Homology Theory and the Borsuk-Ulam Theorem

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1 Notes and acknowledgements

This text provides an introduction to homology theory from an axiomatic point of view. References are provided where appropriate. The text assumes knowledge of the basic ideas of algebra algebraic topology.s In particular, the reader is assumed to be aware of *groups, group homomorphisms, homeomorphisms, homotopies, homotopy equivalences* and *quotient spaces*.

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2 Introduction

One of the goals of topology is classifying spaces up to some definition of equivalence, typically homotopy equivalence or the stricter homeomorphism. While establishing equivalence requires the explicit construction of a homotopy equivalence/homeomorphism, inequivalence can in many cases be determined more easily by calculating *topological invariants*. These are properties of a space that are preserved by our definition of equivalence, thus if two spaces have different invariants, they cannot be equivalent. One of the first major homotopy invariances the undergraduate student encounters are the homotopy groups $\pi_n(X)$, the groups of loops $f : \S n \to X$. While these groups carry a lot of geometric information, they can be very difficult to compute; the problem of computing $\pi_n(\S m)$ for $n, m \in \mathbb{Z}$ remains unsolved. As of 2021, only a handful of these groups have been calculated, and this only after the development of several new branches of topology (REF NEEDED).

This motivates the investigation of other homotopy invariances. These should be much easier to compute, and ideally, not at the cost of too much geometric information. The homology groups $H_n(X)$ are an example of such a homotopy invariant. They are best explained in the language of category theory and exact sequences, discussed in sections MISSING and MISSING, so we will fall short of giving a definition just yet.

Historically, homology groups were calculated using one of a number of geometric methods (REF MISSING). It was in YEAR MISSING, that Eilenberg and Steenrod noticed a common thread between these different theories, and defined a set of axioms for what a homology theory should be. In this text we will take an axiomatic approach to homology, setting out the Eilenberg-Steenrod axioms and proving results directly. We will take on faith that at least one homology theory satisfying these axioms exist, which the reader is invited to confirm for themselves in (REF MISSING). While this approach comes at the cost of some geometric intuition, it comes with several advantages:

- For many homology theories, the geometric constructions are mainly used to give proofs of the Eilenberg-Steenrod axioms. In proving individual theorems, the axioms are often used more often than the individual geometric definitions.

- Individual homology theories are equivalent up to which axioms they satisfy and a certain set of "initial conditions" (REF NEEDED). Therefore, in some sense the axioms are not only necessary but also sufficient for defining (a class of) homology theories.

- There are now a large number of homology theories, which can all be understood from the axioms.

- Proving that a homology theory satisfies the axioms is a major undertaking worthy of a project of its own.

A brief outline of the project is as follows: In sections REF and REF we give the key definitions and theorems of category theory and exact sequences, which will serve as the language of homology theory. In section REF, we give the Eilenberg-Steenrod axioms and show some immediate results. Techniques for further calculation are given in sections REF and REF, along with examples of results. This includes the Borsuk-Ulam Theorem, which initially served as a motivator for the project.

3 Category theory

The language of category theory was invented to make homology theory easier to describe (Source MISSING), but has since been used for many areas inside and outside mathematics (Source MISSING). It is a very general construction, as many familiar constructions can be understood as categories, such as groups, rings, vector spaces and topological spaces. Informally, a category is a collection of "objects" and "maps between them".

In many cases we will study the objects are sets and the maps are functions that satisfy some structure-preserving property, however it is important to note that the definition of a category goes beyond this. There are categories whose objects look nothing like sets and whose maps look nothing like functions. Even for a category of sets and functions, the category may have no notion of "elements of a set" or "evaluation a point".

Definition 3.1. *An object $\mathscr{A}$ is a of objects $ob(\mathscr{A})$, for each $A,B \in ob(\mathscr{A})$ a collection of maps $\mathscr{A}(A,B)$, and a composition rule $\circ$ satisfying the following properties:*

1. *If $f \in \mathscr{A}(A,B)$ and $g \in \mathscr{A}(B,C)$ then $g \circ f \in \mathscr{A}(A,C)$*
2. *For each $A \in ob(\mathscr{A})$ there is a unique map $id_A \in \mathscr{A}(A,A)$ with the property that $id_A \circ f = f$ and $g \circ id_A = g$ for every $f : B \to A$ and $g : A \to B$*

The term is similar to a set, but with the important distinction that MISSING.

Definition 3.2. *A chain complex is a sequence of algebraic objects and homomorphisms between them, such that for any consecutive f_i, f_{i+1} we have that $im(f_i) \subset ker(f_{i+1})$. If we have equality instead of inclusion, the construction is called an exact sequence.*

We will often use "exact sequence" to describe both sequences and families indexed by integers of algebraic objects. We also distinguish between *short* and *long exact sequences*, where the former is finite sequences of three or fewer nonzero elements, and the latter is all other sequences.

Example 3.3. *For any group A the following is a long exact sequence:* $\dots \xrightarrow{0} A \xrightarrow{id} A \xrightarrow{0} A \xrightarrow{id}$

Remark 3.4. *The requirement that $im(f_i) \subseteq ker(f_{i+1})$ for all $i \in \mathbb{Z}$ is equivalent to the requirement that $f_{i+1} \circ f_i = 0$ for all $i \in \mathbb{Z}$. This is clear from the definition; if $im(f_i) \subseteq ker(f_{i+1})$ then $f_{i+1} \circ f_i(A_i) = f_{i+1}(im(f_i)) = 0$. Conversely, if $f_{i+1} \circ f_i = 0$ then $f_{i+1}(im(f_i)) = 0$.*

It is important to build up intuition for how exact sequences behave.

Example 3.5. *Suppose the following is a short exact sequence of groups.* $A \xrightarrow{0} B \xrightarrow{f} C \xrightarrow{0} D$

We will often draw the following diagram instead, as it carries the same information. $0 \longrightarrow B \xrightarrow{f} C \longrightarrow 0$ *Since $0 = im(0) = ker(f)$, f is injective. Since $im(f) = ker(0) = C$, f is surjective. Therefore f is an isomorphism.*

Let the following be a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$
Then $B \cong A \bigoplus C$.

Proof. Since $0 = im(0) = ker(f)$, f is injective. It follows that $im(f) \cong A$. Since $im(g) = ker(0) = C$, g is surjective. g defines an injective function $[g] : B/ker(g) \to C$, $[b] \mapsto [g(b)]$. This gives an isomorphism $B/ker(g_= B/im(g) \cong C$. Therefore $B \cong im(g) \bigoplus C \cong A \bigoplus C$. (POSSIBLY ELABORATE) □

We finish this section with a technical lemma that will become useful in the future.

Lemma 3.6 (Braid lemma)**.** *Suppose three long exact sequences and a chain complex make the following commutative diagram.*

$$
\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[rr, bend left, "g_1"] & & D \arrow[dr, "g_2"] \arrow[rr, bend left, "h_3"] & & G \arrow[dr, "h_4"] \arrow[r, bend left, "j_4"] & {} \\
O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \\
& B \arrow[ur, "h_1"] \arrow[rr, bend right, "j_1"] & & E \arrow[ur, "j_2"] \arrow[rr, bend right, "f_3"] & & H \arrow[ur, "f_4"] \arrow[r, bend right, "g_4"] & {}
\end{tikzcd}
$$

Then the chain complex is also a long exact sequence.

Proof. By symmetry of the diagram, it does not matter which sequence is the chain complex. We can assume it is the sequence with homomorphisms f_i. We are given that $im(f_i) \subseteq ker(f_{i+1})$, and need to show that $ker(fi+1) \subseteq im(f_i)$. By the symmetry of the diagram, it is enough to show this for $i = 1,2,3$. We will show that $ker(f_2) \subseteq im(f_1)$ here, and do the other two cases in Appendix (MISSING).

