

Spectral sequences

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1 Introduction

This is an informal and evolving set of notes on spectral sequences. The goal is to flesh out details and complete exercises from the reference material, mainly [1] and [2], with a particular focus on applications to computing homotopy groups of spheres. After setting up the common construction of spectral sequences, we will start with the Serre spectral sequence, a classical and relatively simple spectral sequence ideal for getting comfortable with the calculus of spectral sequences. We will then focus on the Adams sequence, which remains one of the most powerful computational tools available for computing stable homotopy groups of spheres.

2 Spectral sequences

We start with a gentle introduction to spectral sequences, following [1].

Definition 2.1. An **exact couple** is a diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \searrow k & \swarrow j \\ & E & \end{array}$$

exact at every corner.

Given an exact couple, we can form the **derived couple** as follows.

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ & \searrow k' & \swarrow j' \\ & E' & \end{array}$$

We call $d = jk$ the **differential** and note $d^2 = 0$. We then define $E' = \ker(d)/\text{im}(d)$, $A' = i(A)$, $j'(ia) = [ja]$ and all other maps are canonically defined. By a diagram chase, one can show the derived couple is exact. We can therefore take derived couples indefinitely. This will, in particular, form a sequence (E^r, d_r) of objects and morphisms, where $E^{r+1} = \ker(d_r)/\text{im}(d_r)$ and $d_r^2 = 0$. This is called a **spectral sequence**.

Let A^1 and E^1 be bigraded groups, and let $|i| = (0, 1)$, $|j| = (0, 0)$ and $|k| = (-1, -1)$. For example, we could have a CW-complex X with p -skeleta X_p , and let

$$A = \bigoplus_{n,p} H_n(X_p), \quad E = \bigoplus_{n,p} H_n(X_p, X_{p-1}),$$

$i = i^*, j = j^*$ (the canonical inclusions), $k = \partial_n$.

In this case, the derived couple comes from the long exact sequence in homology

$$\cdots \rightarrow H_{n+1}(X_p) \xrightarrow{j} H_{n+1}(X_p, X_{p-1}) \xrightarrow{\partial_{n+1}} H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \rightarrow \cdots$$

After taking the derived couple, we have $|i'| = |i|$ and $|k'| = |k|$ but $|j'| = (0, 1) + |j|$. It follows that $d_r : E_{n,p}^r \rightarrow E_{n-1,p-r+1}^r$ defined on components.

Next, we will make two simplifying assumptions. Note, for example, that these are satisfied by the above homological example.

- (i) For each p , only finitely many of the $E_{n,p}^1$'s are nonzero. Equivalently, all but finitely many of the maps $i : A_{n,p-1}^1 \rightarrow A_{n,p}^1$ are isomorphisms.
- (ii) Defining $A_{-\infty,p}^1$ to be the common bottom value of each column, assume $A_{n,-\infty}^1$.

Since the differentials d_r go up by $r-1$, they are eventually all the zero map, so the $E_{n,p}^r$'s stabilize to some groups $E_{n,p}^\infty$. Under these two assumptions, we have the following convergence result.

Proposition 2.2. *Under (i) and (ii), $E_{n,p}^\infty$ is isomorphic to the quotient F_n^p / F_n^{p-1} for the filtration*

$$\cdots \subset F_n^{p-1} \subset F_n^p \subset \cdots \subset A_{n,\infty}^1$$

by the subgroups $F_n^p = \text{Im}(A_{n,p}^1 \rightarrow A_{n,\infty}^1)$.

Proof. Consider the exact sequence

$$E_{n+1,p+r-1}^r \rightarrow A_{n,p+r-2}^r \xrightarrow{i} A_{n,p+r-1}^r \rightarrow E_{n,p}^r \rightarrow A_{n-1,p-1}^r \rightarrow A_{n-1,p}^r \rightarrow E_{n-1,p-r+2}^r$$

For large r , the first and last E terms are zero by condition (i), and the last two A terms are zero by condition (2). This expresses $E_{n,p}^r$ as the quotient

$$A_{n,p+r-1}^r / i(A_{n,p+r-2}^r) = i^{r-1}(A_{n,p}^1) / i^r(A_{n,p-1}^1)$$

of subgroups of $A_{n,p+r-1}^1 = A_{n,\infty}^1$. [1] □

It is often more useful to define $n = p + q$, $E_{p,q} := E_{p+q,p}$ and vary p and q instead. In this view, the differentials are $d_r : E_{p,q} \rightarrow E_{p+r,q+r-1}$. This allows us to equivalently define a spectral sequence as a sequence of "pages" E^r where each page consists of a grid of groups $E_{p,q}^r$ with differentials $d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$, and where the E^{r+1} page is formed from the E^r page by taking $\ker(d_r)/\text{im}(d_r)$ at each grid element. It is common to start with the E^2 page, where we note

differentials go two units to the left and one unit up. The differentials then get one unit wider and longer after passing to each successive page. If the grid elements are cohomology groups, it is typical to redefine $n = -n$ such that differentials to r units right and $r - 1$ units down. We will use this notation for the rest of these notes.

