## The ultrafilter monad

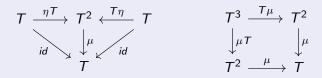
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$$11+11=22$$

## Monads

#### **Definition**

A **monad** on C is an endofunctor T, a **unit** natural transformation  $\eta: id \Rightarrow T$  and a **multiplication** natural transformation  $\mu: T^2 \Rightarrow T$  that is unital and associative, i.e. so the following diagrams commute.



Here  $T\eta$  and  $\eta T$  are defined componentwise by **whiskering**:  $(T\eta)_c = T(\eta_c)$  and  $(\eta T)_c = \eta_{T(c)}$ .

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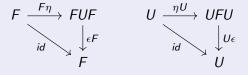
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#### **Definition**

### An adjunction

$$C \stackrel{F}{\underset{U}{\longleftarrow}} D$$

is a pair of functors and a pair of natural transformations  $\eta:1_C\to UF$  and  $\epsilon:FU\to 1_D,$  called the **unit** and **counit** respectively, satisfying the triangle identities:



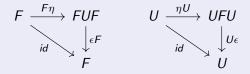
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#### Definition

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is a pair of functors and a pair of natural transformations  $\eta: 1_C \to UF$  and  $\epsilon: FU \to 1_D$ , called the **unit** and **counit** respectively, satisfying the triangle identities:



#### Lemma

An adjunction gives rise to a monad  $T = UF : C \rightarrow C$ .

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## Example

There is an adjunction

Set 
$$\bigcup_{U}^{\mathbb{Z}[-]}$$
 Ab

where U is the forgetful functor, and  $\mathbb{Z}[X]$  is the **free abelian group** generated by X.

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## Example

There is an adjunction

Set 
$$\bigcup_{U}^{\mathbb{Z}[-]}$$
 Ab

where U is the forgetful functor, and  $\mathbb{Z}[X]$  is the **free abelian group** generated by X.

The corresponding monad  $\mathbb{Z}[X]: \mathbf{Set} \to \mathbf{Set}$  maps X to the set of finite formal sums  $\sum_i a_i x_i$ . The unit  $\eta_X: X \to \mathbb{Z}[X]$  maps x to the singleton sum  $\sum^1 x$ . Multiplication is given by distributing coefficients:

$$\mu_X: \mathbb{Z}[\mathbb{Z}[X]] \to \mathbb{Z}[X], \quad \sum_i a_i \sum_j b_j x_j \mapsto \sum_{i,j} a_i b_j x_j$$

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## Adjunctions from monads

Given a monad  $(T, \eta, \mu)$  on C, can we find an adjunction  $C \xrightarrow{\Gamma} D$  that restricts to it?

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# Adjunctions from monads

Given a monad  $(T, \eta, \mu)$  on C, can we find an adjunction  $C \stackrel{F}{\underset{U}{\longleftarrow}} D$  that restricts to it?

#### Lemma

Yes. Furthermore, there is both a final and an initial such adjunction. The final adjunction goes to the category of T-algebras. The initial adjunction goes to the category of free T-algebras.

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## The category of T-algebras

Intiutively, the category consists of **evaluation maps**  $a: TA \to A$  where  $A \in C$ . These are required to play nicely with the unit and multiplication. Maps are maps  $f: A \to B$  in C such that the following diagram commutes:

$$\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow^{a} & & \downarrow^{b} \\
A & \xrightarrow{f} & B
\end{array}$$

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## The category of T-algebras

## Example

Recall the monad  $T=\mathbb{Z}[-]:\mathbf{Set}\to\mathbf{Set}$ . Claim: the category of T-algebras is equivalent to  $\mathbf{Ab}$ . A T-algebra is a set X and an evaluation map  $a:\mathbb{Z}[X]\to X$  carrying a formal sum to its corresponding element. The commutative square

$$\mathbb{Z}[\mathbb{Z}[X]] \xrightarrow{\mu_X} \mathbb{Z}[X]$$

$$\mathbb{Z}[a] \downarrow \qquad \qquad \downarrow a$$

$$\mathbb{Z}[X] \xrightarrow{a} A$$

gives the group axioms.

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# Monadic adjunctions

When an adjunction is equivalent to the adjunction to the category of T-algebras, the adjunction is called **monadic**. The familiar free-forgetful adjunctions between **Set** and **Grp**, **Ab**, **Ring**, **Vect**<sub>k</sub> are all monadic, identifying their categories as **algebraic theories**.

### Generators and relations

Algebraic theories as we know them can be explained in terms of generators and relations. Given an abelian group A, a set of generators G and a set of relations R,  $A = \mathbb{Z}[G]/\mathbb{Z}[R]$ . Equivalently, A is the coequalizer

$$\mathbb{Z}[R] \xrightarrow{ev} \mathbb{Z}[G] \longrightarrow A$$

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This is not well-behaved categorically, as it is not functorial. Instead, we may take *every element* of A to be a generator, and *every* possible equation to be a relation. This identifies A as the coequalizer

$$\mathbb{Z}[\mathbb{Z}[A]] \xrightarrow{\mathbb{Z}[a]} \mathbb{Z}[A] \xrightarrow{a} A$$

# Generators and relations (2)

#### Theorem

In the category of T-algebras for a monad T, an element A is the coequalizer

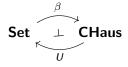
$$T^2A \xrightarrow{T[a]} TA \xrightarrow{a} A$$

We will see a surprising consequence...

