GGT Lecture 4

[Rimling Ghatak]

- Nielsen-Screier : Any subgroup of a free group is free.

Pf : We know : $\boxed{G \text{ is free} \Leftrightarrow G \text{ acts freely on some tree.}}$

Say, G is a free group. , Say H is any subgroup.

Say T. G acts freely on some tree T.

Take the induced action

$ht = t \Rightarrow h = 1$

$\Rightarrow H$ is free. ∎

- Groups as metric spaces :

Say G is group.

$g = \underbrace{s_1 s_2 s_3^{-1} s_4 \cdots}$

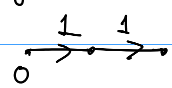
We need : a metric.

Fix a generating set S.

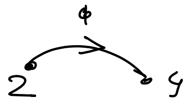
word metric ← $d(g,h) \doteq$ shortest word length of $h^{-1}g$ wrt the generating set S.

$\mathbb{Z}, \{1\}$ $\quad d(2,4) = |4-2|_{\{1\}}$ → shortest word length

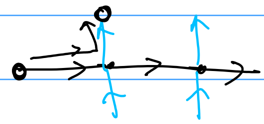
$= |2|_{\{1\}} = 2$



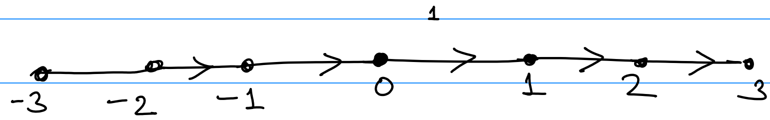
$\mathbb{Z}, \{2, 3\}$ $\quad d(2,4) = 1$



Path metric on the Cayley graph $\Gamma(G, S)$.



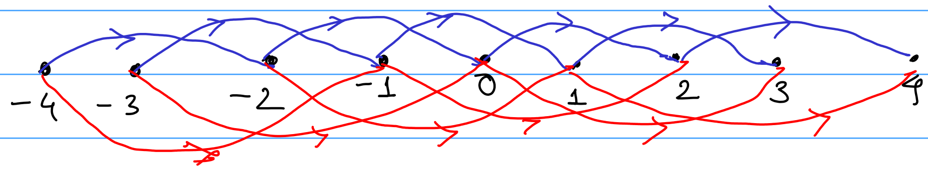
- $\mathbb{Z}$, $\{1\}$.



NOT isometric

- $\mathbb{Z}$, $\{2, 3\}$

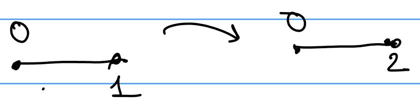
look at 0 and 1

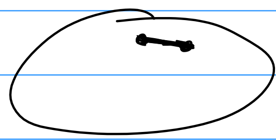


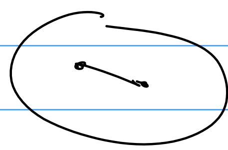
- **Isometry**: A function $f: (X, d_X) \longrightarrow (Y, d_Y)$ b/w metric spaces is called a isometry if $\forall\ x, x' \in X$

$$d_X(x, x') = d_Y(f(x), f(x'))$$

and f is **surjective**.

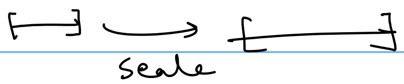






$[0, 1/2]$

$[0, 1]$ $[0, 2]$



scale

- **Def^n**: Suppose $f, g : (X, d_1) \longrightarrow (Y, d_2)$ two functions

We say, f and g are within bdd distance if –

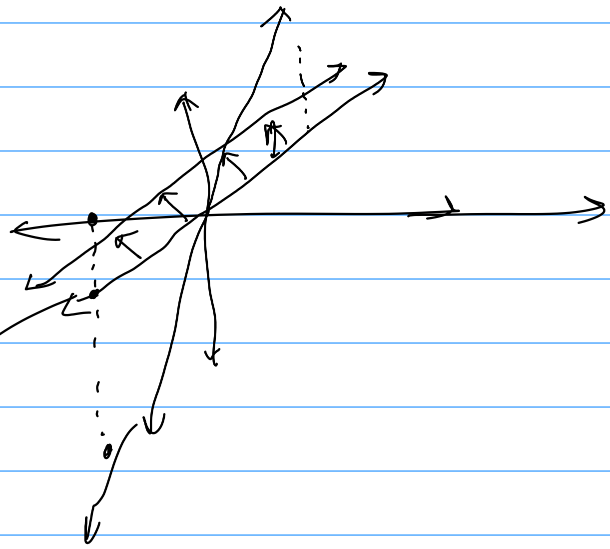
$$\sup d_2(f(x), g(x)) < \infty$$

i.e. $\exists\ c > 0$ s.t $\quad d_2(f(x), g(x)) \leq c \quad \forall x$

- **Example**: $f(x) = x$.

$g(x) = x + 1$ ✓

$g'(x) = 2x$ ✗ ✗



- **Def^n**: (quasi isometry):

Suppose $f : (X, d_1) \longrightarrow (Y, d_2)$ is a function.