Let $x \in ker(f_2)$. Then $0 = f_2(x) = j_2 f_2(x) = g_2 h_2(x)$ by commutativity. It follows that $h_2(x) \in ker(g_2) = im(g_1)$. So $\exists x_1 \in A$ s.t. $g_1 x_1 = h_2(x)$. By commutativity, $g_1 x_1 = h_2 f_1 x_1$. So we have that $0 = g_1(x_1) - h_2(x) = h_2(f_1(x_1) - x)$. So $x_2 := f_1(x_1) - x \in ker(h_2) = im(h_1)$. So $\exists x_3 \in B$ s.t. $h_1(x_3) = x_2$.

Now note that

$$j_1(x_3) = f_2 h_1(x_3) = f_2(x_2) = f_2(f_1(x_1) - x) = 0$$

Where the last equality follows from $f_2 f_1(-) = 0$ and $f_2(x) = 0$. We therefore have that $x_3 \in ker(j_1) = im(j_0)$. So there exists $x_4 \in O$ s.t. $j_0(x_4) = x_3$. Consider $g_0(x_4)$. It satisfies $f_1 g_0(x_4) = h_1 j_0(x_4) = h_1(x_3) = x_2 = f_1(x_1) - x$. Therefore we have

$$x = f_1(x_1 - g_0(x_4))$$

Which shows $x \in im(f_1)$ as required.

[**?**] □

4 Axioms of homology

The axioms for a homology theory were first laid out by Eilenberg and Steenrod [**?**]. However, in this section, we will follow the treatment given in [**?**]. We define $\mathbf{Top}_2$ to be the category of pairs of topological spaces (X,A), where $A,B \in \mathbf{Top}$ and $A \subseteq B$. Morphisms $f : (X,A) \to (Y,B)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$. It is not hard to see this space satisfies the definitions of a category (MISSING). We can similarly define $\mathbf{Top}_3$ as the triples of topological spaces

(X,A,B) where $B \subseteq A \subseteq X$ and where morphisms $f : (X,A,B) \to (Y,C,D)$ are maps $f : X \to Y$ such that $f(A) \subseteq C$ and $f(B) \subseteq D$. We will often make the identification $X \in \mathbf{Top}$ with $(X,\emptyset) \in \mathbf{Top_2}$, and the identification $(X,A) \in \mathbf{Top}_2$ with $(X,A,\emptyset) \in \mathbf{Top}_3$.

Definition 4.1. *A subset $C \subset \mathbf{Top_2}$ is admissible for homology if the following apply:*

(a) C contains all points in **Top**. *In the language of category theory, C contains all final objects in* **Top**, *that is, all objects $\cdot \in \mathbf{Top}$ with the property that there is one and only one morphism from any $X \in \mathbf{Top}$ to $\cdot$.*

(b) For any $(X,A) \in C$, the following commutative diagram lies in C, where all maps are induced by canonical inclusions:

$$\begin{tikzcd} & & (X,\emptyset) \arrow[dr] & & \\ (\emptyset,\emptyset) \arrow[r] & (A,\emptyset) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X \\ & & (A,A) \arrow[ur] & & \end{tikzcd}$$

and also includes all the maps induced on this diagram by a map $f : (X,A) \to (Y,B)$.

(c) For $(X,A) \in C$, C contains the following diagram:

$$\begin{tikzcd} (X,A) \arrow[r, bend left, "t_0"] \arrow[r, bend right, "t_1"'] & (X \times I, A \times I) \end{tikzcd}$$

Where $t_s(x) = (x,s)$.

Remark 4.2. *As noted in [**?**], the definition certifies (1) that C contains all final objects in $\mathbf{Top}_2$, that is all maps $(\cdot,\emptyset) \to (X,A)$, since C contains all maps $(\cdot emptyset) \to (X,\emptyset)$ and also the inclusion $(X,\emptyset) \to (X,A)$. (2) condition (iii) allows us to include all homotopies $h : f\tilde{g}$ between spaces in C purely in the language of category theory as a map $h : (X \times I, A \times I) \to (Y,B)$ such that $ht_0 = f$ and $ht_1 = g$:*

$$\begin{tikzcd} (X,A) \arrow[r, bend left, "t_0"] \arrow[r, bend right, "t_1"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend right=50, "g"'] & (X \times I, A \times I) \arrow[r, "h"] & (Y,B) \end{tikzcd}$$

We now give the axioms of an (ordinary) homology theory for an admissible category.

Definition 4.3. *A homology theory on an admissible category C is a family of functors $H_n : \mathbf{C} \to \mathbf{A}$*

In this text we will use "space" to denote "topological space", and assume all spaces and maps are admissible unless otherwise stated. We will also use D^n to denote the closed n-disk.

4.1 Basic results

Proposition 4.4. *If $A \subset X$ is a deformation retract, then $H_n(X,A) = 0$.*

Proof. If $A \subset X$ is a deformation retract, then the inclusion $i : A \to X$ is a homotopy equivalence. Let $r : X \to A$ be the retraction. Then $ir \simeq id_X$ and $ri \simeq id_A$. By homotopy invariance of H_n and the identity property of functors, $(ir)_* = (id_X)_* = id_{H_nX}$ and $(ri)_* = (id_A)_* = id_{H_nA}$. Since $(ri)_* = r_*i_*$ and vice-versa, we have that i_* is an isomorphism.

Now consider the long exact homology chain:

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_nA \xrightarrow{i_*} H_nX \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}A \longrightarrow \dots$$

Since i_* is an isomorphism, and the chain is exact, $H_nX = im(i_*) = ker(j_*)$ so $0 = im(j_*) = ker(\partial)$

However, on the left we also have $0 = ker(i_*) = im(\partial)$ since i_* is an isomorphism. It follows that $H_n(X,A) = 0$.

$\square$

Remark 4.5. *As a special case of this result, $H_n(X,X) = 0$, as X is a deformation retract of itself. We are also interested in special case where $A = x$ is a single point, i.e. X is contractible. Then we have $H_n(X,x) = 0$.*

4.2 Reduced homology

It is possible to reduce the abelian objects H_nX into simpler objects $\tilde{H}_nX$ by in some sense factoring out the object H_n1. Furthermore, this can be done without losing any information, i.e. the transformation $H_nX \to \tilde{H}_nX$ is reversible.

First we will need some facts about abelian objects.

Definition 4.6. *If there exist maps $f : A \to B$ and $g : B \to A$ between abelian objects such that $g \circ f = id_A$ then g is called a retraction of f, and f is called a something of g.*

g can be thought of as a one-sided inverse of f, as there is no requirement that $g \circ f = id_B$.

Lemma 4.7. *If $g : B \to A$ is a retraction of $f : A \to B$, then $B \cong im(f) \bigoplus ker(g)$*

Proof. The isomorphism is given by

$$h : B \to im(f) \bigoplus ker(g)$$

$$x \mapsto (f \circ g(x), x - f \circ g(x))$$

This is well-defined as $f\circ g(x)\in im(f)$ and

$$g(x-f\circ g(x)) = g(x) - g\circ f\circ g(x) = g(x) - g(x) = 0$$

by the associativity of homomorphisms, and since $g\circ f = id_A$. Hence $x - f\circ g(x) \in ker(g)$ The inverse is

$$h^{-1}: im(f)\bigoplus ker(g) \to B$$
$$(a,b) \mapsto a+b$$

One quickly checks that

$$h\circ h^{-1}(a,b) = h(a+b) = (f\circ g(a+b), a+b-f\circ g(a+b))$$
$$= (f\circ g(a)+0, a+b-f\circ g(a))$$

since $b\in ker(g)$. However, $a = f(c)$ for some $c\in A$, so

$$(f\circ g(a)+0, a+b-f\circ g(a)) = (f\circ g\circ f(c), a+b-f\circ g\circ f(c))$$
$$= (f(c), a+b-f(c)) = (a,b)$$

since $f\circ g = id_B$. Additionally,

$$h^{-1}\circ h(x) = f\circ g(x) + x - f\circ g(x) = x$$

So h and h^{-1} are indeed inverse homomorphisms, and $B\cong im(f)\bigoplus ker(g)$. □

We will use this Lemma on the following construction.

