- Today's Goal:

→ Some Graph Theory

→ Generating set, Presentations

→ Cayley Graph ($\Gamma(G,S)$)

→ Free Groups ($\mathbb{F}_n$)

→ Action on trees ($\mathbb{F}_2 \curvearrowright \Gamma(\mathbb{F}_2, \{a,b\})$)

→ Review Group Action

→ Induced Homomorphism.

→ Metric Spaces & Isometries.

→ Diff types of Grp actions.

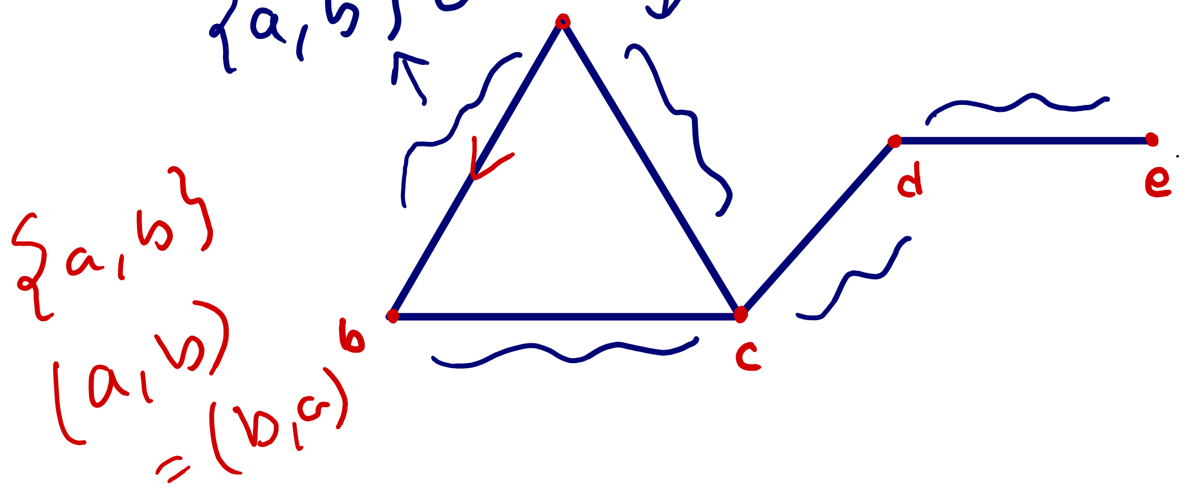
- Some Basic Graph Theory:

<u>Defⁿ</u>: A finite simple undirected graph G is a pair of set (V, E), where V denotes vertex set, E denotes edge set, $E \subseteq \{ T \mid T \subseteq V, |T| = 2 \}$

Ex: $G = (V, E)$, $V = \{a, b, c, d, e\}$

$E = \{\{a,b\}, \{c,a\}, \{c,b\}, \{c,d\}, \{d,e\}\}$

$\{a,b\} \in E$

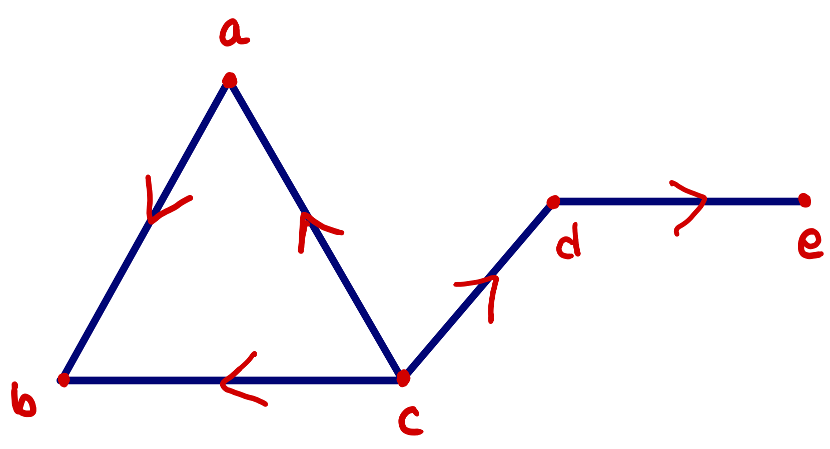


Qn. Can we give directions?

Yes!, then E will be ordered tuples.

Ex: $G = (V, E)$, $V = \{a, b, c, d, e\}$

$E = \{(a,b), (c,a), (c,b), (c,d), (d,e)\}$



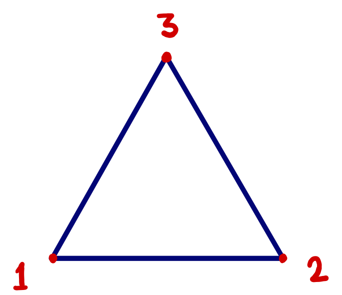
Some nice graphs:

- Complete Graph (K_n):

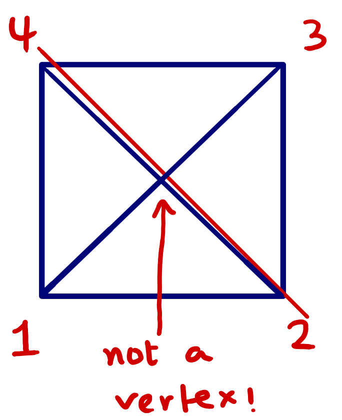
$V = \{1, 2, \dots n\}$, $E = \{\{i,j\} : 1 \le i < j \le n\}$

Ex:

K_3:



K_4:

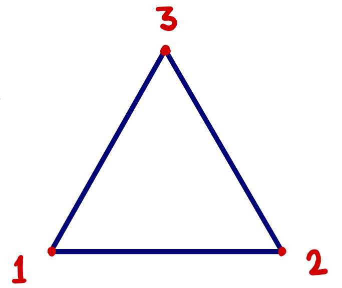


• nth cycle (C_n) $(n \geq 3)$

$V = \{1, 2, \dots n\}$

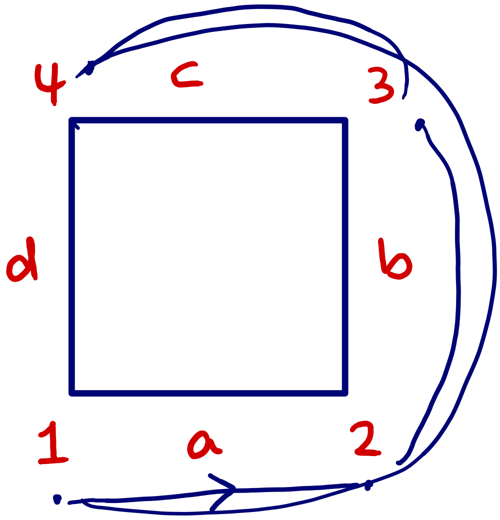
$E = \{\{i, i+1\} : i \in \{1, \dots n-1\}\} \cup \{\{1, n\}\}$

Ex: C_3 :



C_3

C_4 :



