Chromatic Homotopy Theory

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Abstract

The chromatic filtration on the stable homotopy groups of spheres is a reflection of the height filtration on the moduli stack of commutative, 1-dimensional formal group laws. Taking this idea seriously means forging a link between arithmetic geometry and homotopy theory. Periodicity phenomena deep within stable homotopy theory reflect structures found on the moduli stack of formal groups (or, perhaps better, p-divisible groups).

Many special objects in mathematics meet in chromatic homotopy theory: complex orientations, the Adams-Novikov spectral sequence, formal groups, p-divisible groups, modular forms, and even (conjecturally) certain quantum field theories. We will be particularly enthused about using the Fargues-Fontaine curve to reimagine the geometry controlling the chromatic filtration.

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INTRODUCTION

"Desar's chosen field in mathematics was so esoteric that nobody in the institute or the Maths Federative could really check his progress"
The Dispossessed, Ursula K. Le Guin

HOMOTOPY GROUPS OF SPHERES

Let (X, x_0) be a pointed topological space - we will often leave the basepoint implied. A major goal of algebraic topology is to study the topology of X via homotopy invariants. One of the first invariants we meet are the **homotopy groups** of based maps from spheres[1] into X. For $n \geq 0$, these are defined as $\pi_n(X) = [S^n, X]_* = Map_*(S^n, X)/\sim$, where $f \sim g \iff$ there exists a homotopy $h : f \simeq g$. The quotient map $h : S^m \to S^m/\sim \cong S^m \wedge S^m$ that identifies the "equator" of S^m with a point gives rise to a map

$$\pi_n(X) \times \pi_n(X) \cong Map_*(S^n \wedge S^n, X) \to \pi_m(X)$$

by precomposition with h. This gives a group structure on $\pi_n(X)$ when $n \geq 1$, which is abelian when $n \geq 2$. Even more pleasantly, this construction is functorial, yielding functors $\pi_n : \mathrm{Top}_* \to Ab$ when $n \geq 2$.

A reasonable first problem is to study the homotopy groups of spheres $\pi_n(S^m)$. When $n \leq m$, these have a satisfying structure:

THEOREM 1

Let $0 < m \in \mathbb{N}$. Then

$$\pi_n(S^m) \cong \begin{cases} 0, & n < m \\ \mathbb{Z}, & m = n \end{cases}$$

The maps in $\pi_1 S^1$ are classified, up to homotopy, by degree, with $d \in \mathbb{Z}$ corresponding to the map $s_d : S^1 \to S^1 \quad z \mapsto z^d$. The isomorphisms $\pi_1(S^1) \cong \pi_n(S^n)$ can be identified via the Hurewicz homomorphism, which we will study later. It is intuitive, but not trivial, to show $\pi_n(S^m) = 0$ when $n < m$. In the case $n = 1, m = 2$ it can be thought of as the statement "a rubber band on a ball can always be slid off". It follows from the cellular approximation thereom, which states any map $f : X \to Y$ between CW-complexes is homotopic to a cellular map, that is, one which takes the $n-$skeleta of X to the $n-$skeleta of Y. Since S^m has a CW-structure with one 0-cell and one m-cell, it follows that any map $f : S^n \to S^m$ for $n < m$ is homotopic to a constant map. Lastly, let's consider the groups $\pi_n S^1$ for $n > 1$. A map $X \to Y$ where X is simply connected has a lift to a cover $\tilde{Y}$ of Y. This shows any map $f : S^n \to S^1$ where $n \geq 2$ is null-homotopic, since it factors through the universal cover of S^1, $\mathbb{R}$, which is contractible. We therefore have a full classification of $\pi_n(S^1)$.

THEOREM 2

$$\pi_n(S^1) \cong \begin{cases} 0, & n \neq 1 \\ \mathbb{Z}, & n = 1 \end{cases}$$

We might hope (and expect) that the higher homotopy groups of all spheres are this simple. After all, *it's just spheres* - how complicated can it be? As it turns out, very complicated. The first indication of this was found by Hopf. The CW-complex structure of $\mathbb{CP}^2$ is given by adjoining a 4-cell to $\mathbb{CP}^1 \simeq S^2$ in the following way: recall $\mathbb{CP}^2 = \mathbb{C}^3/\sim$ where $[x, y, z] \sim \lambda[x, y, z]$ for $\lambda \in \mathbb{C}$, and notice the subset given by $z \neq 0$ is homeomorphic to $\mathrm{int}(D^4)$ via the homeomorphism $[x, y, z] = [x/z, y/z, 1] \mapsto (x/z, y/z)$, while the subset given by $z = 0$ is homeomorphic to $\mathbb{CP}^1$. We therefore adjoin a $4-$cell via the quotient map $S^3 \subset \mathbb{C}^2 \to \mathbb{CP}^1$. This is a fibration, and the fiber over each point is given by $\{\lambda \in \mathbb{C} : |\lambda| = 1\} \cong S^1$. This gives a fibration

$$S^1 \to S^3 \to S^2$$

[1] What's so special about spheres? You'll see...

called the **Hopf fibration**. The corresponding long exact sequence in homotopy reads

$$\dots \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \dots$$

so by our previous calculations we have $\pi_3(S^2) \cong \mathbb{Z}$!