3 The Serre spectral sequence

The Serre spectral sequence arises from a fibration

$$F \rightarrow E \rightarrow B$$

where B is a path-connected CW-complex such that $\pi_1(B)$ acts trivially on $H_*(F; G)$ ¹. For example, B could be simply-connected. Alternatively, the fibration could arise (as Eilenberg-MacLane spaces [1]) from a short exact sequence of groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

where $A \subset Z(B)$.

We have a spectral sequence of the following form:

why this converges

Theorem 3.1 (Serre spectral sequence). *Given a fibration*

$$F \rightarrow E \rightarrow B$$

where $\pi_1(B)$ acts trivially on $H_*(F)$, there is a spectral sequence $(E_{p,q}^r, d_r)$ with $E_{p,q}^2 = H_p(B, H_q(F; G))$ and whose E^∞ terms $E_{p,n-p}^\infty$ are quotients F_n^p / F_n^{p-1} of a filtration $0 \subset F_n^0 \subset \cdots \subset F_n^n = H_n(X; G)$.

A proof of the theorem is given in [1].

As demonstrated in [1] as Examples 5.4 and 5.5, the path-space fibration $\Omega X \rightarrow PX \rightarrow X$ of a simply connected space X give rise to very nice Serre spectral sequences. Since PX is contractible, the only nonzero entry on the E^∞ page is $E_{0,0}^\infty$, allowing us to deduce a lot about which differentials are isomorphisms, or 0. If X is furthermore a Eilenberg-MacLane space $K(A, n)$, then $\Omega X = K(A, n-1)$. If the homology of $K(A, n-1)$ is well-known, the spectral sequence can then be used to compute the homology of X . The aforementioned examples demonstrate computations using path-space fibrations in the case $X = K(\mathbb{Z}, 2)$ and $X = S^n$.

The following example is Exercise 1 from Chapter 5 of [1].

¹For an explanation of how $\pi_1(B)$ acts, see [1].

Example 3.2. Let $f : S^k \rightarrow S^k$ be a map of degree n where $n, k > 1$. We can turn this into a fibration (up to homotopy) by taking the homotopy fiber $F \rightarrow S^k \rightarrow S^k$. Note S^k is simply-connected. By the universal coefficient theorem, the E^2 entries are given by

$$E_{p,q}^2 = H_p(S^k; H_q F) \cong (H_p(S^k) \otimes H_q F) \oplus (\text{Tor}_1(H_{p-1}(S^k), H_q F))$$

Since $H_p(S^k) = \mathbb{Z}$ for $p = 0, k$ and is 0 otherwise, the only non-zero entries on the E^2 page are in the $p = 0, k$ columns, with value $H_q F$. The only entries that can survive to the E^∞ page lie on the 0^{th} and k^{th} diagonal, as these will be quotients in a filtration of the only non-zero homology groups of the total space S^k ; $H_0 S^k \cong \mathbb{Z}$ and $H_k S^k \cong \mathbb{Z}$. In particular, $H_0 F = \mathbb{Z}$. The only possible non-trivial differentials are on the E^k page, and in particular $H_i F = 0$ for $0 < i < k - 1$. The only possible non-trivial differential is the surjection $d_k : E_{k,0}^k \cong \mathbb{Z} \rightarrow E_{0,k-1}^k \cong H_{k-1} F$. All other non-trivial differentials are isomorphisms, giving $H_j F \cong H_{j+k-1}$ for $j < 1$. This is summarized in the following diagram. where all unspecified entries are 0.

of a filtration

$$0 \subset H_k F \subset F_k^1 \cdots \subset F_k^{k-1} \subset H_k S^k = \mathbb{Z}$$

The only subquotient of \mathbb{Z} which is isomorphic to \mathbb{Z} is the trivial one, giving $0 = F_n^{k-1} = \cdots = H_k F$.

In summary,

$$H_j F = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}/n\mathbb{Z} & j = ik - i, i > 0 \\ 0 & \text{otherwise} \end{cases}$$

References

- [1] Allen Hatcher, *Spectral sequences*, Available at <https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf> (as of January 17, 2023), 2004.
- [2] John McCleary, *A user's guide to spectral sequences*, 2 ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2000.