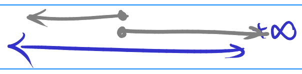
Geometric Group Theory : Ends

Def. A *ray* in (X, d) is just a cont map $[0,\infty) \xrightarrow{\gamma} X$.

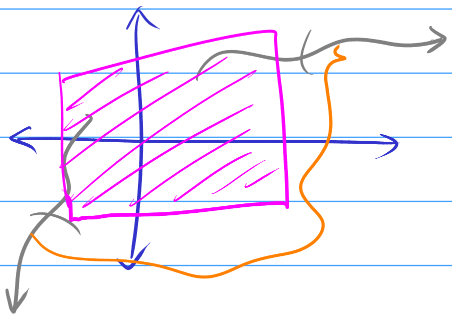
Def. A ray γ is *proper* if for $K \subseteq X$ compact, $\gamma^{-1}(K)$ is cpt.

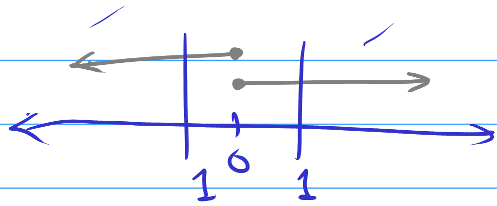


$\mathbb{R}$

$[0,\infty) \to \mathbb{R}$, $t \mapsto t$ $\qquad$ $[0,\infty) \to \mathbb{R}$, $t \mapsto -t$

Def. γ, γ' are two proper rays in X. We say γ and γ' converge to the same end if $\forall K \subseteq X$ cpt, $\exists N \in \mathbb{Z}_{>0}$ s.t. $\gamma[N,\infty)$ and $\gamma'[N,\infty)$ lie in the same path component of $X \setminus K$.





$\mathrm{end}(\gamma) = $ equiv class of γ

$\mathrm{Ends}(X) = \{\mathrm{end}(\gamma) \mid \gamma \text{ proper ray in } X\}$

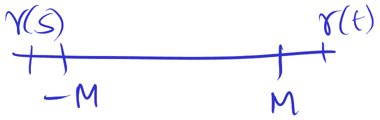
Propn. $\mathbb{R}$ has two ends,

$$\mathrm{Ends}(\mathbb{R}) = \{\mathrm{end}(\gamma_1), \mathrm{end}(\gamma_2)\}$$

$t \mapsto t$ $\qquad$ $t \mapsto -t$

Proof. Take a proper ray $\gamma : [0,\infty) \to \mathbb{R}$.

- If $\gamma(t) \to \infty$ as $t \to \infty$, then $\mathrm{end}(\gamma) = \mathrm{end}(\gamma_1)$.
  (Pick $K \subseteq \mathbb{R}$ cpt. Find $K \subseteq (-R, R)$. Find $M > 0$ s.t. $\gamma[M,\infty) \subseteq (R,\infty)$.)
- If $\gamma(t) \to -\infty$, then $\mathrm{end}(\gamma) = \mathrm{end}(\gamma_2)$.



- $\gamma(t) \not\to \infty$ as $t \to \infty \Rightarrow \exists M>0$ st. $\forall R>0$ $\exists s>R: \gamma(s)<M$.
  $\gamma(t) \not\to -\infty$ as $t\to\infty \Rightarrow \exists M>0$ st. $\forall R>0$ $\exists t>R: \gamma(t)>-M$.
  wlog

  Either one of $\gamma(s), \gamma(t) \in [-M,M]$ or $\gamma(s) < -M < M < \gamma(t)$.
  In the second case, by IVT, $\exists r \in [s,t]: \gamma(r) \in [-M,M]$.
  In either case, $\exists u > R: \gamma(u) \in [-M,M]$.
  $\Rightarrow \gamma^{-1}[-M,M]$ is unbounded but cpt. $\rightarrow\leftarrow$ $\square$

- $\mathbb{R}^n$ has one end (for $n>1$). $[0,\infty) \to \mathbb{R}^n$
  $t \mapsto (t,0,0,\dots,0)$

Want to give Ends(X) a topology

Defn. Let $\{\text{end}(\gamma_n)\} \subseteq \text{Ends}(X)$ be a sequence, $\text{end}(\gamma) \in \text{Ends}(X)$.
We say $\text{end}(\gamma_n) \to \text{end}(\gamma)$ if $\forall K \subseteq X$ cpt, $\exists \{N_n\}$ st.
$\gamma_n[N_n,\infty)$ and $\gamma[N_n,\infty)$ lie in the same path comp of $X \setminus K$
for sufficiently large N.