In some sense, the nontriviality of higher homotopy groups of spheres comes from the failure of the suspension functor $\Omega : Top_* \to Top_*$ to be an equivalence. If it were an equivalence, we would have $\Omega S^n \simeq S^{n-1}$ and the loop space fibration $\Omega S^n \to PS^n \to S^n$, where PX are the space of paths in X, would read $S^{n-1} \to * \to S^n$ up to homotopy equivalence. The long exact sequence in homotopy would then give $\pi_k S^n \cong \pi_{k-1} S^{n-1}$ for all k. This desire for Ω to be an equivalence will shortly lead to some interesting maths.

Let us now compare with homology. The homology functor $H_* : Top_* \to grAb$ is in a sense a completely algebraic homotopy invariant. They are easy to compute compared to homotopy groups, thanks to the presence of *excision*. A concrete form of the excision axiom says: given a pair (X, A) of topological spaces, with $A \subset X$, and a subset $U \subset A$ such that $\overline{U} \subset int(A)$, the inclusion

$$(X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

induces an isomorphism in (reduced) relative homology. Excising a small disk D^n from the pair (S^n, D^n), of an n-sphere and its upper hemisphere, gives an isomorphism $H_n(S^n, D^n) \cong H_n(D^n, S^{n-1})$. Combining the two long exact sequences in (reduced) relative homology gives isomorphisms

$$\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^n, D^n) \cong \tilde{H}_{k-1}(D^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1}).$$

The reduced homology groups of S^n are therefore the same as those of S^1, shifted by $n-1$ in degree. Clearly we have

$$\tilde{H}_n(S^0) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0 \text{ or } n = m \\ 0 & \text{otherwise} \end{cases}$$

The previous discussion says homology groups of spheres are stable under suspension $\Sigma : H_k(S^n) \xrightarrow{\cong} H_{k+1}(\Sigma S^n)$. Since we don't have excision for homotopy groups, we can't expect a similar result. However, it turns out that we almost have stability, in the following sense:

THEOREM 3: THE FREUDENTHAL SUSPENSION THEOREM

The homomorphism $\sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ induced by suspension is an isomorphism when $n \geq k + 2$.

This theorem states that the homotopy groups of spheres are *eventually* stable under Σ. We have already confirmed this result for $n \leq m$, giving us the first non-trivial stable value $\pi_2 S^2 = \mathbb{Z}$. After calculating $\pi_4 S^3 = \mathbb{Z}/2\mathbb{Z}$ we would then have

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_4 S^3 \cong \pi_5 S^4 \cong \dots$$

We denote the group

$$\pi_n^s := \pi_{2n+2} S^{n+2} = \operatorname{colim}_{k \in \mathbb{N}} \big(\pi_{n+k}(S^k)\big) \text{ for } n \geq 2$$

the n^{th} **stable homotopy group of spheres**.

The following table shows some of the homotopy groups of spheres, with the stable homotopy groups highlighted. Note that in some cases, stability comes earlier than promised. We notice all of the groups are finitely generated, and most of them are finite. This was proven in generality by Serre.

THEOREM 4: SERRE

The homotopy groups $\pi_n(S^m)$ are finite, except for

$$\pi_n(S^n) = \mathbb{Z} \text{ for } n \geq 1$$

and

$$\pi_{4m-1}(S^{2m-1}) = \mathbb{Z} \oplus A$$

where A is finite.

In particular, the stable homotopy groups are finite for $n \geq 1$.

	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Figure 1: Some homotopy groups of spheres, taken from the HoTT book, **HoTT**.

HOMOLOGY THEORIES

Let us cast an old story in a new light.

DEFINITION 5

A **(reduced) homology functor** is a functor

$$\tilde{E}_* : HoS_* \to grAb$$

from the homotopy category of pointed spaces to the category graded abelian groups, along with a natural suspension ismorphism

$$\sigma : \tilde{E}_* \to \tilde{E}_{*+1} \circ \Sigma,$$

such that

(i) (Finitary) $\tilde{E}_*$ preserves filtered colimits.

(ii) (Excisive) Given an inclusion $A \xhookrightarrow{X}$ we obtain an exact sequence

$$\tilde{E}_*(A) \to \tilde{E}_*(X) \to \tilde{E}_*(X/A).$$

We can compare this definition to the classical one by Eilenberg and Maclane (...)

The long exact sequence in homology comes straight from the definition by consecutively taking homotopy quotients:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] & \\ * \arrow[r] & X/A \arrow[r] \arrow[d] & \Sigma A \arrow[r] \arrow[d] & * \arrow[d] \\ & * \arrow[r] & \Sigma X \arrow[r] \arrow[d] & \Sigma(X/A) \\ & & \dots & \end{tikzcd}$$

This gives rise to a LES, using conditions (1) and (2), as well as the suspension isomorphism.

Both $\tilde{H}_*$ and π_*^s are homology functors, as are all your favourite homology theories from algebraic topology. The reader will easily see how the definition can be modified for cohomology functors.

It turns out that all homology functors factor through π_*^s, in the following sense:

$$\begin{tikzcd} HoS_* \arrow[rr, "\tilde{E}_*"] \arrow[dr, "E"'] & & Ab^{\mathbb{Z}} \\ & HoS_* \arrow[ur, "\pi_*^s"'] & \end{tikzcd}$$

This allows us to make the following definition.