Definition 4.8. *$\tilde{H}_n(X) = ker(p^*: H_nX \to H_n1)$ where p^* is the map induced by the initial map $p: X\to 1$. $\tilde{H}_nX)$ is called the reduced homology of X.*

Proposition 4.9. *For any $x\in X$, $H_n(X,A) = \tilde{H}_n(X,A)\bigoplus H_n1 = H_n(X,x)\bigoplus H_n1$*

Proof. For the first equality, consider the following diagram. x exists by assumption MISSING of an admissable category, and p exists since 1 is an initial object. Notice p is a retraction of x.

DIAGRAM

H_n induces the following diagram, and since functors map compositions to compositions and identities to identities, we have that $p^*\circ x^* = id_{H_n1}$, so p^* is a retraction of x^*.

DIAGRAM

By 2.1,

$$H_nX \cong im(x^*)\bigoplus ker(p^*) = H_n1\bigoplus \tilde{H}_nX$$

where the last equality holds because $p^*\circ x^* = id_{H_n1}$ guarantees that x^* is injective.

For the second equality, note any two initial objects are isomorphic. In particular, for $x\in X$, $x\cong 1$. We hence have the following long exact chain

DIAGRAM

From which we can extract a short exact chain

DIAGRAM

By exactness, $H_n 1 \cong im(x^*) \cong ker(j*)$ and $im(j*) \cong ker(\partial) = H_n(X,x)$. Furthermore, by the first isomorphism theorem, $im(j*) \cong H_nX/ker(j*)$. Therefore,

$$H_n(X,x) \cong H_nX/H_n1 \cong \tilde{H}_nX \bigoplus H_n1/H_n1 \cong \tilde{H}_nX$$

where the last few isomorphisms are done somewhat informally, to be made precise at a future point. MISSING $\square$

Corollary 4.9.1. *If X is contractible to $x \in X$, then $H_n(X) = H_n(X,x) \bigoplus H_n1 = 0 \bigoplus H_n1 \cong H_n1$, by 2.2. It follows that $\tilde{H}_nX = 0$*

2.6 shows that $H_n(X,A)$ always carries around a copy of H_n1, which can be safely removed by going to the kernel of p^*. Gluing a copy of H_n1 onto $\tilde{H}_n(X,A)$ recovers the original object.

As we will see, there is a long exact chain for the reduced homology. To define it, we will need the following lemma. Consider an admissable category C and a triple $(X,A,B) \in Top_{(3)}$ such that $(X,A),(X,B),(A,B) \in C$. The long exact sequences, which will be labelled $(1),(3),(4)$ respectively, form a braid diagram as shown below. The sequence (2), called the *long exact homology sequence* for the triple (X,A,B) is defined in such a way that the diagram commutes. Explicitly, it is the sequence

DIAGRAM

where ∂ is the composition of the transformation $\partial : H_{n+1}(X,A) \to H_nA$ and $i^* : H_nA \to H_n(A,B)$. The other two maps are induced by the inclusions $H_n(A,B) \to H_n(X,B)$ and $H_n(X,B) \to H_n(X,A)$.

Proposition 4.10. *For a triple $(X,A,B) \in Top_{(3)}$ such that $(X,A),(X,B),(A,B) \in C$, the sequence*

DIAGRAM

is a long exact sequence.

Proof. We will start by noting that the diagram above commutes. In any subdiagram of the diagram not involving ∂, this follows by noting the subdiagram is induced (via a composition-preserving functor) from a diagram of inclusions which commute by the definition of an admissible category. Any subsquare or triangle involving ∂ commutes since ∂ is a natural transformation, or by the definition of the special ∂ defined above.

After noting the diagram commutes, it becomes easy to show (2) is a chain complex, i.e. that the composition of any two maps is 0. For two out of three compositions, you can via commutativity choose an alternative path which goes via two consecutive maps in a exact sequence, and is hence 0. For the final composition

DIAGRAM

note we have the following commutative diagram (commutative since it is induced by the commutative diagram of inclusions):

Since $H_n(A,A) = 0$, the composition is 0, as required. (https://ncatlab.org/nlab/show/braid+lemma# Munkr
The result is then achieved by an application of the Braid Lemma. $\square$

Corollary 4.10.1. *The sequence*

DIAGRAM

is a long exact sequence.

Proof. First note that if $(X,A) \in C$ then by the requirements of an admissable category, (X,x) and $(A,x) \in C$ as well for any $x \in A$. Hence by the previous proposition, and using the isomorphism $\tilde{H}_n X \cong H_n(X,x)$, we get the result. $\square$

4.3 Homology of $\S n$

We will now perform a calculation of the groups $H_k S^n$ from the axioms. Our approach will be an adaptation of that taken in citeWerndli.

Lemma 4.11. *For all $n \in \mathbb{Z}$,*

$$\tilde{(H)}_k S^n \cong \tilde{H}_{k-1} S^{n-1}$$

Proof. Consider first the pair (D^n, S^{n-1}), where S^{n-1} is the boundary of D^n. Since D^n is contractible, the reduced homology sequence reads: $\ldots \longrightarrow 0 \longrightarrow H_k(D^n, S^{n-1}) \longrightarrow \tilde{H}_{k-1} S^{n-1}$
This gives an isomorphism

$$H_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1} S^{n-1}$$

Additionally, we have the pair (S^n, D^n), where D^n is identified as the closed lower hemisphere of S^n. Since D^n is contractible, $H_k(S^n, D^n) \cong H_k(S^n, \cdot) \cong \tilde{H}_k S^n$. The open disk $D_{1/2}$ of radius $1/2$ is a subset of D^n which can be excised from the pair (S^n, D^n). The resulting space deformation retracts to the pair (D^n, S^{n-1}), where D^n is the upper hemisphere and S^{n-1} its boundary. By the excision axiom,

$$\tilde{H}_k S^n \cong H_k(S^n, D^n) \cong H_k(D^n, S^{n-1})$$

Together with our previous isomorphism, we have

$$\tilde{H}_k S^n \cong H_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1} S^{n-1}$$

as required.

$\square$

For $n > 0$,

$$H_k S^n = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We identify $S^0 = \cdot \sqcup \cdot$. By proposition (MISSING),

$$H_k S^0 = H_k \cdot \bigoplus H_k \cdot = \begin{cases} \mathbb{Z} \times \mathbb{Z} & = 0 \\ 0 & \text{otherwise} \end{cases}$$

. It follows that $\tilde{H}_0 S^0 = \mathbb{Z}$ and $\tilde{H}_k S^0 = 0$ when $k \neq 0$. By Lemma **??**,

$$\tilde{H}_k S^n = \tilde{H}_{k-n} S^0 = \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

This implies the result. $\square$

Remark 4.12. *For the sake of these calculations, the choice of coefficients $H_0 \cdot = \mathbb{Z}$ was arbitrary and played no role. The result could be written more generally as $H_n S^n = \begin{cases} H_0 \cdot & n = k \\ 0 & \text{otherwise} \end{cases}$ for ordinary homology theories. This will be important in section (MISSING), where we use these results for a homology theory with coefficients $H_0 \cdot = \mathbb{Z}/2\mathbb{Z}$.*

5 The Mayer-Vietoris Sequence

When the boundary maps i^*, j^* are not homotopic to the identity and/or the homology groups of A and X are not immediate, it can be difficult to calculate the homology groups directly from the axioms. However, in many cases it is possible to cover X by two spaces A and B whose homology groups and/or inclusion maps are easy to identify. In these cases the following result makes calculation very convenient:

Theorem 5.0.1. *Let A, B be subsets of X whose interiors cover X, Then there is a long exact sequence:*

$$\dots \longrightarrow H_{n+1} X \xrightarrow{\partial_{n+1}} H_n(A \cap B) \xrightarrow{(i^*, j^*)} H_n A \bigoplus H_n B \xrightarrow{k^* - l^*} H_n X \longrightarrow \dots$$

The maps i^*, j^*, k^*, l^* are induced by the inclusions $i : A \to A$, $j : A \to A$, $k : A \to X$, $l : B \to X$.

