Chromatic Homotopy Theory

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Abstract

Chromatic homotopy theory is...

Contents

INTRODUCTION

"Desar's chosen field in mathematics was so esoteric that nobody in the institute or the Maths Federative could really check his progress"
The Dispossessed, Ursula K. Le Guin

The chromatic filtration on the stable homotopy groups of spheres is a reflection of the height filtration on the moduli stack of commutative, 1-dimensional formal group laws. Taking this idea seriously means forging a link between arithmetic geometry and homotopy theory. Periodicity phenomena deep within stable homotopy theory reflect structures found on the moduli stack of formal groups (or, perhaps better, p-divisible groups).

Many special objects in mathematics meet in chromatic homotopy theory: complex orientations, the Adams-Novikov spectral sequence, formal groups, p-divisible groups, modular forms, and even (conjecturally) certain quantum field theories. One theme I'll be particularly enthused about is using the Fargues-Fontaine curve to reimagine the geometry controlling the chromatic filtration.

PRELIMINARIES

HIGHER CATEGORIES, HOMOTOPY (CO)LIMITS AND SPACES

HOMOTOPY (CO)LIMITS

We restrict our attention to the category Top of topologcial spaces. We would like a notion of (co)limit which is invariant under homotopy. For instance, if $X \simeq X'$ then we would like the pushouts $X \sqcup_Z Y$ and $X' \sqcup_Z Y$ to be homotopy equivalent for all Y and Z. This typically dramatically fails, for instance, if we choose $X = Y = \mathrm{pt}$ then the pushout $X \sqcup_{S^1} Y = \mathrm{pt} \sqcup_Z \mathrm{pt} = \mathrm{pt}$, but if we replace the point with $X = \mathrm{pt} \simeq D^2$, the unit disk in $\mathbb{R}^2$, then in this case we have that $D^2 \sqcup_{S^1} D^2 \simeq S^2$, the sphere, and certainly $S^2 \not\simeq \mathrm{pt}$.

This leads us to the notion of a *homotopy (co)limit*. We'll give a few examples of homotopy (co)limit construcitons in the category of topological spaces, then scetch a framework for giving general cases of these so-called 'homotopy-coherent' constructions - the framework of ∞-categories.

EXAMPLE 1: HOMOTOPY PUSHOUTS

Let X, Y and Z be (pointed) CW-complexes, and consider a diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

We can replace the map $f : Z \to Y$ by its *mapping cylinder*[a] $M(f) := \Big(([0,1] \times X) \sqcup Y\Big)/\sim$ where $\forall x \in X; (0, x) \sim f(x)$, so we think of $M(f)$ as gluing $\{0\} \times X$ to Y via f and we have an inclusion $i_f : X \cong \{1\} \times X \subseteq M(f)$. We can take the homotopy pushout of the above diagram by simply replacing f and g with their mapping cylinders and taking the ordinary pushout;

$$\begin{tikzcd} Z \arrow[r, "i_f"] \arrow[d, "i_g"'] & M(f) \arrow[d] \\ M(g) \arrow[r] & X \sqcup^h_Z Y := M(f) \sqcup_Z M(g) \end{tikzcd}$$

If we choose $Z = S^1$ and $X = Y = \mathrm{pt}$ then we can see that $\mathrm{pt} \sqcup_{S^1} \mathrm{pt} = \mathrm{pt}$ whereas $\mathrm{pt} \sqcup^h_{S^1} \mathrm{pt} = S^2$, since $M(S^1 \to \mathrm{pt})$ is (homotopic to) the inclusion $S^1 \hookrightarrow D^2$.

[a] In the language of model categories we could say $M(f)$ is a 'cofibrant replacement' of f, we will use no such language here.

EXAMPLE 2: HOMOTOPY PULLBACKS

Similarly if we have a diagram of (pointed) CW-complexes

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

we can define the homotopy pullback by the formula

$$X \times^h_Z Y := \Big\{(x, y, \gamma) \in X \times Y \times Z^{[0,1]} \,:\, \gamma(0) = f(x), \gamma(1) = g(y)\Big\}.$$

Thus the homotopy pullback is much like the ordinary pullback, except instead of requiring that $f(x) = g(y)$ we simply require a path between them and, importantly, that this path is part of the *data* of $X \times^h_Z Y$.

