Spectral sequences

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1 Introduction

This is an informal and evolving set of notes on spectral sequences. The goal is to flesh out details and complete exercises from the reference material, mainly [1] and [2], with a particular focus on applications to computing homotopy groups of spheres. Higher categorical formalism will be emphasised where possible. After setting up the common construction of spectral sequences, we will start with the Serre spectral sequence, mostly as it is a classical and relatively simple spectral sequence, ideal for getting comfortable with the calculus of spectral sequences. We will then focus on the Adams and Adams-Novikov sequences, which remain some of the most powerful computational tools available to date, particularly for computing stable homotopy groups of spheres.

2 Spectral sequences

3 The Serre spectral sequence

The Serre spectral sequence arises from a fibration

$$F \to E \to B$$

where B is a path-connected CW-complex such that $\pi_1(B)$ acts trivially on $H_*(F;G)$. For example, B could be simply-connected. Alternatively, the fibration could arise (as Eilenberg-MacLane spaces) from a short exact sequence of groups

$$0 \to A \to B \to C \to 0$$

where $A \subset Z(B)$ as shown in Example **??**. In either case, we have a spectral sequence of the following form:

Theorem 3.1 (Serre spectral sequence)**.** *Given a fibration*

$$F \to E \to B$$

where $\pi_1(B)$ acts trivially on $H_*(F)$, there is a spectral sequence $(E^r_{p,q}, d_r)$ with $E^2_{p,q} = H_p(B, H_q(F;G))$ and whose E^∞ terms $E^\infty_{p,n-p}$ are quotients F^p_n / F^{p-1}_n of a filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X;G)$.

Note we have made a change of notation from the previous section by setting $n = p+q$. A proof of the theorem is given in [1].

As demonstrated in [1] as Examples 5.4 and 5.5, the path-space fibration $\Omega X \to PX \to X$ of a simply connected space X give rise to very nice Serre spectral sequences. Since PX is contractible, the only nonzero entry on the E^∞ page is

$E^{\infty}_{0,0}$, allowing us to deduce a lot about which differentials are isomorphisms, or 0. If X is furthermore a Eilenberg-MacLane space $K(A,n)$, then $\Omega X = K(A,n-1)$. If the homology of $K(A,n-1)$ is well-known, the spectral sequence can then be used to compute the homology of X. The aforementioned examples demonstrate computations using path-space fibrations in the case $X = K(\mathbb{Z},2)$ and $X = S^n$.

The following example is Exercise 1 from Chapter 5 of [1].

Example 3.2. Let $f : S^k \to S^k$ be a map of degree n where $n,k > 1$. The homotopy fiber F gives a fibration $F \to S^k \to S^k$ and we note S^k is simply-connected. By the universal coefficient theorem, the E^2 entries are given by

$$E^2_{p,q} = H_p(S^k;H_qF) \cong \big(H_p(S^k) \otimes H_qF\big) \oplus \big(Tor_1(H_{p-1}(S^k),H_qF\big)$$

Since $H_p(S^k) = \mathbb{Z}$ for $p = 0,k$ and is 0 otherwise, the only non-zero entries on the E^2 page are in the $p = 0,k$ columns, with value H_qF. The only entries that can survive to the E^∞ page lie on the 0^{th} and k^{th} diagonal, as these will be quotients in a filtration of the only non-zero homology groups of the total space S^k; $H_0S^k \cong \mathbb{Z}$ and $H_kS^k \cong \mathbb{Z}$. In particular, $H_0F = \mathbb{Z}$. The only possible non-trivial differentials are on the E^k page, and in particular $H_iF = 0$ for $0 < i < k-1$. The only possible non-trivial differential is the surjection $d_k : E^k_{k,0} \cong \mathbb{Z} \to E^k_{0,k-1} \cong H_{k-1}F$. All other non-trivial differentials are isomorphisms, giving $H_jF \cong H_{j+k-1}$ for < 1. This is summarized in the following diagram. where all unspecified entries are 0.

$$
\begin{tikzcd}
 & {} & & \\
\vdots & {} & & \\
2k-1 & H_{2k-1}F & & H_{2k-1}F \arrow[uull, "\cong"] \\
2k-2 & H_{2k-2}F & & H_{2k-2}F \arrow[uull, "\cong"] \\
\vdots & & & \\
k & H_kF & & H_kF \arrow[uuull, "\cong"] \\
k-1 & H_{k-1}F & & H_{k-1}F \arrow[uuull, "\cong"] \\
\vdots & & & \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[uull, two heads] \\
 & 0 & \cdots & k
\end{tikzcd}
$$

The long exact sequence in homotopy for the fibration $F \to S^k \xrightarrow{s_n} S^k$, together with the observation $\pi_{k-1}S^k = 0$, gives a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \pi_{k-1}F \to 0$$

giving $\pi_{k-1}F = \mathbb{Z}/n\mathbb{Z}$. The long exact sequence also gives $\pi_j F = 0$ for $j < k-1$, since $\pi_j S^k = 0$ for these values. The Hurewicz theorem therefore gives an isomorphism $H_{k-1}F \cong \pi_{k-1}F \cong \mathbb{Z}/n\mathbb{Z}$.

Returning to the spectral sequence, we find that the only interesting differential is the surjection $d_k : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. The kernel $\ker(d_k) = n\mathbb{Z} \cong \mathbb{Z}$ is the final quotient of a filtration

$$0 \subset H_k F \subset F_k^1 \cdots \subset F_k^{k-1} \subset H_k S^k = \mathbb{Z}$$

The only subquotient of $\mathbb{Z}$ which is isomorphic to $\mathbb{Z}$ is the trivial one, giving $0 = F_n^{k-1} = \cdots = H_k F$.

In summary,

$$H_j F = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}/n\mathbb{Z} & j = ik - i, i > 0 \\ 0 & \text{otherwise} \end{cases}$$

References

[1] Allen Hatcher, *Spectral sequences*, Available at `https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf` (as of January 17, 2023), 2004.

[2] John McCleary, *A user's guide to spectral sequences*, 2 ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2000.