There is furthermore a long exact sequence for reduced homology: (MISSING)

Before diving into the proof, let us consider a few examples showcasing how convenient this theorem can be.

Example 5.1. *The theorem gives a very easy proof that $H_n(X \sqcup Y) = H_n X \bigoplus H_n Y$ via the obvious cover $A = X$, $B = Y$, $A \cap B = \emptyset$. Since $H_n \emptyset = 0$ for all n, Mayer-Vietoris gives the isomorphism $H_n X \bigoplus H_n Y \cong H_n(X \sqcup Y)$.*

Example 5.2 (One-point wedges of circles)**.** *Consider the figure eight $X_2 = S^1 \wedge S^1$ where $\wedge$ is the wedge product. We can cover X by $A = (S^1 \setminus D^1_+) \wedge S^1$ and $B = S^1 \wedge (S^1 \setminus D^1_-)$ where D^1_+ and D^1_- are the open top and bottom semicircles of S^1.*

It is easy to confirm this covering satisfies the requirements of the Mayer-Vietoris sequence. $A \cap B \simeq \cdot$, as $A \cap B$ is star-shaped, hence contractible. Additionally, $A \simeq B \simeq S^1$. Since $\tilde{H}_n \cdot = 0$, the reduced Mayer-vietoris sequence gives

$$\tilde{H}_n(S^1 \wedge S^1) \cong \tilde{H}_n S^1 \bigoplus \tilde{H}_n$$

It follows that

$$H_n(S^1 \wedge S^1) = \begin{cases} \mathbb{Z} \bigoplus \mathbb{Z} & n = 0, 1 \\ 0 & otherwise \end{cases}$$

This argument can easily be extended by induction to the wedge of m circles (wedged at the same point) has homology group

$$H_n(\bigwedge_m S^1) = \begin{cases} \mathbb{Z}^m & n = 0, 1 \\ 0 & otherwise \end{cases}$$

Remark 5.3. *One could hope that we could extend this method to the the wedge of $\mathbb{Z}$ copies of S^1, $X_\infty \bigwedge_{\mathbb{Z}} S^1 := \bigsqcup_{i \in \mathbb{Z}} S^1_i / \sim$ where $N_i \sim N_j$ for $i, j \in \mathbb{Z}$. It also has a Mayer-Vietoris covering $A = S^1_0 \cup_{i \neq 0} D^1_{i,+} \cong S^1$, $B = X_\infty \setminus D^1_{1,+} \cong X_\infty$, $A \cap B \cong \cdot$, giving the isomorphism $\tilde{H}_n S^1 \bigoplus \tilde{H}_n X_\infty \cong \tilde{H}_n X_\infty$. For $n = 0, 1$ this gives $\mathbb{Z} \bigoplus H_n X_\infty \cong H_n X_\infty$. It seems obvious that the homology groups should be $\mathbb{Z}^\infty$ for $n = 0, 1$ and 0 otherwise, however, this does not follow directly from Mayer-Vietoris.*

Before we resume giving calculations from Mayer-Vietoris, it is only right that we examine the proof.

5.1 Proof of Mayer-Vietoris

5.2 Calculations

Example 5.4. *Another space whose homology groups are easily calculated from Mayer-Vietoris is the quotient of the torus $T^2 := S^1 \times S^1$ under the quotient identifying one copy of S^1 to a point, i.e. $(N, x) \sim (N, y)$ for all $x, y \in S^1$. It can be visualised as in the MISSING image.*

A cover that can easily be seen to satisfy Mayer-Vietoris is $A = D^1_+ \times S^1 \cong \cdot$, $B = (T^2 / \sim) \setminus \{[N, x]\} \cong S^1$ with $A \cap B \cong S^1 \sqcup S^1$, as in the MISSING image. The deformation retraction $B \simeq S^1$ composes with the inclusion $i : A \cap B \to B$ such that $i|_A = i|_B = id_{S^1}$. It should be clear that the inclusion induces the map $i^* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (x, y) \mapsto x + y$, but for sake of completeness this can be seen from the following commutative diagram, where the inclusions i_1, i_2 are as one expects:

$$\begin{tikzcd}
 & S^1 & \\
 & S^1 \sqcup S^1 \arrow[u, "i"] & \\
S^1 \arrow[uur, "id"] \arrow[ur, "i_1"] & & S^1 \arrow[uul, "id"'] \arrow[ul, "i_2"']
\end{tikzcd}$$

It induces the following commutative map in homology (for n=0,1), from which the formula for i^* can be easily read.

$$\begin{tikzcd} & \mathbb{Z} & \\ & \mathbb{Z}\times\mathbb{Z} \arrow[u, "i"] & \\ \mathbb{Z} \arrow[uur, "x"] \arrow[ur, "{(x,0)}"'] & & \mathbb{Z} \arrow[uul, "y"'] \arrow[ul, "{(y,0)}"] \end{tikzcd}$$

The fact that $i_1^* = (x,0)$ (and the similar statement for i_2) follows from the direct sum theorem (REFERENCE).

The Mayer-Vietoris sequence reads as follows:

$$0 \longrightarrow H_2X \xrightarrow{\partial_2} \mathbb{Z}\times\mathbb{Z} \xrightarrow{i^*=x+y} \mathbb{Z} \xrightarrow{k^*} H_1X \xrightarrow{\partial_1} \dots$$

Now $ker(i^*) =< (x,-x) >\cong \mathbb{Z} = im(\partial_2)$ by exactness. Since ∂_2 is injective, $H_2X \cong \mathbb{Z}$. Additionally, since i^* is surjective, $k^* = 0$. Therefore, the rest of the sequence reads:

$$0 \longrightarrow H_1X \xrightarrow{\partial_1} \mathbb{Z}\times\mathbb{Z} \xrightarrow{i^*=x+y} \mathbb{Z} \xrightarrow{k^*} H_0X \longrightarrow 0$$

This diagram is similar to the previous diagram, giving us $H_1X = \mathbb{Z}$ and $H_0X = 0$. $H_nX = 0$ for all other values of x.

6 Cellular homology

A common class of spaces is the class of *cell complexes* or *CW complexes*, which are constructed by iteratively gluing copies of D^n in a manner defined by a function defined on the boundary $\partial D^n \cong \S n-1$. As will become apparent, CW-complexes have the right structure for establishing a very practical way of calculating their homology groups.

Definition 6.1. *A CW-complex X is the union of a sequence of spaces X^n, called the n-skeleta of X^n defined as follows: X^0 is a discrete set, and for each X^{n-1}, X^n is obtained by gluing copies of D^n_α, called n-cells, along a map $\phi_\alpha : S^{n-1} \to X^{n-1}$ defined on the boundary of D^n_α. Explicitly, X^n is the quotient of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the identification $x \sim \phi_\alpha(x)$ for $x \in S^{n-1} \subset D^n_\alpha$. If $X = X^n$ for some $n \in \mathbb{N}^*$, then X is called a finite cell complex. Otherwise, X is given the weak topology: a subset $A \subset X$ is open iff A is open in X^n for every $n \in \mathbb{N}^*$.*

Many familiar spaces naturally arise as cell complexes.

