Spectral sequences

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1 Introduction

This is an informal and evolving set of notes on spectral sequences. The goal is to flesh out details and complete exercises from the reference material, mainly [1] and [2], with a particular focus on applications to computing homotopy groups of spheres. After setting up the common construction of spectral sequences, we will start with the Serre spectral sequence, a classical and relatively simple spectral sequence ideal for getting comfortable with the calculus of spectral sequences. We will then focus on the Adams sequence, which remains one of the most powerful computational tools available for computing stable homotopy groups of spheres.

2 Spectral sequences

We start with a gentle introduction to spectral sequences, following [1].

Definition 2.1. An **exact couple** is a diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

exact at every corner.

Given an exact couple, we can form the **derived couple** as follows.

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

We call $d = jk$ the **differential** and note $d^2 = 0$. We then define $E' = ker(d)/im(d)$, $A' = i(A)$, $j'(ia) = [ja]$ and all other maps are canonically defined. By a diagram chase, one can show the derived couple is exact. We can therefore take derived couples indefinitely. This will, in particular, form a sequence (E^r, d_r) of objects and morphisms, where $E^{r+1} = ker(d_r)/im(d_r)$ and $d_r^2 = 0$. This is called a **spectral sequence**.

Let A^1 and E^1 be bigraded groups, and let $|i| = (0,1), |j| = (0,0)$ and $|k| = (-1,-1)$. For example, we could have a CW-complex X with $p-$skeleta X_p, and let

$$A = \bigoplus_{n,p} H_n(X_p), \quad E = \bigoplus_{n,p} H_n(X_p, X_{p-1}),$$

$$i = i^*,\, j = j^* \text{ (the canonical inclusions)},\, k = \partial_n.$$

In this case, the derived couple comes from the long exact sequence in homology

$$\cdots \to H_{n+1}(X_p) \xrightarrow{j} H_{n+1}(X_p, X_{p-1}) \xrightarrow{\partial_{n+1}} H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \to \dots$$

After taking the derived couple, we have $|i'| = |i|$ and $|k'| = |k|$ but $|j'| = (0,1) + |j|$. It follows that $d_r : E^r_{n,p} \to E^r_{n-1,p-r+1}$ defined on components.

Next, we will make two simplifying assumptions. Note, for example, that these are satisfied by the above homological example.

(i) For each p, only finitely many of the $E^1_{n,p}$'s are nonzero. Equivalently, all but finitely many of the maps $i : A^1_{n,p-1} \to A^1_{n,p}$ are isomorphisms.

(ii) Defining $A^1_{-\infty,p}$ to be the common bottom value of each column, assume $A^1_{n,-\infty}$.

Since the differentials d_r go up by $r-1$, they are eventually all the zero map, so the $E^r_{n,p}$'s stabilize to some groups $E^\infty_{n,p}$. Under these two assumptions, we have the following convergence result.

Proposition 2.2. *Under (i) and (ii), $E^\infty_{n,p}$ is isomorphic to the quotient F^p_n / F^{p-1}_n for the filtration*

$$\cdots \subset F^{p-1}_n \subset F^p_n \subset \cdots \subset A^1_{n,\infty}$$

by the subgroups $F^p_n = Im(A^1_{n,p} \to A^1_{n,\infty})$.

Proof. Consider the exact sequence

$$E^r_{n+1,p+r-1} \to A^r_{n,p+r-2} \xrightarrow{i} A^r_{n,p+r-1} \to E^r_{n,p} \to A^r_{n-1,p-1} \to A^r_{n-1,p} \to E^r_{n-1,p-r+2}$$

For large r, the first and last E terms are zero by condition (i), and the last two A terms are zero by condition (2). This expresses $E^r_{n,p}$ as the quotient

$$A^r_{n,p+r-1} / i(A^r_{n,p+r-2}) = i^{r-1}(A^1_{n,p}) / i^r(A^1_{n,p-1})$$

of subgroups of $A^1_{n,p+r-1} = A^1_{n,\infty}$. [1] □

It is often more useful to define $n = p+q$, $E_{p,q} := E_{p+q,p}$ and vary p and q instead. In this view, the differentials are $d_r : E_{p,q} \to E_{p+r,q+r-1}$ This allows us to equivalently define a spectral sequence as a sequence of "pages" E^r where each page consists of a grid of groups $E^r_{p,q}$ with differentials $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, and where the E^{r+1} page is formed from the E^r page by taking $ker(d_r)/im(d_r)$ at each grid element. It is common to start with the E^2 page, where we note differentials go two units to the left and one unit up. The differentials then get one unit wider and longer after passing to each successive page. If the grid elements are cohomology groups, it is typical to redefine $n = -n$ such that differentials to r units right and $r-1$ units down. We will use this notation for the rest of these notes.

3 The Serre spectral sequence

3.1 ...in homology

The Serre spectral sequence arises from a fibration

$$F \to E \to B$$

where B is a path-connected CW-complex such that $\pi_1(B)$ acts trivially on $H_*(F;G)$[1]. For example, B could be simply-connected. Alternatively, the fibration could arise (as Eilenberg-MacLane spaces [1]) from a short exact sequence of groups

$$0 \to A \to B \to C \to 0$$

where $A \subset Z(B)$. We can filter X by the subspaces $X_p = \pi^{-1}(B_p)$.

(B, B_p) is $p-$connected, and since π has the homotopy lifting property, (X, X_p) is also connected[2]. Then by Hurewicz, $H_n(X, X_p) = pi_n(X, X_p) = 0$ for $n \le p$ so by the long exact sequence the inclusions $X_p \to X$ induce isomorphisms $H_n(X_p) \to H_n(X)$ for $n < p$. Note furthermore $H_n(X_p) = 0$ for $p < 0$. The bigraded groups $A = \bigoplus H_n(X_p)$ and $E = \bigoplus H_n(X_p, X_{p-1})$ therefore satisfy conditions (i) and (ii), so the associated spectral sequence converges to $H_*(X)$.