We say f is a (λ, ε) quasi isometric embedding if $\exists\ \varepsilon \geq 0$, $\lambda \geq 1$ s.t –

$$\frac{1}{\lambda} d_1(x, x') - \varepsilon \leq d_2(f(x), f(x')) \leq \lambda d_1(x, x') + \varepsilon$$

$\lambda > \lambda$

$\lambda > \lambda_2$

$\varepsilon > \varepsilon_1$

$> \varepsilon_2$

If $\exists\ c$ st $\forall y \in Y$, $\exists\ x \in X$ s.t $d(f(x), y) \leq C$ then f is a (λ, ε) quasi isometry.

X and Y are called quasi isometric spaces if there exists a quasi isometry between them.

Want to look at : $f: X \longrightarrow X$ forms a grp.

- Prop / Observation :

(1) If f is a (λ, ε) quasi isometry.

Then it is also a (λ', ε') q.i for $\lambda' \geq \lambda$, $\varepsilon' \geq \varepsilon$

(2) $f: X \to Y$ is a isometry

$\forall y \in Y$, $\{f^{-1}(y)\}$ is a single point.

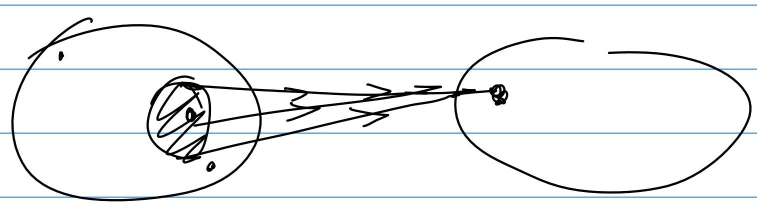
for quasi isometries :- $f^{-1}(y)$ has bdd diameter

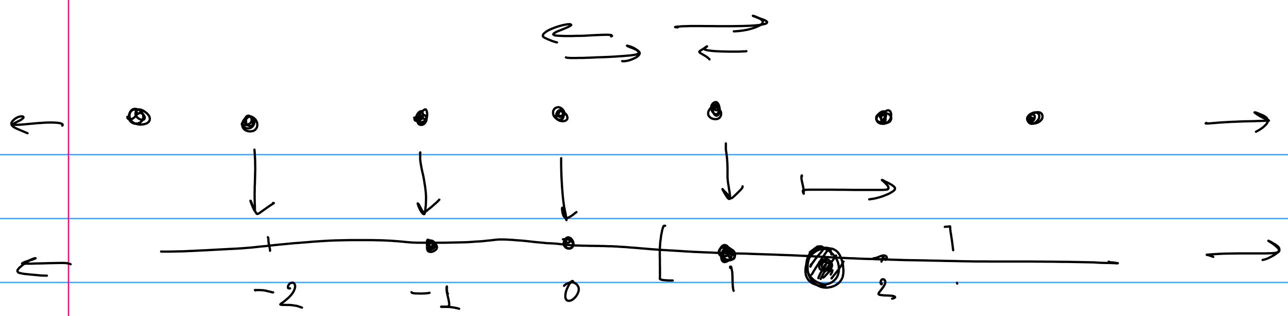
▶ Pf : Suppose $A = f^{-1}(y)$ say f is a (λ, ε) q.i

$a, a' \in A$, then $\frac{1}{\lambda} d(a, a') - \varepsilon \leq \underbrace{d(f(a), f(a'))}_{0}$

$\Rightarrow d(a, a') \leq \lambda \varepsilon$

$\Rightarrow \operatorname{diam}(A) \leq \lambda \varepsilon$





$\mathbb{Z}$, $\mathbb{R}$ are quasi isometric

- Example: $(\mathbb{Z}, d_{\mathbb{Z}}) \longrightarrow (\mathbb{R}, d)$

$\{1\}$

inclusion map is a quasi isometric embedding.

ⓒ

$d_{\mathbb{Z}}(m,n) = |n-m|$

$d(m,n) = |n-m|$

X set
Y subset.
Then a map $f: Y \to X$
$y \mapsto y$
is called a inclusion.

- Exercise: Show any isometry is a quasi isometry.

$(\lambda = 1, \ \varepsilon = 0, \ c = 0)$

This map is also quasi isometry. } CHECK.
Take $c = 1$.

This shows $\mathbb{R}$ and $\mathbb{Z}$ are quasi isometric

- Example: $f: (\mathbb{R}, d) \longrightarrow (\mathbb{Z}, d_{\mathbb{Z}})$
$x \longmapsto [x]$

$2.5 \mapsto 2$
$2 \mapsto 2$
$3.1 \mapsto 3$

Exercise: Check if is $(1,2)$ quasi isometric embedding

- Is this a quasi isometry? Yes. Surjective ✓

$$f : \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$x \longmapsto x+1$$

$f : X \to \cancel{X}$

let $W = \{ f : (X, d_X) \to (\cancel{X}, d_X) \ : f \text{ is a quasi isometry} \}$

define a relation $f \sim g$ if f and g are within bdd distance.

$\sim$ is a equivalence relation

$W/\sim$ = set of equivalence classes.

this forms a group.

$= \underline{QI(X)}$

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$g \circ f$ is a quasi isometry.

We will need a identity.

We will need a inverse.