# Chromatic Homotopy Theory

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**Abstract**

Chromatic homotopy theory is…

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# Introduction

"Desar's chosen field in mathematics was so esoteric that nobody in the institute or the Maths Federative could really check his progress"
*The Dispossessed, Ursula K. Le Guin*

The chromatic filtration on the stable homotopy groups of spheres is a reflection of the height filtration on the moduli stack of commutative, 1-dimensional formal group laws. Taking this idea seriously means forging a link between arithmetic geometry and homotopy theory. Periodicity phenomena deep within stable homotopy theory reflect structures found on the moduli stack of formal groups (or, perhaps better, p-divisible groups).

Many special objects in mathematics meet in chromatic homotopy theory: complex orientations, the Adams-Novikov spectral sequence, formal groups, p-divisible groups, modular forms, and even (conjecturally) certain quantum field theories. One theme I'll be particularly enthused about is using the Fargues-Fontaine curve to reimagine the geometry controlling the chromatic filtration.

# PRELIMINARIES

## HIGHER CATEGORIES, HOMOTOPY (CO)LIMITS AND SPACES

We restrict our attention to the category Top of topologcial spaces. We would like a notion of (co)limit which is invariant under homotopy. For instance, if $X \simeq X'$ then we would like the pushouts $X \sqcup_Z Y$ and $X' \sqcup_Z Y$ to be homotopy equivalent for all $Y$ and $Z$. This typically dramatically fails, for instance, if we choose $X = Y = \mathrm{pt}$ then the pushout $X \sqcup_{S^1} Y = \mathrm{pt} \sqcup_Z \mathrm{pt} = \mathrm{pt}$, but if $X = \mathrm{pt} \simeq D^2$, the unit disk in $\mathbb{R}^2$, and in this case we have that $D^2 \sqcup_{S^1} D^2 \simeq S^2$, the sphere, and certainly $S^2 \not\simeq \mathrm{pt}$.

This leads us to the notion of a *homotopy (co)limit*. We'll give a few examples of homotopy (co)limit construcitons in the category of topological spaces, then scetch a framework for giving general cases of these so-called 'homotopy-coherent' constructions - the framework of $\infty$-categories.

### EXAMPLE 1: HOMOTOPY PUSHOUTS

Let $X, Y$ and $Z$ be (pointed) CW-complexes, and consider a diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

We can replace the map $f : Z \to Y$ by its *mapping cylinder*[a] $M(f) := \Big(([0,1] \times X) \sqcup Y\Big)/\sim$ where $\forall x \in X; (0, x) \sim f(x)$, so we think of $M(f)$ as gluing $\{0\} \times X$ to $Y$ via $f$ and we have an inclusion $i_f : X \cong \{1\} \times X \subseteq M(f)$. We can take the homotopy pushout of the above diagram by simply replacing $f$ and $g$ with their mapping cylinders and taking the ordinary pushout;

$$\begin{tikzcd} Z \arrow[r, "i_f"] \arrow[d, "i_g"'] & M(f) \arrow[d] \\ M(g) \arrow[r] & X \sqcup^h_Z Y := M(f) \sqcup_Z M(g) \end{tikzcd}$$

If we choose $Z = S^1$ and $X = Y = \mathrm{pt}$ then we can see that $\mathrm{pt} \sqcup_{S^1} \mathrm{pt} = \mathrm{pt}$ whereas $\mathrm{pt} \sqcup^h_{S^1} \mathrm{pt} = S^2$, since $M(S^1 \to \mathrm{pt})$ is (homotopic to) the inclusion $S^1 \hookrightarrow D^2$.

[a] In the language of model categories we could say $M(f)$ is a 'cofibrant replacement' of $f$, we will use no such language here.