Example 6.2. *$\S n$ is a CW-complex with 1 0-cell and 1 $n-cell$ when $n > 0$.*

Example 6.3. *$\mathbb{RP}^2$ is a CW-complex with 1 0-cell, 1 1-cell and 1 2-cell, and where the attaching map $\phi : S^1 \to X^1$ is the double cover $z \mapsto z^2$. To see this, note that $\mathbb{RP}^2$ can be thought of as the upper dome of the 2-sphere, with antipodal points on the equator identified. The upper dome is homeomorphic to D^2 and for $z \in \S 1$ the condition $z \sim -z$ is equivalent to the condition $z \sim z^2$. This is immediately obvious* *from noting that z and $-z$ are both mapped to z^2, and conversely if $z^2 = y^2$, then $z = \pm y$.*

Iteratively, $\mathbb{RP}^n$ can be understood as a copy of D^n glued to a copy of $\mathbb{RP}^{n-1}$ via the projection map $z \mapsto [z]$ along the boundary ∂D^n.

Example 6.4. *The torus T^2 has a cellular structure composed of 1 0-cell, 2 1-cells and 1 2-cell. We can use the familiar identification of T^2 as a quotient of the square, noting that the two maps a and b are in fact 1-cells, as the corners are identified as a single point. The identification map $f : S^1 \to X^1$ is then given by the concatenation $a \bullet b \bullet -a \bullet -b$. The Klein bottle, which also arises as a quotient of the square, can be defined in a similar way.*

IMAGE MISSING

Remark 6.5. *The cell structure of a CW-complex need not be unique. For example an alternative cell structure of S^2 is one 0-cell, 1 1-cell, and two 2-cells (the upper and lower hemispheres) both glued via the identity map on their boundaries.*

We will show that the homology groups of CW-complexes can be identified as the homology groups of a certain chain complex of relative homology group $H_n(X^n, X^{n-1})$. This method of "taking homology twice" in essence trims away a lot of fat, making calculations easier. By "homology group of a chain complex" we mean the following:

Let the following diagram be a chain complex of abelian groups.

$$\dots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots$$

Then the abelian group $\tilde{H}_n(A_n) := Ker(d_n)/Im(d_{n+1})$ is called the *homology group of the chain complex.*

Remark 6.6. *Note both $Ker(d_n)$ and $Im(d_{n+1})$ are abelian subgroups, and $Im(d_{n+1}) \subseteq Ker(d_n)$ as we are dealing with a chain complex. (FINISH PROOF)*

We will restrict our study to finite cell complexes, but the reader is invited to confirm that the established results also hold for general cell complexes (REF). The proof is adapted from (REF HATCHER), with axiomatic replacements to references to singular homology. We first establish some basic results

Lemma 6.7. *The following hold for finite cell complexes X:*

(i) *$H_n(X^n, X^{n-1}) = \mathbb{Z}^m$ where m is the number of n-cells of X and $n \in \mathbb{N}^*$.*

(ii) *$H_m(X^n) = 0$ for $m > n$.*

(iii) *The inclusion $X^n iX$ gives rise to an isomorphism $H_k(X^n) \cong H_k(X)$ whenever $k > n$.*

Proof. The statement is trivial for $n = 0$. For $n > 0$, notice that X^{n-1} is a deformation retract of $A := X^n \setminus \sqcup_m \cdot$, where the m copies of $\cdot$ are the centers of the m n-cells of X. The subset $B := X^n \setminus \sqcup_m D^n_{1/2}$ satisfies the conditions of excision, where $D^n_{1/2}$ is the disk of radius $1/2$ sitting inside an n-cell. It follows that

$$H_k(X^n, X^{n-1}) \cong H_k(X^n, A) \cong H_k(X^n, A \setminus B) \cong H_k(\sqcup_m D^n, \sqcup_m S^{n-1})$$

These homology groups are familiar: $\mathbb{Z}^m$ if $k = n$, and 0 otherwise, proving (i).

$0 \longrightarrow H_m X^{n-1} \longrightarrow H_m(X^n) \longrightarrow 0$ whenever $m \neq n, n-1$. Therefore, for $m > n$ we have $H_m X^n \cong H_m X^{n-1} \cong \ldots \cong H_m X^0 = 0$, proving (ii). When $m < n$ we have $H_m(X^n) i H_m(X^{n+1})$ is an isomorphism. By induction, we get a chain of inclusions, all of which are isomorphisms:

$$H_m(X^n) i H_m(X^{n+1}) i \ldots i H_m(X^m)$$

Since the inclusion $H_m(X^n) i H_m(X^m)$ is the composition of the above inclusions, it too is an isomorphism. For finite cell complexes, $X = X^m$ for some m, proving (ii). □

Theorem 6.7.1. *The following is a chain complex, where the map d_n is defined as the composition $H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) i H_{n-1}(X^{n-1}, X^{n-2})$.*

$\ldots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \ldots$

We have an isomorphism $Ker(d_n)/Im(d_{n+1}) \cong H_n(X)$

Proof. The relative homology sequences of X^n and X^{n-1} for all natural numbers n fit into a "braid-diagram:"

$$\begin{tikzcd}
 & & & & & H_n(X^n,X^{n-1}) & \\
 & & H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] & & \\
 & & & H_n(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & & \\
\cdots \arrow[rr] & & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & {} \\
 & \cdots \arrow[ur] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "l"] & \\
 & & & & H_{n-1}(X^{n-2}) \arrow[ur] & &
\end{tikzcd}$$

Via Lemma **??**, we can reduce some groups to 0, and make an identification with $H_n(X)$:

$$\begin{tikzcd}
 & & & & 0 & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[r, "\cong"] & H_n(X) & \\
 & & H_n(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & {} \\
\cdots \arrow[ur] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "l"] & \\
 & & & 0 \arrow[ur] & &
\end{tikzcd}$$

Since the composition of any two d_n and d_{n-1} includes the compositions of two successive maps of an exact sequence, the composition is 0. The horizontal sequence is therefore a chain complex.

We have

$$im(d_{n+1}) = im(j\partial_{n+1}) = im(\partial n+1) = ker(i)$$

where the second equality comes from the injectivity of j, and the third equality from the exact sequence. Similarly,

$$ker(d_n) = ker(l\partial_n) = ker(\partial_n) = im(j) \cong H_n(X^n)$$

The second equality follows from the injectivity of l, the third from the exact sequence, and the fourth from the injectivity of j. We therefore have that $ker(d_n)/im(d_{n+1}) \cong H_n(X^n)/ker(i)$. Now by the surjectivity of i, $im(i) = H_nX$. Without loss of surjectivity, i defines an injective function i^* on $H_n(X^n)/ker(i)$, so that $H_n(X^n)/ker(i) \cong im(i^*) = H_nX$.

REF HATCHER ☐

Cellular boundary formula...

Theorem 6.7.2.

6.1 Immediate consequences

We now give a number of easy calculations of homology groups of CW-complexes. The following corollary becomes very useful.