C_4

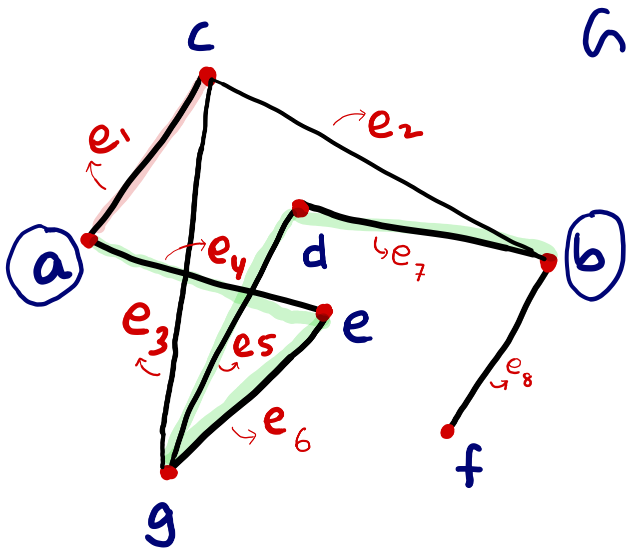
• Path in a graph: let G be a graph then a path is a seq of edges & vertices in which none of the edge or vertices repeat.

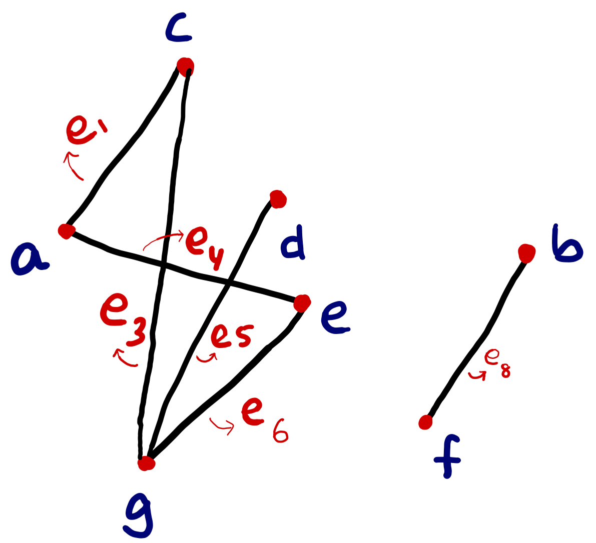
Ex: 1 a 2 b 3 c 4 ✓

1 a 2 b 3 c 4 d 1 ✗

Connected graphs: A graph G is said to be conn. if for $a, b \in V$, a path b/w a and b.

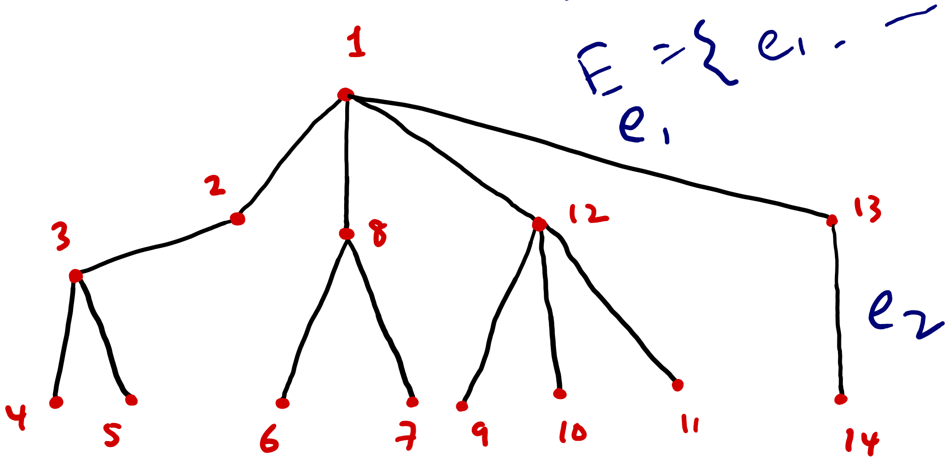
Ex:





Tree: A conn. graph with no cycle.

$V = \{1, \ldots 14\}$.
$E = \{e_1, \ldots\}$.



• Symmetries of a Graph?

Yes! $\varphi : V \to V$

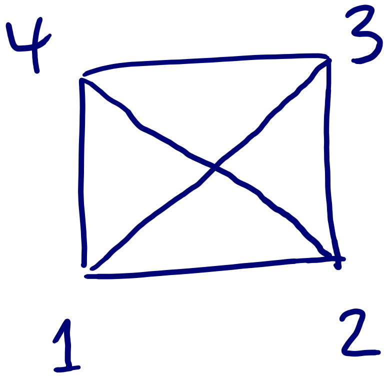
Def$^{\underline{n}}$: A bij. map $\varphi : V \to V$, is an automorphism of V, if for $a, b \in V$,

$$\{a,b\} \in E \iff \{\varphi(a), \varphi(b)\} \in E$$

Set of all these 'φ's forms a group, $(Aut(G))$ Γ

$Aut(\Gamma) = \{\varphi \mid \varphi : V \to V\}$

Ex: K_n:



$\text{Aut}(K_4) = ?$

$f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$

Exc:

$\text{Aut}(K_n) =$

Aut (G) = set of all automorphisms of G.

(Aut (G), ∘) ← fn. comp.

$f_1 \circ f_2 \in$

• Back to group theory:-

→ Group Actions.

→ Def$^{\underline{n}}$: (words) A word in the letter a, b is a string of finite length (but arbitarily long) which is made up of symbols 'a', 'b', 'a^{-1}', 'b^{-1}'.

set of finite tuples (a, b, a, a^{-1}, a, b)

just a notation!

Ex: $aba^{-1}bba^{-1}aa$, $aabb^{-1}bbba^{-1}$,

there's empty word too... ε = " ".

" __ "

Qn. How to multiply two words?

$\underset{w_1}{(aba^{-1})} \cdot \underset{w_2}{(abaa)} = aba^{-1}abaa$

Def$^{\underline{n}}$: (reduced words) A word with an additional prop. that we never see

a & a^{-1} together (same for b) $\left\{ \begin{matrix} aa^{-1} = \varepsilon \\ bb^{-1} = \varepsilon \end{matrix} \right\}$

Ex: $ab\underbrace{a^{-1}a}_{\varepsilon}baa = abbaa$

Don't want this!

Exc: Show that this process of reducing is unique.

Free grp on 2 letters. $\{a,b\}$ x, y

Thus, $\mathbb{F}_2 = \{$ all "reduced" words using a and $b\}$

forms a group using "$\cdot$" where "$\cdot$" stands for string concatenation & then reduction.

$(ab^{-1}) \in \mathbb{F}_2$, $(baa) \in \mathbb{F}_2$ $\Big\}$ $ab^{-1}baa = aaa \in \mathbb{F}_2$ $\varepsilon \in \mathbb{F}_2$

- Generators & Relations.

