What will be covered

1. Define Farey Graph
2. Action of $\mathbb{SL}_2(\mathbb{Z})$ on Farey Graph
3. Construction of Farey Graph
4. Define Farey Tree using Farey Complex

Defining Farey Graph

Intuition: $\frac{p}{q}$

Consider $(p,q) \in \mathbb{Z}^2$
Here we can find 2 problem

1. as $\frac{p}{q} = \frac{-p}{-q}$ so lets consider both $(p,q),(-p,-q)$ as one denoted by $\pm(p,q)$
2. $\frac{p}{q} = \frac{2p}{2q} = \frac{ap}{aq}$ so let include only p,q st $\gcd(|p|,|q|) = 1$

and formally define the vertices $V = \left\{(p,q) \in \mathbb{Z}^2 | \gcd(|p|,|q|) = 1\right\}/\sim$
where $\sim$ is a equivalence relation defined as
$(p,q) \sim (r,s) \iff (p,q) = (r,s)$ or $(p,q) = (-r,-s)$

Lets formalise representation of elements in V
if $\pm(p,q) \in V$ then $q > 0$ exception is when $q = 0$ p retains the sign of its other operand
so $(4,-3) \to \pm(-4,3)$ $(-3,-1) \to \pm(3,1)$

we connect $\pm(p_1,q_1)$ and $\pm(p_2,q_2)$ if

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1$$

or equivalently

$$p_1 q_2 - q_1 p_2 \quad = \pm 1$$

Action of $\mathbb{SL}_2(\mathbb{Z})$ on Farey Graph

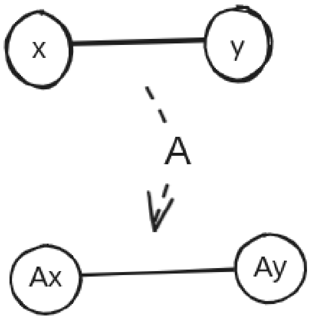
let $\pm(p,q) \in V$ and $A \in \mathbb{SL}_2(\mathbb{Z})$

$$\pm(p,q) * A = \pm A \begin{pmatrix} p \\ q \end{pmatrix}$$

We need to show that this is a valid group action on Farey Graph

1. Group operation well defined
   - $\pm A \begin{pmatrix} p \\ q \end{pmatrix} \in V$
2. Show $Ix = x$ where $x \in V$
3. Show Associativity $e.i$, $AB(\pm x) = A(\pm Bx) = \pm ABx$
   - This is guaranteed by associativity of matrix multiplication

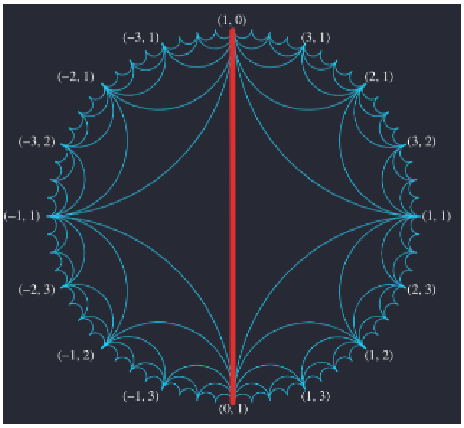
4. Show that adjacency is maintained that is if x, y are connected by an edge then Ax, Ay are also connected



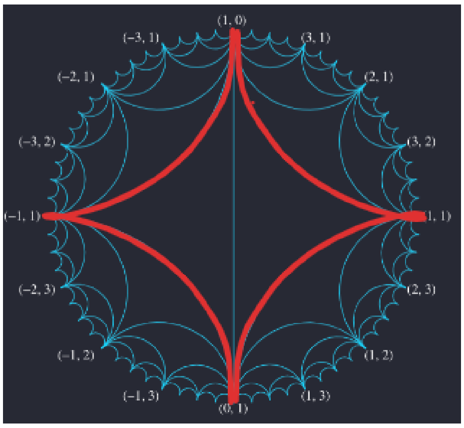
- Recall x , y are connected if $|x\ y| = \pm 1$