Corollary 6.7.1. *If a CW-complex X has no cells in adjacent dimensions, then H_nX is the free abelian group generated by the $n-$cells of X.*

Proof. Suppose X has $m > 0$ n-cells. The chain complex reads:

$0 \xrightarrow{d_{n+1}} \mathbb{Z}^m \xrightarrow{d_n} 0$

It follows that $H_n(X) = Ker(d_n)/Im(d_{n+1}) = \mathbb{Z}^m$. □

Remark 6.8. *MISSING:* $d_1 = 0$

Example 6.9 (§n)**.** *Cellular homology gives a very easy calculation of the homology groups of* §n *for* $n > 1$. S^n *has a cell structure of 1 0-cell and 1 n-cell, and therefore has no cells in adjacent dimension. By Corollary* **??**,

$$H_m S^n = \begin{cases} \mathbb{Z} & m = 0, n \\ 0 & otherwise \end{cases}$$

Example 6.10 (Orientable surface of genus g, M_g)**.** M_g *is the orientable surface with g "holes". It is known that it can be identified as a quotient of the regular 4g-sided polygon, where side i is identified with the opposite of side* $i+2$ *(counted clockwise). In pictures,* M_1 *is the torus* T^2*, identified as the familiar quotient of the square.* M_2 *is the shown quotient of the 8-sided polygon. It is clear that* M_g *should be given a CW-complex structure with* 1 0 *cell,* $2g$ 1*-cells and* 1 $2-cell$ *glued along the map* $f : a_1 \cdot 2 \cdot -a_1 \cdot -a_2 \cdot a_3 \cdot \cdots \cdot -a_{2g-1} \cdot -a_{2g}$. *According to Theorem* **??**, *we should calculate the degree of the composition of f with the map collapsing everything but the circle* a_i. *This is the map* $a_i \cdot -a_i \simeq 0$, *so* $d_2 = 0$. *The chain complex is:*

$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$

Since all maps are 0, *the homology groups can be read as:*

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{2g} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & otherwise \end{cases}$$

Example 6.11 (Klein bottle)**.** *The Klein bottle K is the following quotient of the* 2*-disk:*

$$\begin{array}{ccc} x & \xrightarrow{a} & x \\ {\scriptstyle b}\uparrow & & \downarrow{\scriptstyle b} \\ x & \xrightarrow{a} & x \end{array}$$

We can therefore give it a CW-complex structure of $1 0$*-cell (the point x),* $2 1$*-cells corresponding to the paths a and b, and one 2-cell glued along the path* $f = a \cdot b \cdot -a \cdot b$. *By Theorem* **??**, *we should calculate the degree of f composed with the quotient collapsing respectively b and a to a point. The first of these is* $a \cdot -a \simeq 0$, *and the second is* $b \cdot b \simeq z^2$. *It follows that* $d_2(x) = (0, 2x)$. *The chain complex reads:*

$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{(0,2x)} \mathbb{Z} \times \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$

We simply read off:

$$H_2X = ker(d_2) = 0$$

$$H_1X = ker(d_1)/im(d_2) \cong \mathbb{Z} \times \mathbb{Z}/0 \times 2\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$H_0X = ker(d_0) = 0$$

and $H_nX = 0$ for all other values of n.

6.2 Real projective space $\mathbb{RP}^n$

Cellular homology makes the calculation of $\mathbb{RP}^n$ very easy. We use the cell structure with 1 k-cell for $0 \leq k \leq n$, where the $k-th$ glue map is projection onto $\mathbb{RP}^{k-1}$.

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ and } k \text{ odd.} \end{cases}$$

Proof. □

Now we will look at the homologies of $\mathbb{RP}^n$ with respect to an arbitrary (ordinary) homology. Recall that the puncture of D^n is homotopy equivalent to S^{n-1}. Recall also that the puncture of $\mathbb{RP}^n$ is homotopy equivalent to $\mathbb{RP}^{n-1}$. Now consider the following diagram:

$$\begin{tikzcd} H_k(D^n, S^{n-1}) \arrow[r, "f^*"] \arrow[d, "i^*"] & H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \arrow[d, "j^*"] \\ H_k(D^n, D^n \setminus \{*\}) & H_k(\mathbb{RP}^n, \mathbb{RP}^n \setminus \{*\}) \arrow[l, "h^*"'] \end{tikzcd}$$

$*$ is defined as the center of D^n when considering $\mathbb{RP}^n$ as the glue of $\mathbb{RP}^{n-1}$ and D^n. i^* and j^* are the maps induced by the canonical inclusions. By the previous comment, i and j are homotopy equivalences, so i^* and j^* are isomorphisms. h is the isomorphism guaranteed by the Excision Axiom after noting that $(D^n, D^n \setminus \{*\})$ can be found by excising everything but a smaller, closed n-disk in $D^n \subset \mathbb{RP}^n$ from the other set. This open set then satisfies all the requirements of excision. We can therefore define f^* as the unique isomorphism that makes the diagram commute. By a previous result, we have

$$H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = H_k(D^n, S^{n-1}) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}$$

From this observation we can deduce a number of facts about $H_k(\mathbb{RP}^n)$.

Lemma 6.12. $H_k(\mathbb{RP}^n) \cong H_k(\mathbb{RP}^{n-1})$ *whenever* $k \neq n, n-1$.

Proof. By assumption,

$$H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = H_{k+1}(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = 0.$$

Therefore, the long exact homology sequence reads as

$$0 \longrightarrow H_k(\mathbb{RP}^{n-1}) \longrightarrow H_k(\mathbb{RP}^n) \longrightarrow 0$$

which induces the necessary isomorphism. $\square$

Corollary 6.12.1. *$H_k(\mathbb{RP}^n) = 0$ when $k > n$ and $n \neq 0$.*

Additionally, $H_0(\mathbb{RP}^n = H_0 1 := G$.

Proof. By the previous lemma, we have, for $k > n, n \neq 1$

$$H_k(\mathbb{RP}^n) \cong H_k(\mathbb{RP}^{n-1}) \cong \ldots \cong H_k(\mathbb{RP}^1) \cong H_k(S^1) = 0$$

The second result has already been verified for $n = 0, 1$. For $n > 1$ we can use the previous lemma again with $k = 0$ to get

$$H_0(\mathbb{RP}^n) \cong H_0(\mathbb{RP}^{n-1}) \cong \ldots \cong H_0(\mathbb{RP}^1) \cong H_0(S^1) = G$$

$\square$

Lemma 6.13. *i If $H_n(\mathbb{RP}^n) = 0$ then $H_{n+1}(\mathbb{RP}^{n+1}) = G$ and $H_n(\mathbb{RP}^{n+1}) = 0$.*

ii If $H_n(\mathbb{RP}^n) = G$ and $n > 0$, then $H_{n+1}(\mathbb{RP}^{n+1}) = 0$.

Proof. (i) Consider the following section of the long homology sequence for $(\mathbb{RP}^{n+1}, \mathbb{RP}^n)$:

$$H_{n+1}(\mathbb{RP}^n) \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}) \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}, \mathbb{RP}^n) \longrightarrow H_n(\mathbb{RP}^n) \longrightarrow H_n(\mathbb{RP}^{n+1}) \longrightarrow$$

By assumption, 4.1.1 and 4.1, this reduces to:

$$0 \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}) \longrightarrow G \longrightarrow 0 \longrightarrow H_n(\mathbb{RP}^{n+1}) \longrightarrow 0$$

This induces the two required isomorphisms.

(ii) In this case, the same diagram reduces to

$$0 \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}) \longrightarrow G \longrightarrow G \longrightarrow H_n(\mathbb{RP}^{n+1}) \longrightarrow 0$$

If I can show the map $\partial : G \to G$ is injective, I have my result. (... MISSING) $\square$

6.3 Complex projective space $\mathbb{CP}^n$

The complex projective space, $\mathbb{CP}^n$ is defined similarly to $\mathbb{RP}^n$ as the space of complex lines in $\mathbb{C}^{n+1}$. Explicitly it is the quotient $\mathbb{C}^{n+1} \setminus \{0\}/ where z\mathrm{w} for \in \mathbb{C}$.

