Geometric Group Theory UDGRP
Presentation on Ends

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Today, we are going to formalise the notion of "ends" of a metric space. Roughly, "the number of ends" of a metric space is the number of different infinities, upto a suitable notion of equivalence. We will capture these ways using rays.

Definition. A *ray* in a metric space (X, d) is a continuous map $\gamma\colon [0, \infty) \to X$.

Definition. A ray γ is *proper* if for any compact subset $K \subseteq X$, the inverse image $\gamma^{-1}(K)$ is compact.

Definition. Suppose γ and γ' are two proper rays in X. We say that γ and γ' *converge to the same end* iff for every compact set $K \subseteq X$, there is a positive integer N such that $\gamma[N, \infty)$ and $\gamma'[N, \infty)$ lie in the same path component of $X \setminus K$.

The idea is, if in the complement of any compact set, one could still join the two rays with a path, then we say that they converge to the same end. Now, the above defines an equivalence relation on the set of proper rays in X. Denote the equivalence class of γ by $\mathrm{end}(\gamma)$, and the set of equivalence classes by $\mathrm{Ends}(X)$.

For example, $\mathbb{R}$ has two infinities, captured by the ends $[0, \infty) \to \mathbb{R}$ sending $t \mapsto t$ and $t \mapsto -t$. To see this, first note that they are clearly proper. Also, these two rays don't converge to the same end, since removing the compact set $[-1, 1]$, we see that the two rays cannot be joined by a path. To see that there are no other ends, let $\gamma\colon [0, \infty) \to \mathbb{R}$ be a proper ray. If $\gamma(t) \to \pm\infty$ as $t \to \infty$, then we are done. (Say if $\gamma(t) \to \infty$ as $t \to \infty$, given a compact set K, pick a real number $R > 0$ such that $K \subseteq (-R, R)$. Then, pick N such that $\gamma[N, \infty) \subseteq (R, \infty)$. Thus, γ converges to the same end as the ray $t \mapsto t$.) Otherwise, there is $M > 0$ such that for any $R > 0$, there is $s, t > R$ such that $\gamma(t) < M$ and $\gamma(s) > -M$. Then, using IVT if needed, we find that for all $R > 0$, there is a $r > R$ with $\gamma(r) \in [-M, M]$. Thus, $\gamma^{-1}[-M, M]$ is unbounded, a contradiction as γ is proper.

For another example, one can verify that $\mathbb{R}^n$ has only one end for $n \geq 2$, and any proper ray converges to the same end as $(0, \infty) \to \mathbb{R}^n$ sending $t \mapsto (t, 0, \ldots, 0)$.

We will now endow $\mathrm{Ends}(X)$ with a topology. First, we need a notion of convergence:

Definition. Let $\mathrm{end}(\gamma_n)$ be a sequence of points in $\mathrm{Ends}(X)$, and $\mathrm{end}(\gamma) \in \mathrm{Ends}(X)$. We say that $\mathrm{end}(\gamma_n)$ *converges to* $\mathrm{end}(\gamma)$ if for all compact sets K, there is a sequence N_n of integers such that $\gamma_n[N_n, \infty)$ and $\gamma[N_n, \infty)$ lie in the same path component of $X \setminus K$ after a sufficiently large n.

Essentially, a sequence $\mathrm{end}(\gamma_n)$ converges to $\mathrm{end}(\gamma)$ if for any compact set, one can find a large n after which one can find tails of each γ_n in the same path component as a tail of γ_n.

One can easily do a sanity check that the definition above does not depend on the choice of representatives: suppose $\mathrm{end}(\gamma_n) = \mathrm{end}(\gamma_n')$ and $\mathrm{end}(\gamma_n)$ converges to $\mathrm{end}(\gamma)$ and fix a compact set K. One can find N_ns such that $\gamma_n[N_n, \infty)$ and $\gamma[N_n, \infty)$ all lie in the same path component of $X \setminus K$, after a large n. Then, pick integers M_n with $\gamma_n[M_n, \infty)$ and $\gamma_n'[M_n, \infty)$ lie in the same path component of $X \setminus K$. Take $K_n = \max\{M_n, N_n\}$, then $\gamma_n[K_n, \infty)$, $\gamma_n'[K_n, \infty)$ and $\gamma[K_n, \infty)$ all lie in the same path component for large n.

