Axiomatic Homology Theory and the Borsuk-Ulam Theorem

Malthe Sporring

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1 Axioms

An admissable category...

The axioms of (generic) homology theory...

2 Basic results

Proposition 2.1. *If $A \subset X$ is a deformation retract, then $H_n(X,A) = 0$.*

Proof. If $A \subset X$ is a deformation retract, then the inclusion $i : A \to X$ is a homotopy equivalence. Let $r : X \to A$ be the retraction. Then $ir \simeq id_X$ and $ri \simeq id_A$. By homotopy invariance of H_n and the identity property of functors, $(ir)_* = (id_X)_* = id_{H_nX}$ and $(ri)_* = (id_A)_* = id_{H_nA}$. Since $(ri)_* = r_*i_*$ and vice-versa, we have that i_* is an isomorphism.

Now consider the long exact homology chain:

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_nA \xrightarrow{i_*} H_nX \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}A \longrightarrow \dots$$

Since i_* is an isomorphism, and the chain is exact, $H_nX = im(i_*) = ker(j_*)$ so $0 = im(j_*) = ker(\partial)$

However, on the left we also have $0 = ker(i_*) = im(\partial)$ since i_* is an isomorphism. It follows that $H_n(X,A) = 0$.

□

Remark 2.2. *As a special case of this result, $H_n(X,X) = 0$, as X is a deformation retract of itself. We are also interested in special case where $A = x$ is a single point, i.e. X is contractible. Then we have $H_n(X,x) = 0$.*

It is possible to reduce the abelian objects H_nX into simpler objects $\tilde{H}_nX$ by in some sense factoring out the object H_n1. Furthermore, this can be done without losing any information, i.e. the transformation $H_nX \to \tilde{H}_nX$ is reversible.

First we will need some facts about abelian objects.

Definition 2.3. *If there exist maps $f : A \to B$ and $g : B \to A$ between abelian objects such that $g \circ f = id_A$ then g is called a retraction of f, and f is called a something of g.*

g can be thought of as a one-sided inverse of f, as there is no requirement that $g \circ f = id_B$.

Lemma 2.4. *If $g : B \to A$ is a retraction of $f : A \to B$, then $B \cong im(f) \bigoplus ker(g)$*

Proof. The isomorphism is given by

$$h : B \to im(f) \bigoplus ker(g)$$

$$x \mapsto (f \circ g(x), x - f \circ g(x))$$

This is well-defined as $f \circ g(x) \in im(f)$ and

$$g(x - f \circ g(x)) = g(x) - g \circ f \circ g(x) = g(x) - g(x) = 0$$

by the associativity of homomorphisms, and since $g \circ f = id_A$. Hence $x - f \circ g(x) \in ker(g)$ The inverse is

$$h^{-1} : im(f) \bigoplus ker(g) \to B$$

$$(a, b) \mapsto a + b$$

One quickly checks that

$$h \circ h^{-1}(a, b) = h(a + b) = (f \circ g(a + b), a + b - f \circ g(a + b))$$

$$= (f \circ g(a) + 0, a + b - f \circ g(a))$$

since $b \in ker(g)$. However, $a = f(c)$ for some $c \in A$, so

$$(f \circ g(a) + 0, a + b - f \circ g(a)) = (f \circ g \circ f(c), a + b - f \circ g \circ f(c))$$

$$= (f(c), a + b - f(c)) = (a, b)$$

since $f \circ g = id_B$. Additionally,

$$h^{-1} \circ h(x) = f \circ g(x) + x - f \circ g(x) = x$$

So h and h^{-1} are indeed inverse homomorphisms, and $B \cong im(f) \bigoplus ker(g)$. □

We will use this Lemma on the following construction.

Definition 2.5. *$\tilde{H}_n(X) = ker(p^* : H_nX \to H_n1)$ where p^* is the map induced by the initial map $p : X \to 1$. $\tilde{H}_nX)$ is called the reduced homology of X.*

Proposition 2.6. *For any $x \in X$, $H_n(X, A) = \tilde{H}_n(X, A) \bigoplus H_n1 = H_n(X, x) \bigoplus H_n1$*

Proof. For the first equality, consider the following diagram. x exists by assumption MISSING of an admissable category, and p exists since 1 is an initial object. Notice p is a retraction of x.

DIAGRAM

H_n induces the following diagram, and since functors map compositions to compositions and identities to identities, we have that $p^* \circ x^* = id_{H_n 1}$, so p^* is a retraction of x^*.

DIAGRAM

By 2,

$$H_n X \cong im(x^*) \bigoplus ker(p^*) = im(x^*) \bigoplus H_n 1 \bigoplus \tilde{H}_n X$$

where the last equality holds because $p^* \circ x^* = id_{H_n 1}$ guarantees that x^* is injective.

For the second equality, note any two initial objects are isomorphic. In particular, for $x \in X$, $x \cong 1$. We hence have the following long exact chain

DIAGRAM

From which we can extract a short exact chain

DIAGRAM

By exactness, $H_n 1 \cong im(x^*) \cong ker(j*)$ and $im(j*) \cong ker(\partial) = H_n(X,x)$. Furthermore, by the first isomorphism theorem, $im(j*) \cong H_n X / ker(j*)$. Therefore,

$$H_n(X,x) \cong H_n X / H_n 1 \cong \tilde{H}_n X \bigoplus H_n 1 / H_n 1 \cong \tilde{H}_n X$$

where the last few isomorphisms are done somewhat informally, to be made precise at a future point. MISSING $\square$

Corollary 2.6.1. *If X is contractible to $x \in X$, then $H_n(X) = H_n(X,x) \bigoplus H_n 1 = 0 \bigoplus H_n 1 \cong H_n 1$, by 2.2. It follows that $\tilde{H}_n X = 0$*

2.6 shows that $H_n(X,A)$ always carries around a copy of $H_n 1$, which can be safely removed by going to the kernel of p^*. Gluing a copy of $H_n 1$ onto $\tilde{H}_n(X,A)$ recovers the original object.

We would like to do manipulations using $\tilde{H}_n X$ instead of $H_n X$, as these spaces are simpler. To justify this, we should show that $\tilde{H}_n X$ forms a homology theory whenever H_n does.

Theorem 2.6.1. *If H_n and ∂ form a homology theory over some category C, then for $\tilde{H}_n X$ there exist $\tilde{\partial}$ such that they form a homology theory as well.*