$e \notin S$
$S \cap S^{-1} = \phi$

Def^n (Generating Set) A gen. set $S \subseteq G$ of a group is a "subset" of G s.t. every element of G can be expressed as a combination of finitely many elements of the subset & their inverses. Ex: $G = \mathbb{Z}$, $S = \{1\}$

- Group Presentation $\bar{1}^n = \bar{0}$

$$\mathbb{Z}/n\mathbb{Z} = \underbrace{\{\bar{0}, \bar{1}, \dots \overline{n-1}\}} = \langle \bar{1} \mid \bar{1}^n = 0 \rangle$$

$$= \langle \underbrace{1} \mid \underbrace{1^n = 0} \rangle$$

gene. id.

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n = \bar{0} \rangle$$

$$G = \langle \underbrace{s_1 \dots s_n} \mid \underbrace{R_1 \dots R_k} \rangle$$

$$\mathbb{Z} = \langle 1 \rangle$$

$n = \underbrace{1 + \dots + 1}$ $\mathbb{Z} = \langle 1 \mid \dots \rangle$

$$\mathbb{F}_2 = \langle a, b \mid \text{no relation} \rangle$$

let (G, S) be a grp with $G = \langle S \rangle$

We make a graph using vertices of group G & edges to be directed & labelled as: (G, S)

$V = \{g_1, \dots g_n\}$.

$E =$

$$g \xrightarrow{s} gs = g_1$$

where $s \in S$.

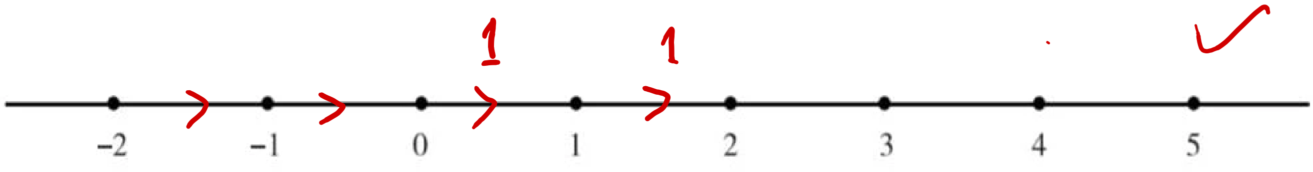
$g_0 \xrightarrow{s}$ $g_1 = g_0 s$, $s \in S$

$g_1 = g_0 s$.