Construction of Farey Graph

1. First consider $\pm(1,0), \pm(0,1)$ and obviously they are connected.

- 

2. Now, try to find vertices connected to both $\pm(1,0), \pm(0,1)$ we find $\pm(1,1), \pm(-1,1)$

- 

3. We repeat the process such that if x and y are connect then we find out the the points connected to both x and y it turn out that exactly 2 vertices that are connect to both x and y

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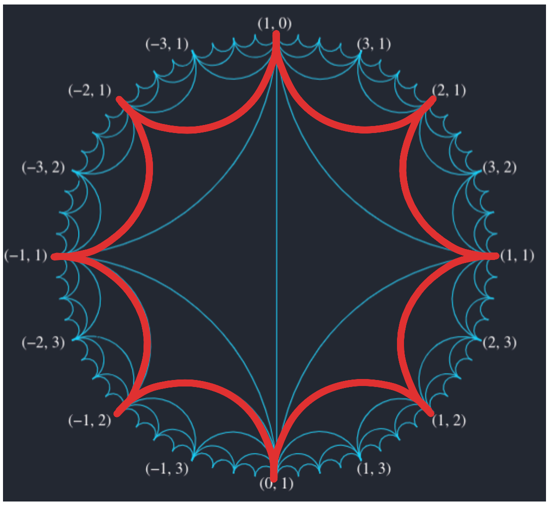

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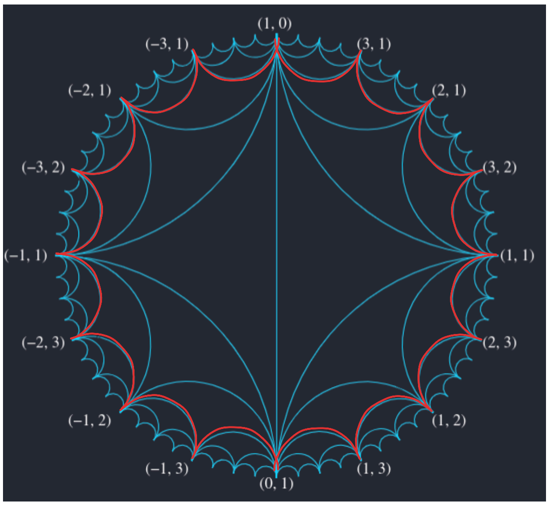


Exercise 1
If $\pm(p_1, q_2), \pm(p_2, q_2)$ are connected then show that vertices connected to both of them are $\pm(p_1 + p_2, q_1 + q_2), \pm(p_1 - p_2, q_1 - q_2)$

From the above exercise we know vertices common to both x and y are $x + y, x - y$ but x,y would already be connected to $x - y$ as if $x - y = z($ say $)$ then $x = z + y$ so, vertex z was connected before x
Hence only new vertices connected is $x + y$

Finding a new vertex to connect both the vertex of an edge is called mediant rule. lets denote applying mediant rule on x,y as $x \oplus y = x + y$
note: mediant rule is only applied only between adjacent vertices

By this we can say that if $x + y = z$ and z was connected in the n^{th} step then x, y was connected in or before $n - 1$ step

