Axiomatic Homology Theory and the Borsuk-Ulam Theorem

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1 Axioms

Definition 1.1. *A chain complex is a sequence of algebraic objects and homomorphisms between them, such that for any consecutive f_i, f_{i+1} we have that $im(f_i) \subset ker(f_{i+1})$. If we have equality instead of inclusion, the construction is called an exact sequence.*

**MISSING: Not actually sequences, as they can be infinite in both directions

We also distinguish between *short* and *long exact sequences*, where the former is finite sequences of three or fewer nonzero elements, and the latter is all other sequences.

Example 1.2. *For any objects $A_1, A_2, \dots$ the following is a long exact sequence: DIAGRAM MISSING*

Remark 1.3. *The requirement that $im(f_i) \subseteq ker(f_{i+1})$ for all $i \in \mathbb{Z}$ is equivalent to the requirement that $f_{i+1} \circ f_i = 0$ for all $i \in \mathbb{Z}$. This is clear from the definition; if $im(f_i) \subseteq ker(f_{i+1})$ then $f_{i+1} \circ f_i(A_i) = f_{i+1}(im(f_i)) = 0$. Conversely, if $f_{i+1} \circ f_i = 0$ then $f_{i+1}(im(f_i)) = 0$.*

Lemma 1.4 (Braid lemma)**.**

An admissable category...

The axioms of (generic) homology theory...

2 Basic results

Proposition 2.1. *If $A \subset X$ is a deformation retract, then $H_n(X,A) = 0$.*

Proof. If $A \subset X$ is a deformation retract, then the inclusion $i : A \to X$ is a homotopy equivalence. Let $r : X \to A$ be the retraction. Then $ir \simeq id_X$ and $ri \simeq id_A$. By homotopy invariance of H_n and the identity property of functors, $(ir)_* = (id_X)_* = id_{H_nX}$ and $(ri)_* = (id_A)_* = id_{H_nA}$. Since $(ri)_* = r_*i_*$ and vice-versa, we have that i_* is an isomorphism.

Now consider the long exact homology chain:

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_nA \xrightarrow{i_*} H_nX \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}A \longrightarrow \dots$$

Since i_* is an isomorphism, and the chain is exact, $H_nX = im(i_*) = ker(j_*)$ so $0 = im(j_*) = ker(\partial)$

However, on the left we also have $0 = ker(i_*) = im(\partial)$ since i_* is an isomorphism. It follows that $H_n(X,A) = 0$.

□

Remark 2.2. *As a special case of this result, $H_n(X,X) = 0$, as X is a deformation retract of itself. We are also interested in special case where $A = x$ is a single point, i.e. X is contractible. Then we have $H_n(X,x) = 0$.*

2.1 Reduced homology

It is possible to reduce the abelian objects H_nX into simpler objects $\tilde{H}_nX$ by in some sense factoring out the object H_n1. Furthermore, this can be done without losing any information, i.e. the transformation $H_nX \to \tilde{H}_nX$ is reversible.

First we will need some facts about abelian objects.

Definition 2.3. *If there exist maps $f : A \to B$ and $g : B \to A$ between abelian objects such that $g \circ f = id_A$ then g is called a retraction of f, and f is called a something of g.*

g can be thought of as a one-sided inverse of f, as there is no requirement that $g \circ f = id_B$.

Lemma 2.4. *If $g : B \to A$ is a retraction of $f : A \to B$, then $B \cong im(f) \bigoplus ker(g)$*

Proof. The isomorphism is given by

$$h : B \to im(f) \bigoplus ker(g)$$

$$x \mapsto (f \circ g(x), x - f \circ g(x))$$

This is well-defined as $f \circ g(x) \in im(f)$ and

$$g(x - f \circ g(x)) = g(x) - g \circ f \circ g(x) = g(x) - g(x) = 0$$

by the associativity of homomorphisms, and since $g \circ f = id_A$. Hence $x - f \circ g(x) \in ker(g)$ The inverse is

$$h^{-1} : im(f) \bigoplus ker(g) \to B$$

$$(a,b) \mapsto a + b$$

One quickly checks that

$$h \circ h^{-1}(a,b) = h(a+b) = (f \circ g(a+b), a+b - f \circ g(a+b))$$

$$= (f \circ g(a) + 0, a + b - f \circ g(a))$$

since $b \in ker(g)$. However, $a = f(c)$ for some $c \in A$, so

$$(f \circ g(a) + 0, a + b - f \circ g(a)) = (f \circ g \circ f(c), a + b - f \circ g \circ f(c))$$
$$= (f(c), a + b - f(c)) = (a, b)$$

since $f \circ g = id_B$. Additionally,

$$h^{-1} \circ h(x) = f \circ g(x) + x - f \circ g(x) = x$$

So h and h^{-1} are indeed inverse homomorphisms, and $B \cong im(f) \bigoplus ker(g)$. □

We will use this Lemma on the following construction.