If $\underset{\to \text{end}(\gamma)}{\text{end}(\gamma_n)} = \text{end}(\gamma_n')$, claim $\text{end}(\gamma_n') \to \text{end}(\gamma)$.
Fix $K \subseteq X$ cpt. Find N_n st. $\gamma_n[N_n,\infty)$, $\gamma[N_n,\infty)$ lie in the
same path comp." Find M_n st. $\gamma_n[M_n,\infty)$, $\gamma_n'[M_n,\infty)$ lie ——
————————. Take $K_n = \max\{M_n, N_n\}$ then
$\gamma_n'[K_n,\infty)$, $\gamma[K_n,\infty)$ lie in the same path comp for large n.

$F \subseteq \text{Ends}(X)$ is closed iff whenever $\underset{F}{\underset{\cap}{\text{end}(\gamma_n)}} \to \text{end}(\gamma)$
then $\text{end}(\gamma) \in F$.

$F_1 \cup F_2$

$\underset{n>1}{\text{Ends}(\mathbb{R}^n)} = \{*\}$, $\text{Ends}(\mathbb{R}) = \{a,b\}$ Top $= \emptyset, \{a,b\}$.

$\mathrm{Ends}(G) := \mathrm{Ends}(C(G;X))$

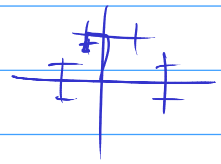
Thm. If X, Y (proper, geodesic) are q.i., then $\mathrm{Ends}(X) \simeq \mathrm{Ends}(Y)$

$\mathbb{Z}, \mathbb{R}$ are q.i. $\Rightarrow \mathrm{Ends}(\mathbb{Z}) \simeq \mathrm{Ends}(\mathbb{R}) \Rightarrow \mathbb{Z}$ has 2 ends

$\mathbb{Z}^n, \mathbb{R}^n$ are q.i. $\Rightarrow \mathrm{Ends}(\mathbb{Z}^n) \simeq \mathrm{Ends}(\mathbb{R}^n) \simeq \{*\}$
$n > 1$ $\Rightarrow \mathbb{Z}^n$ has 1 end

G fin group $\quad \gamma : [0,\infty) \to C(G)$ proper

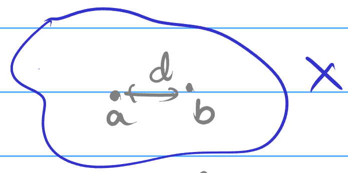
$\Rightarrow \gamma^{-1}(C(G))$ is cpt
$= [0,\infty)$



$\mathrm{Ends}(G) \simeq \emptyset$

$\mathrm{Ends}(F_2) \simeq$ Cantor set

$\Rightarrow F_2$ has ∞ly many ends



$[0,d] \xrightarrow{\varphi} X$

$0 \mapsto a$

$d \mapsto b$

$d(\varphi(t_1), \varphi(t_2)) = |t_1 - t_2|$

for

Thm. If X, Y are proper geodesic metric spaces, and $f : X \to Y$ q.i. embedding, then it induces a $f_* : \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$.
$\mathrm{end}(\gamma) \mapsto \mathrm{end}(f_\#(\gamma))$

$f_\#(\gamma)$ = concatenate geodesic segments $[f \circ \gamma(n), f \circ \gamma(n+1)]$

Metric Spaces of Non-positive Curvature, Bridson-Haefliger
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Theorem. Let G be fin gen.

1. $|\mathrm{Ends}(G)| = 0, 1, 2$ or ∞
2. $\mathrm{Ends}(G)$ is compact, and is either finite or uncountable.
3. G has 0 ends iff G is finite.
4. G has 2 ends iff G is virtually $\mathbb{Z}$ $(\mathbb{Z} \overset{f.i.}{\hookrightarrow} G)$
5. G has ∞ly many ends iff $G = A *_C B$, $[A:C] \geq 3$, $[B:C] \geq 2$
   or $G = A *_C$, $[A:C] \geq 3$

$\mathbb{Z}_2 * \mathbb{Z}_3$ has inf many ends

$\mathbb{Z}_2 * \mathbb{Z}_2$ is q.i. to $\mathbb{Z}$ $\Rightarrow$ $\mathbb{Z}_2 * \mathbb{Z}_2$ has 2 ends $\Rightarrow$ $\mathbb{Z}_2 * \mathbb{Z}_2$ is virtually $\mathbb{Z}$.

Amalgamated Free Product

$A = \langle X_A \mid R_A \rangle$

$B = \langle X_B \mid R_B \rangle$

$i_A : C \hookrightarrow A \qquad i_B : C \hookrightarrow B$

$$A *_C B = \langle X_A, X_B \mid R_A, R_B, i_A(c)\, i_B(c)^{-1} \;\; c \in C \rangle$$

HNN extension

$A = \langle X \mid R \rangle$

$C \leq A, \quad C \overset{\phi}{\hookrightarrow} A$

$$A *_C = \langle X, t \mid R, \; t x t^{-1} = \phi(x) \;\; x \in C \rangle$$