Axiomatic Homology Theory

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1 Notes and acknowledgements

This text provides an introduction to homology theory from an axiomatic point of view. The reader is assumed to have knowledge of undergraduate level Algebra and Algebraic Topology, particularly *groups, homotopies, homotopy equivalences* and *quotient spaces*.

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2 Introduction

One of the goals of algebraic topology is classifying spaces up to some definition of equivalence, typically homotopy equivalence. Establishing equivalence can be tedious, as it often requires the construction of an explicit homotopy equivalence. However, *inequivalence* is often easier to establish by calculating **homotopy invariants**. These are properties of a space that are preserved by homotopy equivalences, hence if two spaces have different invariants, they cannot be homotopy equivalent. One of the first major homotopy invariances the undergraduate student encounters are the homotopy groups $\pi_n(X), n \in \mathbb{N} \cup \{0\}$: the groups of maps $f : S^n \to X$ with some base point. The homotopy groups carry a lot of geometric information, but are notoriously difficult to compute. Even for a simple space like the 2-sphere S^2, it is not obvious that there are non-trivial maps $S^3 \to S^2$ (we show that such a map exists in section 8.5), and it is even less obvious what the higher homotopy groups are. The groups $\pi_n(S^2)$ do not seem to follow a pattern, and remain an active area of research. Only recently in 2015 was it proven that $\pi_n(S^2)$ is not zero for all $n \geq 2$ [Ivanov et al., 2015].

To avoid these complications, we would like to find homotopy invariances that are easier to compute, and ideally, not at the cost of too much geometric information. The homology groups $H_n(X)$ are an example of such a homotopy invariant. Like the homotopy groups, they are a sequence of groups, one for each $n \in \mathbb{Z}$, but they are much simpler and easier to compute. For example, the spheres have the simple structure

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0, m \\ 0 & \text{otherwise} \end{cases}$$

Part of the reason why they are easier to compute is that they come with extra structure, most importantly a **long exact sequence**, which, broadly speaking, relates the homology groups H_nX to each other and to the homology groups H_nA of a subspace $A \subseteq X$. Therefore, the more you know about some of the homology groups of a space and a subspace, the easier it is to calculate the rest.

Historically, homology groups were produced from a number of geometric methods. It was Eilenberg and Steenrod who united the different homology theories by laying out a set of axioms that all homology theories satisfy [Eilenberg and Steenrod, 1952]. In this text, we will take such an axiomatic approach, proving all results directly from the axioms. In some ways this approach more directly shows the point of homology theory. As will become clear, we will very rarely want for the geometric construction, and in fact all (ordinary) homology theories which satisfy the same Eilenberg-Steenrod axioms and initial conditions are equivalent; the geometric construction contributes no more than can be captured in the axioms (REFERENCE). It is therefore believable that the main role of a geometric construction of homology theory is to prove the Eilenberg-Steenrod Axioms for that theory. For the results to be true we have to take given that there exists at least one homology theory which satisfies the axioms. Proving this is a major undertaking worth its own project. Singular Homology is an example of a homology theory which satisfies our assumptions, and the reader is invited to confirm that it satisfies the axioms in [Hatcher, 2002].

Homology theory is best understood in the language of category theory and chain complexes, which sections 3 and 4 are devoted to. In section 5, we lay out the Eilenberg-Steenrod axioms and prove some immediate results for a general ordinary homology theory, most importantly the homology groups of the n-sphere. In the following sections we make the choice $H_0\bullet = \mathbb{Z}$, which corresponds to Singular Homology. In sections 6, 7 and 8 we lay out three practical methods for calculating homology groups; The Mayer-Vietoris Sequence, degree maps and Cellular Homology, and use them to prove some fascinating results.

3 Category theory

The language of category theory was invented specifically for homology theory, but has since then become an entire field of itself with many applications [Marquis, 2020]. It is a very general construction, as many familiar constructions can be understood as categories, such as groups, rings, vector spaces and topological spaces. Informally, a category is a collection of "objects" and "maps between them". In these cases, the objects are sets with some structure, and the maps are functions that satisfy some structure-preserving property. However, the definition of a category is more general and this, and there are categories whose objects look nothing like sets and whose maps look nothing like functions between sets.

Definition 3.1. A category $\mathscr{A}$ is

- a **collection**[1] of objects $ob(\mathscr{A})$,
- for each $A, B \in ob(\mathscr{A})$ a collection $\mathscr{A}(A,B)$ of maps from A to B,
- a composition function $\circ : \mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{A}(A,C)$,

which satisfy the following properties:

(a) **Associativity**: $(f \circ g) \circ h = f \circ (g \circ f)$,

(b) **Identity laws**: For each $A \in ob(\mathscr{A})$ there is a unique map $id_A \in \mathscr{A}(A,A)$ with the property that $id_A \circ f$=f and $g \circ id_A = g$ for every $f \in \mathscr{A}(B,A)$ and $g \in \mathscr{A}(A,B)$

Note we often write $A \in \mathscr{A}$ to mean $A \in ob(\mathscr{A})$, and fg to mean $f \circ g$.

Categories are best understood through examples, of which we give a few. Many more examples are given in [Leinster, 2016].

Example 3.2. The following are categories.

(a) The empty category $\emptyset$ with no objects and homomorphisms.

(b) A one-object category $\{A\}$ with only the identity map id_A.

(c) **Top** is a category, where spaces are topological spaces, and the maps are all continuous maps. This is because the identity map is always continuous, and compositions of continuous maps are continuous.

(d) We have similar constructions for other "sets with structure" and "structure-preserving maps" such as **Vec** the category of vector spaces and linear maps, and **Grp**, the category of groups and homomorphisms.

(e) Many categories can be found as subcategories of previous examples. For example, we have the subcategory **Ab** $\subset$ **Grp** of all abelian groups and homomorphisms between them.

(f) Finally, here is an example of a category that cannot be interpreted as "sets" and "structure-preserving maps":

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[loop above, "gf"] \arrow[loop below, "id_A"'] & B \arrow[l, bend left, "g"] \arrow[loop below, "id_B"'] \end{tikzcd}$$

[1] The term **collection** is similar to a set, with some technical distinguishing properties laid out in [Leinster, 2016].

If we tried to interpret this as two one-element sets A and B together with the only maps f,g between them, we run into trouble, as there are two distinct maps from A to itself! However this category satisfies the definitions: we have identities, and for every two maps we have defined their compositions

$$fg = id_B, f(gf) = f, (gf)g = g, (gf)(gf) = gf$$

These compositions are forced upon us by the associativity requirement, for example $f(gf) = (fg)f = (id_A)f = f$, and $(gf)(gf) = g(fg)f = gf$. One might think gf contradicts the uniqueness of the identity requirement, but it does not, as $(gf)id_A = gf \neq id_A$.

Two definitions that will play a big role in our study are notions of "maps between categories" and "maps between maps between categories". These are called **functors** and **natural transformations**, respectively.

Definition 3.3. A **functor** $F : \mathscr{C} \to \mathscr{H}$ between two categories is a function assigning each $X \in ob(\mathscr{C})$ to some $F(X) \in ob(\mathscr{H})$, and each $f \in \mathscr{C}(A,B)$ to some $F(f) \in \mathscr{C}(F(A),F(B))$, such that

(a) $F(f \circ g) = F(f) \circ F(g)$

(b) $F(id_A) = id_{F(A)}$

The definition of a functor is set up such that it takes commutative diagrams to commutative diagrams. I.e. if this commutative diagram is in $\mathscr{C}$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "g"] \\ D \arrow[r, "h"] & C \end{tikzcd}$$

then the following commutative diagram is in $\mathscr{H}$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "F(j)"] & F(B) \arrow[d, "F(g)"] \\ F(D) \arrow[r, "F(h)"] & F(C) \end{tikzcd}$$

Definition 3.4. A **natural transformation** $\alpha : F \to G$ between functors $F,G : \mathscr{A} \to \mathscr{B}$ is a family of functions $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ between objects in $\mathscr{B}$ such that any map $f : A \to B$ in $\mathscr{A}$ gives rise to a commutative diagram

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\alpha_A} & G(A) \\
\Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\
F(B) & \xrightarrow{\alpha_B} & F(B)
\end{array}
$$

Functors and natural transformations are explored in depth in [Leinster, 2016]. For now we will make do with the definitions. The properties we will use the most are that *functors preserve commutative diagrams*, and that *natural transformations commute with functors* in the way specified by the above diagram.

4 Chain Complexes

Definition 4.1. A **chain complex** is a family $(A_i)_{i\in\mathbb{Z}}$ of abelian groups, as well as a family $(f_i : A_i \to A_{i+1})_{i\in\mathbb{Z}}$ of homomorphisms between consecutive groups, such that $im(f_i) \subset ker(f_{i+1})$. If we have equality instead of inclusion, the family is called an **exact sequence**.

We distinguish between **short** and **long exact sequences**, where short exact sequences are exact sequences with three or fewer consecutive non-zero groups, i.e. sequences of the form

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$$

All other exact sequences are called long exact sequences.

Example 4.2. For any abelian group A the following is a long exact sequence:

$$\dots \xrightarrow{0} A \xrightarrow{id} A \xrightarrow{0} A \xrightarrow{id} \dots$$

Remark 4.3. The requirement that $im(f_i) \subseteq ker(f_{i+1})$ is equivalent to the requirement that $f_{i+1} \circ f_i = 0$. This is clear from the definition: if $im(f_i) \subseteq ker(f_{i+1})$ then

$$\forall x \in A_i, f_{i+1} \circ f_i(x) \in f_{i+1}(im(f_i)) = \{0\}.$$

Conversely, if $\forall x \in A_i, f_{i+1} \circ f_i = 0$ then $f_{i+1}(im(f_i)) = \{0\}$.

Example 4.4. Suppose the following is part of a long exact sequence of groups.

$$\dots \longrightarrow A \xrightarrow{0} B \xrightarrow{f} C \xrightarrow{0} D \longrightarrow \dots$$

We say this **gives rise to** the following short exact sequence, as they carry the same information.

$$0 \longrightarrow B \xrightarrow{f} C \longrightarrow 0$$

We often omit specifying any homomorphisms out of or into 0, as there is only one: the 0 homomorphism. Since $0 = im(0) = ker(f)$, f is injective. Since $im(f) = ker(0) = C$, f is surjective. Therefore f is an isomorphism.

Definition 4.5. If there exist maps $f : A \to B$ and $g : B \to A$ between abelian groups such that $g \circ f = id_A$ then g is called a **retraction** of f. If alternatively $f \circ g = id_B$, then f is called a **section** of f.

This definition formalises the idea of a "one-sided inverse".

Proposition 4.6. *Let the following be a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*(a) If there exists a **section** $s : C \to B$ of g, then f and s define an isomorphism*

$$(f+s) : A \oplus C \cong B$$

$$(a,c) \mapsto f(a) + s(c).$$

*(b) If there exists a **retraction** $r : B \to A$ of f, then r and g define an isomorphism*

$$(r,g) : B \to A \oplus C$$

$$b \mapsto (r(b), g(b))$$

Proof. (a) First we show that $(f+s)$ is injective. By exactness we have that f is injective, as $0 = im(0) = ker(f)$, and that g is surjective, as $im(g) = ker(0) = C$.