It is trivial to see that $\mathbb{CP}^0 = \cdot$, as any $z\tilde{1}$ via multiplication by $\frac{1}{z}$. To understand $\mathbb{CP}^1$, we can first see it as a quotient of S^3, thought of as sitting inside $\mathbb{C}^2$. This follows from the identification

$$\begin{bmatrix} z \\ w \end{bmatrix} \mid \begin{bmatrix} z \\ w \end{bmatrix} \mid \begin{bmatrix} z \\ w \end{bmatrix} := \begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix}$$

Here we are using the typical norm

$$|\begin{bmatrix} x+iy \\ a+ib \end{bmatrix}| = |\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}|$$

Next note that each $\begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix} \in S^3$ is equivalent to exactly all elements in a circle $S^1 \in S^3$. This is because $\{\lambda \begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix}, |\lambda| = 1\}$ is the set of all rotations of $\begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix}$ along some direction.

Consider the case $w \neq 0$. Then, by multiplying by $\frac{\bar{w}^*}{|\bar{w}|}$ we can choose a unique representative $\begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix} := \frac{\bar{w}^*}{|\bar{w}|} \begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix}$ such that $\tilde{w} > 0$. This can then be identified as an element of the (open) upper disk of the equator $S^2 \in S^3$, as the only restriction on $\tilde{z}$ is that $|\begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix}| = 1$. For example the element $\begin{bmatrix} \tilde{a} \\ \tilde{w} \end{bmatrix} \in S^2$ where $\tilde{w} > 0$ is achieved in the previous construction by setting $z = a/(\frac{w^*}{|w|})$. When $w = 0$, $\{\begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix}\} = \mathbb{CP}^0 \cong \cdot$. It follows that $\mathbb{CP}^1$ is the (closed) upper hemisphere of S^2, quotient its equator, which is homeomorphic to S^2. So as a cell complex, $\mathbb{CP}^1$ has 1 0-cell and 1 2-cell.

We can use this cell structure to define $\mathbb{CP}^2$. As before, we can find a representative $\begin{bmatrix} \bar{z} \\ \bar{w} \\ \bar{u} \end{bmatrix} \in S^5$. For $u = 0$ this is $\mathbb{CP}^1$. For $|u| > 0$ we can multiply by $\frac{u^*}{|u|}$ to find a representative in the (open) upper hemisphere of the equator $S^4 \in S^5$. As previously, this covers the whole open upper hemisphere of S^4, as there are no other restrictions on $\bar{z}, \bar{w}$. Therefore $\mathbb{CP}^2$ has a $2n$-cell for $0 \leq n \leq 2$. The additional gluing map is the projection $S^3 \subset \mathbb{C}^2 \to \mathbb{CP}^1$. Since $\mathbb{CP}^2$ has no cells in adjacent dimensions, its homology groups can be directly read of as

$$H_n(\mathbb{CP}^2) = \begin{cases} \mathbb{Z} & n = 0,2,4 \\ 0 & \text{otherwise} \end{cases}$$

The general case can be done by induction.

Proposition 6.14. *As a cell complex, $\mathbb{CP}^n$ has a $2m$-cell for $0 < m \leq n$. As a consequence,*

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k \text{ even}, 0 \leq k \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We have proved the case $n = 2$. Suppose the statement is true for some n. We will then prove it is true for $n+1$. As before, we can identify elements of $\mathbb{CP}^{n+1}$ by representatives of S^{2n+1} by dividing by norm. These can furthermore be identified as either elements of $\mathbb{CP}^n$ as a quotient of the equator of S^{2n} or elements of the open upper $2n$-cell of S^{2n}. The gluing map is the projection of ∂D^{2n} onto $\mathbb{CP}^n$. Therefore $\mathbb{CP}^{n+1}$ has the cell structure of $\mathbb{CP}^n$, with an additionall $2n$-cell. This proves the inductive case. The homology groups follow from noting $\mathbb{CP}^n$ has no n-cells in adjacent dimensions. $\square$

Remark 6.15. *It is worth looking at the gluing map of the 4-cell of* $\mathbb{CP}^2$ *onto* $\mathbb{CP}^1 \cong \S 2$. *Via the homeomorphism, this is a map* $\S 3 \to \S 2$ *with the property that the pre-image of every point is a great circle of* S^3. *This is because* $[\begin{bmatrix} z \\ w \end{bmatrix}] \in \mathbb{CP}^1$ *is mapped to by* $\{\lambda \begin{bmatrix} z \\ w \end{bmatrix} : \lambda \in \mathbb{C}, |\lambda| = 1\}$, *which is a great circle of* $\S 3$. *This is exactly what characterises the Hopf map (REFERENCE).*

6.4 Quaternionic projective space $\mathbb{HP}^n$ and beyond

The 2-dimensionality of $\mathbb{C}$ ensured that the n-cells of $\mathbb{CP}^n$ were well spread in dimension, leading to an easy homology calculation. One can wonder if the same method can be applied to the higher dimensional extension of $\mathbb{R}$. Indeed we can!

Definition 6.16.

Proposition 6.17. $\mathbb{HP}^n$ *has a cell structure with a* $2k$*-cell for* $0 \leq k \leq n$. *As a consequence,*

$$H_k(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & k \text{ even}, 0 \leq k \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Proof. As in previous cases, $\mathbb{HP}^0 = \cdot$, as $z \sim 1$ for every $z \in \mathbb{H}$ by division by z. For general $\mathbb{HP}^n$ we will proceed by induction. Let elements of $\mathbb{HP}^n$ be written as $[z_1, \ldots, z_n] \in \mathbb{HP}^n$. The subset $\{[0, z_2, \ldots, z_n]\} \subset \mathbb{HP}^n$ can be identified with $\mathbb{HP}^{n-1}$. The complement $\mathbb{HP}^{n-1c} \in \mathbb{HP}^n = \{[z_0, z_1, \ldots, z_n] : z_1 \neq 0\} = \{[1, z_1, \ldots, z_n]\} \cong \mathbb{H}^{n-1} \cong D^{4n}$ by multiplication by $1/z_0$ and relabelling. It follows that $\mathbb{HP}^n$ has the cell structure of $\mathbb{HP}^{n-1}$ with an additional $4n$-cell. This completes the identification of the cell structure of $\mathbb{HP}^n$.

As $\mathbb{HP}^n$ has no cells in adjacent dimensions, its $k-th$ homology group is either $\mathbb{Z}$ or 0, depending on whether $\mathbb{HP}^n$ has a k-cell. $\square$

Remark 6.18. *Notice that* $\mathbb{HP}^1$ *has a 0-cell and a 4-cell, and is therefore homeomorphic to* $\S 4$. *As in Remark* **??**, *the gluing map of the 8-cell of* $\mathbb{HP}^2$ *onto* $\mathbb{HP}^1 \cong \S 4$, *therefore gives a "Hopf"-map* $S^7 \to S^4$ *with the property that the preimage of a point is a "great" copy of* S^3. *If we have real division algebras of* $\mathbb{R}$ *for every positive power of two we could repeat this process, yielding "Hopf"-maps from* S^{2^n-1} *to* S^{n-1} *for all* $n > 0$. *However this turns out to be false, as shown by X (REFERENCE)!!! The non-existence of such maps for* $n > 4$ *proves there are no* 2^n*-dimensional division algebras of* $\mathbb{R}$ *for* $n > 4$.

6.5 The Borsuk-Ulam Theorem

The Borsuk-Ulam theorem is a fascinating fixed-point result, which can be proven via cellular homology and a technical homotopy remark. It states that every odd map $f : \S^n \to \mathbb{R}^n$ maps two antipodal points to the same point. A popular interpretation of this statement is that there are always two antipodal points on Earth with the same temperature and barometric pressure. This follows from Borsuk-Ulam, where we are assuming the Earth is a 2-sphere, and temperature and barometric pressure are continuous functions on the surface of Earth (although both of these assumptions are dubious).