We can continue this game of naming a type of limit or colimit and seeing what the homotopy coherent version is, but the game becomes quickly tiresome so we invent new areas of maths instead. Historicall, Quillen invented model categories, which work via similar methods to example 1 by replacing objects and morphisms by corresponding fibrant/cofibrant versions of them. Model categories are excellent in many ways, but their lack of higher data means certain constructions become clunky[1]. We instead opt for ∞-categories, modeled as quasi-categories as per Joyal and Lurie[2].

Before continuing this introduction we offer a quick definition, that of the loop-spaces and suspensions of spaces as homotopy pullbacks/pushouts.

DEFINITION 3: LOOP SPACES AND SUSPENSIONS

Let X be a (pointed) CW-complex. We define the loop-space $\Omega X := \mathrm{pt} \times^h_X \mathrm{pt}$ and suspension $\Sigma X := \mathrm{pt} \sqcup^h_X \mathrm{pt}$.

This definition is certainly equivalent to the classical formulas

$$\Omega X := \mathrm{Map}_*(S^1, X) \text{ and } \Sigma X := (X \times [0,1])/\big((\{0\} \times X) \cup (\{1\} \times X) \cup ([0,1] \times \{*\})\big)$$

where $* \in X$ is the basepoint, but it allows a rather simplified proof of the functors $\Omega, \Sigma : \mathrm{Top} \to \mathrm{Top}$ being adjoint. If you believe us, for a moment, that in the ∞-category of spaces (whatever that may mean) there is an analagous universality property of the homotopy pullback and pushout then the (homotopy) pushout-pullback diagram

$$\begin{tikzcd}
\Omega\Sigma X \arrow[rr] \arrow[dd] & & \mathrm{pt} \arrow[d] \\
& X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{pt} \arrow[r] & \Sigma X
\end{tikzcd}$$

gives us maps $X \to \Omega\Sigma X$ (by the 'universal property' of the pullback and that pt = pt), which is the unit of the adjuntion, a similar diagram gives the counit, and the triangle identities follow from the uniqueness of the universal property.

SIMPLICIAL SETS

Ok, so what is a simplicial set? The plan is as follows; we'll define a category Δ, called the 'simplex' or simply 'delta' category, which allows us to define 'simplicial objects' in a category C. Simplicial sets are then simplicial objects in the category of sets, and they give a particularly nice way of modelling of the homotopy theory of spaces (not dissimilar to CW-complexes, but more 'combinatorial').

DEFINITION 4: SIMPLICIAL OBJECTS

Let $\Delta \subseteq \mathrm{PoSet}$ be the category of *finite, linearly ordered posets* and order-preserving maps, thus each object in Δ is equivalent to a poset $[n] := \{0 < 1 < \cdots < n\}$. If $\mathcal{C}$ is a category then by a *simplicial object* of $\mathcal{C}$ we will mean a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The **category of simplicial objects** $s\mathcal{C}$ is the category $\mathrm{Hom}(\Delta^{\mathrm{op}}, \mathcal{C})$ of functors and natural transformations.

If we choose $\mathcal{C} = \mathcal{S}\mathrm{et}$ in the above definition then we can analyse what data a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ gives us. For each natural number $n \in \mathbb{N}$ we have a set $X_n := X([n])$ and we have maps $X_n \to X_m$ corresponding to the order-preserving maps $[m] \to [n]$ (note the op in Δ^{op} reversing the

[1] Any reader offended by this statement is of course perfectly justified, but the author reccomends a comparison between the ∞-category of spectra and the many (inequivalent) model categories of spectra as justification.

[2] "Aha!" says the prudent model-category officianado, "you see, you use model categories for ∞-categories after all". "Shut up," I say in return.

direction of the arrows). We can actually generate all of the maps $[m] \to [n]$ by looking at two special classes of maps; the *face* δ_i and *degeneracy* σ_i maps given by

$$\delta_i : [n-1] \to [n], \quad k \mapsto \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

$$\delta_i : [n] \to [n-1], \quad k \mapsto \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases}$$

whence it can easily (if tediously) be shown that any map $[m] \to [n]$ can be written as a composition of face and degeneracy maps.