Suppose $0 = f(a) + s(c)$. Then

$$g(0) = g(f(a) + s(c)) = gf(a) + gs(c) = c$$

Which implies $c = 0$. But then $0 = f(a)$, which implies $a = 0$ as f is injective. $ker(f+s) = 0$, so $(f+s)$ is injective.

Now we show (f,s) is surjective. Now let $b \in B$. Let $c = g(b) \in C$ and $a \in A$ be the unique element that maps to $b - sg(b)$. This element is in $im(f) = ker(g)$, since $g(b - s(g(b))) = g(b) - g(b) = 0$, and is unique by injectivity of f. Then $f(a) + s(c) = b - sg(b) + sg(b) = b$. Therefore (f,s) is surjective, so it is an isomorphism.

(b) First we show that (r,g) is injective. Suppose $(r(b),g(b)) = (0,0)$. Then $g(b) = 0$, so $b \in ker(g) = im(f)$. Now r is injective on $im(f)$, since $rf = id_A$. Therefore $r(b) = 0 \implies b = 0$. So (r,g) is injective.

Now we show (r,g) is surjective. Let $(a,c) \in A \oplus C$. Since r is a retraction of f, $r(f(a)) = a$. Now g is surjective, so there exists $b \in B$ such that $c = g(b)$. Now since $im(f) = ker(g)$, $g(f(a)) = 0$.

Consider $fr(b) \in B$. It also has the property $g(fr(b)) = 0$, as $gf(-) = 0$. However, $r(fr(b)) = r(b)$. We can therefore let $y = a + b - fr(b) \in B$. Then $g(y) = g(a) + g(b) + g(fr(b)) = g(b) = c$, and $r(y) = r(f(a)) + r(b) - r(fr(b)) = a$. So (r,g) is surjective. Hence it is an isomorphism.

$\square$

Corollary 4.7. *If $f : A \to B$ admits a retraction $g : B \to A$, then*

$$B \cong im(f) \oplus ker(g)$$

Proof. We can define $r| := r|_{im(f)} : im(f) \to A$, which is injective since rf is an isomorphism. Trivially we can define $f : A \to im(f)$, which by abuse of notation we give the same name as $f : A \to B$. The composition $fr|(x) : im(f) \to B$ is injective, as

$$fr|(x) = 0 \implies r|(x) = r(0) = 0 \implies x = 0$$

as $r|$ is injective.

We can also define a homomorphism

$$h : B \to ker(r)$$

$$x \mapsto x - fr(x)$$

since $r(x - fr(x)) = r(x) - r(x) = 0$. Note h is surjective, as if $r(x) = 0$, then $fr(x) = 0$, so $x = x - fr(x) = h(x)$, where by abuse of notation, we are considering x to be both in $ker(r)$ and B.

Note $h(fr|)(x) = fr|(x) - fr(fr|(x)) = fr|(x) - fr|(x) = 0$, as $rf = id_B$. So $im(f) \subseteq ker(h)$. Additionally, if $x - fr|(x) = 0$, then $x = fr|(x)$, so $ker(h) \subseteq im(f)$. The horizontal part of the following commutative diagram is therefore a short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & im(f) \arrow[r, "fr|"] \arrow[d, "r|"] & B \arrow[r, "h"] \arrow[dl, bend left, "r"] & ker(r) \arrow[r] & 0 \\
& A \arrow[u, bend left, "f"] \arrow[ur, "f"'] & & &
\end{tikzcd}$$

Since fr is a retraction of $fr|$, Proposition 4.6 gives an isomorphism $(fr,h) : B \to im(f) \oplus ker(g)$ $\square$

We finish this section with a technical lemma that will become useful in section 5.

Lemma 4.8 (Braid lemma)**.** *Suppose three long exact sequences and a chain complex make the following commutative diagram.*

$$\begin{tikzcd}
{} \arrow[rr, bend left] & & A \arrow[dr, "f_1"] \arrow[rr, bend left, "g_1"] & & D \arrow[dr, "g_2"] \arrow[rr, bend left, "h_3"] & & G \arrow[dr, "h_4"] \arrow[rr, bend left, "j_4"] & & {} \\
& O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I & \\
{} \arrow[rr, bend right] & & B \arrow[ur, "h_1"] \arrow[rr, bend right, "j_1"] & & E \arrow[ur, "j_2"] \arrow[rr, bend right, "f_3"] & & H \arrow[ur, "f_4"] \arrow[rr, bend right, "g_4"] & & {}
\end{tikzcd}$$

Then the chain complex is also a long exact sequence.

Proof. By symmetry of the diagram, it does not matter which sequence is the chain complex. We can assume it is the sequence with homomorphisms f_i. We are given that $im(f_i) \subseteq ker(f_{i+1})$, and need to show that $ker(f_{i+1}) \subseteq im(f_i)$. By the symmetry of the diagram, it is enough to show this for $i = 1,2,3$. We will show that $ker(f_2) \subseteq im(f_1)$ here, and do the other two cases in the Appendix.

Let $x \in ker(f_2)$. Then $0 = f_2(x) = j_2 f_2(x) = g_2 h_2(x)$ by commutativity. It follows that $h_2(x) \in ker(g_2) = im(g_1)$. So $\exists x_1 \in A$ s.t. $g_1(x_1) = h_2(x)$. By commutativity, $g_1(x_1) = h_2 f_1(x_1)$. So we have that $0 = g_1(x_1) - h_2(x) = h_2(f_1(x_1) - x)$. Let $x_2 := f_1(x_1) - x \in ker(h_2) = im(h_1)$. Then $\exists x_3 \in B$ s.t. $h_1(x_3) = x_2$.

Now note that

$$j_1(x_3) = f_2 h_1(x_3) = f_2(x_2) = f_2(f_1(x_1) - x) = 0$$

Where the last equality follows from $f_2 f_1(-) = 0$ and $f_2(x) = 0$. We therefore have that $x_3 \in ker(j_1) = im(j_0)$. So there exists $x_4 \in O$ s.t. $j_0(x_4) = x_3$. Consider $g_0(x_4)$. It satisfies $f_1 g_0(x_4) = h_1 j_0(x_4) = h_1(x_3) = x_2 = f_1(x_1) - x$. Therefore we have

$$x = f_1(x_1 - g_0(x_4))$$

Which shows $x \in im(f_1)$ as required. [Eilenberg and Steenrod, 1952] $\square$

5 Axioms

5.1 The Eilenberg-Steenrod axioms

The axioms for a homology theory were first laid out by Eilenberg and Steenrod in [Eilenberg and Steenrod, 1952]. In this section, we will follow a simplified treatment given in [Werndli, 2009]. We define $\mathbf{Top_2}$ to be the category of pairs of topological spaces (X,A), where $A,B \in \mathbf{Top}$ and $A \subseteq X$. Maps $\mathbf{Top_2}((X,A),(Y,B)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$. It is not hard to see this space satisfies the definition of a category.

Lemma 5.1. $\mathbf{Top_2}$ *is a category.*

Proof. It is easy to see that the identity map id_X satisfies the requirement for each (X,A), as $id_X(A) = A$. Furthermore, if $f \in \mathbf{Top_2}((X,A),(Y,B)$ and $g \in \mathbf{Top_2}((Y,B),(Z,C)$, then gf is continuous and $gf(A) = g(f(A)) \subseteq g(B) \subseteq C$ as required. The associativity and identity properties are satisfied because $\mathbb{T}\rtimes\imath$ is a category. $\square$

We can similarly define $\mathbf{Top_3}$ as the triples of topological spaces (X,A,B) where $B \subseteq A \subseteq X$ and where morphisms $f : (X,A,B) \to (Y,C,D)$ are maps $f : X \to Y$ such that $f(A) \subseteq C$ and $f(B) \subseteq D$. We identify $X \in \mathbf{Top}$ with $(X,\emptyset) \in \mathbf{Top_2}$, and $(X,A) \in \mathbf{Top_2}$ with $(X,A,\emptyset) \in \mathbf{Top_3}$.

Definition 5.2. A subcategory $\mathscr{C} \subset \mathbf{Top_2}$ is **admissible for homology** if the following apply:

(a) C contains all points in **Top**. In the language of category theory, $\mathscr{C}$ contains all **final objects** in **Top**, that is, all objects $\bullet \in \mathbf{Top}$ with the property that there is one and only one morphism from any $X \in \mathbf{Top}$ to $\bullet$.

(b) For any $(X,A) \in \mathscr{C}$, the following commutative diagram lies in $\mathscr{C}$, where all maps are induced by canonical inclusions:

$$\begin{tikzcd}
 & & (X,\emptyset) \arrow[dr] & & \\
(\emptyset,\emptyset) \arrow[r] & (A,\emptyset) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X) \\
 & & (A,A) \arrow[ur] & &
\end{tikzcd}$$

Furthermore, for any $f \in \mathscr{C}((X,A),(Y,B))$, $\mathscr{C}$ contains all the canonical maps induced by f on the above diagram to the corresponding diagram of (Y,B).

(c) For $(X,A) \in \mathscr{C}$, $\mathscr{C}$ contains the following diagram:

$$\begin{tikzcd} (X,A) \arrow[r, bend left, "\tau_0"] \arrow[r, bend right, "\tau_1"'] & (X \times I, A \times I) \end{tikzcd}$$

Where $\tau_t(x) = (x,t)$.

Remark 5.3. As noted in [Werndli, 2009], the definition certifies that

1. $\mathscr{C}$ contains all final objects in $\mathbf{Top}_2$, that is all maps $(\bullet, \emptyset) \to (X,A)$, where $\bullet$ is some fixed one-point set in $\mathscr{C}$. This is because $\mathscr{C}$ contains all maps $(\bullet, \emptyset) \to (X, \emptyset)$ and also the inclusion $(X, \emptyset) \to (X,A)$,

2. $\mathscr{C}$ contains all homotopies $h : f \simeq g$, for $f, g \in \mathscr{C}((X,A),(Y,B))$. By (iii), we can identify homotopies as maps $h : (X \times I, A \times I) \to (Y,B)$ such that $h\tau_0 = f$ and $h\tau_1 = g$:

$$\begin{tikzcd} (X,A) \arrow[r, bend left, "t_0"] \arrow[r, bend right, "t_1"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend right=50, "g"'] & (X \times I, A \times I) \arrow[r, "h"] & (Y,B) \end{tikzcd}$$

In this text we will assume all spaces and maps are admissible, unless otherwise stated. We will also use "space" to denote "topological space".

We now give the axioms of an (ordinary) homology theory for an admissible category.

Definition 5.4. An **ordinary homology theory** on an admissible category $\mathscr{C}$ is a family of functors $(H_n : \mathscr{C} \to \mathbf{Ab})_{n \in \mathbb{Z}}$ to the category of Abelian groups[2] **Ab** and a family of natural transformations $\partial_n : H_n \to H_{n-1} \circ p$, where p is the functor sending (X,A) to $(A, \emptyset)$ and $f : (X,A) \to (Y,B)$ to $f|_A^B : (A, \emptyset) \to (B, \emptyset)$. We will often write f^* for $H_n f$, ∂ for ∂_n and $H_n X$ for $H_n(X, \emptyset)$, as it is usually obvious what role they play. H_n and ∂ are assumed to hold the following axioms:

[2]The original definition defines functors into an **abelian category**, which generally speaking is a category where homomorphisms can be added and wehere we can define the kernel and image of a homomorphism. In this text we will only deal with the category of abelian groups.