S

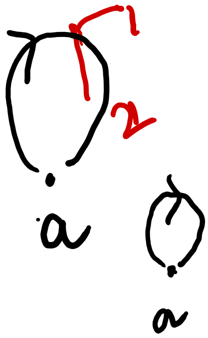
Ex: $(\mathbb{Z}, \{1\})$

Γ_1



$0+1$ $1+1$

$0 \in S$



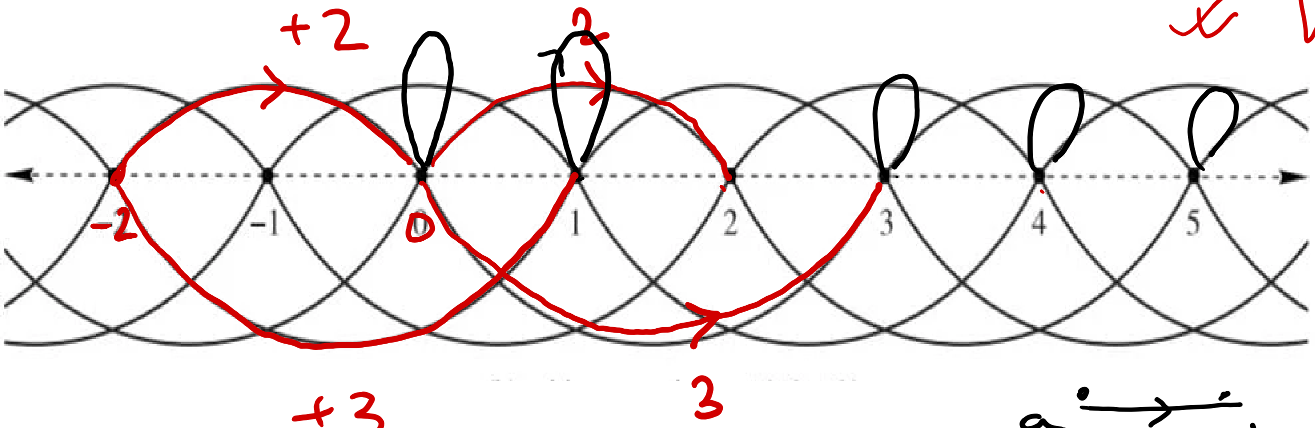
$0 \in S$

$\cong$

F_1

$(\mathbb{Z}, \{2, 3\})$

+2



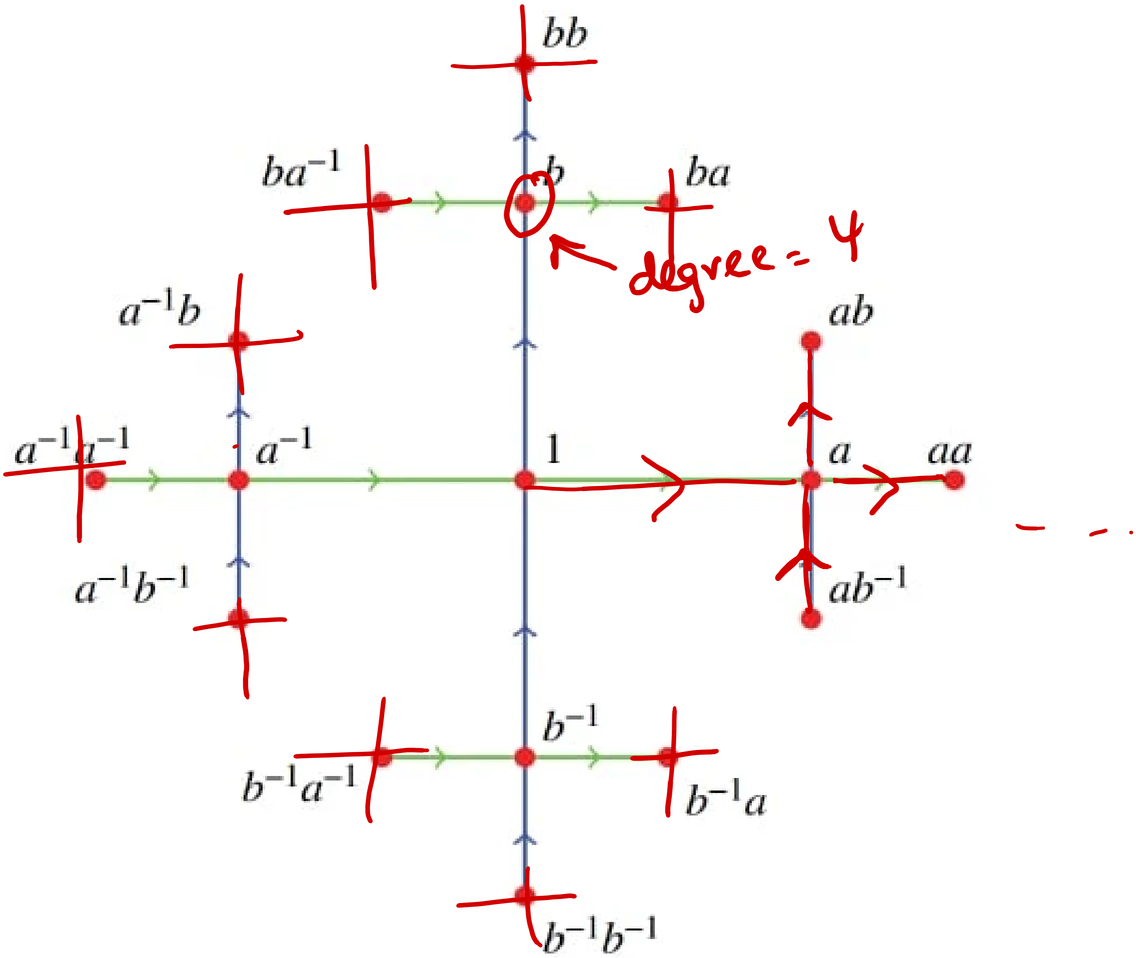
+3

$a \xrightarrow{s} b = as$

Γ_1 & Γ_2

$\Gamma_1 \overset{Q.I}{\sim} F_2$ $\Gamma_1 \not\cong \Gamma_2$

$$(\mathbb{F}_2, \{a, b\})$$ with b^{-1} written above b



$\Gamma \leftarrow$ cayley graph of a

Exc: Let $\Gamma = \Gamma(G, S)$ $S \subseteq G$ where G is a group & $S \subseteq G$, then: $S \subseteq G$

(i) Γ is conn. iff S generates G.

(ii) Γ contains a self loop iff $e \in S$

(iii) Γ has atmost one edge b/w 2 vers, iff $S \cap S^{-1} = \phi$.

iff $\langle S \rangle = G$

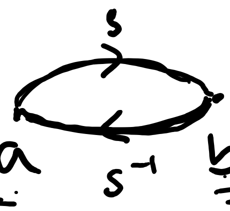
$\Gamma \leftarrow$ conn.

$\langle S \rangle =$ "grp gen by S."

$S = \{2, 3\}$, $\mathbb{Z}$

$S' = \{0, 2, 3\}$ $2S$

$S = \{s, s^{-1}\}$



$bs^{-1} = a$ $b = as$

$b = as \Rightarrow a = bs^{-1}$

Cayley $=$ $\mathrm{Cay}(G,S)$.

$\Gamma(G,S)$

$=$

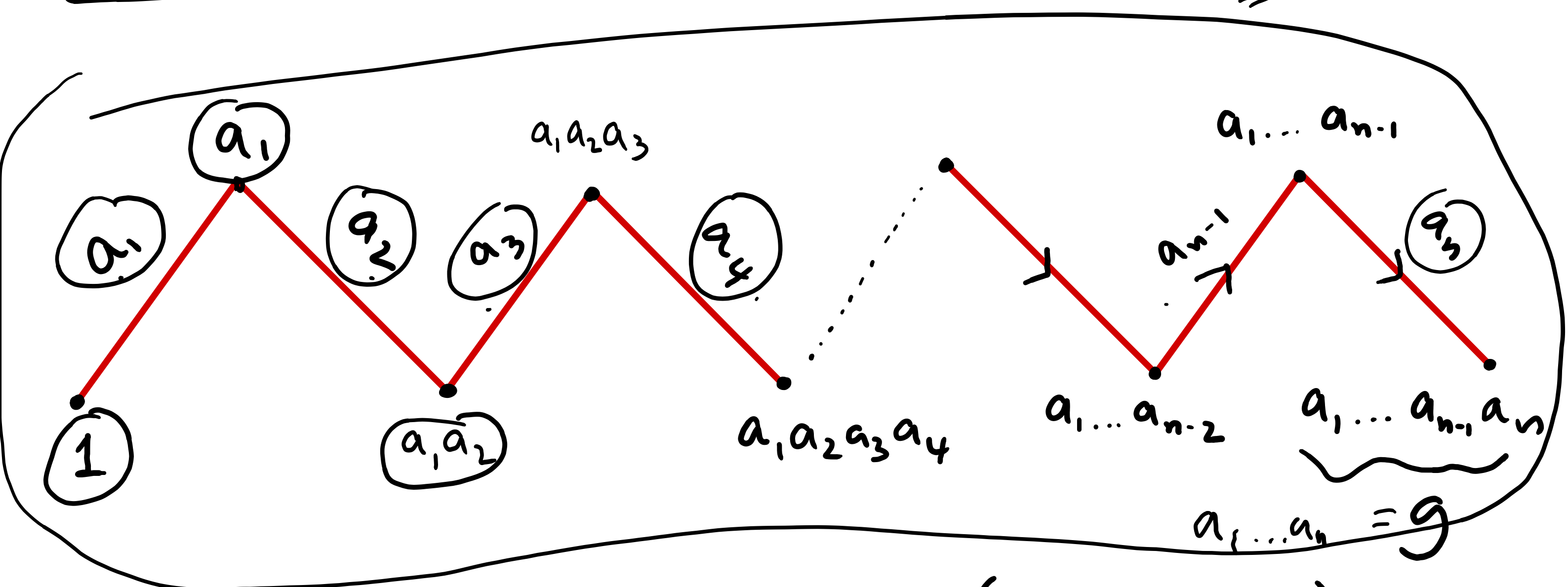
$G \curvearrowright \Gamma$

$$\begin{tikzcd} G \arrow[r, "\text{th.}"] \arrow[r, bend right] & \Gamma(G) \end{tikzcd}$$

$\Gamma(G)$

$\Gamma(G)$
$\parallel$
(V, E)

<u>Observation</u> :



$a_1, \dots, a_n$ are labels (in order) on an edge path of $\Gamma(G,S)$ from say 1 to g, then $g = \underline{\underline{a_1 \dots a_n}}$

Conversely if $g = a_1 \dots a_n$ then $\exists$ an edge path with labels $a_1 \dots a_n$.

$G \curvearrowright \Gamma(G,S)$ Grp act on a grp $\Gamma(G,S)$

$G \curvearrowright X$ Grp action on set X.