If $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a simplicial set then we can apply X to the face and degenerace maps in the category Δ above to get face $d_i := X(\delta_i) : X_n \to X_{n-1}$ and degeneracy $s_i := X(\sigma_i) : X_{n-1} \to X_n$ maps for X (again, note the reversal of arrows after aplying X). We write the following diagram for a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, where the maps

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_3 \cdots$$

Figure 1: A simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

In figure 1 the maps going left are the face maps and the maps going right are the degenaracy maps. The intuition here is that the face maps take an 'n-cell' x (an element $x \in X_n$) and give the i^{th} face $d_i(x)$ of x, whereas the degeneracy maps give a degenerate n-cell from an $n-1$ cell. We'll give two examples of simplicial sets now, these are in fact our prototypical examples of ∞-categories.

EXAMPLE 5: NERVE OF A CATEGORY

Let $\mathcal{C}$ be an ordinary category, we can form a simplicial set $N(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by sending $[n]$ to the set $N(\mathcal{C})_n$ of sequences of n composable maps

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} x_n. \tag{1}$$

The i^{th} face map send a sequence of n maps to a sequence of $n-1$ maps by composing the i^{th} and the $(i+1)^{\text{th}}$ map, so the above sequence would get sent to the $n-1$ sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \ldots x_{i-1} \xrightarrow{f_i \circ f_{i-1}} x_{i+1} \ldots \xrightarrow{f_{n-1}} x_n.$$

under d_i. The degeneracy map s_i is given by inserting an identity map to get a sequence of length $n+1$, so the n-sequence in equation 1 would get sent to the $n+1$ sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \ldots x_i \xrightarrow{\mathrm{id}_{x_i}} x_i \ldots \xrightarrow{f_{n-1}} x_n.$$

EXAMPLE 6: HOMOTOPY TYPES

Let Y be a CW-complex, then we can get a simplicial set $\mathrm{Sing}(Y) : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by sending $[n] \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^n|, Y)$ where

$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;:\; \sum_{k=0}^{n} t_k = 1 \right\}$$

is the so-called *topological n-simplex*. Thus $\mathrm{Sing}(Y)_1$ is the set of paths in Y. The face maps send an n-simplex to its i^{th} face, for example there are two maps $\mathrm{Sing}(Y)_1 \to \mathrm{Sing}(Y)_0$ sending a path to either one of its endpoints. The degeneragy maps send an $n-1$ simplex to the constant n-simplex, so a point $y \in Y = \mathrm{Sing}(Y)_0$ would be sent to the constant path at y under the degeneracy map $\mathrm{Sing}(Y)_0 \to \mathrm{Sing}(Y)_1$.

The two preceeding examples show that the theory of simplicial sets subsumes both category theory and homotopy theory. They also show the typical approach to getting ∞-categories from some other mathematical object - we have to find some kind of nerve construction. This 'nerve' was N for categories and Sing for topological spaces, and later we will see that the ∞-category of ∞-categories will be given by constructing a 2-category of ∞-categories and taking its *coherent nerve*, which is simply an analogue of N or Sing for 2-categories instead of 1-categories or topological spaces. The ∞-category of cdgas[3] is constructed using a differential-graded nerve N^{dg} in Lurie, 2017, but the category of spectra is formed by taking a limit of a diagram of ∞-categories, and so this will be our first example of an ∞-category which isn't contructed via a nerve (though it certainly could be).

We have seen how to get a simplicial set $\text{Sing}(Y)$ from a topological space Y, but we can also get a topoplogical space $|X|$ from a simplicial set $X : \Delta^{\text{op}} \to \mathcal{S}\text{et}$, called the *geometric realization*. To do this we start by defining what will become our analogues of disks for CW-complexes;

Definition 7: Standard Simplex

The **standard n-simplex** is the representable functor

$$\Delta^n : \Delta^{\text{op}} \to \mathcal{S}\text{et}, \qquad [n] \mapsto \text{Hom}_\Delta(-, [n]).$$

Note that if $X : \Delta^{\text{op}} \to \mathcal{S}\text{et}$ is another simplicial set then the Yoneda lemma gives us that $\text{Hom}_{s\mathcal{S}\text{et}}(\Delta^n, X) \cong X_n$, so in some sense these standard n-simplicies are 'detecting' all the n-simplicies of X. Note the similarity between Δ^n and the role that disk D^n plays for CW-complexes. In particular, note that $\text{Hom}_{s\mathcal{S}\text{et}}(\Delta^0, \text{Sing}(Y)) \cong \text{Sing}(Y)_0$ recovers the points of our space Y. This leads us to the idea of realization.