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## An unrelated story?

There is an adjunction



between sets and compact Hausdorff spaces.  $\beta$  is given by **Stone–Čech compactification** on the set with the discrete topology. The induced monad has an explicit description as the set  $\beta(X)$  of **ultrafilters** on X.

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### **Ultrafilters**

#### Definition

An **ultrafilter**  $\mathcal{F}$  on a set X is a set of subsets such that whenever we write  $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$  as a finite disjoint union, exactly one of the  $X_i$ 's is in  $\mathcal{F}$ .

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### Example

Given any  $x \in X$  its **principal ultrafilter**  $\mathcal{F}_x$  is the set of all subsets containing x. Note given any partition  $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ , exactly one  $X_i$  contains x.

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### Limits of ultrafilters

Ultrafilters classify convergence on a topological space.

#### **Definition**

 $x \in X$  is a **limit** of an ultrafilter  $\mathcal{F}$  if every open neighbourhood of x is contained in  $\mathcal{F}$ .

For example, x is a limit of the principal ultrafilter  $\mathcal{F}_x$ .

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#### Lemma

If a topological space X is compact, then every ultrafilter on X has at least one limit. If X is Hausdorff, then every ultrafilter on X has at most one limit. Therefore, X is compact Hausdorff if and only if every ultrafilter has exactly one limit.

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## Algebras over the ultrafilter monad

A  $\beta$ -algebra is a map lim :  $\beta(X) \to X$  sending each ultrafilter  $\mathcal F$  to a unique lim  $\mathcal F \in X$ . By the previous lemma, this hints that  $\beta$ -algebras are exactly compact Hausdorff spaces. And indeed...

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#### Lemma

**CHaus** is equivalent to the category of  $\beta$ -algebras, i.e. the adjunction  $\beta \dashv U$  is monadic.

This means compact Hausdorff spaces are an algebraic theory!

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## Generators and relations on compact Hausdorff spaces

A compact Hausdorff space X is the coequaliser

$$\beta^2 X \xrightarrow{\beta[\lim]} \beta X \xrightarrow{\lim} X$$

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For example of a relation, fix an ultrafilter  $\mathcal{F}$  on X and take the principal ultrafilter of ultrafilters  $\mathcal{G}_{\mathcal{F}} \in \beta^2 X$ . The multiplication takes an ultrafilter of ultrafilters  $\mathcal{G}$  to the ultrafilter  $\{A \in X : [A] \in \mathcal{G}\}$  where [A] is the set of all ultrafilters containing A. In particular,  $\mu_X(\mathcal{G}_{\mathcal{F}}) = \mathcal{F}$ .

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$$\lim \mathcal{G}_{\lim \mathcal{F}} = \lim \mathcal{F}$$

as expected.

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## Thank you for listening!

#### References



- nLab authors, *ultrafilter*, https://ncatlab.org/nlab/show/ultrafilter, 2022, Revision 43.
- Emily Riehl, Category theory in context, Dover Publications, 2017.

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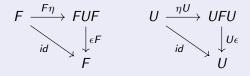
# Adjunctions

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# The category of T-algebras

#### **Definition**

The category  $C^T$  of T-algebras has as objects pairs  $(A \in C, a : TA \to A)$  such that the following diagrams commute:

$$\begin{array}{cccc}
A & \xrightarrow{\eta_A} & TA & & T^2A & \xrightarrow{\mu_A} & TA \\
\downarrow a & & \downarrow a & & \downarrow a \\
& A & & TA & \xrightarrow{a} & A
\end{array}$$

Maps are maps  $f: A \rightarrow B$  in C such that the following diagram commutes:

$$TA \xrightarrow{Tf} TB$$

$$\downarrow a \qquad \qquad \downarrow b$$

$$A \xrightarrow{f} B$$

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## Topology of $\beta$

 $\beta(X) \in \mathbf{CHaus}$  has a topology generated by the open sets

$$\mathcal{F}^A := \{ \mathcal{F} \in \beta(X) : A \in \mathcal{F} \} \text{ where } A \subset X.$$

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# $\beta$ -homomorphisms

A  $\beta-$ homomorphism is a map  $X \to Y$  such that the following square commutes

$$\beta(X) \xrightarrow{\beta f} \beta(Y)$$

$$\lim_{X \longrightarrow f} \qquad \lim_{Y \longrightarrow Y} Y$$

where  $\beta f(\mathcal{F})$  is the ultrafilter that takes a partition  $Y = Y_1 \sqcup \cdots \sqcup Y_n$ , and picks out the unique  $Y_i$  such that  $f^{-1}(Y_i) \in \mathcal{F}$ .

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