$\Gamma(G,S)$

E, V

X

$\Phi : G \times X \to X$

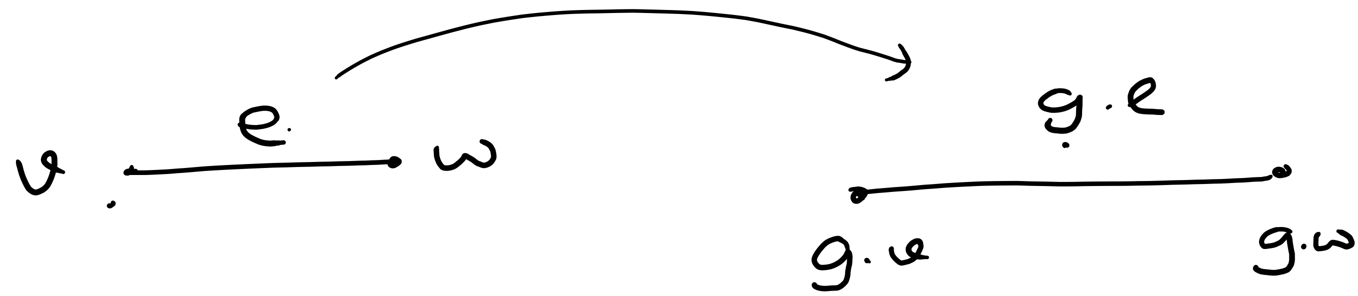
$$\begin{cases} e \cdot x = x \\ g \cdot (h \cdot x) = (gh) \cdot x \end{cases}$$

$$\begin{tikzcd}
x \arrow[rr, dash, "gh"] \arrow[dr, dash, "h"'] & \circlearrowright & \tilde{x} \\
& x' \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\left.\begin{aligned} &e\cdot x = x \\ &gh\cdot x = g\cdot(h\cdot x) \end{aligned}\right\} \quad \underline{\underline{G \curvearrowright X}}$$

$$\underline{\underline{G \curvearrowright \Gamma(V,E)}}$$

$$\left.\begin{cases} \underline{\underline{e\cdot x = x}}, & \forall\, \underline{\underline{v}}, e \\ (g\cdot h)\cdot x = (gh)\cdot x & \forall\, x \in \underline{\underline{V}} \text{ or } \underline{\underline{E}}. \end{cases}\right\} \text{ Grp action on } \underline{\text{grp}}.$$



Exc: left mul. (here) is a grp.

$$\underline{G \curvearrowright G}.$$

$$G \curvearrowright \Gamma(G,S)$$

$\Gamma(G,S) \leftarrow$ cayley graph.

$$\mathrm{Aut}(\Gamma(G,S))$$

$$\|$$

$$\Phi : V(\Gamma) \to V(\Gamma)$$

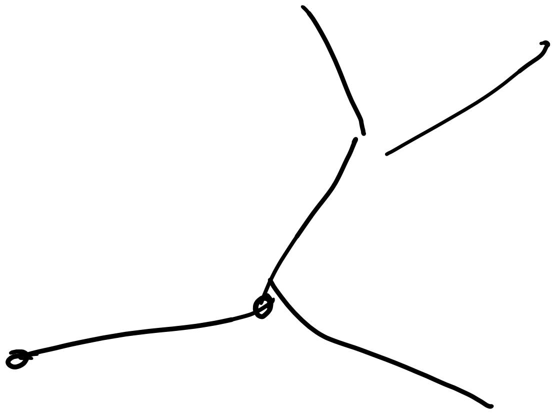
$g \in G$

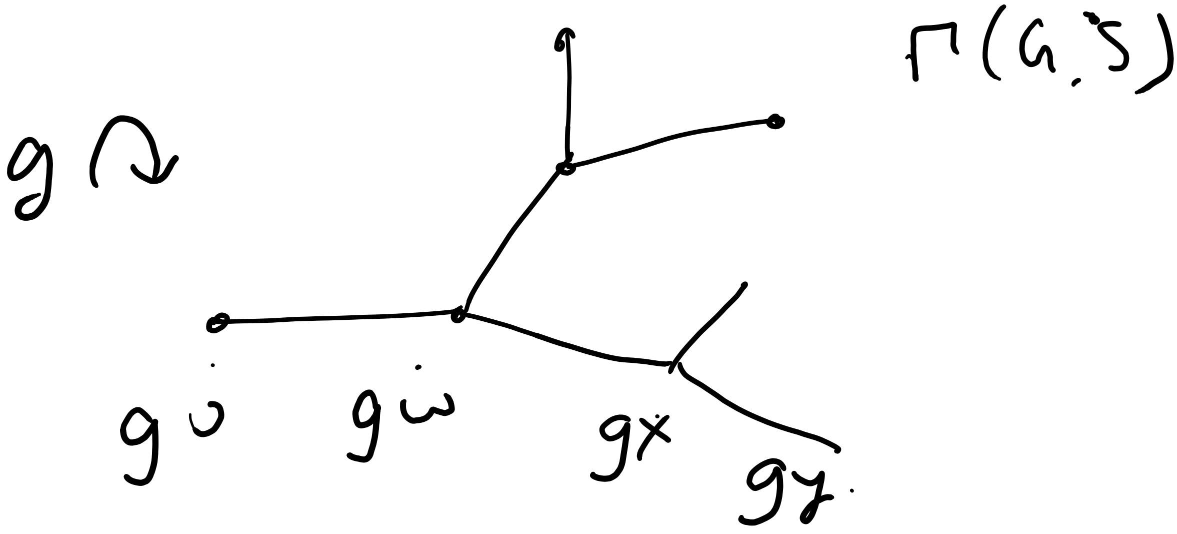
$$\Gamma$$

$$g \longmapsto \Phi_g(v) = g \cdot v = gv.$$

g

$$\begin{tikzcd}
v \arrow[r, "s"] \arrow[d, "\Phi_g"'] & vs \arrow[d, "\Phi_g"] \\
gv \arrow[r] & gvs
\end{tikzcd}$$





$$g \longmapsto \Phi_g$$

$$G \longrightarrow \underbrace{\mathrm{Aut}(\Gamma(G,S))}_{\Phi}.$$

Φ $\quad g \xrightarrow{\Phi_g}$

g'

$g \in G$

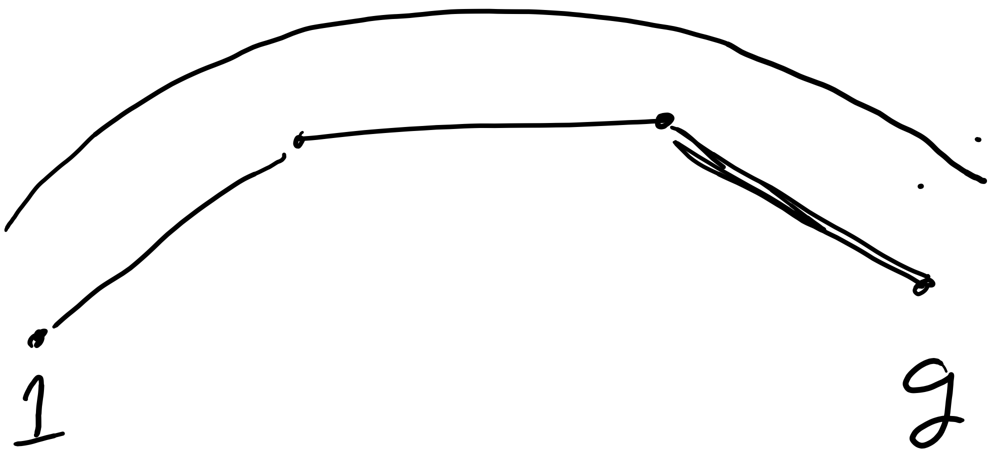
$\ell(g) :=$ length of the word g

$=$ len. of the shortest word in $S \cup S^{-1}$ which is equal to g.