Definition 8: Geometric Realization

The **geometric realization** of a simplicial set $X : \Delta^{\text{op}} \to \mathcal{S}\text{et}$ is the topological space

$$|X| := \operatorname*{colim}_{\Delta^n \to X} |\Delta^n|$$

where $|\Delta^n|$ is the topological n-simplex of example 6.

Any reader familiar with Kan-extensions might be interested to know that the geometric realization functor $|-| : s\mathcal{S}\text{et} \to \mathcal{T}\text{op}$ was the first example of a Kan-extension.

Theorem 9

The functors $|-| : s\mathcal{S}\text{et} \leftrightarrows \mathcal{T}\text{op} : \text{Sing}$ form an adjunction.

This adjunction can be realized to be a 'Quillen adjunction', which says that the homotopy theory of simplicial sets is in some sense equivalent to the homotopy theory of topoplogical spaces. It turns out that the objects representing homotopy types are exactly the ∞-groupoids and it is this that justifies calling an ∞-groupoid a 'space'.

[3] Commutative differential graded algebras

Higher categories

Finally, we move onto the definition of an ∞-category, which are simplicial sets $X : \Delta^{\text{op}} \to \mathcal{S}\text{et}$ satisfying some particular coniditions, and example 5 defining the nerve $N(C)$ of a category already hints at the idea. We will want X_0 to be our objects and X_1 to be our morphisms, the fact that we have an identity map $1_x : x \to x$ for every object $x \in X_0$ is already satisfied by setting $1_x := s_0(x)$, the degenerate 1-simplex corresponding to our 0-simplex x. The n-simplicies X_n should be thought of as sequences of maps

$$x_0 \xrightarrow{f_0} x_1 \to \cdots \to x_{n-1} \xrightarrow{f_{n-1}} x_n$$

where the domain of f_k agrees with the codomain of f_{k-1}. We want to say 'composable' maps here but we run into a problem. We want our theory of ∞-categories to subsume the theory of thr homotopy theory of spaces, but if we think of X as a space instead of a category we realise that paths in a space don't have a unique composition. So the property defining an ∞-category should give a not-neccecarily unique way to compose 1-simplicies to live somewhere between the theory of categories and the homotopy theory of spaces. Luckily that's not actually particularly hard.

We define the n, k-horns Λ^n_k by the formula

$$\Lambda^n_k : \Delta^{\text{op}} \to \mathcal{S}\text{et}, \qquad \Lambda^n_k := \bigcup_{k \in E \subsetneq [n]} \Delta^E.$$

which upon first sight confuses anyone and everyone, which is why no higher category theorist ever uses this formula and just uses pictures instead, and forunately the 2-horns give essentially all the intution you'll need.

Let's look at the $(n, k) = (2, 1)$ case first, i.e. Λ^2_1. In any case, the union runs over the subsets $E \subsetneq [n]$ which contain $k = 1$, the possible options for E are thus $E_1 := \{1\}, E_0 := \{0 < 1\}$ and $E_2 := \{1 < 2\}$.

- The 0-simplicies $\Lambda^2_1([0])$ are given by the union of the $\Delta^E([0]) = \text{Hom}([0], E)$. Evaluating this for E_1, E_2 and E_3 gives

$$\begin{aligned} \Delta^{E_0}([0]) &= \text{Hom}([0], E_2) = \{0\} \\ \Delta^{E_1}([0]) &= \text{Hom}([0], E_1) = \{1\} \\ \Delta^{E_2}([0]) &= \text{Hom}([0], E_2) = \{1, 2\} \end{aligned}$$

  and so the union over all of these gives $\Lambda^2_1([0]) = \{0, 1, 2\}$.
- The 1-simplicies $\Lambda^2_1([1])$ are given similarly by the union of the $\Delta^E([1]) = \text{Hom}([1], E)$, noting that any non-injective maps correspond to degenerate simplicies we are free to only look at injective maps $\text{Hom}([1], E_i)$, these turn out to be given by just $\Lambda^2_1([1]) = \{\text{id}_{E_0}, f\}$ where $f : [1] \to E_2$ sends $f(0) = 1$ and $f(1) = 2$.