(a) (Homotopy invariance) If $f \simeq g$, then $f^* = g^*$.

(b) (Long exact sequence) The inclusions

$$(A, \emptyset) \xrightarrow{\ i\ } (X, \emptyset) \xrightarrow{\ j\ } (X, A)$$

give rise to a long exact sequence

$$\longrightarrow H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n A \xrightarrow{i^*} H_n X \xrightarrow{j^*} H_n(X,A) \longrightarrow$$

(c) (Excision) If $U \subset A \subset X$ is open in X and satisfies $\overline{U} \subseteq int(A)$, then the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ gives rise to an isomorphism $H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$.

(d) (Dimension) For any one-point set $\bullet$,

$$H_n \bullet = \begin{cases} G & n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where G is some fixed abelian group.

Remark 5.5. As H_n is a functor, it preserves commutative diagrams. That is, any commutative diagram of spaces in $\mathbf{Top}_2$ gives rise to a commutative diagram in homology. Furthermore $id^* = id$ by the identity property of functors. Together with axiom (a), this asserts that homology theory can be used to distinguish between homotopy equivalent spaces: if $f : X \to Y$ is a homotopy equivalence with homotopy inverse $g : Y \to X$, then as $fg \simeq id$, $f^* g^* = (fg)^* = id$, so f^* is surjective. Similarly as $gf \simeq id$, $g^* f^* = id$, so f^* is injective. Hence $f^* : H_n X \to H_n Y$ is an isomorphism.

The extra dimension axiom is what defines an ordinary homology theory as opposed to a (general) homology theory. It is essential in our study. The choice $H_0 = G$, called a "choice of coefficients" distinguishes homology theories from each other. For the remainder of this text, we will make the choice $H_0 \bullet = \mathbb{Z}$, which corresponds to Singular Homology Theory [Hatcher, 2002]. However, it should be clear how many of the proofs apply to an arbitrary choice of coefficients.

5.2 Basic results

Proposition 5.6. *If $A \subset X$ is a deformation retract, then $H_n(X,A) = 0$.*

Proof. If $A \subset X$ is a deformation retract, then the inclusion $i : A \to X$ is a homotopy equivalence. By Remark 5.5, $i : H_nA \to H_nX$ is an isomorphism. Now consider the homology sequence for (X,A):

$$H_nA \xrightarrow{\cong} H_nX \xrightarrow{j^*} H_n(X,A) \xrightarrow{\partial} H_{n-1}A \xrightarrow{\cong} H_{n-2}X$$

By exactness, $ker(j^*) = H_nX$, so $j^* = 0$. Similarly, $im(\partial) = 0$, so $\partial = 0$. However $im(j^*) = 0 = ker(\partial)$, so $H_n(X,A) = 0$. [Werndli, 2009] □

Remark 5.7. As a special case of this result, $H_n(X,X) = 0$, as X is a deformation retract of itself. We are also interested in special case where $A = x$ is a single point, i.e. X is contractible. Then we have $H_nX \cong H_n\bullet = \mathbb{Z}$ and $H_n(X,x) = 0$.

Next, we would like to show how to calculate the homology of a disconnected space.

Lemma 5.8. *Consider the following commutative diagram.*

$$\begin{tikzcd}
B \arrow[rr, "g"] \arrow[dr, "i'"] & & B' \\
& X \arrow[ur, "j"] \arrow[dr, "j'"] & \\
A \arrow[ur, "i"] \arrow[rr, "f"] & & A'
\end{tikzcd}$$

If f and g are isomorphisms, and the diagonals are exact, then there exist isomorphisms $(i+i') : A \oplus B \to X$ and $(j', j) : X \to A' \oplus B'$.

Proof. By commutativity, $ji' = g$, so $ji'g^{-1} = id_{B'}$. Therefore $i'g^{-1}$ is a section of j. By Proposition 4.6, we have an isomorphism

$$(i + i'g^{-1}) : A \oplus B' \to X$$

$$(a,b) \mapsto i(a) + i'g^{-1}(b)$$

Since g^{-1} is an isomorphism, $(i+i') : A \oplus B \to X$ is also an isomorphism in the obvious way.

Similarly, $f^{-1}j'$ is a retraction of i. Again, by Proposition 4.6, we have an isomorphism

$$(f^{-1}j', j) : X \to A \oplus B'$$

$$x \mapsto (f^{-1}j'(x), j(x))$$

which since f^{-1} is an isomorphism also gives an isomorphism

$$(j', j) : X \to A' \oplus B'$$

□

Proposition 5.9. *The inclusions $i_A : A \to A \sqcup B$, $j_B : B \to A \sqcup B$ induce an isomorphism $(i_A^* + i_B^*) : H_n A \oplus H_n B \to H_n(A \sqcup B)$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
(A,\emptyset) \arrow[rr, "f"] \arrow[dr, "i_A"] & & (X,B) \\
& (X,\emptyset) \arrow[ur, "j_B"] \arrow[dr, "j_A"] & \\
(B,\emptyset) \arrow[rr, "g"] \arrow[ur, "i_B"] & & (X,A)
\end{tikzcd}$$

It induces the following commutative diagram in homology, which satisfies the previous lemma, as the diagonals are part of the exact sequences of (X,A) and (X,B) respectively, and f^* and g^* are isomorphisms by the excision axiom.

$$\begin{tikzcd}
H_n A \arrow[rr, "f^*"] \arrow[dr, "i_A^*"] & & H_n(X,B) \\
& H_n X \arrow[ur, "j_B^*"] \arrow[dr, "j_A^*"] & \\
H_n B \arrow[rr, "g^*"] \arrow[ur, "i_B^*"] & & H_n(X,A)
\end{tikzcd}$$

The result then follows from the lemma. [Werndli, 2009] □

5.3 Reduced homology

Results in homology are easy when the associated homology sequence has maps that are isomorphisms or 0-maps. Therefore, if it is possible to "simplify" a homology sequence by for example replacing some groups with 0, this is often advantageous. As we will show, it is often possible to factor out $H_n\bullet$ from a homology sequence of (X,A), resulting in a simpler sequence. Furthermore, this transformation is reversible.

Definition 5.10. For non-empty X, $\tilde{H}_nX := ker(p^* : H_nX \to H_n\bullet)$ where p^* is induced by the initial map $p : X \to \bullet$. $\tilde{H}_nX$ is called the **n-th reduced homology group** of X.

Proposition 5.11. *For non-empty X,*

$$H_nX \cong \tilde{H}_nX \oplus H_n\bullet$$

and for any $x \in X$

$$\tilde{H}_nX \cong H_n(X,x)$$

Proof. Consider the homomorphism $p^* : H_nX \to H_n\bullet$

Note p^* is a retraction of $i^* : H_n\bullet \to H_nX$ induced by the inclusion, as $pi : \bullet \to \bullet$ is trivially the identity, and H_n is a functor. By Corollary 4.7,

$$H_nX \cong ker(p^*) \oplus im(i^*) \oplus \cong \tilde{H}_nX \oplus im(i^*).$$

If we can show i^* is injective, we have our first equality. By the naturality of ∂, the following diagram commutes:

$$\begin{array}{ccc} H_{n+1}(X,\bullet) & \xrightarrow{\partial} & H_n\bullet \\ \big\downarrow{\scriptstyle p^*} & & \big\downarrow{\scriptstyle p^*} \\ H_{n+1}(\bullet,\bullet) & \xrightarrow{\partial} & H_n\bullet \end{array}$$

Note $p^* : H_n\bullet \to H_n\bullet$ is the identity, as $p : \bullet \to \bullet$ is the identity. Therefore ∂ factors through $H_{n+1}(\bullet,\bullet) = 0$. By the exactness of the homology sequence for $(X,\bullet)$, $0 = im(\partial) = ker(i^*)$, so i^* is injective as required, and

$$H_nX \cong \tilde{H}_nX \oplus H_n\bullet.$$

For the second isomorphism, note that the long exact sequence of $(X,\bullet)$

$$\ldots \longrightarrow H_{n+1}(X,\bullet) \xrightarrow{0} H_n\bullet \xrightarrow{i^*} H_nX \xrightarrow{j^*} H_n(X,\bullet) \xrightarrow{0} \ldots$$

gives a short exact sequence

$$0 \longrightarrow H_n\bullet \xrightarrow{i^*} H_nX \xrightarrow{j^*} H_n(X,\bullet) \longrightarrow 0$$

Since $p^* : H_nX \to H_n\bullet$ is a retraction of i^*, Proposition 4.6 gives that $H_nX \cong H_n(X,\bullet) \oplus H_n\bullet$. Since this is isomorphic to $\tilde{H}_nX \oplus H_n\bullet$,

$$H_n(X,x) \cong \tilde{H}_nX.$$

$\square$

Corollary 5.12. *If X is contractible to $x \in X$, then $\tilde{H}_n(X) = H_n(X,x) = 0$ by the previous result and Remark 5.7.*

As we will see, the reduced homology group also come with a long exact sequence. To define it, we will need a lemma. Consider an admissible category C and a triple $(X,A,B) \in \mathbf{Top}_3$ such that $(X,A),(X,B),(A,B) \in C$. The homology sequences of these three pairs, which will be labelled $(1),(3),(4)$ respectively, form the following braid diagram:

$$
\begin{tikzcd}
\cdot \arrow[dr, "(1)"] & & & & & & \cdot \\
& H_{n+1}(X,A) \arrow[dr] \arrow[rr, bend left, "\partial"] & & H_n(A,B) \arrow[dr, "i^*"] \arrow[rr, bend left] & & H_{n-1}B \arrow[ur] \arrow[dr] & \\
\cdot \arrow[ur, "(2)"] \arrow[dr, "(3)"] & & H_nA \arrow[ur] \arrow[dr] & & H_n(X,B) \arrow[ur] \arrow[dr, "j^*"] & & \cdot \\
& H_nB \arrow[ur] \arrow[rr, bend right] & & H_nX \arrow[ur] \arrow[rr, bend right] & & H_n(X,A) \arrow[ur] \arrow[dr] & \\
\cdot \arrow[ur, "(4)"] & & & & & & \cdot
\end{tikzcd}
$$

The sequences $(1),(3),(4)$ commute with each other. The sequence (2) is called the **long exact homology sequence** for the triple (X,A,B). The map $\partial : H_{n+1}(X,A) \to H_n(A,B)$ is defined so the diagram commutes, and all other maps in the sequence are induced by the canonical inclusions, which it is easy to see also commute with the diagram, either by looking at inclusions or the naturality of ∂.

Proposition 5.13. *For a triple $(X,A,B) \in \mathbf{Top}_3$, the sequence*

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_n(A,B) \xrightarrow{i^*} H_n(X,B) \xrightarrow{j^*} H_n(X,A) \xrightarrow{\partial} \dots$$

is a long exact sequence.

