GGT lecture 6

- (X,d) metric space.

Take the set $Y = \{$ quasi isometries $f : X \to X \}$

equiv relation $f \sim g$ if they are within bdd distance of each other

Exercise: SHOW

i.e $\sup_x d(f(x), g(x)) < \infty$

$Y/\sim$ $\}\to$ The equivalence classes.

$=: QI(X)$

$QI(X)$ is a group wrt composition.

(1) Suppose $f : (X, d_1) \to (Y, d_2)$ is a q.i embedding. and say $g : (X, d_1) \to (Y, d_2)$ is equivalent to f (→ within bdd distance)

Then, g is also a q.i embedding.

▸ Pf: $f \sim g \Rightarrow \exists C$ s.t $d_2(f(x), g(x)) \leq C \quad \forall x$

Say f is a (λ, ε)-q.i embedding

Now, for $x, x' \in X$

$$d_2(g(x), g(x')) \leq d_2(g(x), f(x)) + d_2(f(x), f(x')) + d_2(f(x'), g(x'))$$

$$\leq 2C + d_2(f(x), f(x')) \quad [\because \text{bdd distance}]$$

$$\leq 2C + \lambda d(x, x') + \varepsilon \quad [\because f \text{ is q.i embed}]$$

$$\frac{1}{\lambda} d(x,x') - \varepsilon \leq d_2(f(x), f(x')) \leq d_2(f(x), g(x)) + d_2(g(x), g(x')) + d_2(g(x'), f(x'))$$

$$\leq 2c + d_2(g(x), g(x'))$$

$$\Rightarrow \frac{1}{\lambda} d(x,x') - \varepsilon \leq 2c + d_2(g(x), g(x'))$$

$$\Rightarrow \frac{1}{\lambda} d(x,x') - (\varepsilon + 2c) \leq d_2(g(x), g(x'))$$

So, g is a $(\lambda, 2c+\varepsilon)$ - q.i embedding.

② Exercise: If f is q.i and $g \sim f$, then g is a q.i

③ Composition of q.i embeddings is a q.i embedding.

Pf:

$$X \xrightarrow[\lambda, \varepsilon]{f} Y \xrightarrow[\lambda', \varepsilon']{g} Z$$

To show: $g \circ f$ is q.i embedding.

$$d_Z(g\circ f(x), g\circ f(x')) \leq \lambda' d_Y(f(x), f(x')) + \varepsilon' \quad [\because g \text{ is q.i embedding}]$$

$$\leq \lambda'(\lambda d_X(x,x') + \varepsilon) + \varepsilon'$$

$$\leq \lambda\lambda' d_X(x,x') + (\varepsilon' + \varepsilon\lambda')$$

$$\frac{1}{\lambda} d(x,x') - \varepsilon \leq d(f(x), f(x'))$$

$$\Rightarrow d(x,x') \leq \lambda d(f(x), f(x')) + \lambda\varepsilon$$

$$\leq \lambda(\lambda' d(g\circ f(x), g\circ f(x')) + \varepsilon') + \varepsilon$$

$$\leq \lambda\lambda' d(g\circ f(x), g\circ f(x')) + (\lambda\varepsilon' + \varepsilon)$$

(4) f, g q.i $\Rightarrow$ $g \circ f$ is a q.i.

Pf:- Use (3) +

Take $z \in Z$

As g is a q.i, $\exists\ c \geq 0$ and $y \in Y$ s.t $d(g(y), z) \leq c$

Now, $\exists\ x \in X$ and c' s.t $d(y, f(x)) \leq c'$ [$\because f$ is a q.i]

Now, $d(z, g\circ f(x)) \leq d(z, g(y)) + d(g(y), g\circ f(x))$

$\leq c + \lambda d(y, f(x)) + \varepsilon$

$\leq c + \lambda c' + \varepsilon$

- Towards the inverse :-

(5) Suppose $f : X \longrightarrow Y$ is q.i. Then $\exists$ q.i $g : Y \longrightarrow X$ and a constant k s.t

$$\begin{cases} d(f\circ g(y), y) \leq k \\ d(g\circ f(x), x) \leq k \end{cases} \quad \forall x \in X\ \forall y \in Y$$

Pf:- f is (λ, ε) q.i embedding.
$\exists\ c$ s.t $\forall y \in Y$, $\exists\ x \in X$ s.t $d(f(x), y) \leq c$

$g : Y \longrightarrow X$
$y \longmapsto x$

Claim :- g is the req map.

Step:- Check q.i

step 2 :

Exercise

Q: Multiple inverses ?

(6) Ans: Any two inverses are at a bdd distance from each other.

Exercise

$\Rightarrow$ $QI(X)$ is a group with composition.

$\downarrow$

quasi-isometry group of X

• Def^n: Two metric spaces (X, d_1) and (Y, d_2) are called quasi-isometric if $\exists$ a q.i $f : (X, d_1) \to (Y, d_2)$.

Imp!! Thm: Suppose (X, d_1) and (Y, d_2) are quasi isometric.