Putting this all together we have three 0-simplicies $\{0, 1, 2\}$ and two 1-simplicies $\{0 < 1, 1 < 2\}$ which can be put into a pretty picture like so

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Doing the same thing for Λ^2_0 and Λ^2_2 will give you the following pictures.

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

The way to easily remember these pictures is to say that 'Λ^2_k is the triangle with the edge oppsite the k-vertex removed'. Similarly Λ^n_k is the n-simplex with the face opposite the k-vertex removed.

Definition 10

Let Λ_k^n be the horns as above. We call the horns with $k = 0$ or $k = n$ (i.e. Λ_n^n and Λ_0^n) the **outer horns**. The horns with $0 < k < n$ are called the **inner horns**.

The inner horn Λ_1^2 looks a lot like a pair of composable arrows. In fact, finding a composition will correspond to finding an arrow $0 \to 2$ with a 2-simplex with the arrows $0 \to 1$, $1 \to 2$ and $0 \to 2$ as its boundary. This motivates the following.

Definition 11: ∞-category

An **∞-category** is a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that any map $\Lambda_k^n \to X$ with $k \neq 0, n$ (so Λ_k^n is an inner horn) has a lift to a map $\Delta^n \to X$. In other words, X is an ∞-category if the restriction map

$$\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Lambda_k^n, X) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, X)$$

is a surjection for $k \neq 0, n$.

Definition 12: ∞-groupoid

An **∞-groupoid** is a simplicial set $X : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that any map $\Lambda_k^n \to X$ (so Λ_k^n is an inner *or outer* horn) has a lift to a map $\Delta^n \to X$. In other words, X is an ∞-groupoid if the restriction map

$$\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Lambda_k^n, X) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, X)$$

is a surjection for all k.

What homology sees

What homotopy sees

Homology Functors and Spectra

07th Feb

We see the sphere spectrum $\mathbb{S}$ is a homology theory,

Definition 1: Homology

A **Homology theory** is a functor $\mathbb{E} : S_* \to S_*$ which is

- **Finitary;** $\mathbb{E}$ commutes with filtered colimits
- **Reduced;** $\mathbb{E}(*) = *$, and
- **Excisive** $\mathbb{E}$ takes (homotopy) pushouts to (homotopy) pullbacks.

The excisive property gives us that $\mathbb{E}(U) \simeq \Omega\mathbb{E}(\Sigma U)$.

Theorem 2: Spectra

We call thus subcategory $\mathrm{Sp} \subseteq \mathrm{Fun}_*^{\mathrm{fin}}(S_*, S_*)$ the category of **spectra**. We fet a left adjoint to the inclusion $D : \mathrm{Fun}_*^{\mathrm{fin}}(S_*, S_*) \to \mathrm{Sp}$, given by $DF := \mathrm{colim}_{n\to\infty} \Omega^n F \Sigma^n$. This enforces excisiveness.

The hope is that we can use Hurewicz-type maps $\mathbb{S} \to \mathbb{E}$ to gather information about the (stable) homotopy groups of spheres.

14th Feb - Localizations and Récollements

Recall from our previous discussion we have a monoidal structure on the category of spectra with unit given by $\mathbb{S} := \Sigma^\infty S^0$ and symmetric-monoidal product given by

$$E \otimes F := D(E \circ F) \simeq D(E \boxtimes F)$$

where $(E \boxtimes F)(X) := \operatorname{colim}_{A \wedge B \to X} E(A) \wedge F(B)$ is the Day convolution and $DF := \operatorname{colim}_{n \to \infty} \Omega^n F \Sigma^n$ is the Goodwillie differential. Indeed, we then get a functor $E \otimes - : \mathrm{Sp} \to \mathrm{Sp}$ which preserves colimits, and since Sp is a presentable category (see Lurie, 2017) we get that it has a right adjoint, the *function spectra* $E \otimes - \dashv \mathbb{F}(E, -)$ which acts as an internal hom for the ∞-category Sp of spectra. It has homotopy groups

$$\pi_n \mathbb{F}(E, F) = \operatorname{Ext}^{-i}_{\mathrm{ho(Sp)}}(E, F).$$

All of this structure was used to justify referring to Sp as some 'homotopical' analogue of the category of abelian groups.