With this, we can put a topology on $\mathrm{Ends}(X)$ by declaring those subsets F to be closed, for which whenever there is a sequence $\mathrm{end}(\gamma_n) \in F$ with $\mathrm{end}(\gamma_n) \to \mathrm{end}(\gamma)$, then $\mathrm{end}(\gamma) \in F$.

Thus, $\mathrm{Ends}(\mathbb{R}^n)$ is the point space for $n \geq 2$, and it is not too hard to see that $\mathrm{Ends}(\mathbb{R})$ has the trivial topology. Now, we will go to group theory again, and as usual, one would like to define for a finitely generated group G, the space $\mathrm{Ends}(G)$ as $\mathrm{Ends}(\mathcal{C}(G; X))$ where $\mathcal{C}(G; X)$ is the Cayley graph of G with the generating set X. This has an obvious problem: $\mathcal{C}(G; X)$ depends on X. But one might be able to guess,

Theorem. If X and Y are quasi-isometric, then $\mathrm{Ends}(X)$ and $\mathrm{Ends}(Y)$ are homeomorphic.

This allows us to define $\mathrm{Ends}(G)$ unambiguously. Now, we can see that $\mathrm{Ends}(\mathbb{Z})$ is homeomorphic to $\mathrm{Ends}(\mathbb{R})$, the trivial topology on two points, and $\mathrm{Ends}(\mathbb{Z}^n)$ is homeomorphic to the point space. For a finite group G, $\mathrm{Ends}(G)$ is empty, since no ray $\gamma\colon [0, \infty) \to \mathcal{C}(G)$ is proper, for the inverse image of the compact set $\mathcal{C}(G)$ is $[0, \infty)$, which is not compact. An interesting example, though a little hard to prove, is that $\mathrm{Ends}(\mathrm{F}_2)$ is homeomorphic to the Cantor set, and in particular, F_2 has infinitely many ends.

The above result comes from the general functoriality of Ends for proper geodesic spaces: if X and Y are proper geodesic metric spaces and $f\colon X \to Y$ is a quasi-isometry, then we get a continuous map $f_*\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ sending $\mathrm{end}(\gamma) \mapsto \mathrm{end}(f_\#(\gamma))$ where $f_\#(\gamma)$ is defined by concatenating geodesic segments $[f(\gamma(n)), f(\gamma(n+1))]$. Showing this stronger statement is quite difficult and an outline of a proof can be found in the book "Metric Spaces of Non-Positive Curvature" by Bridson and Haefliger, pages 145-146.

To summarise this talk, I will now give a big result about determining the number of ends of a group.

Theorem. Let G be a finitely generated group.

1. G has 0, 1, 2 or infinitely many ends.
2. $\mathrm{Ends}(G)$ is compact, and is either finite or uncountable.
3. G has 0 ends iff G is finite.
4. G has 2 ends iff G is virtually $\mathbb{Z}$.
5. G has infinitely many ends iff it is either $A *_C B$ where $[A : C] \geq 3$ and $[B : C] \geq 2$, or $A*_C$ where $[A : C] \geq 3$.

The last line needs an explanation on the notation. For groups $A = \langle X_A \,|\, R_A \rangle$, $B = \langle X_B \,|\, R_B \rangle$, C with embeddings $i_A\colon C \to A$ and $i_B\colon C \to B$, the amalgamated free product $A *_C B$ is given by the presentation $\langle X_A, X_B \,|\, R_A, R_B, i_A(c) i_B(c)^{-1} \rangle$. For a group $A = \langle X \,|\, R \rangle$ with $C \leq A$ and an embedding $\phi\colon C \to A$, the HNN extension $A*_C$ is given by the presentation $\langle X, t \,|\, R, txt^{-1} = \phi(x) \rangle$. More details can be found in any standard text including Bridson-Haefliger.

For applications of this theorem, we can use it to conclude that $\mathbb{Z}_2 * \mathbb{Z}_3$ (in fact, quasi-isometric to F_2) has infinitely many ends. We could also show things like $\mathbb{Z}_2 * \mathbb{Z}_2$ is virtually $\mathbb{Z}$, since it is quasi-isometric to $\mathbb{Z}$ and hence has 2 ends.

This completes my talk, and I hope you learnt something! I would like to thank Rinkiny Ghatak and Treanungkur Mal for this amazing UDGRP, and giving me the opportunity to present this here. Thank you for attending this talk!