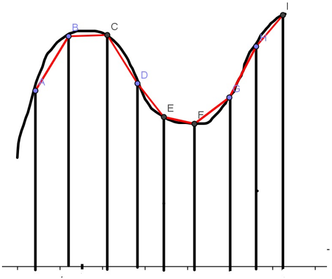

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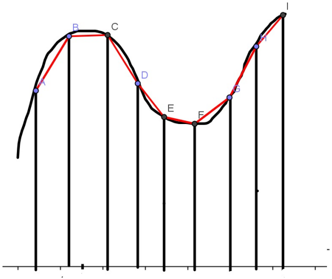

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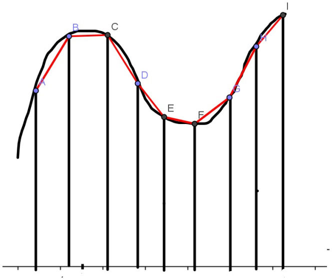

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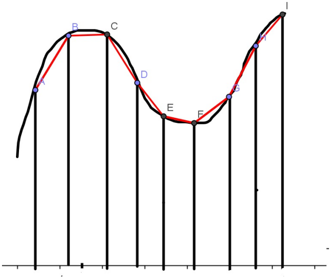
4. hence write down the trapezoidal rule.



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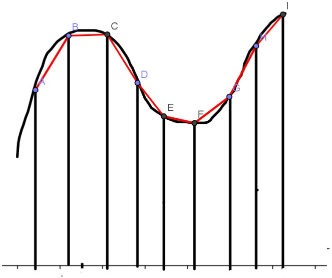

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$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

Where $\Delta x = \dfrac{b-a}{n}$

Example

What is the connction between the trapezoidal rule and the left and the right approximations corresponding to a subdivision.

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3. What is the area bounded by this parabola in the interval $[x_1, x_2]$.

Simpson's Rule

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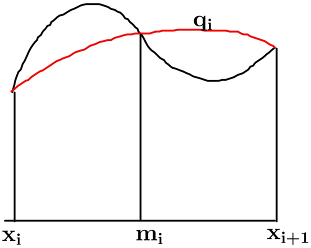
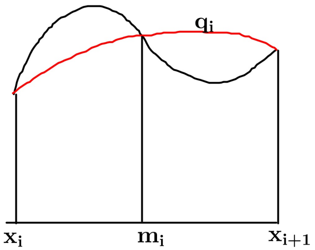
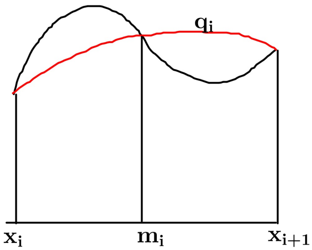

q_i
x_i
m_i
x_{i+1}



Approximate the function in each interval $[x_i, x_{i+1}]$ by the quadratic q_i, passing through x_i, x_{i+1} and the midpoint $m_i = \dfrac{x_i + x_{i+1}}{2}$, for $i = 0, 1, ..., n-1$.



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$$q_i = f(x_i)\frac{(x-m_i)(x-x_{i+1})}{(x_i-m_i)(x_i-x_{i+1})} + f(x_{i+1})\frac{(x-x_i)(x-m_i)}{(x_{i+1}-x_i)(x_{i+1}-m_i)} + \\ + f(m_i)\frac{(x-x_i)(x-x_{i+1})}{(m_i-m_i)(m_i-x_{i+1})}$$

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    - *in parts where the function is varying rapidly, one can use smaller divisions*
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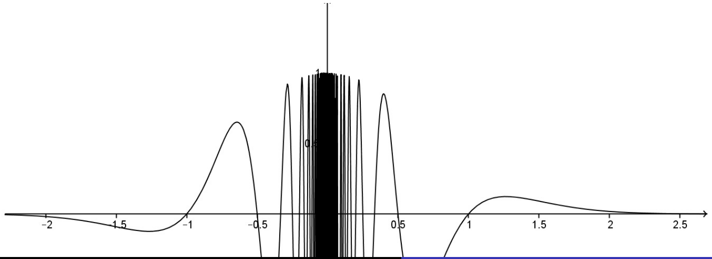

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When using a numerical method to estimate a value, it is the common practice to generate sequence of solutions with decreasing step sizes to while the solution has reached a desired accuracy (while the numerical scheme converge).

Algorithm

Integrate(f, a, b, ϵ)
f : function to integrate
$[a, b]$: *interval*
ϵ : *precision (required accuracy)*
integer $n = 10$: *number of divisions*
sum-new = estimate with n divisions
do
    sum-old = sum-new
    n = 3n : increase the number of divisions
    sum-new = estimate with n divisions : now n has increased!
while | sum-new - sum-old | $< \epsilon$
return sum-new

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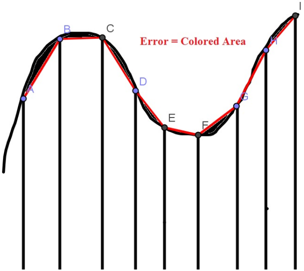

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- We will do the error estimates of the three methods we have discussed so far in the next class.

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Definition (Convergence)

Numerical solution approaches the exact solution as the step size goes to zero. I.e. errors go to zero

- Consistency
- Stability