Proof. We first show that (2) is a chain complex. By commutativity, the compositions $i\partial$ and ∂j factor through two consecutive maps in a long exact sequence, and are hence 0. For j^*i^*, note that i and j factor through (A,A):

$$\begin{tikzcd}
(A,B) \arrow[d, "i"] \arrow[dr, "k"] & \\
(X,B) \arrow[d, "j"] & (A,A) \arrow[dl, "h"] \\
(X,A) &
\end{tikzcd}$$

In homology, this means j^*i^* factors through $H_n(A,A) = 0$, so $j^*i^* = 0$.

$$\begin{tikzcd}
H_n(A,B) \arrow[d, "i^*"] \arrow[dr, "0"] & \\
H_n(X,B) \arrow[d, "j^*"] & 0 \arrow[dl, "0"] \\
H_n(X,A) &
\end{tikzcd}$$

The result then follows by an application of the Braid Lemma (Lemma 4.8). [Werndli, 2009] $\square$

Corollary 5.14. *The sequence*

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} \tilde{H}_n A \xrightarrow{i^*} \tilde{H}_n X \xrightarrow{j^*} H_n(X,A) \xrightarrow{\partial} \dots$$

is an exact sequence, called the ***reduced homology sequence*** *of* (X,A).

Proof. This is simply the long exact sequence for a triple (X,A,x), where $x \in A$:

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H(A,x) \xrightarrow{i^*} H(X,x) \xrightarrow{j^*} H_n(X,A) \xrightarrow{\partial} \dots$$

Using the isomorphisms $\tilde{H}_n X \cong H_n(X,x)$ and defining the homomorphisms in the reduced sequence as appropriate gives the result. For example, we define $i^* : \tilde{H}_A \to \tilde{H}_n X$ as the composition

$$\tilde{H}_n A \xrightarrow{\cong} H_n(A,x) \xrightarrow{i^*} H_n(X,x) \xrightarrow{\cong} \tilde{H}_n X.$$

$\square$

5.4 Homology of S^n

We will now perform a calculation of the groups $H_k S^n$ from the axioms. Our approach will be an adaptation of that taken in [Werndli, 2009]. We write S^n for the n-sphere,

$$S^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\},$$

and D^n for the closed n-disk,

$$D^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}.$$

Lemma 5.15. *For every $n \in \mathbb{Z}$,*

$$\tilde{H}_k S^n \cong \tilde{H}_{k-1} S^{n-1}$$

Proof. Consider first the pair (D^n, S^{n-1}), where S^{n-1} is the boundary of D^n. Since D^n is contractible, the reduced homology sequence reads:

$$0 \longrightarrow H_k(D^n, S^{n-1}) \longrightarrow \tilde{H}_{k-1} S^{n-1} \longrightarrow 0$$

This gives an isomorphism

$$H_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1} S^{n-1}$$

Additionally, we have the pair (S^n, D^n), where D^n is identified as the closed lower hemisphere of S^n. Since $\tilde{H}_k D^n = 0$, the reduced homology sequence for this pair reads

$$0 \longrightarrow \tilde{H}_k S^n \longrightarrow H_k(S^n, D^n) \longrightarrow 0$$

which gives an isomorphism $\tilde{H}_k S^k \cong H_k(S^n, D^n)$.

The open disk $D_{1/2}$ of radius $1/2$ is a subset of D^n which can be excised from the pair (S^n, D^n). The resulting space deformation retracts to the pair (D^n, S^{n-1}), where D^n is the upper hemisphere and S^{n-1} its boundary. By the excision axiom,

$$H_k(S^n, D^n) \cong H_k(D^n, S^{n-1})$$

All in all,

$$\tilde{H}_k S^n \cong H_k(S^n, D^n) \cong H_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1} S^{n-1}.$$

□

Proposition 5.16. *For $n > 0$,*

$$H_k S^n = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We identify $S^0 = \bullet \sqcup \bullet$. By Proposition 5.9,

$$H_k S^0 = H_k \bullet \oplus H_k \bullet = \begin{cases} \mathbb{Z}\times\mathbb{Z} & k=0 \\ 0 & \text{otherwise} \end{cases}$$

It follows that $\tilde{H}_0 S^0 = \mathbb{Z}$ and $\tilde{H}_k S^0 = 0$ when $k \neq 0$. By Lemma 5.15,

$$\tilde{H}_k S^n = \tilde{H}_{k-n} S^0 = \begin{cases} \mathbb{Z} & k=n \\ 0 & \text{otherwise} \end{cases}$$

By Proposition 5.11,

$$H_k S^n = \tilde{H}_k S^n \oplus H_k \bullet = \begin{cases} \mathbb{Z} & k=0,n \\ 0 & \text{otherwise} \end{cases}$$

[Werndli, 2009] □

6 The Mayer-Vietoris Sequence

6.1 Statement and proof

In the previous section, we relied heavily on identifying homomorphisms in a long exact sequence as either isomorphisms of the 0-map. This is not the case in general, and it can feel like the axioms give very little help with identifying homology groups and their maps when they are not trivial. However, in many cases we can cover a space X by the interiors two spaces A and B whose homology groups we know. In this case we say that (A,B) is a Mayer-Vietoris cover of X. The excision axiom gives us a very convenient method for relating the homology groups of X to the homology groups of its Mayer-Vietoris cover:

Theorem 6.1. *Let A and B be closed subsets of X whose interiors cover X. Suppose furthermore that*

$$\overline{A\setminus(A\cap B)} \cap \overline{B\setminus(A\cap B)} = \emptyset.$$

Then there is a long exact sequence:

$$\dots \longrightarrow H_{n+1}X \xrightarrow{\partial_{n+1}} H_n(A\cap B) \xrightarrow{(i^*,j^*)} H_nA \oplus H_nB \xrightarrow{k^*-l^*} H_nX \longrightarrow \dots$$

called the ***Mayer-Vietoris sequence*** *of* $(X;A,B)$*. The maps* i^*, j^*, k^*, l^* *are induced by the inclusions:*

$$\begin{tikzcd}
A \arrow[r, "k"] & X \\
A \cap B \arrow[u, "i"] \arrow[r, "j"] & B \arrow[u, "l"]
\end{tikzcd}$$

Remark 6.2. The definition is set up such that the inclusions $(B, A\cap B) \to (X,A)$ and $(A, A\cap B) \to (X,B)$ are excisions, and hence induce isomorphisms in homology. In the first case, $A \setminus A \cap B$ is open since its complement, is closed. This set satisfies excision, since $\overline{A \setminus (A\cap B)} \subseteq (\overline{B \setminus (A\cap B)})^{\mathsf{c}} = int(A)$ by assumption. A similar argument shows that the second inclusion is an excision.

To prove this theorem, we follow the approach laid out in [Eilenberg and Steenrod, 1952], first proving a lemma.

Lemma 6.3. *Consider the following diagram.*

$$\begin{tikzcd}
 & G_0 \arrow[dl, "l_1"'] \arrow[dr, "l_2"] \arrow[dd, "i_0"] & \\
G_1' & & G_2' \\
 & G \arrow[ul, "j_1"] \arrow[ur, "j_2"'] \arrow[dd, "j_0"] & \\
G_2 \arrow[uu, "k_1"] \arrow[ur, "i_2"] \arrow[dr, "h_1"] & & G_1 \arrow[uu, "k_2"'] \arrow[ul, "i_1"'] \arrow[dl, "h_2"] \\
 & G_0' &
\end{tikzcd}$$

Suppose each triangle commutes, k_1, k_2 *are isomorphisms, and the diagonals are exact. Then* $h_1 k_1^{-1} l_1 = -h_2 k_2^{-1} l_2$.

Proof. By Lemma 5.8, $(i_1 + i_2) : G_1 \oplus G_2 \to G$ is an isomorphism, and so is $(j_1, j_2) : G \to (G_1', G_2')$. What does this tell us? Every $x \in G$ is identified with a unique $(j_1(x), j_2(x)) \in G_1' \oplus G_2'$, and there are unique $x_2, x_1 \in G_2 \oplus G_1$ such that $i_2(x_2) + i_1(x_1) = x$. These two representations are related by an isomorphism (k_1^{-1}, k_2^{-1}). It is not immediately obvious that this isomorphism maps $(j_1(x), j_2(x))$ to (x_2, x_1). However, by commutativity, k_1^{-1} is the inverse of $j_1 i_2$, and $j_1(i_2(x_2)) = j_1(x)$, so k_1^{-1} maps $j_1(x)$ to x_2. A similar argument shows k_2^{-1} maps $j_2(x)$ to x_1.

It follows that

$$x = (i_2 + i_1)(k_1^{-1}, k_2^{-1})(j_1, j_2)x = i_2 k_1^{-1} j_1(x) + i_1 k_2^{-1} j_2(x).$$

In particular, letting $g \in G_0$ and $x = i_0(g)$,

$$i_0(g) = i_2 k_1^{-1} j_1 i_0(g) + i_1 k_2^{-1} j_2 i_0(g)$$

Applying j_0 to both sides and noting that $j_0 i_0(g) = 0$ by exactness, we get that

$$j_0 i_2 k_1^{-1} j_1 i_0(g) = -j_0 i_1 k_2^{-1} j_2 i_0(g).$$

By commutativity in the diagram, this also gives

$$h_1 k_1^{-1} l_1(g) = -h_2 k_2^{-1} l_2.$$

[Eilenberg and Steenrod, 1952] □

We will use this lemma to prove the main result.

Proof of Theorem 6.1. The proof will use a certain diagram, where all maps are induced by canonical inclusions, except for the boundary map d (which is the boundary map of the homology sequence of $(X, A \cap B)$), and the maps ∂_i (which are defined to commute in their triangle).

$$\begin{tikzcd}
 & & H_n(A \cap B) \arrow[dll, "j^*"'] \arrow[drr, "i^*"] \arrow[dd, "m^*"] & & \\
H_n B \arrow[drr, "l^*"] & & & & H_n A \arrow[dll, "k^*"] \\
 & & H_n X \arrow[dll, "l_1^*"'] \arrow[drr, "l_2^*"] \arrow[dd, "n^*"] & & \\
H_n(X,A) & & & & H_n(X,B) \\
 & & H_n(X, A \cap B) \arrow[ull, "j_1^*"] \arrow[urr, "j_2^*"'] \arrow[dd, "d"] & & \\
H_n(B, A \cap B) \arrow[uu, "k_1^*"] \arrow[urr, "i_2^*"] \arrow[drr, "\partial_1"] & & & & H_n(A, A \cap B) \arrow[uu, "k_2^*"'] \arrow[ull, "i_1^*"'] \arrow[dll, "\partial_2"] \\
 & & H_{n-1}(A \cap B) & &
\end{tikzcd}$$

First we need to show that the lower diagram satisfies the requirements of Lemma 6.3. The diagonals are exact because they come from homology sequences of triples (or pairs, in the case of the vertical sequence). The upper four triangles commute because they are induced by commutative triangles of inclusions, and the lower two triangles commute by definition. Finally k_1 and k_2 are isomorphisms by Remark 6.2. We can therefore define $\partial : H_nX \to H_{n-1}(A \cap B)$ as the composition $\partial_1 k_1^{*-1} l_1^* = -\partial_2 k_2^{*-1} l_2^*$. The remainder of the proof is showing that the Mayer-Vietoris sequence is exact. This results to a diagram-chasing challenge. We will show that $H_nA \oplus H_nB \to H_nX \to H_{n-1}(A \cap B)$ is exact, and refer to [Eilenberg and Steenrod, 1952] for the rest of the pairs.