DEFINITION 6

A **homology theory** is a functor

$$E : HoS_* \to HoS_*$$

such that E is...

(i) (Reduced) $E(*) = *$

(ii) (Finitary) E preserves filtered colimits

(iii) (Excisive) The natural map $E \to \Omega \circ E \circ \Sigma$ is an equivalence.

A **homology theory** E gives rise to a **homology functor** $\pi_*^s E$ as one can check. The advantage of homology theories is that they are strictly homotopical.

The category of homology theories is equivalent to that of *spectra*, which we can think of as objects that represent homology functors. One definition of a spectrum is the following, although we will see other equivalent definitions later.

DEFINITION 7

A **spectrum** is a family $\big(E(i)\big)_{i\in\mathbb{N}}$ of pointed spaces indexed over the natural numbers with equivalences $\sigma : E(n) \to \Omega E(n+1)$.

Spectra have been studied in classical algebraic topology since the 1960's, but are naturally studied in the more recent formalism of ∞-categories. By the excision property of a homology theory, we get a spectrum

$$(E(n) = E(S^n))_{n\in\mathbb{N}}$$

One can show that this is in fact an equivalence of categories. This leads us to studying the category of spectra, which we will do in detail in Section 2.2.

WHERE ARE WE HEADED?

Our study of the category of spectra will show it has a lot of algebraic properties. There is a strong analogue between the category of spectra **Sp** in homotopy theory and the category of abelian groups **Ab** in algebra. In spectra, the spectrum representing stable homotopy theory is the sphere spectrum

$\mathbb{S}$, and it plays the role of the integers $\mathbb{Z}$ in spectra. In particular, it is initial, giving unique maps $\mathbb{S} \to E$ into any other spectrum, which are generalizations of the Hurewicz map for other homology theories. Additionally, one can use the symmetric monoidal structure on **Sp** to give $\mathbb{S}$ a commutative ring structure, in a way to be suitably defined.

Our ultimate goal is to understand $\mathbb{S}$ and compute $\pi_*\mathbb{S}$. After localising at a prime (to be suitably defined), we will see there is a filtration of $\mathbb{S}$ by localisations

$$\mathbb{S} \to \cdots \to L_k\mathbb{S} \to L_{k-1}\mathbb{S} \to \cdots \to L_0\mathbb{S} = H\mathbb{Z},$$

exhibiting an infinite "chromatic tower" of increasingly complex (co)homology theories that approximate the stable homotopy groups π_*. The first homology theory is ordinary (co)homology, the second is complex topological K-Theory, the third is topological modular forms (tmf), and so on. Each theory "sees more" of the topology of a space, but gets rapidly gets harder to compute - already at height 3 we know very little. The upshot of the chromatic perspective is that we can put on a pair of lenses that are "suitably fine" for the problem we are studying.

The layers of the chromatic tower are roughly the $K(n)$-local spheres, which we will define. They relate to localisations via the cartesian square, called the chromatic fracture square

$$\begin{tikzcd}
L_k\mathbb{S} \arrow[r] \arrow[d] & L_{K(k)}\mathbb{S} \arrow[d] \\
L_{k-1}\mathbb{S} \arrow[r] & L_{k-1}L_{K(k)}\mathbb{S}
\end{tikzcd}$$

The chromatic fracture square says we can understand the localisations $L_k\mathbb{S}$ by (1) computing $\pi_*L_{K(n)}\mathbb{S}$ and (2) understanding the maps in the above diagram. For (2), we a conjecture called the **chromatic splitting conjecture**, which says the bottom map is a split monomorphism.

It will take a lot of work to define and justify everything in this introduction. Firstly, we will need to set up some of the ∞-categorical language, which we do in Section 2.1. In Section 2.2 we will study the category of spectra and its analogy with the category of abelian groups, and in 2.3 we will see how to localise a spectrum. We will also need to study formal groups and their heights, as they control the chromatic tower, and this will be done in Section **??**. This will finally let us justify the claims in Section 1.3.

Preliminaries

Higher categories, homotopy (co)limits and Spaces

Homotopy (co)limits

We restrict our attention to the category Top of topologcial spaces. We would like a notion of (co)limit which is invariant under homotopy. For instance, if $X \simeq X'$ then we would like the pushouts $X \sqcup_Z Y$ and $X' \sqcup_Z Y$ to be homotopy equivalent for all Y and Z. This typically dramatically fails, for instance, if we choose $X = Y = \mathrm{pt}$ then the pushout $X \sqcup_{S^1} Y = \mathrm{pt} \sqcup_Z \mathrm{pt} = \mathrm{pt}$, but if we replace the point with $X = \mathrm{pt} \simeq D^2$, the unit disk in $\mathbb{R}^2$, then in this case we have that $D^2 \sqcup_{S^1} D^2 \simeq S^2$, the sphere, and certainly $S^2 \not\simeq \mathrm{pt}$.

This leads us to the notion of a *homotopy (co)limit*. We'll give a few examples of homotopy (co)limit construcitons in the category of topological spaces, then scetch a framework for giving general cases of these so-called 'homotopy-coherent' constructions - the framework of ∞-categories.