Definition 2.5. *$\tilde{H}_n(X) = ker(p^* : H_nX \to H_n1)$ where p^* is the map induced by the initial map $p : X \to 1$. $\tilde{H}_nX)$ is called the reduced homology of X.*

Proposition 2.6. *For any $x \in X$, $H_n(X,A) = \tilde{H}_n(X,A) \bigoplus H_n1 = H_n(X,x) \bigoplus H_n1$*

Proof. For the first equality, consider the following diagram. x exists by assumption MISSING of an admissable category, and p exists since 1 is an initial object. Notice p is a retraction of x.

DIAGRAM

H_n induces the following diagram, and since functors map compositions to compositions and identities to identities, we have that $p^* \circ x^* = id_{H_n1}$, so p^* is a retraction of x^*.

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By 2.1,
$$H_nX \cong im(x^*) \bigoplus ker(p^*) = H_n1 \bigoplus \tilde{H}_nX$$
where the last equality holds because $p^* \circ x^* = id_{H_n1}$ guarantees that x^* is injective.

For the second equality, note any two initial objects are isomorphic. In particular, for $x \in X$, $x \cong 1$. We hence have the following long exact chain

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From which we can extract a short exact chain

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By exactness, $H_n1 \cong im(x^*) \cong ker(j*)$ and $im(j*) \cong ker(\partial) = H_n(X,x)$. Furthermore, by the first isomorphism theorem, $im(j*) \cong H_nX/ker(j*)$. Therefore,

$$H_n(X,x) \cong H_nX/H_n1 \cong \tilde{H}_nX \bigoplus H_n1/H_n1 \cong \tilde{H}_nX$$

where the last few isomorphisms are done somewhat informally, to be made precise at a future point. MISSING □

Corollary 2.6.1. *If X is contractible to $x \in X$, then $H_n(X) = H_n(X,x) \bigoplus H_n1 = 0 \bigoplus H_n1 \cong H_n1$, by 2.2. It follows that $\tilde{H}_nX = 0$*

2.6 shows that $H_n(X,A)$ always carries around a copy of $H_n 1$, which can be safely removed by going to the kernel of p^*. Gluing a copy of $H_n 1$ onto $\tilde{H}_n(X,A)$ recovers the original object.

As we will see, there is a long exact chain for the reduced homology. To define it, we will need the following lemma. Consider an admissable category C and a triple $(X,A,B) \in Top_{(3)}$ such that $(X,A),(X,B),(A,B) \in C$. The long exact sequences, which will be labelled $(1),(3),(4)$ respectively, form a braid diagram as shown below. The sequence (2), called the *long exact homology sequence* for the triple (X,A,B) is defined in such a way that the diagram commutes. Explicitly, it is the sequence

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where ∂ is the composition of the transformation $\partial : H_{n+1}(X,A) \to H_n A$ and $i^* : H_n A \to H_n(A,B)$. The other two maps are induced by the inclusions $H_n(A,B) \to H_n(X,B)$ and $H_n(X,B) \to H_n(X,A)$.

Proposition 2.7. *For a triple $(X,A,B) \in Top_{(3)}$ such that $(X,A),(X,B),(A,B) \in C$, the sequence*

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is a long exact sequence.

Proof. We will start by noting that the diagram above commutes. In any subdiagram of the diagram not involving ∂, this follows by noting the subdiagram is induced (via a composition-preserving functor) from a diagram of inclusions which commute by the definition of an admissible category. Any subsquare or triangle involving ∂ commutes since ∂ is a natural transformation, or by the definition of the special ∂ defined above.

After noting the diagram commutes, it becomes easy to show (2) is a chain complex, i.e. that the composition of any two maps is 0. For two out of three compositions, you can via commutativity choose an alternative path which goes via two consecutive maps in a exact sequence, and is hence 0. For the final composition

DIAGRAM

note we have the following commutative diagram (commutative since it is induced by the commutative diagram of inclusions):

Since $H_n(A,A) = 0$, the composition is 0, as required. (https://ncatlab.org/nlab/show/braid+lemma# Munkr
The result is then achieved by an application of the braid lemma. □

Corollary 2.7.1. *The sequence*

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is a long exact sequence.