We start by showing $\partial(k^* - l^*) = 0$. This follows from the two decompositions of ∂:

$$\partial k^* - \partial l^* = \partial_1 k_1^{*-1} l_1^* k^* + \partial_2 k_2^{*-1} l_2^* l^*.$$

Both summands contain two consecutive maps in an exact sequence, and are hence 0.

Next, we show that $ker(\partial) \subseteq im(k^* - l^*)$. Suppose $x \in H_nX$ is such that $\partial(x) = 0$. Then

$$0 = \partial_1 k_1^{*-1} l_1^*(x) = d i_2^* k_1^{*-1} l_1^*(x).$$

We define $y := i_2^* k_1^{*-1} l_1^*(x) \in ker(d)$. Note that

$$(j_1^*(y), j_2^*(y)) = (l_1^*(x), 0)$$

where the first component follows from $j_1^* i_2^* k_1^{*-1} = id$, and the second component from $j_2^* i_2^* = 0$ by exactness. By exactness, $ker(d) = im(n^*)$, so there exists some $x_1 \in H_nX$ such that $n^*(x_1) = y$. By what we have just shown,

$$l_1^*(x_1) = j_1^* n^*(x_1) = j_1^*(y) = l_1^*(x),$$

and

$$l_2^*(x_1) = j_2^* n^*(x_2) = j_2^*(y) = 0.$$

Now write $x = (x - x_1) + x_1$. Since $l_1^*(x - x_1) = l_1^*(x) - l_1^*(x_1) = 0$, $(x - x_1) \in ker(l_1^*) = im(k^*)$ so $\exists a \in H_nA$ such that $k^*(a) = (x - x_1)$. Additionally, $x_1 \in ker(l_2^*) = im(l^*)$, so $\exists b \in H_nB$ such that $l^*(b) = x_1$. Therefore

$$x = (x - x_1) + x_1 = k^*(a) + l^*(b) = k^*(a) - l^*(-b),$$

so $x \in im(k^* - l^*)$.

We have shown $im(k^* - l^*) \subseteq ker(\partial)$ and $ker(\partial) \subseteq im(k^* - l^*)$, so they are in fact equal. [Eilenberg and Steenrod, 1952] □

We would like to also show, in a similar vein to regular homology sequences, that maps between Mayer-Vietoris covers induce maps between Mayer-Vietoirs sequences. We say that $f:(X;X_1,X_2)\to(Y;Y_1,Y_2)$ is a **map between the Mayer-Vietoris covers of X and Y** if $f(X_1)\subseteq Y_1$ and $f(X_2)\subseteq Y_2$. Note this necessarily implies $f(X_1\cap X_2)\subseteq Y_1\cap Y_2$.

Proposition 6.4. *Let $f:(X;X_1,X_2)\to(Y;Y_1,Y_2)$ be map between the Mayer-Vietoris covers of X and Y. Then f induces maps between their Mayer-Vietoris sequences, which commute with the sequence.*

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}X \arrow[r] \arrow[d, "f^*"] & H_n(X_1\cap X_2) \arrow[r] \arrow[d, "f^*"] & H_nX_1\oplus H_nX_2 \arrow[r] \arrow[d, "f^*"] & H_nX \arrow[r] \arrow[d, "f^*"] & {} \\
{} \arrow[r] & H_{n+1}Y \arrow[r] & H_n(Y_1\cap Y_2) \arrow[r] & H_nY_1\oplus H_nY_2 \arrow[r] & H_nY \arrow[r] & {}
\end{tikzcd}$$

The map $f^*:(X_1\cap X_2)\to(Y_1\cap Y_2)$ is induced by the restriction $f|_{X_1\cap X_2}$, and $f^*:H_nX_1\oplus H_nX_2\to H_nY_1\oplus H_nY_2$ is induced by $(f|_{X_1},f|_{X_2})$.

Proof. We need to confirm three conditions.

(a) $\big(f^*(i^*,j^*)=(i^*,j^*)f^*\big)$. This trivially holds because f commutes with (i,j) in the diagram

$$\begin{tikzcd}
(X_1\cap X_2) \arrow[r, "(i,j)"] \arrow[d, "f|_{X_1\cap X_2}"] & X_1\oplus X_2 \arrow[d, "(f|_{X_1},f|_{X_2})"] \\
(Y_1\cap Y_2) \arrow[r, "(i,j)"] & Y_1\oplus Y_2
\end{tikzcd}$$

(b) $\big(f^*(k^*-l^*)=(k^*-l^*)f^*\big)$. This will follow from showing that $f^*k^*=k^*f^*$ and $f^*l^*=l^*f^*$. However this trivially holds because f commutes with k in the diagram:

$$\begin{tikzcd}
X_1 \arrow[r, "k"] \arrow[d, "f|_{X_1}"] & X \arrow[d, "f"] \\
Y_1 \arrow[r, "k"] & Y
\end{tikzcd}$$

and commutes with l in a similar diagram.

(c) $\big(f^*\partial=\partial f^*.\big)$ Note that f induces a map between the homology sequences of $(X,X_1\cap X_2)$ and $(Y,Y_1\cap Y_2)$. By naturality of the boundary map d (the same boundary map from the definition of ∂), the following diagram commutes:

$$\begin{array}{ccc} H_n(X, X_1 \cap X_2) & \xrightarrow{f^*} & H_n(Y, Y_1 \cap Y_2) \\ \Big\downarrow d & & \Big\downarrow d \\ H_{n-1}(X_1 \cap X_2) & \xrightarrow{f^*} & H_{n-1}(Y_1 \cap Y_2) \end{array}$$

We therefore have that

$$f^* \partial = f^* d i_2^* k_1^{*-1} l_1^* = d f^* i_2^* k_1^{*-1} l_1^*.$$

This reduces the problem to showing

$$f^* i_2^* k_1^{*-1} l_1^* = i_2^* k_1^{*-1} l_1^* f^*.$$

Just as in (a) and (b), we can see that f commutes with i_2^*, j_1^* and l_1^*, by noting f commutes with i_2, j_1, l_1. To see that f^* commutes with k_1^{*-1}, note that f^* commutes with $k_1^* = j_1^* i_2^*$, as f commutes with j_1 and i_2. We therefore have $f^* k^* = k^* f^*$. Since k^* is an isomorphism, we can left-multiply and right-multiply by k^{*-1} on both sides to get $k^{*-1} f^* = f^* k^{*-1}$. It follows that

$$f^* i_2^* k_1^{*-1} l_1^* = i_2^* k_1^{*-1} l_1^* f^*,$$

as required.

□

Finally, as with homology sequences of pairs, we can safely remove copies of $H_n\bullet$ (or $H_n \bullet \oplus H_n \bullet$ for $H_n A \oplus H_n B$) from the Mayer-Vietoris sequence to get a sequence in reduced homology.

Proposition 6.5 (Reduced Mayer-Vietoris Sequence)**.** *If $A \cap B \neq \emptyset$, there is a long exact sequence in reduced homology, called the **reduced Mayer-Vietoris sequence**:*

$$\ldots \longrightarrow \tilde{H}_{n+1} X \xrightarrow{\partial_{n+1}} \tilde{H}_n(A \cap B) \xrightarrow{(i^*, j^*)} \tilde{H}_n A \oplus H_n B \xrightarrow{k^* - l^*} \tilde{H}_n X \longrightarrow \ldots$$

Proof. Omitted. See [Spanier, 1981]. □

6.2 Examples

In this section we show a few examples of how convenient the Mayer-Vietoris sequence can be for calculating homology groups.

Example 6.6 (One-point wedges of circles)**.** Consider the figure eight $8 = S^1 \wedge S^1$ where $\wedge$ is the wedge product identifying the south-pole S of the top circle with the north-pole N of the lower circle. Let U_N be a small open neighbourhood of N, the north-pole of the top circle.We can cover 8 by $A = 8 \setminus U_N \simeq S^1$, and $B = D^+ \simeq \bullet$, the upper half circle of the top circle. Then $A \cap B = D^+ \setminus U_N \simeq \bullet \sqcup \bullet$. It is easy to see that this cover satisfies Mayer-Vietoris. Now note that

$$H_n(\bullet \sqcup \bullet) = \begin{cases} \mathbb{Z} \times \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

by Proposition 5.9. Since $\tilde{H}_0 S^1 \oplus \tilde{H}_0 \bullet = 0$, the reduced Mayer-Vietoris sequence reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{k^*} \tilde{H}_1 8 \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \longrightarrow \tilde{H}_0 8 \longrightarrow 0$$

Therefore $\tilde{H}_0 8 = 0$, so by Proposition 5.11, $H_0 8 = \mathbb{Z}$. Furthermore, note that the fold $f : 8 \to S^1$, which is the identity on the lower circle, and the reflection on the upper circle, is a retraction of the inclusion $k : S^1 \to 8$ of the lower circle. Since H_n is a functor, f^* is a retraction of k^*. By Proposition 4.7, $H_1 8 \cong \mathbb{Z} \times \mathbb{Z}$. The rest of the Mayer-Vietoris sequence easily gives $H_n 8 = 0$ whenever $n \neq 0, 1$.

This argument can easily be extended by induction to show the wedge of m circles (wedged at the same point) $\bigwedge_m S^1$ has homology groups $H_1 \bigwedge_m S^1 = \mathbb{Z}^m$ and $H_0 \bigwedge_m S^1 = \mathbb{Z}$. Let $A = \bigwedge_m S^1 \setminus U_x \simeq \bigwedge_{m-1} S^1$, where U_x is a small open neighbourhood of some point in one of the circles (not the wedge point), and $B = I \simeq \bullet$, a small interval of the punctured circle containing the puncture in its interior and not intersecting the wedge point. Then A and B easily satisfy Mayer-Vietoris, and $A \cap B \simeq \bullet \sqcup \bullet$. By induction, the reduced Mayer-Vietoris sequence reads:

$$0 \longrightarrow \mathbb{Z}^{m-1} \xrightarrow{k^*} \tilde{H}_1 8 \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \longrightarrow \tilde{H}_0 8 \longrightarrow 0$$

This shows $\tilde{H}_0 \bigwedge_m S^1 = 0$. Again $k : S^1 \to \bigwedge_m S^1$ admits a retraction $r : \bigwedge_m S^1 \to S^1$ which is the identity on every circle except the one that had U_x removed, and which folds final circle onto any other circle. Therefore Proposition 4.7 gives that $H_1 \bigwedge_m S^1 \cong \mathbb{Z}^m$.