The proof of Borsuk-Ulam demonstrates the advantage of an axiomatic approach to homology. Since in section (MISSING), we proved the homology groups of the spheres in terms of the homology group of $H_0\cdot$, we immediately get their homology groups in any other coefficients. In this section we will work with a homology theory over the coefficients $\mathbb{Z}/2\mathbb{Z}$. We will assume there is a homology theory with these coefficients. One piece of work left is to show that our work on degrees can be extended to these new coefficients.

If $f : S^n \to S^n$ has degree m, then the induced map in $\mathbb{Z}/p\mathbb{Z}$ coefficients, where p is a prime, is multiplication by $m \pmod p$.

Proof. MISSING □

Our proof of Borsuk-Ulam follows the approach in [**?**]. We will need the following technical result.

Theorem 6.18.1 (Cellular approximation)**.** *Every map $f : X \to Y$ is homotopic to a cellular map g, that is a map such that $g(X^n) \subseteq Y^n$ for each n. If g is already cellular on a subcomplex of X, the homotopy can be taken to be relative on the subcomplex.*

Proof. Omitted. See [**?**]. □

Instead of proving Borsuk-Ulam, we prove the following stronger result.

Theorem 6.18.2. *Odd maps $f : S^n \to S^n, n \geq 1$ have odd degree.*

Proof. We prove this by induction on n. For $n > 0$, an odd map $f : S^{n-1} \to S^{n-1}$ has odd degree if and only if it induces an isomorphism. We can take this to be the induction property, and note that it hold trivially for $n = 0$. Suppose the statement holds for $n-1$. We can give S^n a cell structure with 1 0-cell and 1 $n-1$-cell corresponding to the equatorial sphere S^{n-1}, and 2 n-cells corresponding to the upper and lower hemispheres of S^n, glued by the identity map on the boundary onto S^{n-1}. Note this is not the usual cell-structure of S^n with 1 0-cell and 1 n-cell. By Theorem **??**, $f \simeq g$ where g is cellular.$f^* = g^*$, f^* inherits the properties of both an odd map and a cellular map. We may therefore assume f is both odd and cellular.

$f(\S n-1) \subseteq \S n-1$, f gives a map from the homology sequence of $(S^n, S^{n-1}$ to itself. This looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & H_n S^n \arrow[r, "i"] \arrow[d, "f^*"] & H_n(S^n, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "f^*"] & H_{n-1} S^{n-1} \arrow[r] \arrow[d, "\cong"] & \dots \\
0 \arrow[r] & H_n S^n \arrow[r, "i"] & H_n(S^n, S^{n-1}) \arrow[r, "\partial"] & H_{n-1} S^{n-1} \arrow[r] & \dots
\end{tikzcd}$$

The homology groups $H_n(S^n, S^{n-1}$ are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generators corresponding to the upper and lower hemispheres, as S^n is a CW-complex with two n-cells glued onto a copy of S^{n-1}.

In coefficients, the above diagram becomes:
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "i"] \arrow[d, "f^*"] & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[r, "\partial"] \arrow[d, "f^*"] & H_{n-1} S^{n-1} \arrow[r] \arrow[d, "\cong"] & {} \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "i"] & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[r, "\partial"] & H_{n-1} S^{n-1} \arrow[r] & {}
\end{tikzcd}$$

$f^* \neq (0,0)$ by commutativity as ∂ is injective, hence not 0. f commutes with the antipodal map by oddness: $f(-id) = (-id)f$. It follows that $f^*(-id)^* = (-id)^* f^*$. Note that the antipodal map also gives a map between these sequences, and that it swaps the hemispheres (REFERENCE): $(-id)^*(x,y) = (y,x)$. So $f^*(x,0) = (-id)^* f^*(0,y)$. Since $f^* \neq (0,0)$, $f^*(x,0) = (x,0)$ or $(0,x)$ and $f^*(0,y)$ is the other one. So f^* is an isomorphism.

By commutativity in the left square, since i is injective, and $f^* i$ is injective, $f^* : S^n \to S^n$ is also injective, hence is an isomorphism. This implies the degree m of $f : S^n \to S^n$ is s.t. $m mod 2 = 1$, i.e. m is odd.

[**?**] □

7 Appendix

7.1 Proof of Braid Lemma

In this section we give the rest of the proof of Lemma **??**.

Proof. Recall the following commutative braid lemma diagram, where we have assumed the sequence indexed by f_i is a chain complex, and the other sequences are exact sequences.

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[rr, bend left, "g_1"] & & D \arrow[dr, "g_2"] \arrow[rr, bend left, "h_3"] & & G \arrow[dr, "h_4"] \arrow[rr, bend left, "j_4"] & & J \\
O \arrow[ur, "g_0"] \arrow[dr, "j_0"'] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \arrow[ur, "f_5"] \arrow[dr, "h_5"] & \\
& B \arrow[ur, "h_1"] \arrow[rr, bend right, "j_1"'] & & E \arrow[ur, "j_2"] \arrow[rr, bend right, "f_3"'] & & H \arrow[ur, "f_4"] \arrow[rr, bend right, "g_4"'] & & K
\end{tikzcd}$$

We have shown that $ker(f_2) \subseteq im(f_1)$ and need to show that $ker(f_3) \subseteq im(f_1)$ and $ker(f_4) \subseteq im(f_3)$.

(a) $ker(f_3) \subseteq im(f_1)$.

Let $x \in E$ be s.t. $f_3(x) = 0$. By commutativity, $g_3 j_2(x) = 0$, so $j_2(x) \in ker(g_3) = im(g_2)$. Then $\exists x_1 \in D$ s.t. $g_2(x_1) = j_2(x)$. It satisfies $h_3(x_1) = j_3 g_2(x_1) = j_3 j_2(x) = 0$, since (j_i) is a chain complex. So $x_1 \in ker(h_3) = im(h_2)$. Therefore there exists $x_2 \in C$ s.t. $h_2(x_2) = x_1$. This element is such that $j_2 f_2(x_2) = g_2 h_2(x_2) = g_2(x_1) = j_2(x)$. We therefore have $j_2(f_2(x_2) - x) = 0$.

Let $x_3 := f_2(x_2) - x$. Then $x_3 \in ker(j_2) = im(j_1)$. Let $x_4 \in B$ be s.t. $j_1(x_4) = x_3$. x_4 is such that $f_2 h_1(x_4) = j_1(x_4) = x_3 = f_2(x_2) - x$. Finally, we see that $x = f_2(x_2 - h_1(x_4)$, so $x \in im(f_2)$ as required.

(b) $ker(f_4 \subseteq im(f_3)$.

Let $x \in H$ be s.t. $f_4(x) = 0$. Then $0 = h_5 f_4(x) = g_4(x)$. So $x \in ker(g_4) = im(g_3)$. Let $x_1 \in F$ be s.t. $g_3(x_1) = x$. Then $h_4 j_3(x_1) = f_4 g_3(x_1) = f_4(x) = 0$ So $j_3(x_1) \in ker(h_4) = im(h_3)$. Let $x_2 \in D$ be s.t. $h_3(x_2) = j_3(x_1)$. Then $j_3(x_1) = j_2 g_2(x_2)$, s.t. $x_3 := g_2(x_2) - x_1 \in \mathrm{ker}(j_3) = im(j_2)$. Let $x_4 \in E$ be s.t. $j_2(x_4) = x_3$. Then $f_3(x_4) = g_3 j_2(x_4) = g_3(x_3) = g_3(g_2(x_2) - x_1) = -g_3(x_1) = -x$. Therefore $x = f_3(-x_4)$, and $x \in im(f_3)$ as required.

$\square$