$S =$

$\ell(g) < \infty$

$\ell(e) = 0$

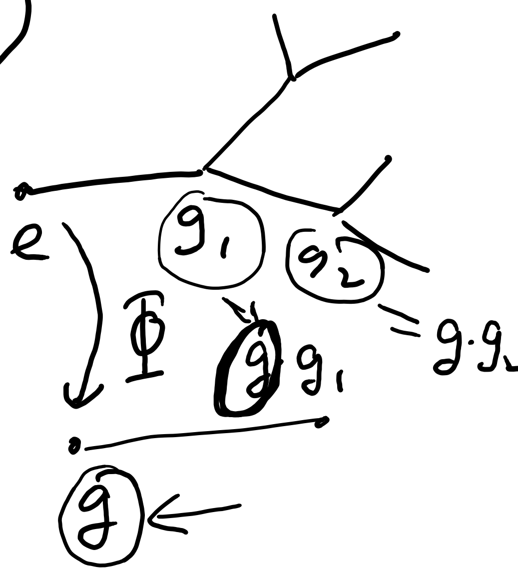


$\Phi \in \mathrm{Aut}(\Gamma(G,S))$

$=$

$\Phi = \Phi_{\tilde{g}}$ //

$\Phi(e) = g$



$\tilde{g} \stackrel{?}{=} g$ //

Base Case $(\underbrace{\ell(g) = 0}_{e})$

$\Phi(e) \stackrel{?}{=} \Phi_g(e)$

$\Phi(e) = g$

$\Phi_g(e) = g \cdot e = ge = g$

$\ell(v) = n$, $\Phi(v) = \Phi_g(v)$.

$\ell(w) = n+1$,

$w = v \cdot s$, $s \in S \cup S^{-1}$

$s \in S$

$\Phi(w) = \Phi_g(w)$ a b iff $S \cap S^{-1} = \phi$

$$\begin{tikzcd}
\Phi(v) \arrow[r, "s"] & \Phi(v)s \arrow[dd, leftrightarrow, bend left] \\
v \arrow[u, "\Phi"] \arrow[r, "s"] \arrow[d, "\Phi_g"'] & w \\
\Phi_g(v) \arrow[r, "s"'] & w
\end{tikzcd}$$

$\Phi_g = \Phi$

$\Phi_g(v) \cdot s = \Phi(v) \cdot s$

Exc: $\Phi \in Aut(\Gamma(G,S))$, Φ preserves labels.

$$\Phi_g(\omega) = \Phi(\omega).$$

$$\Phi \in \mathrm{Aut}(\Gamma(G,S))$$

$$\downarrow$$

$$g$$

$$G \overset{\sim}{\longleftrightarrow} \mathrm{Aut}(\Gamma(G,S))$$

$$f : G \longrightarrow \mathrm{Aut}(\Gamma(G,S))$$
$$g \longmapsto \Phi_g$$

$$\Phi$$

Ex: $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$

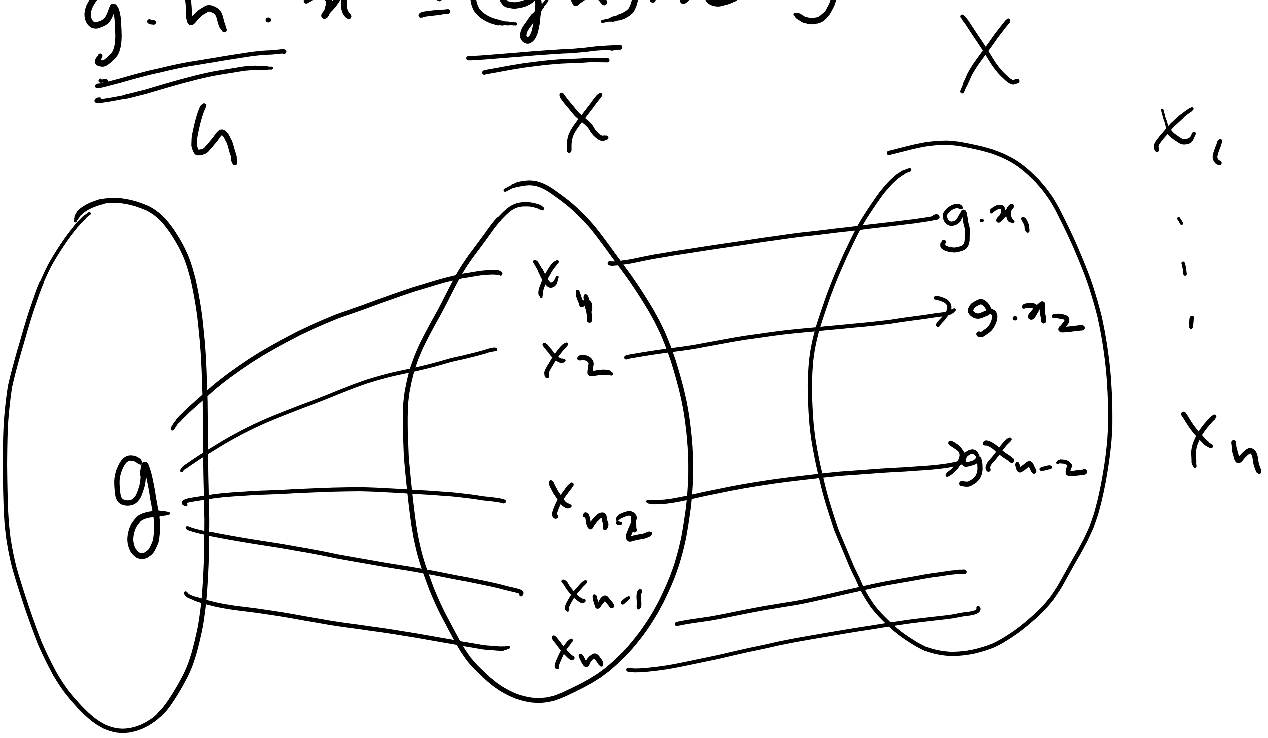
$$\mathrm{Aut}(\Gamma(G, S)) \cong G.$$

$$\begin{tikzcd}
G \arrow[rr, dash, "\cong"] \arrow[dr] & & \mathrm{Aut}(\Gamma(G,S)) \\
& \Gamma(G,S) \arrow[ur] \arrow[dl] \arrow[d, mapsto] & \\
s_1 & s_2 &
\end{tikzcd}$$

$G \curvearrowright X$

$$\left.\begin{array}{l} e \cdot x = x \\ g \cdot h \cdot x = (gh) \cdot x \end{array}\right\} \text{Formal}$$