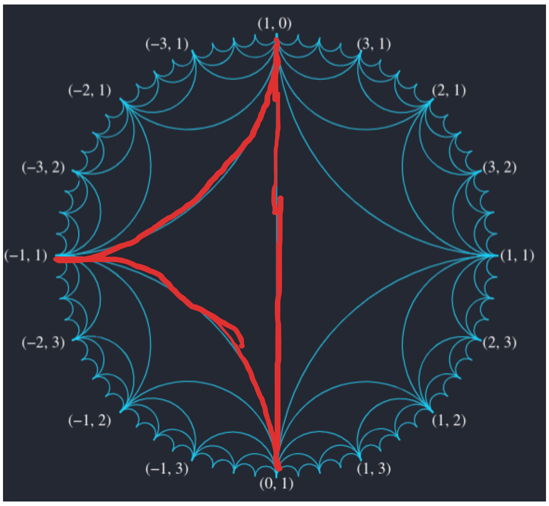
Exercise 2

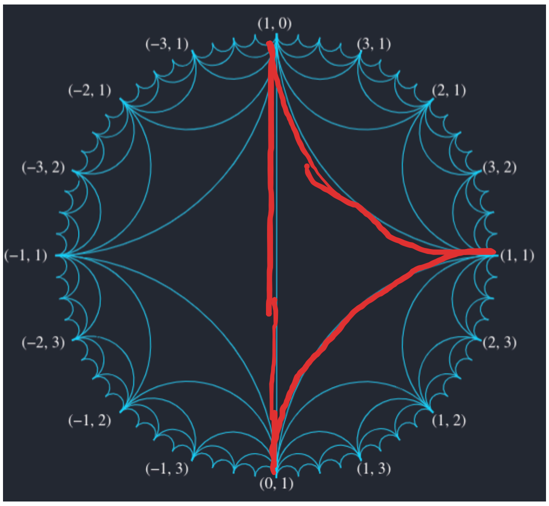
Show that this the above construction indeed constructs whole of Farey Graph
Hint: If (a,b) and (c, d) are connected then say (a, b) was constructed first then we can get (c, d) by mediant rule between (a, b) and $(c - a, d - b)$. Induct on this fact

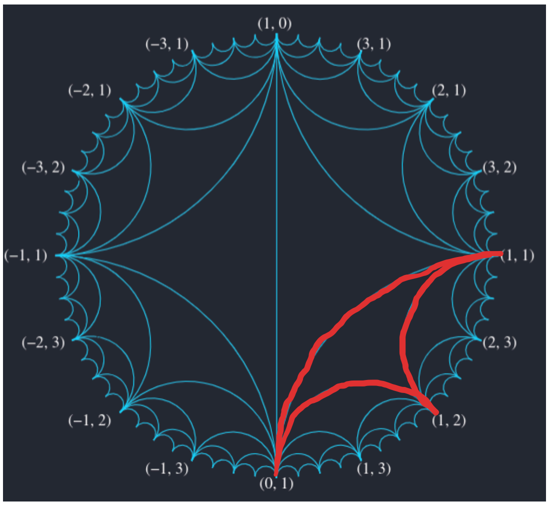
Definition of Farey Complex

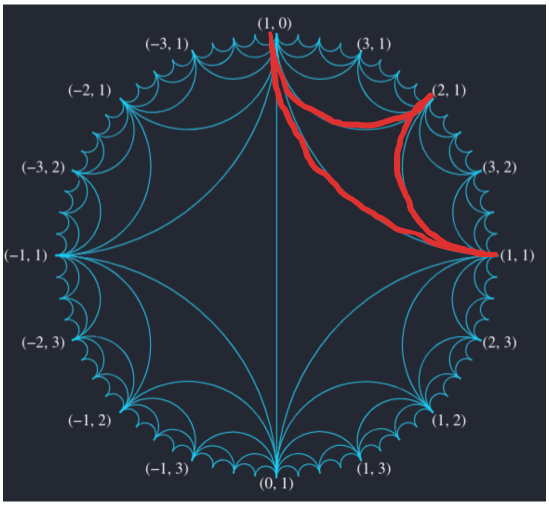
We can observe triangles forming in Farey Graph such that x,y,$z \in V$ and pair wise connected.

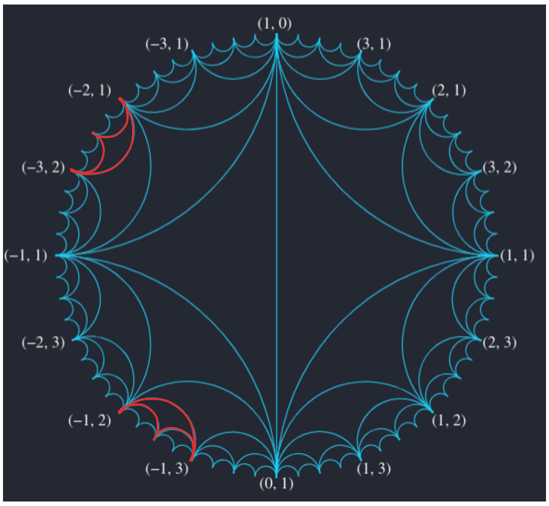
Lets glue up these vertices together and treat it as a single object call it Farey Complex
The group action from $\mathbb{SL}_2(\mathbb{Z})$ can still be induced on The Farey complex
If (x, y, z) is a Farey complex then $(x, y, z)A \rightarrow (xA, yA, zA)$