Example 1: Homotopy pushouts

Let X, Y and Z be (pointed) CW-complexes, and consider a diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

We can replace the map $f : Z \to Y$ by its *mapping cylinder*[a] $M(f) := \Big(([0,1] \times X) \sqcup Y\Big)/\sim$ where $\forall x \in X; (0, x) \sim f(x)$, so we think of $M(f)$ as gluing $\{0\} \times X$ to Y via f and we have an inclusion $i_f : X \cong \{1\} \times X \subseteq M(f)$. We can take the homotopy pushout of the above diagram by simply replacing f and g with their mapping cylinders and taking the ordinary pushout;

$$\begin{tikzcd} Z \arrow[r, "i_f"] \arrow[d, "i_g"'] & M(f) \arrow[d] \\ M(g) \arrow[r] & X \sqcup^h_Z Y := M(f) \sqcup_Z M(g) \end{tikzcd}$$

If we choose $Z = S^1$ and $X = Y = \mathrm{pt}$ then we can see that $\mathrm{pt} \sqcup_{S^1} \mathrm{pt} = \mathrm{pt}$ whereas $\mathrm{pt} \sqcup^h_{S^1} \mathrm{pt} = S^2$, since $M(S^1 \to \mathrm{pt})$ is (homotopic to) the inclusion $S^1 \hookrightarrow D^2$.

[a] In the language of model categories we could say $M(f)$ is a 'cofibrant replacement' of f, we will use no such language here.

Example 2: Homotopy pullbacks

Similarly if we have a diagram of (pointed) CW-complexes

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

we can define the homotopy pullback by the formula

$$X \times^h_Z Y := \Big\{(x, y, \gamma) \in X \times Y \times Z^{[0,1]} \ : \ \gamma(0) = f(x), \gamma(1) = g(y)\Big\}.$$

Thus the homotopy pullback is much like the ordinary pullback, except instead of requiring that $f(x) = g(y)$ we simply require a path between them and, importantly, that this path is part of the *data* of $X \times^h_Z Y$.

We can continue this game of naming a type of limit or colimit and seeing what the homotopy coherent version is, but the game becomes quickly tiresome so we invent new areas of maths instead. Historicall, Quillen invented model categories, which work via similar methods to example 1 by replacing objects and morphisms by corresponding fibrant/cofibrant versions of them. Model categories are excellent in many ways, but their lack of higher data means certain constructions become clunky[2]. We instead opt for ∞-categories, modeled as quasi-categories as per Joyal and Lurie[3].

Before continuing this introduction we offer a quick definition, that of the loop-spaces and suspensions of spaces as homotopy pullbacks/pushouts.

DEFINITION 3: LOOP SPACES AND SUSPENSIONS

Let X be a (pointed) CW-complex. We define the loop-space $\Omega X := \mathrm{pt} \times^h_X \mathrm{pt}$ and suspension $\Sigma X := \mathrm{pt} \sqcup^h_X \mathrm{pt}$.

This definition is certainly equivalent to the classical formulas

$$\Omega X := \mathrm{Map}_*(S^1, X) \text{ and } \Sigma X := (X \times [0,1])/\big((\{0\} \times X) \cup (\{1\} \times X) \cup ([0,1] \times \{*\})\big)$$

where $* \in X$ is the basepoint, but it allows a rather simplified proof of the functors $\Omega, \Sigma : \mathrm{Top} \to \mathrm{Top}$ being adjoint. If you believe us, for a moment, that in the ∞-category of spaces (whatever that may mean) there is an analagous universality property of the homotopy pullback and pushout then the (homotopy) pushout-pullback diagram

$$\begin{tikzcd}
\Omega\Sigma X \arrow[rr] \arrow[dd] & & \mathrm{pt} \arrow[d] \\
 & X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{pt} \arrow[r] & \Sigma X
\end{tikzcd}$$

gives us maps $X \to \Omega\Sigma X$ (by the 'universal property' of the pullback and that pt = pt), which is the unit of the adjuntion, a similar diagram gives the counit, and the triangle identities follow from the uniqueness of the universal property.

SIMPLICIAL SETS

Ok, so what is a simplicial set? The plan is as follows; we'll define a category Δ, called the 'simplex' or simply 'delta' category, which allows us to define 'simplicial objects' in a category C. Simplicial sets are then simplicial objects in the category of sets, and they give a particularly nice way of modelling of the homotopy theory of spaces (not dissimilar to CW-complexes, but more 'combinatorial').

DEFINITION 4: SIMPLICIAL OBJECTS

Let $\Delta \subseteq \mathrm{PoSet}$ be the category of *finite, linearly ordered posets* and order-preserving maps, thus each object in Δ is equivalent to a poset $[n] := \{0 < 1 < \cdots < n\}$. If $\mathcal{C}$ is a category then by a *simplicial object* of $\mathcal{C}$ we will mean a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The **category of simplicial objects** $s\mathcal{C}$ is the category $\mathrm{Hom}(\Delta^{\mathrm{op}}, \mathcal{C})$ of functors and natural transformations.

