The ultrafilter monad

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1 Introduction

We give an exposition on monads by exploring a key example: the ultrafilter monad. We will learn that every adjoint pair gives rise to a monad, and moreover that every monad arises from an adjunction. In fact, more is true: there is always a final and initial such adjunction. The final adjunction goes to the Eilenberg-Moore category, which classifies the algebras over the monad. Familiar algebraic categories, such as groups and rings, arise as the categories of algebras over a monad. Indeed, it is sensible to *define* an algebraic category in this way. This has a surprising and enlightening consequence: the category of compact hausdorff spaces is an algebraic category over the ultrafilter monad.

From another perspective, we will see how sometimes a single functor can give rise to a monad: the codensity monad. In particular, the codensity monad of the inclusion **Fin** $\to$ **Set** is the ultrafilter monad! The concept of finiteness alone is therefore enough to define compact haussdorff spaces in a routine way.

Knowledge of basic category theory will be assumed, in particular functors, natural transformations and adjoints.

2 Monads

Recall that an **adjunction**

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "G"] \end{tikzcd}$$

is a pair of functors along with natural transformations

$$\eta : 1_C \to GF \text{ and } \varepsilon : FG \to 1_D,$$

called the **unit** and **counit** respectively, satisfying the triangle identities:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "id"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "id"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

Here $(F\eta)_c = F(\eta_c)$ and $(\varepsilon F)_c = \varepsilon_{F(c)}$. We can ask what part of the adjunction is visible from the perspective of C. Now we cannot see F and G, only the composite endofunctor GF, which we will call T. In addition, the counit is no longer visible, only the unit. If we look a little harder, some more structure becomes visible. Applying G to the left triangle identity above reveals a "multiplication" natural transformation $\mu : T^2 \to T$ given by the whiskered counit $G\varepsilon F$. The triangle identities reveal that this multiplication plays nicely with the unit:

$$\begin{tikzcd} GF \arrow[r, "GF\eta"] \arrow[dr, "id"'] & GFGF \arrow[d, "G\varepsilon F"] & GF \arrow[l, "\eta GF"'] \arrow[dl, "id"] \\ & GF & \end{tikzcd}$$

Or, using our new notation,

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "id"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "id"] \\ & T & \end{tikzcd}$$

One can similarly verify that the multiplication is associative in the sense that the following diagram commutes:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

This gives (C, T, μ, η) the structure of a **monad**. It is reminiscent of monoids, as we have a multiplication that is associative and unital. In fact, a monad is the same as a monoid object in the category of endofunctors on C.

Having defined monads from adjunctions, we can ask the dual question: given a monad (T, μ, η), can we find an adjunction

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "G"] \end{tikzcd}$$

that restricts to it? The answer is that not only can we always find such an adjunction, there are *initial* and *final* such choices. These adjunctions live in the category of adjunctions that restrict to the monad, where maps are functors $F : D \to D'$ between codomains such that F commutes with the adjoint functors. The final adjunction is between C and the **Eilenberg-Moore category** of the monad, which classifies the "algebras over the monad". The initial adjunction is between C and the **Kleisli category** of the monad, classifying the "free algebras over the monad". These mottos match our intuitions: there is a monad T on **Set** sending a set X to the (underlying set of) the free abelian group on X. The algebras over T are exactly **Ab**, and the free algebras over T are exactly the category of free abelian groups. The former fact makes the adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

a **monadic adjunction**: it is the Eilenberg-Moore adjunction associated to its monad. It makes a lot of sense to define an algebraic category as one with a monadic adjunction from sets. From this perspective, $\mathbf{Grp}, \mathbf{Ab}, \mathbf{Ring}, \mathbf{Vect}_k$ are all algebraic categories. Notably, **Field** is not an algebraic category, which may or may not be to your taste, but correctly identifies that the category of fields is not as well-behaved as algebraic categories. More surprising is that the category of compact Hausdorff spaces is algebraic, as we will see next.

3 The ultrafilter monad

4 Codensity monads

References