### EXAMPLE 2: HOMOTOPY PULLBACKS

Similarly if we have a diagram of (pointed) CW-complexes

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

we can define the homotopy pullback by the formula

$$X \times^h_Z Y := \Big\{ (x, y, \gamma) \in X \times Y \times Z^{[0,1]} \, : \, \gamma(0) = f(x), \gamma(1) = g(y) \Big\}.$$

Thus the homotopy pullback is much like the ordinary pullback, except instead of requiring that $f(x) = g(y)$ we simply require a path between them and, importantly, that this path is part of the *data* of $X \times^h_Z Y$.

We can continue this game of naming a type of limit or colimit and seeing what the homotopy coherent version is, but the game becomes quickly tiresome so we invent new areas of maths instead. Historicall, Quillen invented model categories, which work via similar methods to example

1 by replacing objects and morphisms by corresponding fibrant/cofibrant versions of them. Model categories are excellent in many ways, but their lack of higher data means certain constructions become clunky[1]. We instead opt for $\infty$-categories, modeled as quasi-categories as per Joyal and Lurie[2].

Before continuing this introduction we offer a quick definition, that of the loop-spaces and suspensions of spaces as homotopy pullbacks/pushouts.

### Definition 3: Loop Spaces and Suspensions

Let $X$ be a (pointed) CW-complex. We define the loop-space $\Omega X := \mathrm{pt} \times_X^h \mathrm{pt}$ and suspension $\Sigma X := \mathrm{pt} \sqcup_X^h \mathrm{pt}$.

This definition is certainly equivalent to the classical formulas

$$\Omega X := \mathrm{Map}_*(S^1, X) \text{ and } \Sigma X := (X \times [0,1])/\big((\{0\} \times X) \cup (\{1\} \times X) \cup ([0,1] \times \{*\})\big)$$

where $* \in X$ is the basepoint, but it allows a rather simplified proof of the functors $\Omega, \Sigma : \mathrm{Top} \to \mathrm{Top}$ being adjoint. If you believe us, for a moment, that in the $\infty$-category of spaces (whatever that may mean) there is an analagous universality property of the homotopy pullback and pushout then the (homotopy) pushout-pullback diagram

$$\begin{tikzcd}
\Omega\Sigma X \arrow[rr] \arrow[dd] & & \mathrm{pt} \arrow[d] \\
& X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{pt} \arrow[r] & \Sigma X
\end{tikzcd}$$

gives us maps $X \to \Omega\Sigma X$ (by the 'universal property' of the pullback and that $\mathrm{pt} = \mathrm{pt}$), which is the unit of the adjuntion, a similar diagram gives the counit, and the triangle identities follow from the uniqueness of the universal property.

[1] Any reader offended by this statement is of course perfectly justified, but the author reccomends a comparison between the $\infty$-category of spectra and the many (inequivalent) model categories of spectra as justification.

[2] "Aha!" says the prudent model-category officianado, "you see, you use model categories for $\infty$-categories after all". "Shut up," I say in return.

## What homology sees

## What homotopy sees

## Homology Functors and Spectra

# 07th Feb

We see the sphere spectrum $\mathbb{S}$ is a homology theory,

**Definition 1: Homology**

A **Homology theory** is a functor $\mathbb{E} : S_* \to S_*$ which is

- **Finitary;** $\mathbb{E}$ commutes with filtered colimits
- **Reduced;** $\mathbb{E}(*) = *$, and
- **Excisive** $\mathbb{E}$ takes (homotopy) pushouts to (homotopy) pullbacks.

The excisive property gives us that $\mathbb{E}(U) \simeq \Omega\mathbb{E}(\Sigma U)$.

**Theorem 2: Spectra**

We call thus subcategory $\mathrm{Sp} \subseteq \mathrm{Fun}_*^{\mathrm{fin}}(S_*, S_*)$ the category of **spectra**. We fet a left adjoint to the inclusion $D : \mathrm{Fun}_*^{\mathrm{fin}}(S_*, S_*) \to \mathrm{Sp}$, given by $DF := \mathrm{colim}_{n\to\infty}\Omega^n F \Sigma^n$. This enforces excisiveness.

The hope is that we can use Hurewicz-type maps $\mathbb{S} \to \mathbb{E}$ to gather information about the (stable) homotopy groups of spheres.