If we choose $\mathcal{C} = \mathcal{S}\mathrm{et}$ in the above definition then we can analyse what data a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ gives us. For each natural number $n \in \mathbb{N}$ we have a set $X_n := X([n])$ and we have maps $X_n \to X_m$ corresponding to the order-preserving maps $[m] \to [n]$ (note the op in Δ^{op} reversing the

[2]Any reader offended by this statement is of course perfectly justified, but the author reccomends a comparison between the ∞-category of spectra and the many (inequivalent) model categories of spectra as justification.

[3]"Aha!" says the prudent model-category officianado, "you see, you use model categories for ∞-categories after all". "Shut up," I say in return.

direction of the arrows). We can actually generate all of the maps $[m] \to [n]$ by looking at two special classes of maps; the *face* δ_i and *degeneracy* σ_i maps given by

$$\delta_i : [n-1] \to [n], \quad k \mapsto \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

$$\delta_i : [n] \to [n-1], \quad k \mapsto \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases}$$

whence it can easily (if tediously) be shown that any map $[m] \to [n]$ can be written as a composition of face and degeneracy maps.

If $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a simplicial set then we can apply X to the face and degenerace maps in the category Δ above to get face $d_i := X(\delta_i) : X_n \to X_{n-1}$ and degeneracy $s_i := X(\sigma_i) : X_{n-1} \to X_n$ maps for X (again, note the reversal of arrows after aplying X). We write the following diagram for a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, where the maps

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] & X_2 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & X_3 \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] \\ \arrow[from=1-1, to=1-2] \arrow[from=1-2, to=1-3, shift left=2] \arrow[from=1-2, to=1-3, shift right=2] \arrow[from=1-3, to=1-4, shift left=4] \arrow[from=1-3, to=1-4] \arrow[from=1-3, to=1-4, shift right=4] \end{tikzcd}$$

Figure 2: A simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

In figure 2 the maps going left are the face maps and the maps going right are the degenaracy maps. The intuition here is that the face maps take an 'n-cell' x (an element $x \in X_n$) and give the i^{th} face $d_i(x)$ of x, whereas the degeneracy maps give a degenerate n-cell from an $n-1$ cell. We'll give two examples of simplicial sets now, these are in fact our prototypical examples of ∞-categories.

Example 5: Nerve of a Category

Let $\mathcal{C}$ be an ordinary category, we can form a simplicial set $N(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by sending $[n]$ to the set $N(\mathcal{C})_n$ of sequences of n composable maps

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} x_n. \tag{1}$$

The i^{th} face map send a sequence of n maps to a sequence of $n-1$ maps by composing the i^{th} and the $(i+1)^{\mathrm{th}}$ map, so the above sequence would get sent to the $n-1$ sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \ldots x_{i-1} \xrightarrow{f_i \circ f_{i-1}} x_{i+1} \ldots \xrightarrow{f_{n-1}} x_n.$$

under d_i. The degeneracy map s_i is given by inserting an identity map to get a sequence of length $n+1$, so the n-sequence in equation 1 would get sent to the $n+1$ sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \ldots x_i \xrightarrow{\mathrm{id}_{x_i}} x_i \ldots \xrightarrow{f_{n-1}} x_n.$$

Example 6: Homotopy Types

Let Y be a CW-complex, then we can get a simplicial set $\mathrm{Sing}(Y) : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by sending $[n] \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^n|, Y)$ where

$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \sum_{k=0}^{n} t_k = 1 \right\}$$

is the so-called *topological n-simplex*. Thus $\mathrm{Sing}(Y)_1$ is the set of paths in Y. The face maps send an n-simplex to its i^{th} face, for example there are two maps $\mathrm{Sing}(Y)_1 \to \mathrm{Sing}(Y)_0$ sending a path to either one of its endpoints. The degeneragy maps send an $n-1$ simplex to the constant n-simplex, so a point $y \in Y = \mathrm{Sing}(Y)_0$ would be sent to the constant path at y under the degeneracy map $\mathrm{Sing}(Y)_0 \to \mathrm{Sing}(Y)_1$.

The two preceeding examples show that the theory of simplicial sets subsumes both category theory and homotopy theory. They also show the typical approach to getting ∞-categories from some other mathematical object - we have to find some kind of nerve construction. This 'nerve' was N for categories and Sing for topological spaces, and later we will see that the ∞-category of ∞-categories will be given by constructing a 2-category of ∞-categories and taking its *coherent nerve*, which is simply an analogue of N or Sing for 2-categories instead of 1-categories or topological spaces. The ∞-category of cdgas[4] is constructed using a differential-graded nerve N^{dg} in Lurie, 2017, but the category of spectra is formed by taking a limit of a diagram of ∞-categories, and so this will be our first example of an ∞-category which isn't contructed via a nerve (though it certainly could be).

We have seen how to get a simplicial set $\mathrm{Sing}(Y)$ from a topological space Y, but we can also get a topoplogical space $|X|$ from a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, called the *geometric realization*. To do this we start by defining what will become our analogues of disks for CW-complexes;

DEFINITION 7: STANDARD SIMPLEX

The **standard n-simplex** is the representable functor

$$\Delta^n : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}, \qquad [n] \mapsto \mathrm{Hom}_\Delta(-, [n]).$$

Note that if $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is another simplicial set then the Yoneda lemma gives us that $\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, X) \cong X_n$, so in some sense these standard n-simplicies are 'detecting' all the n-simplicies of X. Note the similarity between Δ^n and the role that disk D^n plays for CW-complexes. In particular, note that $\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^0, \mathrm{Sing}(Y)) \cong \mathrm{Sing}(Y)_0$ recovers the points of our space Y. This leads us to the idea of realization.

