Axiomatic Homology Theory and the Borsuk-Ulam Theorem

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July 6, 2021

1 Axioms

An admissable category...

The axioms of (generic) homology theory...

2 Basic results

Proposition 2.1. *If $A \subset X$ is a deformation retract, then $H_n(X,A) = 0$.*

Proof. If $A \subset X$ is a deformation retract, then the inclusion $i : A \to X$ is a homotopy equivalence. Let $r : X \to A$ be the retraction. Then $ir \simeq id_X$ and $ri \simeq id_A$. By homotopy invariance of H_n and the identity property of functors, $(ir)_* = (id_X)_* = id_{H_nX}$ and $(ri)_* = (id_A)_* = id_{H_nA}$. Since $(ri)_* = r_*i_*$ and vice-versa, we have that i_* is an isomorphism.

Now consider the long exact homology chain:

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_nA \xrightarrow{i_*} H_nX \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}A \longrightarrow \dots$$

Since i_* is an isomorphism, and the chain is exact, $H_nX = im(i_*) = ker(j_*)$ so $0 = im(j_*) = ker(\partial)$

However, on the left we also have $0 = ker(i_*) = im(\partial)$ since i_* is an isomorphism. It follows that $H_n(X,A) = 0$.

□

Remark 2.2. *As a special case of this result, $H_n(X,X) = 0$, as X is a deformation retract of itself.*