### The ultrafilter monad

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#### 1 Introduction

We give an exposition on monads by exploring a key example: the ultrafilter monad. We will learn that every adjoint pair gives rise to a monad, and moreover that every monad arises from an adjunction. In fact, more is true: there is always a final and initial such adjunction. The final adjunction goes to the Eilenberg-Moore category, which classifies the algebras over the monad. Familiar algebraic categories, such as groups and rings, arise as the categories of algebras over a monad. Indeed, it is sensible to *define* an algebraic category in this way. This has a surprising and enlightening consequence: the category of compact hausdorff spaces is an algebraic category over the ultrafilter monad.

From another perspective, we will see how sometimes a single functor can give rise to a monad: the codensity monad. In particular, the codensity monad of the inclusion  $\mathbf{Fin} \to \mathbf{Set}$  is the ultrafilter monad! The concept of finiteness alone is therefore enough to define compact haussdorff spaces in a routine way.

Knowledge of basic category theory will be assumed, in particular functors, natural transformations and adjoints.

#### 2 Monads

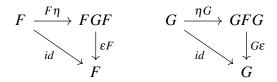
Recall that an adjunction

$$C \xrightarrow{F} D$$

is a pair of functors along with natural transformations

$$\eta: 1_C \to GF$$
 and  $\varepsilon: FG \to 1_D$ ,

called the unit and counit respectively, satisfying the triangle identities:



Here  $(F\eta)_c = F(\eta_c)$  and  $(\varepsilon F)_c = \varepsilon_{F(c)}$ . We can ask what part of the adjunction is visible from the perspective of C. Now we cannot see F and G, only the composite endofunctor GF, which we will call T. In addition, the counit is no longer visible, only the unit. If we look a little harder, some more structure becomes visible. Applying G to the left triangle identity above reveals a "multiplication" natural transformation  $\mu: T^2 \to T$  given by the whiskered counit  $G\varepsilon F$ . The triangle identities reveal that this multiplication plays nicely with the unit:

$$GF \xrightarrow{GF\eta} GFGF \xleftarrow{\eta GF} GF$$

$$\downarrow_{id} \qquad \downarrow_{G\varepsilon F} \qquad \downarrow_{id}$$

$$GF$$

Or, using our new notation,

$$T \xrightarrow[id]{\eta T} T^2 \xleftarrow{T\eta} T$$

$$\downarrow \mu \qquad \qquad \downarrow id$$

One can similarly verify that the multiplication is associative in the sense that the following diagram commutes:

$$T^{3} \xrightarrow{T\mu} T^{2}$$

$$\downarrow^{\mu T} \qquad \downarrow^{\mu}$$

$$T^{2} \xrightarrow{\mu} T$$

This gives  $(C, T, \mu, \eta)$  the structure of a **monad**. It is reminiscent of monoids, as we have a multiplication that is associative and unital. In fact, a monad is the same as a monoid object in the category of endofunctors on C.

Having defined monads from adjunctions, we can ask the dual question: given a monad  $(T, \mu, \eta)$ , can we find an adjunction

$$C \xrightarrow{F} D$$

that restricts to it? The answer is that not only can we always find such an adjunction, there are *initial* and *final* such choices. These adjunctions live in the category of adjunctions that restrict to the monad, where maps are functors  $F:D\to D'$  between codomains such that F commutes with the adjoint functors. The final adjunction is between C and the **Eilenberg-Moore category** of the monad, which classifies the "algebras over the monad". The initial adjunction is between C and the **Kleisli category** of the monad, classifying the "free algebras over the monad". These mottos match our intuitions: there is a monad T on **Set** sending a set X to the (underlying set of) the free abelian group on X. The algebras over T are exactly **Ab**, and the free algebras over T are exactly the category of free abelian groups. The former fact makes the adjunction

$$\mathbf{Set} \overset{F}{\underset{U}{\smile}} \mathbf{Ab}$$

a **monadic adjunction**: it is the Eilenberg-Moore adjunction associated to its monad. It makes a lot of sense to define an algebraic category as one with a monadic adjunction from sets. From this perspective,  $Grp, Ab, Ring, Vect_k$  are all algebraic categories. Notably, **Field** is not an algebraic category, which may or may not be to your taste, but correctly identifies that the category of fields is not as well-behaved as algebraic categories. More surprising is that the category of compact Hausdorff spaces is algebraic, as we will see next.

#### 3 The ultrafilter monad

## 4 Codensity monads

# References