DEFINITION 8: GEOMETRIC REALIZATION

The **geometric realization** of a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is the topological space

$$|X| := \operatorname*{colim}_{\Delta^n \to X} |\Delta^n|$$

where $|\Delta^n|$ is the topological n-simplex of example 6.

Any reader familiar with Kan-extensions might be interested to know that the geometric realization functor $|-| : s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ was the first example of a Kan-extension.

THEOREM 9

The functors $|-| : s\mathcal{S}\mathrm{et} \leftrightarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$ form an adjunction.

This adjunction can be realized to be a 'Quillen adjunction', which says that the homotopy theory of simplicial sets is in some sense equivalent to the homotopy theory of topoplogical spaces. It turns out that the objects representing homotopy types are exactly the ∞-groupoids and it is this that justifies calling an ∞-groupoid a 'space'.

[4] Commutative differential graded algebras

Higher categories

Finally, we move onto the definition of an ∞-category, which are simplicial sets $X : \Delta^{\text{op}} \to \mathcal{S}\text{et}$ satisfying some particular coniditions, and example 5 defining the nerve $N(C)$ of a category already hints at the idea. We will want X_0 to be our objects and X_1 to be our morphisms, the fact that we have an identity map $1_x : x \to x$ for every object $x \in X_0$ is already satisfied by setting $1_x := s_0(x)$, the degenerate 1-simplex corresponding to our 0-simplex x. The n-simplicies X_n should be thought of as sequences of maps

$$x_0 \xrightarrow{f_0} x_1 \to \cdots \to x_{n-1} \xrightarrow{f_{n-1}} x_n$$

where the domain of f_k agrees with the codomain of f_{k-1}. We want to say 'composable' maps here but we run into a problem. We want our theory of ∞-categories to subsume the theory of thr homotopy theory of spaces, but if we think of X as a space instead of a category we realise that paths in a space don't have a unique composition. So the property defining an ∞-category should give a not-neccecarily unique way to compose 1-simplicies to live somewhere between the theory of categories and the homotopy theory of spaces. Luckily that's not actually particularly hard.

We define the n,k-horns Λ^n_k by the formula

$$\Lambda^n_k : \Delta^{\text{op}} \to \mathcal{S}\text{et}, \qquad \Lambda^n_k := \bigcup_{k \in E \subsetneq [n]} \Delta^E.$$

which upon first sight confuses anyone and everyone, which is why no higher category theorist ever uses this formula and just uses pictures instead, and forunately the 2-horns give essentially all the intution you'll need.

Let's look at the $(n,k) = (2,1)$ case first, i.e. Λ^2_1. In any case, the union runs over the subsets $E \subsetneq [n]$ which contain $k = 1$, the possible options for E are thus $E_1 := \{1\}, E_0 := \{0 < 1\}$ and $E_2 := \{1 < 2\}$.

- The 0-simplicies $\Lambda^2_1([0])$ are given by the union of the $\Delta^E([0]) = \text{Hom}([0], E)$. Evaluating this for E_1, E_2 and E_3 gives

$$\begin{aligned} \Delta^{E_0}([0]) &= \text{Hom}([0], E_2) = \{0\} \\ \Delta^{E_1}([0]) &= \text{Hom}([0], E_1) = \{1\} \\ \Delta^{E_2}([0]) &= \text{Hom}([0], E_2) = \{1, 2\} \end{aligned}$$

  and so the union over all of these gives $\Lambda^2_1([0]) = \{0, 1, 2\}$.
- The 1-simplicies $\Lambda^2_1([1])$ are given similarly by the union of the $\Delta^E([1]) = \text{Hom}([1], E)$, noting that any non-injective maps correspond to degenerate simplicies we are free to only look at injective maps $\text{Hom}([1], E_i)$, these turn out to be given by just $\Lambda^2_1([1]) = \{\text{id}_{E_0}, f\}$ where $f : [1] \to E_2$ sends $f(0) = 1$ and $f(1) = 2$.

Putting this all together we have three 0-simplicies $\{0, 1, 2\}$ and two 1-simplicies $\{0 < 1, 1 < 2\}$ which can be put into a pretty picture like so

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Doing the same thing for Λ^2_0 and Λ^2_2 will give you the following pictures.

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

The way to easily remember these pictures is to say that 'Λ^2_k is the triangle with the edge oppsite the k-vertex removed'. Similarly Λ^n_k is the n-simplex with the face opposite the k-vertex removed.

DEFINITION 10

Let Λ^n_k be the horns as above. We call the horns with $k = 0$ or $k = n$ (i.e. Λ^n_n and Λ^n_0) the **outer horns**. The horns with $0 < k < n$ are called the **inner horns**.

The inner horn Λ^2_1 looks a lot like a pair of composable arrows. In fact, finding a composition will correspond to finding an arrow $0 \to 2$ with a 2-simplex with the arrows $0 \to 1$, $1 \to 2$ and $0 \to 2$ as its boundary. This motivates the following.