$\Downarrow$

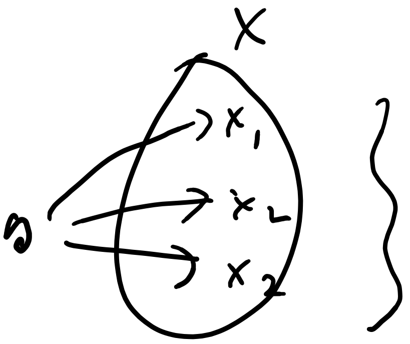


$g \quad X \longrightarrow X$

$\Phi : G \times X \longrightarrow X$

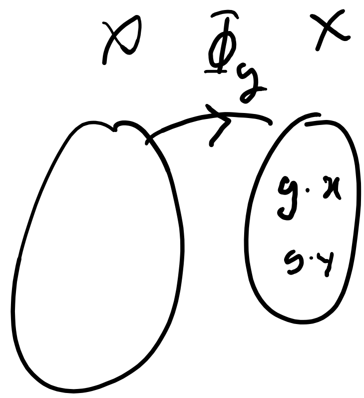
$g : X \longrightarrow X$

$g \in G$ (fix)



$\Phi_g : X \longrightarrow X$

$x \longmapsto g \cdot x$



Φ_g is bij ✓

$$\underbrace{g \cdot x}_{\in X} = \underbrace{g \cdot y}_{\in X}$$

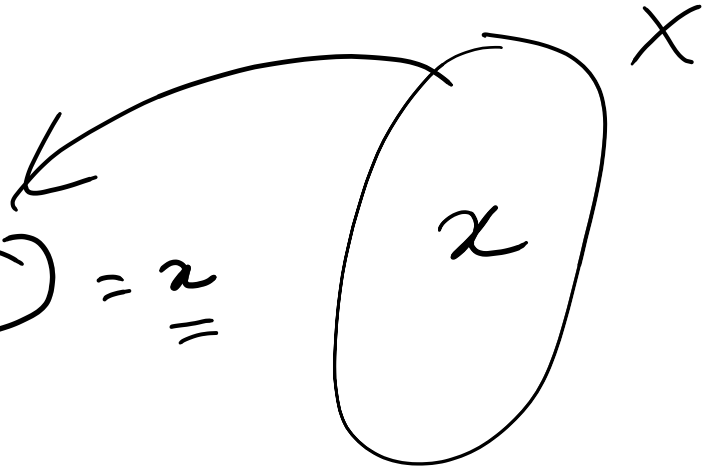
$$\Rightarrow \underbrace{g^{-1} \cdot g \cdot x}_{(g^{-1}g)\cdot x} = g^{-1} \cdot g \cdot y$$

$$(g^{-1}g)\cdot x = (g^{-1}g)\cdot y$$

$$\Rightarrow e \cdot x = e \cdot y$$

$$\Rightarrow x = y \quad //$$

Φ_g one-one.



$$g \cdot (\;) = x$$

$$y = g^{-1} \cdot x$$

$$x \in X$$

$$g^{-1} \cdot x \in X$$

$$g \in G \Rightarrow g^{-1} \in G.$$

$$\Phi : G \times X \to X$$

$$g^{-1} \cdot x \to y$$

$$g \cdot y = g \cdot (g^{-1} \cdot x)$$
$$= (g g^{-1}) \cdot x$$
$$= e \cdot x$$
$$= x_{\shortparallel}$$

$$\Phi : G \times X \longrightarrow X$$
$$g : X \longrightarrow X \quad (\Phi_g)$$

$$S_{1 \times 1}$$

$$\varphi : \underline{G} \longrightarrow \underline{\underline{\mathrm{Sym}(X)}}$$
$$g \longmapsto \Phi_g$$
$$g \longmapsto \Phi_g$$

Exc: φ is <u>homomorphism</u>.

$$gh \longmapsto \varphi_{gh}$$

$$g \cdot h \longmapsto \Phi_h$$

$$\downarrow$$

$$\Phi_g \qquad \boxed{\Phi_g \circ \Phi_h = \Phi_{gh}}$$

fn

G.

$\varphi : G \longrightarrow \mathrm{Sym}(X)$ is homomorphism.

$$\ker \varphi = \{ g \in G : \Phi_g = \mathrm{Id} \}.$$

$$\Phi_g(x) = \mathrm{Id}(x)$$

$$\ker \varphi = \{ g \in G : g \cdot x = x \}$$

$X = G.$ $\qquad \Phi : G \times X \to X$

via left mul

$\varphi : G \to \mathrm{Sym}(G)$

$$\ker \varphi = \{ g \in G : gx = x \ \underline{\underline{\forall x \in G}} \}$$

$$= \{ g \in G : g = e \}$$

$$= \{e\}.$$

//

Homo $\longleftrightarrow$ Isom.

$$G/_{\mathrm{Ker}\,\varphi} \cong \mathrm{Im}\,\varphi \qquad G \xleftrightarrow{\sim} G/_{\{e\}}$$

$$\Rightarrow \quad G/_{\underset{\{e\}}{\underset{\shortparallel}{\mathrm{Ker}\,\varphi}}} \cong \mathrm{Im}\,\varphi$$

$\cong$

$$\Rightarrow \quad \underbrace{G \cong \mathrm{Im}\,\varphi \leq \mathrm{Sym}(G)}_{\text{Cayley's Thm.}}$$

$$G \cong K \leq \mathrm{Sym}(G) = S_n.$$

Grp Action

$\Big\downarrow$ induces

$|X|!$

$\varphi : G \to \underline{\underline{Sym(X)}}$

$=$

$\underline{G \curvearrowright X}$ $\left.\begin{array}{l} \cdot\ e \cdot x = x \\ \cdot\ g \cdot h \cdot x = gh \cdot x \end{array}\right\}$

$X = (G, *)$

$Sym(\underline{\underline{X}})$ G

$G \curvearrowright G$, $\underline{\underline{\varphi : G \to Sym(G)}}$

$\varphi : G \to \underline{\underline{Aut(G)}}$

$\leq Sym(G)$

- Left
- Right

$= \underline{\underline{gxg^{-1}}}$ (conj.)

Exc !

Let $g \in G$ (fix).

Right/Left

$$\left.\begin{array}{l}\varphi_g : G \to G \\ x \mapsto gxg^{-1}\end{array}\right\}$$ Conj

$g \in G$

$\Phi_g \leftarrow$ Aut.

$\varphi_g(xy)$

$= gxyg^{-1}$

$= \underbrace{gxg^{-1}}_{\varphi_g(x)} \; \underbrace{gyg^{-1}}_{\varphi_g(y)}$

$X = V$

$G \curvearrowright V$ via lin. trans

$G \curvearrowright V$

$$\varphi : G \to GL_n(V) \leq \underline{\underline{Sym(V)}}$$

inv. lin trasf on V.

bij fn on V.