Example 6.7. Let X be the quotient of the 2-sphere which identifies the North and South poles: $S^2/\sim$, $S \sim N$. The following is easily seen to be a Mayer-Vietoris cover: $A = (D^2_{2\varepsilon,+} \cup D^2_{2\varepsilon,-})/\sim \simeq \bullet$, two small closed disks in the north and south hemispheres with identified centers $S \sim N$, and $B = S^2 \setminus (int(D^2_{\varepsilon,+}) \cup$

$int(D^2_{\varepsilon,-}))/\sim\simeq S^1$, a large closed belt of S^2. For this cover, $A\cap B\cong S^1\sqcup S^1$, as in the MISSING image. The deformation retraction $B\simeq S^1$ composes with the inclusion $i: A\cap B\to B$ such that $i|_A = i|_B = id_{S^1}$. We expect that inclusion induces the map $i^*:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},(x,y)\mapsto x+y$, and indeed this can be shown from the following commutative diagram, where the inclusions i_1,i_2 are as one expects:

$$\begin{tikzcd}
 & S^1 & \\
 & S^1\sqcup S^1 \arrow[u, "i"] & \\
S^1 \arrow[uur, "id"] \arrow[ur, "i_1"'] & & S^1 \arrow[uul, "id"'] \arrow[ul, "i_2"]
\end{tikzcd}$$

It induces the following commutative map in homology (for $n=0,1$), from which the formula for i^* can be easily read.

$$\begin{tikzcd}
 & \mathbb{Z} & \\
 & \mathbb{Z}\times\mathbb{Z} \arrow[u, "i"] & \\
\mathbb{Z} \arrow[uur, "x"] \arrow[ur, "{(x,0)}"'] & & \mathbb{Z} \arrow[uul, "y"'] \arrow[ul, "{(y,0)}"]
\end{tikzcd}$$

The fact that $i_1^*(x)=(x,0)$ and $i_2^*(y)=(0,y)$ follows from Proposition 5.9.

The Mayer-Vietoris sequence reads as follows:

$$0\longrightarrow H_2X\xrightarrow{\partial_2}\mathbb{Z}\times\mathbb{Z}\xrightarrow{i^*=x+y}\mathbb{Z}\xrightarrow{k^*}H_1X\xrightarrow{\partial_1}\dots$$

Now $ker(i^*) =<(x,-x)>\cong\mathbb{Z}=im(\partial_2)$ by exactness. Since ∂_2 is injective, $H_2X\cong\mathbb{Z}$. Additionally, since i^* is surjective, $k^*=0$. Therefore, the rest of the sequence reads:

$$0\longrightarrow H_1X\xrightarrow{\partial_1}\mathbb{Z}\times\mathbb{Z}\xrightarrow{i^*=x+y}\mathbb{Z}\xrightarrow{k^*}H_0X\longrightarrow 0$$

This diagram is similar to the previous diagram, giving us $H_1X=\mathbb{Z}$ and $H_0X=0$. $H_nX=0$ for all other values of x.

7 Degree maps

The only group homomorphisms $g:\mathbb{Z}\to\mathbb{Z}$ are multiplication by an integer m, called the degree of g. For maps $f:S^n\to S^n, n>0$, we can therefore define the

degree $deg(f)$ as the degree of the induced map $f^* : \mathbb{Z} \to \mathbb{Z}$. It is clear from the definition that $deg(id) = 1$, as $id^* = id$. Additionally, if $f \simeq g$, then $f^* = g^*$, so $deg(f) = deg(g)$. Furthermore, $deg(fg) = deg(f)deg(g)$, as $(fg)^* = f^*g^*$. We are interested in some special cases of f.

Proposition 7.1. *$deg(r_i) = -1$ where $r_i : S^n \to S^n$ is the reflection in the $i-th$ coordinate:*

$$(x_1, x_2, \ldots, x_i, \ldots x_{n+1}) \mapsto (x_1, x_2, \ldots, -x_i, \ldots x_{n+1})$$

Proof. By composing with a rotation, we can assume r_i is the reflection in the $(n+1)$th degree which maps the north-pole N to the south-pole S, keeping the equator S^{n-1} fixed. We denote this map by r. Consider the following Mayer-Vietoris cover of S^n: $A = S^n \setminus D^n_+ \simeq \bullet$, $B = S^n \setminus D^n_- \simeq \bullet$ and $A \cap B \simeq S^{n-1}$. r gives a map between the Mayer-Vietoris sequence of $(S^n; A, B)$ and $(S^n; B, A)$. For $n > 0$, this gives the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_nS^n \arrow[r, "\partial"] \arrow[d, "r^*"] & H_{n-1}S^{n-1} \arrow[r] \arrow[d, "id"] & \ldots \\
0 \arrow[r] & H_nS^n \arrow[r, "\partial'"] & H_{n-1}S^{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Recall that $\partial = \partial_1 k_1^{*-1} l_1^* = -\partial_2 k_2^{*-1} l_2^*$ in the following diagram.

$$\begin{tikzcd}
 & & H_nX \arrow[dll, "l_1^*"'] \arrow[drr, "l_2^*"] \arrow[dd, "n^*"] & & \\
H_n(X,A) & & & & H_n(X,B) \\
 & & H_n(X,A\cap B) \arrow[ull, "j_1^*"] \arrow[urr, "j_2^*"'] \arrow[dd, "d"] & & \\
H_n(B,A\cap B) \arrow[uu, "k_1^*"] \arrow[urr, "i_2^*"] \arrow[drr, "\partial_1"] & & & & H_n(A,A\cap B) \arrow[uu, "k_2^*"'] \arrow[ull, "i_1^*"'] \arrow[dll, "\partial_2"] \\
 & & H_{n-1}(A\cap B) & &
\end{tikzcd}$$

Swapping the order of A and B corresponds to a horizontal reflection in the diagram, which therefore changes the sign of the boundary map. It follows that $\partial = -\partial'$. By commutativity, $r^* = -id$, so $deg(r) = -1$.

□

Corollary 7.2. *$deg(-id) = (-1)^{n+1}$ where $(-id) : S^n \to S^n$ is the antipodal map.*

Proof. Since $(-id)$ is the composition $(-id) = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$,

$$deg(-id) = deg(r_1)deg(r_2)\ldots deg(r_n+1) = (-1)^{n+1}.$$

[Hatcher, 2002] □

Corollary 7.3. *If $f : S^n \to S^n$ has no fixed points, then $deg(f) = (-1)^{n+1}$*

Proof. Since $f(x) \neq x$, the line segment from $f(x)$ to $-x$ does not pass through 0. We can therefore define a homotopy $f(x) \simeq (-id)$, which takes each $f(x)$ to x along an arc segment. Therefore $deg(f(x)) = deg(-id) = (-1)^{n+1}$.[Hatcher, 2002] □

Remark 7.4. Note that the antipodal map commutes with the projection $p : S^n \to \mathbb{RP}^n$:

$$\begin{array}{ccc} S^n & \xrightarrow{p} & \mathbb{RP}^n \\ \downarrow{\scriptstyle -id} & & \downarrow{\scriptstyle id} \\ S^n & \xrightarrow{p} & \mathbb{RP}^n \end{array}$$

When n is odd, this gives rise to a commutative map in homology

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{p^*} & H_n\mathbb{RP}^n \\ \downarrow{\scriptstyle -id} & & \downarrow{\scriptstyle id} \\ \mathbb{Z} & \xrightarrow{p^*} & H_n\mathbb{RP}^n \end{array}$$

But this means $p^* = -p^*$, so $p^* = 0$. This is the first hint that there is something fundamentally different about $\mathbb{RP}^n$ when n is odd and when n is even.

Next, we show how the famous Brouwer's fixed point theorem can be proven from degree theory.

Theorem 7.5 (Brouwer's fixed-point theorem)**.** *Every continuous $f : D^n \to D^n$ has a fixed point.*

Proof. Suppose f has no fixed points. Define the map $g : S^n \to S^n$ as the composition

$$S^n \xrightarrow{q} D^n \xrightarrow{f} D^n \xrightarrow{i} S^n$$

where q is the quotient map of the identification $(x_1,\ldots,x_n) \sim (y_1,\ldots,-x_n)$, i.e. the map that folds the lower hemisphere onto the upper hemisphere, where we have identified the codomain as D^n. i is the inclusion identifying D^n as the upper hemisphere of S^n. Note that g also has no fixed points. By Corollary 7.3, $deg(g) = (-1)^{n+1}$. We could alternatively have made the identification $i' : D^n \to S^n$ with the lower hemisphere, and g would still have no fixed points. However, this corresponds to composing g with the reflection $r_{(n+1)}$, which has degree -1. This implies

$$(-1)^{n+1} = deg(rg) = deg(r)deg(g) = (-1)^n$$

which is a contradiction. □

There is a convenient way for calculating degrees of maps $f : S^n \to S^n$. Suppose that f has the property that for some $x \in S^n$, $f^{-1}(x)$ is a finite set $\{y_i\}_{i \in J}$. We define the **local degree** of f, $deg(f)|y_i$ at y_i as the degree of the map $f : U_i \setminus \{y_i\} \to V \setminus \{x\}$, where U_i are disjoint open neighbourhoods of respectively the y_i's and V is an open neighbourhood they are all mapped into. By the metric space structure of S^n we may take the U_i's to be sufficiently small n-disks, and V some small n-disk containing their images. This shows that $f : U_i \setminus \{y_i\} \to V \setminus \{x\}$ gives a map $f : S^{n-1} \to S^{n-1}$ as desired.

Proposition 7.6. *If f is as proposed, then*

$$deg(f) = \sum_{i \in J} deg(f)|y_i.$$

Proof. Omitted. See [Hatcher, 2002] □

8 Cellular homology

8.1 CW-complexes

A common class of spaces is the class of **cell complexes** or **CW complexes**, which are constructed by iteratively gluing copies of D^n in a manner defined by a "gluing map" on the boundary $\partial D^n \cong S^{n-1}$. As will become apparent, CW-complexes have the right structure for establishing a very practical way of calculating their homology groups.

Definition 8.1. A CW-complex X is the union of a sequence of spaces X^n, called the **n-skeleta** of X^n defined as follows: X^0 is a discrete set, and for each X^{n-1}, X^n is obtained by gluing copies of D^n_α, called n-cells, along a **gluing map** $\phi_\alpha : S^{n-1}_\alpha \to X^{n-1}$ defined on the boundary of D^n_α. Explicitly, X^n is the quotient of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha / \sim$ under the identification $x \sim \phi_\alpha(x)$ for $x \in S^{n-1}_\alpha \subset D^n_\alpha$. If $X = X^n$ for some $n \in \mathbb{N} \cup \{0\}$, then X is called a **finite cell complex**. Otherwise, X is given the weak topology: a subset $A \subset X$ is open iff A is open in X^n for every $n \in \mathbb{N} \cup \{0\}$.

Many familiar spaces naturally arise as cell complexes.

Example 8.2. S^n is a CW-complex with 1 0-cell and 1 n-cell when $n > 0$.

Example 8.3. $\mathbb{RP}^2$ is a CW-complex with 1 0-cell, 1 1-cell and 1 2-cell, and where the gluing map $\phi : S^1 \to X^1 = \mathbb{RP}^1$ is the projection $z \mapsto [z]$. Iteratively, $\mathbb{RP}^n$ can be understood as a copy of D^n glued to a copy of $\mathbb{RP}^{n-1}$ via the projection map $z \mapsto [z]$ along the boundary ∂D^n. So $\mathbb{RP}^n$ as a k-cell for $0 \leq k \leq n$.