DEFINITION 11: ∞-CATEGORY

An ∞-**category** is a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that any map $\Lambda^n_k \to X$ with $k \neq 0, n$ (so Λ^n_k is an inner horn) has a lift to a map $\Delta^n \to X$. In other words, X is an ∞-category if the restriction map

$$\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Lambda^n_k, X) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, X)$$

is a surjection for $k \neq 0, n$.

DEFINITION 12: ∞-GROUPOID

An ∞-**groupoid** is a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that any map $\Lambda^n_k \to X$ (so Λ^n_k is an inner *or outer* horn) has a lift to a map $\Delta^n \to X$. In other words, X is an ∞-groupoid if the restriction map

$$\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Lambda^n_k, X) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, X)$$

is a surjection for all k.

THE CATEGORY OF SPECTRA

LOCALISATIONS AND RECOLLEMENTS

Recall from our previous discussion we have a monoidal structure on the category of spectra with unit given by $\mathbb{S} := \Sigma^\infty S^0$ and symmetric-monoidal product given by

$$E \otimes F := D(E \circ F) \simeq D(E \boxtimes F)$$

where $(E \boxtimes F)(X) := \mathrm{colim}_{A \wedge B \to X} E(A) \wedge F(B)$ is the Day convolution and $DF := \mathrm{colim}_{n \to \infty} \Omega^n F \Sigma^n$ is the Goodwillie differential. Indeed, we then get a functor $E \otimes - : \mathrm{Sp} \to \mathrm{Sp}$ which preserves colimits, and since Sp is a presentable category (see Lurie, 2017) we get that it has a right adjoint, the *function spectra* $E \otimes - \dashv \mathbb{F}(E, -)$ which acts as an internal hom for the ∞-category Sp of spectra. It has homotopy groups

$$\pi_n \mathbb{F}(E, F) = \mathrm{Ext}^{-i}_{\mathrm{ho(Sp)}}(E, F).$$

All of this structure was used to justify referring to Sp as some 'homotopical' analogue of the category of abelian groups.

One feature we'd like to mimick from group theory is the idea of splitting a group into its *torsion* and *rational* parts. Note for instance that $H\mathbb{Q} \otimes H\mathbb{Q} = H\mathbb{Q}$, but $H\mathbb{F}_p \otimes H\mathbb{F}_p \not\simeq H\mathbb{F}_p$.

EXAMPLE 13: STEENROD ALGEBRA

What is $H\mathbb{F}_p \otimes H\mathbb{F}_p$, then? Recall that we define the E-homology of F, for two spectra E and F, as the graded abelian group $E_* F := \pi_*(E \otimes F)$, and so the spectrum $H\mathbb{F}_p \otimes H\mathbb{F}_p$ is the $H\mathbb{F}_p$-homology of $H\mathbb{F}_p$. The algebra $\mathcal{A}_p := \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$ is called the *(dual) Steenrod algebra*, and it can be shown that it is the free Hopf algebroid

$$\mathrm{Sym}(\xi_1, \xi_2, \dots, \tau_0, \tau_1, \dots)$$

with comultiplication $\psi : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ given by

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k \qquad \text{and} \qquad \psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \tau_k.$$

The utility in the Steenrod algebra is that the cohomology $H^\bullet(X;\mathbb{F}_p)$ of any space X is naturally a module over $\mathcal{A}_p$, which enables us to encode information about stable cohomology operations, and while the cup-product is not stable[a], we gain the Adams Spectral Sequence, an invaluable tool to stable homotopy theorists, but more on that later.

[a]Thus we can no longer use the cup-product when we think of $H^\bullet(X;\mathbb{F}_p)$ as an $\mathcal{A}_p$-module

Warning 14

We do have that

$$\mathrm{Alg}_{\mathbb{A}_\infty}(\mathrm{Mod}(H\mathbb{F}_p)) \simeq \mathrm{DGA}(\mathbb{F}_p) \qquad \text{and} \qquad \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}(H\mathbb{Q})) \simeq \mathrm{CDGA}(\mathbb{Q}),$$

but we don't have an equivalence $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}(H\mathbb{F}_p)) \not\simeq \mathrm{CDGA}(\mathbb{F}_p)$ - essentially because the rational cohomology $H\mathbb{Q}(\Sigma_n)$ of the symmetric groups is trivial, but $H\mathbb{F}_p(\Sigma_n)$ is not, and the definition of $\mathbb{E}_\infty$-rings uses actions of the symmetric groups Σ_n to permute elements. Thus there is some sense in which the symmetric-monoidal product of spectra really doesn't behave well with $\mathbb{F}_p$-homology, and we'll really need the full generality of $\mathbb{E}_\infty$ algebras to capture everything homotopical algebra, DGAs simply won't do.

The above warning tells us that rational spectra can be handled using the well-understood notion of DGAs, which in the case of $H\mathbb{Q}$ reduces to just studying $\mathbb{Q}$-algebras, but the p-torsion for primes p is somewhat more complicated. Thus the bulk of the work here will be understanding this p-torsion of spectra, and that is why we need the notion of *localizations*. Before defining these we'll go through an example to make sure that any definitions we do make satisfy the right properties. To whit, we will analise the ∞-category of quasicoherent sheaves on a scheme.

