K-Theory

Maltbe Sporring, Adrián Doña Mateo & Will Bevington

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1 Introduction

These are notes from the Winter 2023 eCHT course "K-Theory", based on lectures, discussions, and literature, in particular [1] and [2]. The material presented is not original, but we hope to give a unique perspective. In particular, we are inclined to use categorical arguments whenever available and hope to convince others of the benefits of this approach.

2 Algebraic K-Theory

All rings will be assumed to live in **CRing**, the category of commutative Noetherian rings with unity.

Let us first define what it means for an $R-$module M to be **projective**. The following lemma gives several equivalent conditions. If either (and hence all) hold, we say M is projective.

Lemma 2.1. *The following are equivalent for an $R-$module M:*

(i) Given a module homomorphism $f : M \to N$ and a surjective module homomorphism $g : J \twoheadrightarrow N$, there exists a lift $h : P \to N$ of f:

$$\begin{tikzcd} & J \arrow[d, two heads, "g"] \\ M \arrow[r, "f"'] \arrow[ur, dashed, "\exists h"] & N \end{tikzcd}$$

(ii) Every short exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

splits.

(iii) There exists a module Q such that $M \oplus Q$ is free.

(iv) $Hom_R(M, -)$ is exact.

Proof. $(i) \implies (ii)$. The required section is lifted from the identity $M \xrightarrow{id} M$.

$(ii) \implies (iii)$. Build surjection from a free module $f : F \twoheadrightarrow M$ and consider the short exact sequence

$$0 \longrightarrow ker(f) \longrightarrow F \xrightarrow{f} M \longrightarrow 0$$

Since this splits, there exists an isomorphism $F \cong ker(f) \oplus M$.

$(iii) \implies (i)$. Since free modules have the lifting property, M inherits the lift of the free module $M \oplus Q$:

$$\begin{tikzcd} M \oplus Q \arrow[r, dashed, "\exists h"] \arrow[d, bend left, "pr_1"] & J \arrow[d, two heads, "g"] \\ M \arrow[u, bend left, "i"] \arrow[r, "f"] & N \end{tikzcd}$$

$(i) \iff (iv)$. $Hom_R(M, -)$ is left exact, so it is exact if and only if it preserves surjections. This is exactly condition (i). □

Definition 2.2. Given a ring R, define the **Grothendieck group** of finitely generated R-modules as

$$\mathcal{G}(R) = \mathbb{Z}\{M \in R - \text{Mod} : M \text{ is finitely generated}\} / \sim$$

Where $\sim$ identifies isomorphic modules and such that $M \sim M' + M''$ whenever there exists a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

Example 2.3. Let $R = \Bbbk$ be a field. Two $\Bbbk$-vector spaces are isomorphic if and only if they have the same dimension, so $\dim : G(\Bbbk) \xrightarrow{\cong} \mathbb{Z}$.

Example 2.4. Let $R = \mathbb{Z}$. Any finitely generated abelian group takes the form

$$M \cong \mathbb{Z}^n \times \Pi_{p \text{ prime}} (\mathbb{Z}/p\mathbb{Z})^{i_p}$$

where $n = \text{rank}(M)$ and i_p are non-negative integers, only finitely many of which are non-zero[1]. We canonically build a surjection $f : \mathbb{Z}^{n+r} \twoheadrightarrow M$ where $r = \sum_p i_p$, and note $ker(f) \cong \mathbb{Z}^r$ since $p\mathbb{Z} \cong \mathbb{Z}$. By considering the canonical short exact sequence,

$$[\mathbb{Z}^{n+r}] = [\mathbb{Z}^r] + [M],$$

so

$$[M] = [\mathbb{Z}^{n+r}] - [\mathbb{Z}^r] = (n + r)[Z] - r[Z] = n[Z].$$

It follows that $G(\mathbb{Z})$ is generated freely by $[\mathbb{Z}]$ and $\text{rank} : G(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$.

We have given $\mathcal{G}(R)$ a strictly formal group structure with identity the trivial module, but in particular for modules $[M]$ and $[M']$ we have $[M] + [M'] = [M \oplus M']$. We can therefore hope to define a multiplication by $[M] \cdot [N] = [M \otimes_R N]$, however this multiplication would not be distributive,

[1] We allow all indexes to be 0 in the case $M = 0$.

as the tensor product is not exact[2]. It is, however, exact on the subcategory of projective modules, since any short exact sequence of projectives splits. This motivates the following definition, which is the start of algebraic K-theory.

Definition 2.5. Given a ring R,

$$K(R) := \mathbb{Z}\{M \in R - \text{Mod} : M \text{ is finitely generated and projective}\}$$

Since higher $K-$groups also exist, the notation $K_0(R)$ is often used for the groups defined here, however we will not make that distinction here.