Proof. First note that if $(X,A) \in C$ then by the requirements of an admissable category, (X,x) and $(A,x) \in C$ as well for any $x \in A$. Hence by the previous proposition, and using the isomorphism $\tilde{H}_n X \cong H_n(X,x)$, we get the result. □

3 Homology of $\mathbb{RP}^n$

Now we will look at the homologies of $\mathbb{RP}^n$ with respect to an arbitrary (ordinary) homology. Recall that the puncture of D^n is homotopy equivalent to S^{n-1}. Recall also that the puncture of $\mathbb{RP}^n$ is homotopy equivalent to $\mathbb{RP}^{n-1}$. Now consider the following diagram:

$$\begin{tikzcd} H_k(D^n, S^{n-1}) \arrow[r, "f^*"] \arrow[d, "i^*"] & H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \arrow[d, "j^*"] \\ H_k(D^n, D^n \setminus \{*\}) & H_k(\mathbb{RP}^n, \mathbb{RP}^n \setminus \{*\}) \arrow[l, "h^*"'] \end{tikzcd}$$

$*$ is defined as the center of D^n when considering $\mathbb{RP}^n$ as the glue of $\mathbb{RP}^{n-1}$ and D^n. i^* and j^* are the maps induced by the canonical inclusions. By the previous comment, i and j are homotopy equivalences, so i^* and j^* are isomorphisms. h is the isomorphism guaranteed by the Excision Axiom after noting that $(D^n, D^n \setminus \{*\})$ can be found by excising everything but a smaller, closed n-disk in $D^n \subset \mathbb{RP}^n$ from the other set. This open set then satisfies all the requirements of excision. We can therefore define f^* as the unique isomorphism that makes the diagram commute. By a previous result, we have

$$H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = H_k(D^n, S^{n-1}) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}$$

From this observation we can deduce a number of facts about $H_k(\mathbb{RP}^n)$.

Lemma 3.1. *$H_k(\mathbb{RP}^n) \cong H_k(\mathbb{RP}^{n-1})$ whenever $k \neq n, n-1$.*

Proof. By assumption,

$$H_k(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = H_{k+1}(\mathbb{RP}^n, \mathbb{RP}^{n-1}) = 0.$$

Therefore, the long exact homology sequence reads as

$$0 \longrightarrow H_k(\mathbb{RP}^{n-1}) \longrightarrow H_k(\mathbb{RP}^n) \longrightarrow 0$$

which induces the necessary isomorphism. □

Corollary 3.1.1. *$H_k(\mathbb{RP}^n) = 0$ when $k > n$ and $n \neq 0$.*
Additionally, $H_0(\mathbb{RP}^n = H_0 1 := G$.

Proof. By the previous lemma, we have, for $k > n, n \neq 1$

$$H_k(\mathbb{RP}^n) \cong H_k(\mathbb{RP}^{n-1}) \cong \ldots \cong H_k(\mathbb{RP}^1) \cong H_k(S^1) = 0$$

The second result has already been verified for $n = 0, 1$. For $n > 1$ we can use the previous lemma again with $k = 0$ to get

$$H_0(\mathbb{RP}^n) \cong H_0(\mathbb{RP}^{n-1}) \cong \ldots \cong H_0(\mathbb{RP}^1) \cong H_0(S^1) = G$$

□

Lemma 3.2. *i If $H_n(\mathbb{RP}^n) = 0$ then $H_{n+1}(\mathbb{RP}^{n+1}) = G$ and $H_n(\mathbb{RP}^{n+1}) = 0$.*

ii If $H_n(\mathbb{RP}^n) = G$ and $n > 0$, then $H_{n+1}(\mathbb{RP}^{n+1}) = 0$.

Proof. (i) Consider the following section of the long homology sequence for $(\mathbb{RP}^{n+1}, \mathbb{RP}^n)$:

$$H_{n+1}(\mathbb{RP}^n) \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}) \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}, \mathbb{RP}^n) \longrightarrow H_n(\mathbb{RP}^n) \longrightarrow H_n(\mathbb{RP}^{n+1}) \longrightarrow$$

By assumption, 4.1.1 and 4.1, this reduces to:

$$0 \longrightarrow H_{n+1}(\mathbb{RP}^{n+1}) \longrightarrow G \longrightarrow 0 \longrightarrow H_n(\mathbb{RP}^{n+1}) \longrightarrow 0$$

This induces the two required isomorphisms.

(ii) In this case, the same diagram reduces to

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If I can show the map $\partial : G \to G$ is injective, I have my result. (... MISSING) $\square$

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