EXAMPLE 15

Let X be a quasi-compact and quasi-separated (qcqs) scheme, then its ∞-category of quasi-coherent sheaves is given by

$$D_{qc}(X) := \lim_{\mathrm{Spec}R\to X} \mathrm{Mod}(HR).$$

For instance, if $X = \mathrm{Spec}A$ is affine then $D_{qc}(X)$ is given by $\mathrm{colim}_{A\to R}\mathrm{Mod}(HR)$, and $\mathrm{Mod}(HR) = D(R)$, so $D_{qc}(\mathrm{Spec}A)$ is given by finite projective modules over A, which is exactly the local condition required for a sheaf to be quasi-coherent.

Now suppose that $U \subseteq X$ is a quasi-compact Zariski-open and let $Z = X\backslash U$ so that if U and X are affine this says that Z is given by a finitely generated ideal; it is given by the vanishing of finitely many functions. If we denote the inclusions by $j : U \to X$ and $i : Z \to X$ then we get an adjuntion

$$D_{qc}(X) \underset{j^*}{\overset{j_*}{\leftrightarrows}} D_{qc}(U)$$

with j_* fully-faithful and j^* symmetric-monoidal[a]. Thus we have that j^* is a map of $\mathbb{E}_\infty$-rings, but j_* is not. The projection formula $j_*(j^*\mathcal{E} \otimes_{\mathcal{O}_U} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_X} j_*\mathcal{F}$ gives us, however, that j_* *is* a map of $\mathcal{O}_X$-modules.

The above discussion allows us to make the statement '$U \subseteq X$ is a quasicompact Zariski-open' in purely categorical terms. Notice that $j^*j_* = \mathrm{id}_{D_{qc}(U)}$ gives that j_* is a split monomorphism, already categorifying the notion of 'subset'. Hence understanding $D_{qc}(U) = \{\mathcal{F} \in D_{qc}(X) : \mathcal{F} \xrightarrow{\sim} j_*j^*\mathcal{F}\}$ is the same as understanding when the endofunctor j_*j^* is an equivalence. The projection formula gives that

$$j_*j^*\mathcal{F} = j_*(j^*\mathcal{F} \otimes \mathcal{O}_U) = \mathcal{F} \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U,$$

thus $j_*j^* = - \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U$. Furthermore we that $(j_*j^*)^2 = (j_*j^*)(j_*j^*) = j_*(j^*j_*)j^* = j_*j^*$ (since $j^*j_* = \mathrm{id}$) so that, while j_*j^* isn't the identity, it is idempotent, and the multiplication map $j_*\mathcal{O}_U \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U \xrightarrow{\sim} j_*\mathcal{O}_U$ is an isomorphism (NB we call such a ring R, whose multiplication map $R \otimes R \to R$ is an isomorphism, a **solid ring**). It can be shown that a map $f : Y \to X$ of qcqs shemes is a Zariski-open immersion if and only if $f_*\mathcal{O}_Y$ is a solid ring.

Finally, we see that a subcategory $\mathcal{D} \subseteq D_{qc}(X)$ is of the form $D_{qc}(U)$ for some Zariski-open U if and only if $\mathcal{D}$ is *localizing* and *smashing*. A subcategory $\mathcal{D}$ of a (complete, cocomplete, presentable) stable ∞-category $\mathcal{C}$ is *localizing* if it's a full subcategory and the inclusion $U : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint F such that $L^2 = L$ for $L := UF$. A localizing subcategory is *smashing* if further $\mathcal{C}$ and $\mathcal{D}$ are symmetric-monoidal with F symmetric-monoidal and U a $\mathcal{C}$-module map; $U(b \otimes Fc) = Ub \otimes c$.

[a] Thus we have $j^*\mathcal{O}_X = \mathcal{O}_U$ and $j^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = j^*\mathcal{E} \otimes_{\mathcal{O}_U} j^*\mathcal{F}$.

Note the analogy with $H\mathbb{Q} \otimes_{\mathbb{S}} H\mathbb{Q} = H\mathbb{Q}$ here. We see that $H\mathbb{Q}$ is a solid ring, and in fact $\mathbb{Q}$ is the only solid ring of characteristic zero, and thus the de-Rham–Quillen adjunction between simplicial sets and connective CDGAs is idempotent only over $\mathbb{Q}$ (this is exatly the statement of warning 14) and it is in this way that we should think of $H\mathbb{Q}$ as being an 'open subscheme'[5] of the sphere spectrum $\mathbb{S}$. The $H\mathbb{F}_p$ spectra, however, are *closed* immersions, and we the goal of chromatic homotopy theory is to rebuild $\mathbb{S}$ out of $H\mathbb{Q}$ and the $H\mathbb{F}_p$ as a récollemont, which we will discuss now.

If $U \subseteq X$ is a quasi-compact open as in example 15 above, and $Z := X\backslash U$, then we get a **récollemont**; we can rebuild $D_{qc}(X)$ out of $D_{qc}(U)$ and $D_Z(X) := \ker(j^* : D_{qc}(X) \to D_{qc}(U))$ as follows.

[5] Actually, $\mathbb{S} \to H\mathbb{Q}$ is not finite and so we should more appropriately think of $H\mathbb{Q}$ as a 'pro-open' subscheme.

References

Lurie, J. (2017). Higher algebra.