One feature we'd like to mimick from group theory is the idea of splitting a group into its *torsion* and *rational* parts. Note for instance that $H\mathbb{Q} \otimes H\mathbb{Q} = H\mathbb{Q}$, but $H\mathbb{F}_p \otimes H\mathbb{F}_p \not\simeq H\mathbb{F}_p$.

Example 1: Steenrod Algebra

What is $H\mathbb{F}_p \otimes H\mathbb{F}_p$, then? Recall that we define the E-homology of F, for two spectra E and F, as the graded abelian group $E_*F := \pi_*(E \otimes F)$, and so the spectrum $H\mathbb{F}_p \otimes H\mathbb{F}_p$ is the $H\mathbb{F}_p$-homology of $H\mathbb{F}_p$. The algebra $\mathcal{A}_p := \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$ is called the *(dual) Steenrod algebra*, and it can be shown that it is the free Hopf algebroid

$$\mathrm{Sym}(\xi_1, \xi_2, \dots, \tau_0, \tau_1, \dots)$$

with comultiplication $\psi : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ given by

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k \qquad \text{and} \qquad \psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \tau_k.$$

The utility in the Steenrod algebra is that the cohomology $H^\bullet(X; \mathbb{F}_p)$ of any space X is naturally a module over $\mathcal{A}_p$, which enables us to encode information about stable cohomology operations, and while the cup-product is not stable[a], we gain the Adams Spectral Sequence, an invaluable tool to stable homotopy theorists, but more on that later.

[a] Thus we can no longer use the cup-product when we think of $H^\bullet(X; \mathbb{F}_p)$ as an $\mathcal{A}_p$-module

Warning 2

We do have that

$$\mathrm{Alg}_{\mathbb{A}_\infty}(\mathrm{Mod}(H\mathbb{F}_p)) \simeq \mathrm{DGA}(\mathbb{F}_p) \qquad \text{and} \qquad \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}(H\mathbb{Q})) \simeq \mathrm{CDGA}(\mathbb{Q}),$$

but we don't have an equivalence $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}(H\mathbb{F}_p)) \not\simeq \mathrm{CDGA}(\mathbb{F}_p)$ - essentially because the rational cohomology $H\mathbb{Q}(\Sigma_n)$ of the symmetric groups is trivial, but $H\mathbb{F}_p(\Sigma_n)$ is not, and the definition of $\mathbb{E}_\infty$-rings uses actions of the symmetric groups Σ_n to permute elements. Thus there is some sense in which the symmetric-monoidal product of spectra really doesn't behave well with $\mathbb{F}_p$-homology, and we'll really need the full generality of $\mathbb{E}_\infty$ algebras to capture everything homotopical algebra, DGAs simply won't do.

The above warning tells us that rational spectra can be handled using the well-understood notion of DGAs, which in the case of $H\mathbb{Q}$ reduces to just studying $\mathbb{Q}$-algebras, but the p-torsion for primes p is somewhat more complicated. Thus the bulk of the work here will be understanding this p-torsion of spectra, and that is why we need the notion of *localizations*. Before defining these we'll go through an example to make sure that any definitions we do make satisfy the right properties. To whit, we will analise the ∞-category of quasicoherent sheaves on a scheme.

Example 3

Let X be a quasi-compact and quasi-separated (qcqs) scheme, then its ∞-category of quasi-coherent sheaves is given by

$$D_{qc}(X) := \lim_{\mathrm{Spec}R \to X} \mathrm{Mod}(HR).$$

For instance, if $X = \mathrm{Spec}A$ is affine then $D_{qc}(X)$ is given by $\mathrm{colim}_{A \to R}\mathrm{Mod}(HR)$, and $\mathrm{Mod}(HR) = D(R)$, so $D_{qc}(\mathrm{Spec}A)$ is given by finite projective modules over A, which is exactly the local condition required for a sheaf to be quasi-coherent.