We have a spectral sequence of the following form:

Theorem 3.1 (Serre spectral sequence). *Given a fibration*

$$F \to E \to B$$

where $\pi_1(B)$ acts trivially on $H_*(F)$, there is a spectral sequence $(E^r_{p,q}, d_r)$ with $E^2_{p,q} = H_p(B, H_q(F;G))$ and whose E^∞ terms $E^\infty_{p,n-p}$ are quotients F^p_n / F^{p-1}_n of a filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X;G)$.

A proof of the theorem is given in [1].

As demonstrated in [1] as Examples 5.4 and 5.5, the path-space fibration $\Omega X \to PX \to X$ of a simply connected space X give rise to very nice Serre spectral sequences. Since PX is contractible, the only nonzero entry on the E^∞ page is

[1]For an explanation of how $\pi_1(B)$ acts, see [1].

[2]Elaboration pending.

$E^\infty_{0,0}$, allowing us to deduce a lot about which differentials are isomorphisms, or 0. If X is furthermore a Eilenberg-MacLane space $K(A,n)$, then $\Omega X = K(A,n-1)$. If the homology of $K(A,n-1)$ is well-known, the spectral sequence can then be used to compute the homology of X. The aforementioned examples demonstrate computations using path-space fibrations in the case $X = K(\mathbb{Z},2)$ and $X = S^n$.

The following example is Exercise 1 from Chapter 5 of [1].

Example 3.2. Let $f : S^k \to S^k$ be a map of degree n where $n,k > 1$. We can turn this into a fibration (up to homotopy) by taking the homotopy fiber $F \to S^k \to S^k$. Note S^k is simply-connected. By the universal coefficient theorem, the E^2 entries are given by

$$E^2_{p,q} = H_p(S^k; H_qF) \cong \big(H_p(S^k) \otimes H_qF\big) \bigoplus \big(Tor_1(H_{p-1}(S^k), H_qF\big)$$

Since $H_p(S^k) = \mathbb{Z}$ for $p = 0,k$ and is 0 otherwise, the only non-zero entries on the E^2 page are in the $p = 0,k$ columns, with value H_qF. The only entries that can survive to the E^∞ page lie on the 0^{th} and k^{th} diagonal, as these will be quotients in a filtration of the only non-zero homology groups of the total space S^k; $H_0S^k \cong \mathbb{Z}$ and $H_kS^k \cong \mathbb{Z}$. In particular, $H_0F = \mathbb{Z}$. The only possible non-trivial differentials are on the E^k page, and in particular $H_iF = 0$ for $0 < i < k-1$. The only possible non-trivial differential is the surjection $d_k : E^k_{k,0} \cong \mathbb{Z} \to E^k_{0,k-1} \cong H_{k-1}F$. All other non-trivial differentials are isomorphisms, giving $H_jF \cong H_{j+k-1}$ for < 1. This is summarized in the following diagram. where all unspecified entries are 0.

$$
\begin{tikzcd}
 & & & \\
\vdots & & & \\
2k-1 & H_{2k-1}F & & H_{2k-1}F \arrow[uull, "\cong"] \\
2k-2 & H_{2k-2}F & & H_{2k-2}F \arrow[uull, "\cong"] \\
\vdots & & & \\
k & H_kF & & H_kF \arrow[uuull, "\cong"] \\
k-1 & H_{k-1}F & & H_{k-1}F \arrow[uuull, "\cong"] \\
\vdots & & & \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[uull, two heads] \\
 & 0 & \cdots & k
\end{tikzcd}
$$

The long exact sequence in homotopy for the fibration $F \to S^k \xrightarrow{s_n} S^k$, together with the observations $\pi_{k-1}S^k = 0$ and $ker(\cdot n) = 0$, gives a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \pi_{k-1}F \to 0$$

giving $\pi_{k-1}F = \mathbb{Z}/n\mathbb{Z}$. The long exact sequence also gives $\pi_j F = 0$ for $j < k-1$, since $\pi_j S^k = 0$ for these values. The Hurewicz theorem therefore gives an isomorphism $H_{k-1}F \cong \pi_{k-1}F \cong \mathbb{Z}/n\mathbb{Z}$.

Returning to the spectral sequence, we find that the only interesting differential is the surjection $d_k : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. The kernel $\ker(d_k) = n\mathbb{Z} \cong \mathbb{Z}$ is the final quotient of a filtration

$$0 \subset H_k F \subset F_k^1 \cdots \subset F_k^{k-1} \subset H_k S^k = \mathbb{Z}$$

The only subquotient of $\mathbb{Z}$ which is isomorphic to $\mathbb{Z}$ is the trivial one, giving $0 = F_n^{k-1} = \cdots = H_k F$.

In summary,

$$H_j F = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}/n\mathbb{Z} & j = ik - i, i > 0 \\ 0 & \text{otherwise} \end{cases}$$

3.2 ...in cohomology

A completely analogous spectral sequence exists in cohomology, by replacing everywhere H_n by H^n in theorem 3.1 and reversing the direction of differentials. The filtration is also reordered

$$0 \subset F_n^n \subset \cdots \subset F_0^n = H^n(X)$$

with $E_\infty^{p,n-p}$ now isomorphic to F_p^n / F_{p+1}^n. If cohomology is taken over a ring R, we can now use the cup product for computations.

References

[1] Allen Hatcher, *Spectral sequences*, Available at https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf (as of January 17, 2023), 2004.

[2] John McCleary, *A user's guide to spectral sequences*, 2 ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2000.