Example 8.4. The torus T^2 has a cellular structure composed of 1 0-cell, 2 1-cells and 1 2-cell. We can use the familiar identification of T^2 as a quotient of the square, where the two 1-cells a and b have their endpoints glued to a single 0-cell, as the corners are identified to a single point. The gluing map $\phi : S^1 \to X^1$ is then the concatenation $a \cdot b \cdot -a \cdot -b$, where we are thinking of a and b as paths. The Klein bottle, which also arises as a quotient of the square, can be defined in a similar way.

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Remark 8.5. The cell structure of a CW-complex need not be unique. For example an alternative cell structure of S^2 is one 0-cell, 1 1-cell, and two 2-cells (the upper and lower hemispheres) both glued onto the hemisphere via the identity map on their boundaries.

8.2 Cellular homology groups

We will show that the homology groups of CW-complexes can be identified as the homology groups of a certain chain complex of relative homology group $H_n(X^n, X^{n-1})$. This method of "taking homology twice" trims away some fat, making calculations easier. By "homology group of a chain complex" we mean the following:

Definition 8.6. Let the following diagram be a chain complex of abelian groups.

$$\dots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots$$

Then the abelian group $\tilde{H}_n(A_n) := Ker(d_n)/Im(d_{n+1})$ is called the **homology group of the chain complex**.

Remark 8.7. Note both $Ker(d_n)$ and $Im(d_{n+1})$ are abelian subgroups, and $Im(d_{n+1}) \subseteq Ker(d_n)$ as we are dealing with a chain complex. By [ProofWiki, 2019], a subgroup of an abelian group is normal, so $Ker(d_n)/Im(d_{n+1})$ is well-defined, and this group is also abelian.

We will restrict our study to finite cell complexes, but the reader is invited to confirm that the established results also hold for general cell complexes [Hatcher, 2002]. The proof is adapted from [Hatcher, 2002], with axiomatic replacements to references to singular homology. We first establish some basic results

Lemma 8.8. *The following hold for finite cell complexes X:*

(i) $H_n(X^n, X^{n-1}) = \mathbb{Z}^m$ *where m is the number of n-cells of X and $n \in \mathbb{N}^*$.*

(ii) $H_m(X^n) = 0$ *for $m > n$.*

(iii) The inclusion $X^n \overset{i}{\hookrightarrow} X$ gives rise to an isomorphism $H_k(X^n) \cong H_k(X)$ whenever $k > n$.

Proof. (i) The statement is trivial for $n = 0$. For $n > 0$, notice that X^{n-1} is a deformation retract of $A := X^n \setminus \sqcup_m \bullet$, where the m copies of $\bullet$ are the centers of the m n-cells of X. The subset $B := X^n \setminus \sqcup_m D^n_{1/2}$ satisfies the conditions of excision, where $D^n_{1/2}$ is the disk of radius $\frac{1}{2}$ sitting inside an n-cell. It follows that

$$H_k(X^n, X^{n-1}) \cong H_k(X^n, A) \cong H_k(X^n \setminus B, A \setminus B) \cong H_k(\sqcup_m D^n, \sqcup_m S^{n-1})$$

These homology groups are familiar: they are $\mathbb{Z}^m$ if $k = n$, and 0 otherwise.

(ii) By the previous result, the homology sequence for (X^n, X^{n-1}) reads

$$0 \longrightarrow H_m X^{n-1} \longrightarrow H_m(X^n) \longrightarrow 0$$

whenever $m \neq n, n-1$. Therefore, for $m > n$ we have

$$H_m X^n \cong H_m X^{n-1} \cong \ldots \cong H_m X^0 = 0.$$

(iii) By the previous diagram, when $m < n$ we have that

$$H_m(X^n) \overset{i}{\hookrightarrow} H_m(X^{n+1})$$

is an isomorphism. By induction, we get a chain of inclusions, all of which are isomorphisms:

$$H_m(X^n) \overset{i}{\hookrightarrow} H_m(X^{n+1}) \overset{i}{\hookrightarrow} \ldots \overset{i}{\hookrightarrow} H_m(X^m)$$

Since the inclusion $H_m(X^n) \overset{i}{\hookrightarrow} H_m(X^m)$ is the composition of the above inclusions, it too is an isomorphism. For finite cell complexes, $X = X^m$ for some m, proving (iii).

□

Theorem 8.9. *The following is a chain complex, where the map d_n is defined as the composition $H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xhookrightarrow{i} H_{n-1}(X^{n-1}, X^{n-2})$.*

$$\dots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \dots$$

Furthermore, $Ker(d_n)/Im(d_{n+1}) \cong H_n(X)$

Proof. The relative homology sequences of X^n and X^{n-1} for all natural numbers n fit into the following commutative diagram:

$$\begin{tikzcd}
 & & & & H_n(X^n, X^{n-1}) \\
 & & H_n(X^{n-1}) \arrow[d] & H_n(X^{n+1}) \arrow[ur] & \\
 & & H_n(X^n) \arrow[d, "j"] \arrow[ur, "i"] & & {} \\
{} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "d_{n-1}"] \arrow[ur] & {} \\
{} \arrow[ur] & & H_{n-1}(X^{n-1}) \arrow[ur, "l"] & & \\
 & H_{n-1}(X^{n-2}) \arrow[ur] & & &
\end{tikzcd}$$

Via Lemma 8.8, we can reduce some groups to 0, and make an identification with $H_n(X)$:

$$
\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) \arrow[r, "\cong"] \arrow[ur] & H_nX \\
 & & H_n(X^n) \arrow[ur, "i"] \arrow[d, "j"] & & {} \\
{} \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[r, "d_{n+1}"] \arrow[ur, "\partial_{n+1}"] & H_n(X^n,X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r, "d_{n-1}"] \arrow[ur] & {} \\
{} \arrow[ur] & & H_{n-1}(X^{n-1}) \arrow[ur, "l"] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}
$$

Since the composition of any two d_n and d_{n-1} factors through the compositions of two successive maps of an exact sequence, the composition is 0. The horizontal sequence is therefore a chain complex.

We have

$$im(d_{n+1}) = im(j\partial_{n+1}) = im(\partial_{n+1}) = ker(i)$$

where the second equality comes from the injectivity of j, and the third equality from the exact sequence. Similarly,

$$ker(d_n) = ker(l\partial_n) = ker(\partial_n) = im(j) \cong H_n(X^n)$$

The second equality follows from the injectivity of l, the third from the exact sequence, and the fourth from the injectivity of j.

We therefore have that

$$ker(d_n)/im(d_{n+1}) \cong H_n(X^n)/ker(i).$$

By the first isomorphism theorem, and since i is injective,

$$H_n(X^n)/ker(i) \cong im(i) \cong H_nX.$$

Therefore

$$ker(d_n)/im(d_{n+1}) \cong H_nX.$$

[Hatcher, 2002] $\square$

We also have a method of calculating the boundary maps d_n from degree calculations:

Theorem 8.10. *Let $(e_n^\alpha)_{\alpha \in A}$ represent the generating elements of the α n-cells. Then $d_{n+1}(e_{n+1}^\alpha) = \sum_\beta d_{\alpha,\beta} e_n^\beta$, where $d_{\alpha,\beta}$ is the degree of the map $S^n\alpha \to X^n \to S^n\beta$, that is the composition of the gluing map of e_n^α with the quotient map identifying $X^n \setminus e_n^\beta$. (This can be identified with S^n, as S^n is the one-point compactification of $\mathbb{R}^n \cong e_n^\beta$.*

Proof. Omitted. See [Hatcher, 2002]. □

8.3 Examples

We now give a number of easy calculations of homology groups of CW-complexes. The following corollary becomes very useful.

Corollary 8.11. *If a CW-complex X has no cells in adjacent dimensions, then H_nX is the free abelian group generated by the $n-$cells of X.*

Proof. Suppose X has $m > 0$ n-cells. The chain complex reads:

$0 \xrightarrow{d_{n+1}} \mathbb{Z}^m \xrightarrow{d_n} 0$

It follows that $H_n(X) = Ker(d_n)/Im(d_{n+1}) = \mathbb{Z}^m$. □

Proposition 8.12. *If X is path-connected then $H_0(X) = \mathbb{Z}$.*

Proof. X must *either* be a single point (from which the result is obvious), or have at least one 1-cell, as there is no way to glue an n-sphere for $n > 1$ onto two disconnected spaces, as the former is connected. MISSING □

Corollary 8.13. *If X is path-connected and has only one* 0*-cell, then $d_1 = 0$.*

Proof. By the previous result, $\mathbb{Z} = H_nX = \mathbb{Z}/im(d_1)$. However, the only subgroups of $\mathbb{Z}$ are $m\mathbb{Z}$ for integers m, and the only isomorphism $\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$ is for $m = 0$. Hence $d_1 = 0$. □

Example 8.14 (Orientable surface of genus g)**.** M_g is the orientable surface with g "holes". It is known that it can be identified as a quotient of the regular $4g$-sided polygon, where side i is identified with the opposite of side $i+2$ (counted clockwise). In pictures, M_1 is the torus T^2, identified as the familiar quotient of the square, and M_2 is the given quotient of the 8-sided polygon. It is clear that M_g should be given a CW-complex structure with 1 0-cell, $2g$ 1-cells and 1 $2-cell$ glued along the concatenation $f = a_1 \cdot a_2 \cdot -a_1 \cdot -a_2 \cdot a_3 \cdot \cdots \cdot -a_{2g-1} \cdot -a_{2g}$. According to Theorem 8.10, we should calculate the degree of the composition of

f with the map collapsing everything but the circle a_i. This is the map $a_i \cdot -a_i \simeq 0$, so $d_2 = 0$. The chain complex is:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

Since all maps are 0, the homology groups can be read as:

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{2g} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 8.15 (Klein bottle)**.** The Klein bottle K is the following quotient of the square with its interior:

$$\begin{tikzcd} x \arrow[r, "a"] & x \arrow[d, "b"] \\ x \arrow[u, "b"] \arrow[r, "a"] & x \end{tikzcd}$$

We can therefore give it a CW-complex structure of 10-cell (the point x), 2 1-cells corresponding to the paths a and b, and one 2-cell glued along the path $f = a \cdot b \cdot -a \cdot b$. By Theorem 8.10, we should calculate the degree of f composed with the quotient collapsing respectively b and a to a point. The first of these is $a \cdot -a \simeq 0$, and the second is $b \cdot b \simeq z^2$. It follows that $d_2(x) = (0, 2x)$. The chain complex reads:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{(0,2x)} \mathbb{Z} \times \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

We simply read off:

$$H_2 X = ker(d_2) = 0$$

$$H_1 X = ker(d_1)/im(d_2) \cong (\mathbb{Z} \times \mathbb{Z})/(0 \times 2\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$H_0 X = ker(d_0) = 0$$

and $H_n X = 0$ for all other values of n.

8.4 Real projective space $\mathbb{RP}^n$

Cellular homology makes the full calculation of $\mathbb{RP}^n$ very easy. We use the cell structure with 1 k-cell for $0 \leq k \leq n$, where the $k-th$ glue map is projection onto $\mathbb{RP}^{k-1}$.

Proposition 8.16.