Lemma 2.6. *$K(R)$ is a commutative ring, with multiplication given by*

$$[M] \cdot [N] = [M \otimes_R N].$$

Proof. It is enough to prove that multiplication is well-defined on generators; the rest will follow strictly formally. If M and N are finitely generated, so is $M \otimes_R N$. The tensor product is associative, preserves isomorphisms in both entries and is exact by a previous comment. It follows the product is well-defined on equivalence classes. The multiplicative unit is given by R, and commutativity follows from the commutativity of $\otimes_R$ when R is commutative. All necessary comments about the addition have already been made except this: $M \oplus N$ is projective when M and N are. Indeed, $\oplus$ is exact as a bifunctor, so

$$\mathrm{Hom}_R(M \oplus N, -) = \mathrm{Hom}_R(M, -) \oplus \mathrm{Hom}_R(N, -)$$

is a composition of exact functors and is therefore exact. □

Let's remark that there is nothing special about the subcategory of f.g. projective R-modules, and one can imagine extending this definition to any additive subcategory of an abelian category.

Our immediate goal is to prove that $K(-)$ is a functor. We will make liberal use of the (general) tensor-hom adjunction, recorded below. In general, one has to be careful about module-handedness. For our purposes this will not be important, as the tensor product is commutative on modules over commutative rings.

Lemma 2.7. *There are bijections*

$$Hom_S(Y \otimes_R X, Z) \cong Hom_R(Y, Hom_S(X, Z))$$

[2] A multiplication turning $\mathcal{G}(R)$ into a ring does exist, but this is outside the scope of these notes.

where X is an (R, S) bimodule, and

$$Hom_S(X \otimes_R Y, Z) \cong Hom_R(Y, Hom_S(X, Z))$$

where X is an (S, R) bimodule.

Remark 2.8. In the special case where $X = S$ is a ring with module structure arising from a ring homomorphism $f : R \to S$, the second bijection takes the form

$$\mathrm{Hom}_S(Y \otimes_R S, Z) \cong \mathrm{Hom}_R(Y, f^*Z)$$

where f^* is the restriction of scalars that gives N the $R-$mod structure $r \cdot n = f(r) \cdot n$. This is because $\mathrm{Hom}_S(S, Z) \cong f^*(Z)$ as $R-$modules.

Lemma 2.9. $K(-) : \mathbf{CRing} \to \mathbf{CRing}$ *is a functor.*

Proof. We have already shown $K(R)$ is a ring. It suffices to show a map $f : R \to S$ (giving S an $R-$module structure) induces a map $K(R) \to K(S)$ in a way that preserves composition. To define f^* it is enough to define it on generators of $K(R)$.

Let M be a projective f.g. R-mod. $-\otimes_R M$ is a functor R-mod $\to$ R-mod so f induces an $R-$module homomorphism $f_* : R \otimes_R M = M \to S \otimes_R M$. Note $S \otimes_R M$ also has a canonical S-module structure. If M is finitely generated, so is $S \otimes_R M$ as an S-module with generators $\{(1 \otimes m_i)_{i \in I}\}$. If M is projective as an $R-$mod, so is $S \otimes_R M$ as an S-mod since

$$\mathrm{Hom}_S(S \otimes_R M, -) \cong \mathrm{Hom}_R(M, f^*(-))$$

by adjunction. Since f^* is obviously exact, and M is projective, $S \otimes M$ is projective.

Since $S \otimes_R -$ preserves isomorphisms and split exact sequences (note every short exact sequence of projective modules splits), the induced map on equivalence classes $K(R) \to K(S)$ is well-defined. It is a ring homomorphism because it preserves direct sums by additivity, the multiplicative identity since $S \otimes_R R = S$, and multiplication since

$$S \otimes_R (M \otimes_R N) \cong (S \otimes_R M) \otimes_S (S \otimes_R N),$$

which we can prove using adjunctions and the Yoneda lemma:

$$\mathrm{Hom}_S((S \otimes_R M) \otimes_S (S \otimes_R N), X) \cong \mathrm{Hom}_R(S \otimes_R M, \mathrm{Hom}_S(S \otimes_R N, X))$$

$$\cong \mathrm{Hom}_R(S \otimes_R M, \mathrm{Hom}_R(N, f^*X)) \cong \mathrm{Hom}_R(S \otimes_R M \otimes_R N, f^*X)$$

$$\cong \mathrm{Hom}_S(S \otimes_S (S \otimes_R M \otimes_R N), X) \cong \mathrm{Hom}_S(S \otimes_R M \otimes_R N, X).$$

To show $K(-)$ is functorial, let $R \xrightarrow{f} S \xrightarrow{g} Z$ be ring homomorphisms, identifying S and Z as $R-$modules. Note this makes g an $R-$module map. Since $- \otimes_R M$ is a functor, $g^* f^* = (g \circ f)^*$ as $R-$module maps. We have already checked that these maps canonically define ring homomorphisms on the set of equivalence classes of modules, from which the functoriality of $K(-)$ follows. $\square$

Remark 2.10. $K(-)$ preserves maps between $R-$algebras. Indeed, $f : A \to B$ is a map of $R-$algebras if and only if there exists a commutative diagram in **CRing**:

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where the maps $R \to A$ and $R \to B$ are those exhibiting A and B as $R-$algebras. Applying $K(-)$ to the diagram shows that

$$K(f) : K(A) \to K(B)$$

is a $K(R)-$algebra map.

References

[1] Daniel Dugger, *A geometric introduction to K-theory*, available at `https://pages.uoregon.edu/ddugger/kgeom.pdf` (as of January 25, 2023), n.d.

[2] Allen Hatcher, *Vector bundles & K-theory*, available at `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf` (as of January 25, 2023), 2017.