Then $QI(X) \overset{\text{group isomorphic.}}{\cong} QI(Y)$

▸ Pf: X, Y quasi isometric.

So, $\exists$ function $f : X \to Y$ which is a q.i.

$$f_* : QI(X) \longrightarrow QI(Y)$$
$$[\alpha] \longmapsto [f\alpha f^{-1}]$$

q.i

$Y \to Y$

$$Y \xrightarrow{f^{-1}} X \xrightarrow{\alpha} X \xrightarrow{f} Y$$

composition of q.i is q.i

Claim: f_* is a homomorphism.

Define similarly $(f^{-1})_*$

$f^{-1} : Y \longrightarrow X$

$QI(Y) \longrightarrow QI(X)$

Pf : $f_*([\alpha]\cdot[\beta]) = f_*[\alpha\circ\beta]$

$$= f(\alpha\circ\beta)f^{-1}$$

$$\simeq f\alpha\beta f^{-1}$$

$$= f\alpha f^{-1}\, f\beta f^{-1}$$

$$= f_*([\alpha])\, f_*([\beta])$$

$\blacksquare$

Groups :

Suppose Γ is a group, X is a gen set.

for any $g\in\Gamma$, $g = \underbrace{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}}_{\text{word}} \to k$, $x_i\in X$, $\varepsilon_i = \pm 1$

$|g|$ = min length of words wrt X

$$(\underbrace{a}_{1},\ \underbrace{aa^{-1}a}_{3},)$$

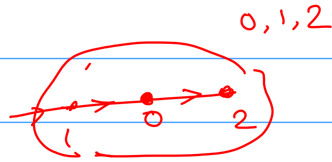
define the word metric, $d_X(g,h) = |g^{-1}h|$

- Path metric on Cayley graph.

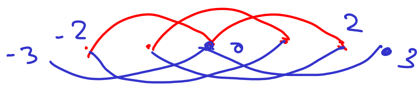
Say group Γ, generating sets X, X'

Then, (Γ, d_X) and $(\Gamma, d_{X'})$ are not isometric.

- $\Gamma = \langle a\rangle$, $X = \{a\}$, $X' = \{a, a^2\}$



$$|B(1,1)|_X = 3, \quad |B(1,1)|_{X'} = 5$$



- **Thm**:- Suppose X, X' are two finite generating sets of Γ. Then (Γ, d_X) and $(\Gamma, d_{X'})$ are q.i.

- **Pf**:- Suppose $g \in \Gamma$, suppose $g = x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$ is a rep^n of g of shortest length in words in X.

Then, $|g|_X = k$

Then, $|g|_{X'} = |x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}|_{X'} \leq |x_1^{\varepsilon_1}|_{X'} + \cdots + |x_k^{\varepsilon_k}|_{X'}$

let, $\max\limits_{x \in X} |x|_{X'} = k_1$, Then $|g|_{X'} \leq k k_1 = k_1 |g|_X$

Similarly, $\exists\, k_2$ s.t $|g|_X \leq k_2 |g|_{X'}$

Take $k > k_1, k_2$

$$\frac{1}{k} d_X(g,h) \leq d_{X'}(g,h) \leq k\, d_X(g,h)$$

$\blacksquare$

- $QI(\Gamma) = QI(\Gamma, d_X)$ wrt some generating set X.

- X is a bdd metric space, then X is q.i to $Y = \{*\}$

$\Rightarrow$ any two bdd spaces are q.i.

False

- If X is bdd and X is q.i to Y, then Y is also bdd.

- If Γ, G are say two diff groups
X, X' of Γ; Y, Y' of G

$(\Gamma, d_X) \xrightarrow{f}$ is q.i (G, d_Y)

then $(\Gamma, d_{X'})$ is q.i to $(G, d_{Y'})$

F_2, F_n, $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^n$, $\mathbb{Z}/n\mathbb{Z}$, ... $\mathbb{Z}_2 * \mathbb{Z}_2$

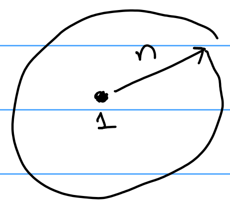
$\mathbb{Z}_2 * \mathbb{Z}_3$

Growth of Groups

Γ is fin gen group and $\underline{\underline{X}}$ is a generating set.

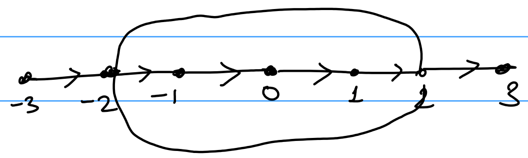
$$f_{\Gamma, X} : \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto \# \{ y \in \Gamma \mid d_X(1, y) \leq n \}$$

growth fun^n of Γ wrt X.