- 





Definition of Farey Tree

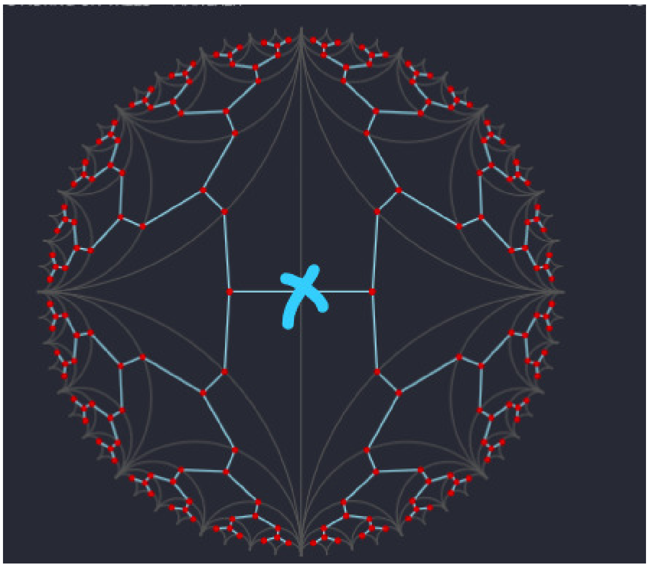
Vertices is the union of both the vertices and edges of Farey complex
An edge is connect to a complex if the edge is in complex

Observations

no 2 edge or 2 vertices are connected
Each edge has exactly 2 connections
Each complex has exactly 3 connections

> **Lemma 3**
>
> *Farey tree is acyclic*

Proof.
Consider any edge and show that if the edge is disconnected then the Farey Tree is separated into 2 different graphs. This shows that there is no Cycle involving that edge. In other words the only vertex connected to the left and right side of the graph is the middle vertex



Now, We can extend this for any arbitrary edge by action of $\mathbb{SL}_2(\mathbb{Z})$.

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Theorem 4

Farey Tree is indeed a tree

Proof.
To show this we need to basically show 2 things

1. The graph is connected
2. The graph is acyclic

Insights on complex
complex has 3 vertices say x y z
x,y are connected and z is then connected to both x,y
so, $x + y = z$ or $x - y = z$
say $x + y = z$ then we call x,y as parents and z as child
if$x - y = z \implies x = y + z$ then $y + z$ are parents and x as child

clearly x, y was connected before z

Finding a path from an arbitrary vertex in Farey Tree to edge joining $\pm(1, 0)$ and $\pm(0, 1)$

Lets find for complex vertex to edge joining $\pm(1, 0)$ and $\pm(0, 1)$
We repeat the following steps until we reach edge joining $\pm(1, 0)$ and $\pm(0, 1)$

1. Find the parent vertex of the complex
    - that is if (x, y, z) is the complex and $x + y = z$ x,y is the parent
2. Move to the edge joining the parent vertex (here edge in Farey complex is a vertex in Farey Tree)
    - here it is edge joining x, y
3. Move to the other complex connected to the edge
   by noting the sequence of vertex we can find a path from any complex to edge joining $\pm(1, 0)$ and $\pm(0, 1)$

After each cycle you move 1 step back in the generation of the graph so after finite number of steps you must reach the end

This can be extend to path starting from an edge (edge in Farey complex is a vertex in Farey Tree)
This shows that Farey tree is complete
we have already shown its acyclic.

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