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n\ odd \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n\ and\ k\ odd. \end{cases}$$

Proof. By theorem 8.10, the degree of $d_{n+1} : (\mathbb{RP}^{n+1}, \mathbb{RP}^n) \to (\mathbb{RP}^n, \mathbb{RP}^{n-1})$ is the degree of the composition

$$S^n \xrightarrow{p} \mathbb{RP}^n \xrightarrow{q} S^n$$

where p is the projection map and q is the quotient map $\mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$. Note that the map applies the identity on the upper hemisphere and the antipodal map on the lower hemisphere, then identifies the equator. The pre-image of a neighbourhood of the north pole N is two neighbourhoods of N and S. The neighbourhood near N is mapped via the identity, and the neighbourhood near S is mapped via the antipodal map. By the Proposition 7.6,

$$deg(d_{n+1}) = deg(qp) = 1 + (-1)^{n+1} = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

The cellular chain complex for $\mathbb{RP}^n$ therefore reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

when n is even, and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

when n is odd. The result can be read off directly. [Hatcher, 2002] $\square$

8.5 Complex projective space $\mathbb{CP}^n$

The complex projective space, $\mathbb{CP}^n$ is defined similarly to $\mathbb{RP}^n$ as the space of complex lines in $\mathbb{C}^{n+1}$. Explicitly it is the quotient $\mathbb{C}^{n+1} \setminus \{0\} \sim$ where $z \sim \lambda w$ for $\forall \lambda \in \mathbb{C}$.

It is trivial to see that $\mathbb{CP}^0 \cong \bullet$, as any $z \sim 1$ via multiplication by $\frac{1}{z}$. $\mathbb{CP}^1$ is the quotient $\mathbb{C}^2/\sim$

$$\begin{bmatrix} z \\ w \end{bmatrix} \sim \lambda \begin{bmatrix} z \\ w \end{bmatrix}.$$

If $w \neq 0$,

$$\begin{bmatrix} z \\ w \end{bmatrix} \sim \frac{1}{w} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z/w \\ 1 \end{bmatrix}$$

We can relabel $u = z/w$, and note that any complex number can be written in this way. So $\{\begin{bmatrix} u \\ 1 \end{bmatrix}\} \cong \mathbb{C} \cong \mathbb{R}^2$. The boundary of this subset of $\mathbb{RP}^1$ is the set $\{\begin{bmatrix} z \\ 0 \end{bmatrix}\}/\sim \cong \mathbb{CP}^0 \cong \bullet$. Therefore $\mathbb{RP}^1$ is the one-point compactification of $\mathbb{R}^2$, which homeomorphic to S^2. We can therefore give it a CW-complex structure of 1 0-cell and 1 2-cell, and its homology groups are the same as that of the 2-sphere.

The general case can be done by induction.

Proposition 8.17. *As a cell complex, $\mathbb{CP}^n$ has a $2m$-cell for $0 < m \leq n$. As a consequence,*

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k \text{ even}, 0 \leq k \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We have proved the case $n = 1$. In general $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$

$$(z_1, z_2, \dots, z_{n+1}) \sim \lambda(z_1, z_2, \dots, z_{n+1}), \lambda \in \mathbb{C}$$

If $z_{n+1} \neq 0$, we can let $\lambda = 1/z_{n+1}$ to find a set of unique representatives

$$\{(u_1, u_2, \dots, 1)\} \cong C^n \cong int(D^{2n})$$

after relabeling $u_k = z_k/z_{n+1}$ for $0 \leq k \leq n$. The boundary of this set in $\mathbb{CP}^n$ is

$$\{(z_1, z_2, \dots, z_n, 0)\}/\sim \cong \mathbb{CP}^n$$

We can therefore give $\mathbb{CP}^n$ the CW-structure of $\mathbb{CP}^{n-1}$, with an additional $2n$-cell glued on the boundary by the projection on its boundary $p : S^{n-1} \to \mathbb{CP}^{n-1}$. The result follows by induction. As $\mathbb{CP}^n$ has no two m-cells in adjacent dimensions, its $m-th$ homology group is the free abelian group generated by its m-cell (or lack thereof). □

Remark 8.18. It is worth looking at the gluing map of the 4-cell of $\mathbb{CP}^2$ onto $\mathbb{CP}^1 \cong S^2$. Via the homeomorphism, this is a map $S^3 \to S^2$ with the property that the pre-image of every point is a great circle of S^3. This is because $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{CP}^1$, chosen as the representative with norm 1, is mapped to by

$$\{\lambda \begin{bmatrix} z \\ w \end{bmatrix} : \lambda \in \mathbb{C}, |\lambda| = 1\},$$

which is a great circle of S^3. This is exactly what characterises the famous Hopf map $h : S^3 \to S^2$. (IMAGE MISSING)

8.6 Quaternionic projective space $\mathbb{HP}^n$ and beyond

The 2-dimensionality of $\mathbb{C}$ ensured that the n-cells of $\mathbb{CP}^n$ were well spread in dimension, leading to an easy homology calculation. One can wonder if the same method can be applied to the higher dimensional extension of $\mathbb{R}$. Indeed we can! The quaternions $\mathbb{H}$ is a four-dimensional extension of $\mathbb{R}$. It is *not* a field, as it is not commutative. However, it retains all the other requirements of a field, importantly does not have zero-divisors. Rings where every nonzero element has a multiplicative inverse are called **division algebras**. The fact that $\mathbb{H}$ is a division algebra is what allows us to use the same argument as before on the Quaternionic projective space $\mathbb{HP}^n$.

Proposition 8.19. *$\mathbb{HP}^n$ has a cell structure with a $4k$-cell for $0 \leq k \leq n$. As a consequence,*

$$H_k(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & k = 0 \; mod \; 4, 0 \leq k \leq 4n \\ 0 & otherwise \end{cases}$$

Proof. As in previous cases, $\mathbb{HP}^0 = \bullet$, as $z \sim 1$ for every $z \in \mathbb{H}$ by division by z. For general $\mathbb{HP}^n$ we will proceed by induction. $\mathbb{HP}^n = \mathbb{H}^{n+1}/\sim$

$$(z_1, \ldots, z_{n+1}) \sim h(z_1, \ldots, z_{n+1}), \forall h \in \mathbb{H}$$

If $z_{n+1} \neq 0$, we can divide by z_{n+1} to find a set of unique representations

$$\{(u_1, \ldots, u_n, 1)\} \cong \mathbb{H}^n \cong int(D^{4n})$$

The boundary of this set in $\mathbb{HP}^n$ is the quotient

$$\{(z_1, \ldots, z_n, 0)\}/\sim \cong \mathbb{HP}^{n-1}$$

We can therefore give $\mathbb{HP}^n$ the CW-structure of $\mathbb{HP}^{n-1}$, with an additional $4n$-cell, mapped onto $\mathbb{HP}^{n-1}$ via the projection on its boundary $p : S^{4n-1} \to \mathbb{HP}^{n-1}$. By induction, $\mathbb{HP}^n$ has the CW-structure stated in the proposition. Since $\mathbb{HP}^n$ has no m-cells in adjacent dimensions, its m-th homology group is the free abelian group generated by its m-cell (or lack thereof). $\square$

Remark 8.20. Notice that $\mathbb{HP}^1$ has a 0-cell and a 4-cell, and is therefore homeomorphic to S^4. As in Remark 8.18, the gluing map of the 8-cell of $\mathbb{HP}^2$ onto $\mathbb{HP}^1 \cong S^4$, therefore gives a "Hopf"-map $S^7 \to S^4$ with the property that the preimage of a point is a "great" copy of S^3. If there was a way to keep extending $\mathbb{R}$ to a division algebra for every positive power of two we could repeat this process, yielding "Hopf"-maps from S^{2^n-1} to S^{n-1} for all $n > 0$. However this turns out to be false, as shown by (REFERENCE)!!! The non-existence of such maps for $n > 4$ proves there are no 2^n-dimensional division algebras of $\mathbb{R}$ for $n > 4$.

8.7 The Borsuk-Ulam Theorem

9 Appendix

9.1 Proof of the Braid Lemma

In this section we give the rest of the proof of Lemma 4.8.

Proof. Recall the following commutative braid lemma diagram, where we have assumed the sequence indexed by f_i is a chain complex, and the other sequences are exact sequences.

$$
\begin{tikzcd}
 & A \arrow[dr, "f_1"] \arrow[rr, bend left, "g_1"] & & D \arrow[dr, "g_2"] \arrow[rr, bend left, "h_3"] & & G \arrow[dr, "h_4"] \arrow[rr, bend left, "j_4"] & & J \\
O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \arrow[ur, "f_5"] \arrow[dr, "h_5"] & \\
 & B \arrow[ur, "h_1"] \arrow[rr, bend right, "j_1"] & & E \arrow[ur, "j_2"] \arrow[rr, bend right, "f_3"] & & H \arrow[ur, "f_4"] \arrow[rr, bend right, "g_4"] & & K
\end{tikzcd}
$$

We have shown that $ker(f_2) \subseteq im(f_1)$ and need to show that $ker(f_3) \subseteq im(f_1)$ and $ker(f_4) \subseteq im(f_3)$.

(a) $ker(f_3) \subseteq im(f_1)$.

Let $x \in E$ be s.t. $f_3(x) = 0$. By commutativity, $g_3 j_2(x) = 0$, so $j_2(x) \in ker(g_3) = im(g_2)$. Then $\exists x_1 \in D$ s.t. $g_2(x_1) = j_2(x)$. It satisfies $h_3(x_1) = j_3 g_2(x_1) = j_3 j_2(x) = 0$, as (j_i) is a chain complex. So $x_1 \in ker(h_3) = im(h_2)$. Therefore there exists $x_2 \in C$ s.t. $h_2(x_2) = x_1$. This element is such that $j_2 f_2(x_2) = g_2 h_2(x_2) = g_2(x_1) = j_2(x)$. We therefore have $j_2(f_2(x_2) - x) = 0$.

Let $x_3 := f_2(x_2) - x$. Then $x_3 \in ker(j_2) = im(j_1)$. Let $x_4 \in B$ be s.t. $j_1(x_4) = x_3$. x_4 is such that $f_2 h_1(x_4) = j_1(x_4) = x_3 = f_2(x_2) - x$. Finally, we see that $x = f_2(x_2 - h_1(x_4)$, so $x \in im(f_2)$ as required.

(b) $ker(f_4 \subseteq im(f_3)$.

Let $x \in H$ be s.t. $f_4(x) = 0$. Then $0 = h_5 f_4(x) = g_4(x)$. So $x \in ker(g_4) = im(g_3)$. Let $x_1 \in F$ be s.t. $g_3(x_1) = x$. Then $h_4 j_3(x_1) = f_4 g_3(x_1) = f_4(x) = 0$ So $j_3(x_1) \in ker(h_4) = im(h_3)$. Let $x_2 \in D$ be s.t. $h_3(x_2) = j_3(x_1)$. Then $j_3(x_1) = j_2 g_2(x_2)$, s.t. $x_3 := g_2(x_2) - x_1 \in \mathrm{ker}(j_3) = im(j_2)$. Let $x_4 \in E$ be s.t. $j_2(x_4) = x_3$. Then $f_3(x_4) = g_3 j_2(x_4) = g_3(x_3) = g_3(g_2(x_2) - x_1) = -g_3(x_1) = -x$. Therefore $x = f_3(-x_4)$, and $x \in im(f_3)$ as required.

$\square$

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