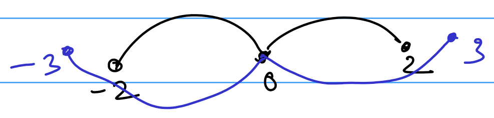
- Example: ① $\mathbb{Z}$, $\{1\}$

$f(1) = 3$
$f(2) = 5$
$\vdots$
$\boxed{f(n) = 2n+1}$



② $\mathbb{Z}$, $\{2, 3\}$

$f(1) = 5$

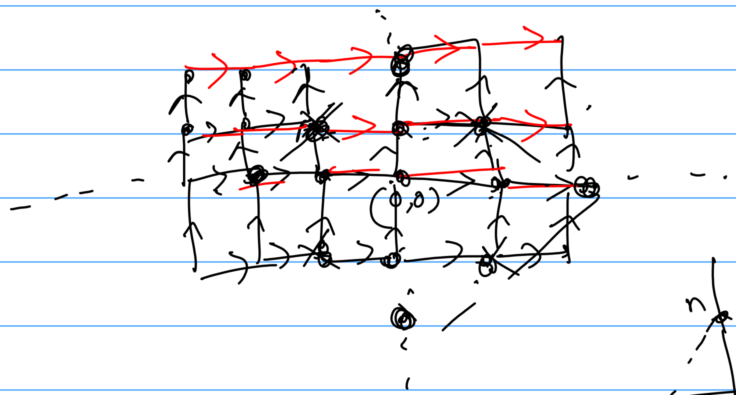


$(0, 2, 3, -2, 3)$

Exercise: $\boxed{f(n) = 6n+1}$ for $n > 1$

③ finite groups, function will be constant after some n.

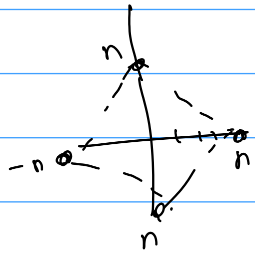
④ $\mathbb{Z}^2$, $\{(0,1),(1,0)\}$



$f(1) = 5$

$f(2) = 13$

$4n + 4(n-1) + 4(n-2) + \dots + 4 + 1$



$2\cancel{4}\cdot\frac{n\cdot(n+1)}{\cancel{2}} + 1$

$$\boxed{f(n) = 2n^2 + 2n + 1}$$

$\mathbb{Z}$ is not q.i to $\mathbb{Z}^2$

- **Def^n:** Suppose $f, g: \mathbb{N} \to \mathbb{N}$ are increasing functions

  Say $f \preceq g$ if $\exists k > 0$ s.t. $f(x) \leq k\, g(kx+k) + k$

  We will say $f \approx g$ if $f \preceq g$ and $g \preceq f$

- **Claim:** $\approx$ is an equivalence relation

  - Symmetry: $f \approx g$, then $g \approx f$.
  - (*) Reflexivity: $(f \approx f)$ $\quad f(x) \leq k\, f(kx+k) + k$

    $f(x) \leq k_1 f(k_2 x + k_3) + k_4$

\# (x)

$k = 1 \Rightarrow f(x) \leq f(x+1) + 1$ for increasing functions.

- Transitivity: $(f \approx g \approx h$, to show: $f \approx h)$

$f(x) \leq k g(kx+k)+k \qquad g(x) \leq k' h(k'x+k')+k'$

$$\leq k\left[k' h\left(k'(kx+k)+k'\right)+k'\right]+k$$

$$= \underline{kk'} h\left(\underline{kk'}x+\underline{(kk'+k')}\right)+\underline{(kk'+k)}$$

Chose a $c >$

Then $f(x) \leq c h(cx+c)+c$ ✓

- Examples :- $6x+1 \approx 2x+1$

Exercise :- ① $x^n \approx a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

② $x^n \approx x^m \iff n=m$

$6x+1, 2x^2+2x+1$ were diff.

Say $n<m$, then $x^n \preceq x^m \qquad x^n \leq x^m$

Say $x^m \preceq x^n$, $\quad x^m \leq k(kx+k)^n+k$

$= k(a_0x^n+a_1x^{n-1} \cdots a_k)+k$

$$x^m/x^n \leq \underline{a_0k} + \cdots$$

$$\underline{x^{m-n}} \leq a_0 k$$

+ve

$x \longrightarrow \infty \quad \Rightarrow \quad x^{m-n} \to \infty$

Contradiction

- Thm :- Γ is a fin generated groups. X and Y finite gen sets.

then $f_{\Gamma,X} \approx f_{\Gamma,Y}$

▸ **Pf:** Γ. , X, Y gen sets

T.S: $f_{\Gamma, X} \approx f_{\Gamma, Y}$.

we have already showed $\frac{1}{K} d_X(1,g) \le d_Y(1,g) \le K d_X(1,g)$

If $d_X(1,g) \le x \Rightarrow d_Y(1,g) \le Kx$

$$g \in \overline{B}_{d_X}(1,x) \Rightarrow g \in \overline{B}_{d_Y}(1,Kx)$$

$$\Rightarrow g \in \overline{B}_{d_Y}(1, Kx+K)$$

$$\Rightarrow f_X(x) = |\overline{B}_{d_X}(1,x)| \le K|\overline{B}_{d_Y}(1; Kx+K)| + K$$

$$= K f_Y(Kx+K) + K$$

$$\Rightarrow f_X \preceq f_Y$$

$$\Rightarrow f_X \approx f_Y.$$

■