Now suppose that $U \subseteq X$ is a quasi-compact Zariski-open and let $Z = X \backslash U$ so that if U and X are affine this says that Z is given by a finitely generated ideal; it is given by the vanishing of finitely many functions. If we denote the inclusions by $j : U \to X$ and $i : Z \to X$ then we get an adjuntion

$$D_{qc}(X) \underset{j^*}{\overset{j_*}{\leftrightarrows}} D_{qc}(U)$$

with j_* fully-faithful and j^* symmetric-monoidal[a]. Thus we have that j^* is a map of $\mathbb{E}_\infty$-rings, but j_* is not. The projection formula $j_*(j^*\mathcal{E} \otimes_{\mathcal{O}_U} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_X} j_*\mathcal{F}$ gives us, however, that j_* *is* a map of $\mathcal{O}_X$-modules.

The above discussion allows us to make the statement '$U \subseteq X$ is a quasicompact Zariski-open' in purely categorical terms. Notice that $j^*j_* = \mathrm{id}_{D_{qc}(U)}$ gives that j_* is a split monomorphism, already categorifying the notion of 'subset'. Hence understanding $D_{qc}(U) = \{\mathcal{F} \in D_{qc}(X) : \mathcal{F} \xrightarrow{\sim} j_*j^*\mathcal{F}\}$ is the same as understanding when the endofunctor j_*j^* is an equivalence. The projection formula gives that

$$j_*j^*\mathcal{F} = j_*(j^*\mathcal{F} \otimes \mathcal{O}_U) = \mathcal{F} \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U,$$

thus $j_*j^* = - \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U$. Furthermore we that $(j_*j^*)^2 = (j_*j^*)(j_*j^*) = j_*(j^*j_*)j^* = j_*j^*$ (since $j^*j_* = \mathrm{id}$) so that, while j_*j^* isn't the identity, it is idempotent, and the multiplication map $j_*\mathcal{O}_U \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U \xrightarrow{\sim} j_*\mathcal{O}_U$ is an isomorphism (NB we call such a ring R, whose multiplication map $R \otimes R \to R$ is an isomorphism, a **solid ring**). It can be shown that a map $f : Y \to X$ of qcqs shemes is a Zariski-open immersion if and only if $f_*\mathcal{O}_Y$ is a solid ring.

Finally, we see that a subcategory $\mathcal{D} \subseteq D_{qc}(X)$ is of the form $D_{qc}(U)$ for some Zariski-open U if and only if $\mathcal{D}$ is *localizing* and *smashing*. A subcategory $\mathcal{D}$ of a (complete, cocomplete, presentable) stable ∞-category $\mathcal{C}$ is *localizing* if it's a full subcategory and the inclusion $U : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint F such that $L^2 = L$ for $L := UF$. A localizing subcategory is *smashing* if further $\mathcal{C}$ and $\mathcal{D}$ are symmetric-monoidal with F symmetric-monoidal and U a $\mathcal{C}$-module map; $U(b \otimes Fc) = Ub \otimes c$.

[a] Thus we have $j^*\mathcal{O}_X = \mathcal{O}_U$ and $j^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = j^*\mathcal{E} \otimes_{\mathcal{O}_U} j^*\mathcal{F}$.

Note the analogy with $H\mathbb{Q} \otimes_{\mathbb{S}} H\mathbb{Q} = H\mathbb{Q}$ here. We see that $H\mathbb{Q}$ is a solid ring, and in fact $\mathbb{Q}$ is the only solid ring of characteristic zero, and thus the de-Rham–Quillen adjunction between simplicial sets and connective CDGAs is idempotent only over $\mathbb{Q}$ (this is exatly the statement of warning 2) and it is in this way that we should think of $H\mathbb{Q}$ as being an 'open subscheme'[4] of the sphere spectrum $\mathbb{S}$. The $H\mathbb{F}_p$ spectra, however, are *closed* immersions, and we the goal of chromatic homotopy theory is to rebuild $\mathbb{S}$ out of $H\mathbb{Q}$ and the $H\mathbb{F}_p$ as a récollemont, which we will discuss now.

If $U \subseteq X$ is a quasi-compact open as in example 3 above, and $Z := X \backslash U$, then we get a **récollemont**; we can rebuild $D_{qc}(X)$ out of $D_{qc}(U)$ and $D_Z(X) := \ker(j^* : D_{qc}(X) \to D_{qc}(U))$ as follows.

[4] Actually, $\mathbb{S} \to H\mathbb{Q}$ is not finite and so we should more appropriately think of $H\mathbb{Q}$ as a 'pro-open' subscheme.

References

Lurie, J. (2017). Higher algebra.