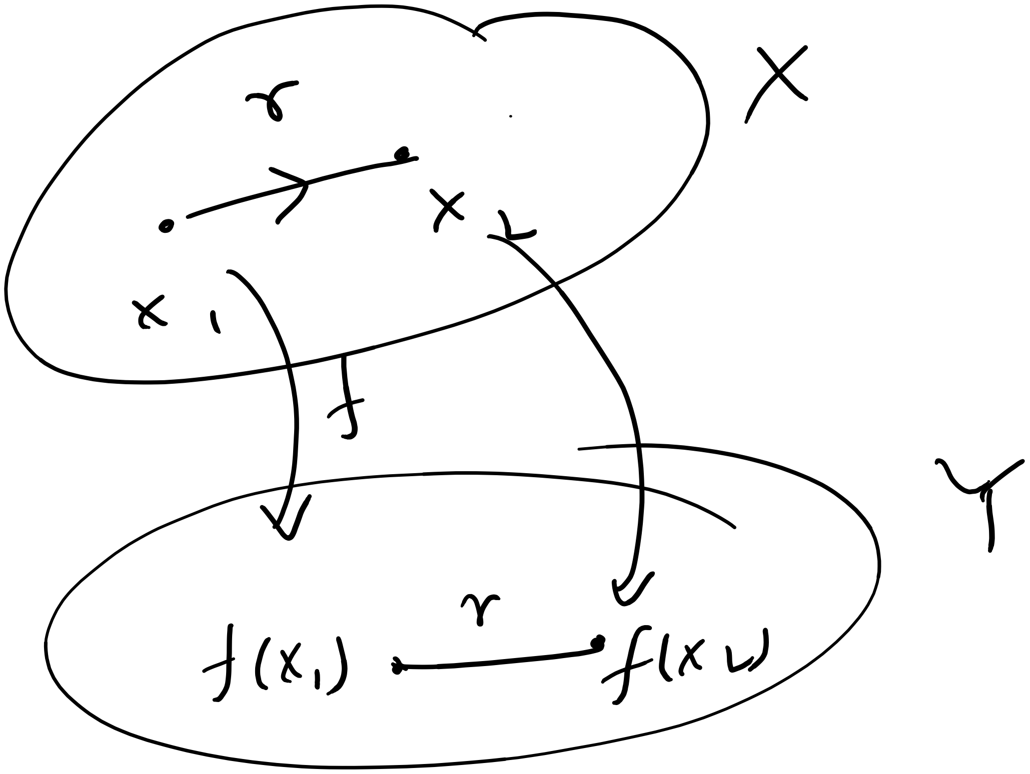
G, V, X

Nice maps on metric spaces.

$(X, \underline{\underline{d_X}})$, $(Y, \underline{\underline{d_Y}})$

$f : X \to Y$, bij

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$



X & Y isometric via 'f'.

$$X \longrightarrow d_X$$

$$\begin{cases} f : X \to X \quad \text{bij} \\ d(x,y) = d(f(x), f(y)) \end{cases}$$

$f \in \mathrm{Isom}(X)$

$X = \mathbb{R}^2$ $\quad f : \mathbb{R}^2 \to \mathbb{R}^2$

grp

v.s

"m.s

$G \curvearrowright X$ "via isometries"

metric space

$$\varphi : G \longrightarrow \mathrm{Sym}(X)$$

$\updownarrow$ $\quad$ Aut $\quad$ $GL_n(\mathbb{R})$

$$\varphi : G \longrightarrow \underbrace{\mathrm{Isom}(X)}_{\text{grp}}$$

.op "fn" comp.

Isom(X) consists of fns. which are isometries for X.

f_1, f_2 $\quad f_1 \circ f_2 \in \mathrm{Isom}(X)$

Free - Action:

Action is free iff

$\boxed{g.x = x \text{ for some } x \in X}$

$\Rightarrow g = e.$

$stab(x) = \{e\}.$

Torsion free grp.

$G \leftarrow$ Torsion free

iff "e" only element of finite order.

$\mathbb{Z} \leftarrow \quad 3 * 3 * 3 *$

$\qquad\quad\; \uparrow \quad \uparrow$

$\qquad\quad\; + \quad +$

$3+3+3+3 \neq 0$

$or(3) = \infty.$

Thm: $G \curvearrowright \mathbb{R}^n$ via isometries, and the action is free then G is torsion free.

Ex: $(G \curvearrowright X, \; H \curvearrowright X)$

$\Phi : G \times X \to X$ $\quad \Phi|_H : H \times X \to X$

Proof: $g \in G$, $\underset{\substack{\| \\ m}}{or(g)} < \infty$,

Claim: $g = e$.

Pf: $H = \langle g \rangle = \{e, g, g^2, g^3, \dots g^{m-1}\}$

$|H| = m < \infty$.

$(g \cdot x)$

$H \curvearrowright \mathbb{R}^n$

$H \curvearrowright \mathbb{R}^n$.

$G \curvearrowright \mathbb{R}^n$

$g \cdot v$ some action

$v \in \mathbb{R}^n$

$\underset{H}{orb}(v) = \{g \cdot v : g \in H\}$

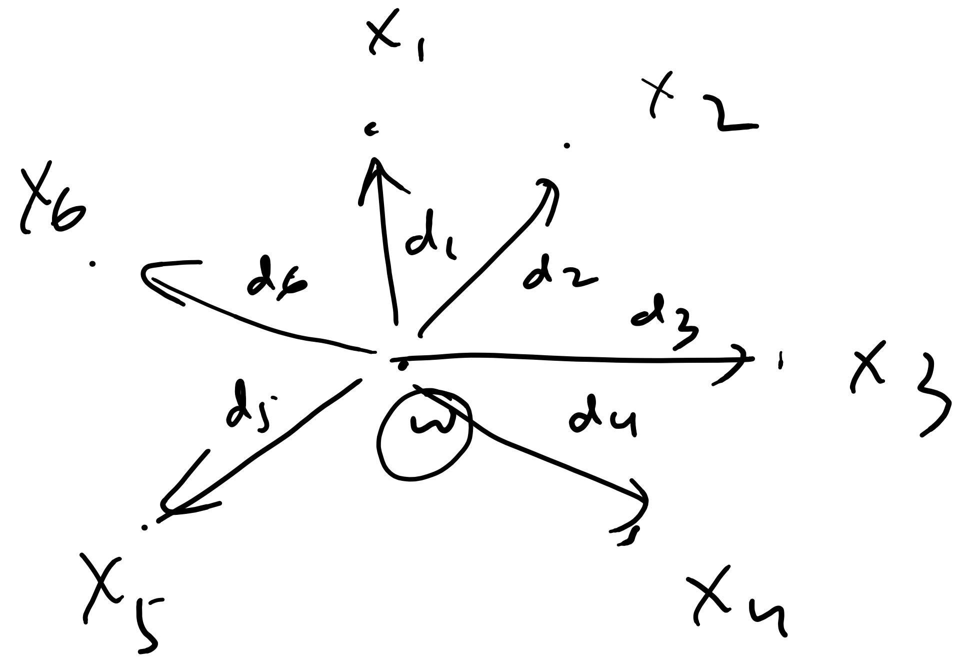
$= \{v, g \cdot v, g^2 \cdot v, \dots g^{m-1} \cdot v\}$

$v, g \cdot v, g^2 \cdot v, \dots$

Exc: Any finite set in $\mathbb{R}^n$ has unique centriod $\{x_1 \dots x_n\}$

convex.

$$w = \left[\sum_{i=1}^{n} d(w, x_i)\right]_{min}$$

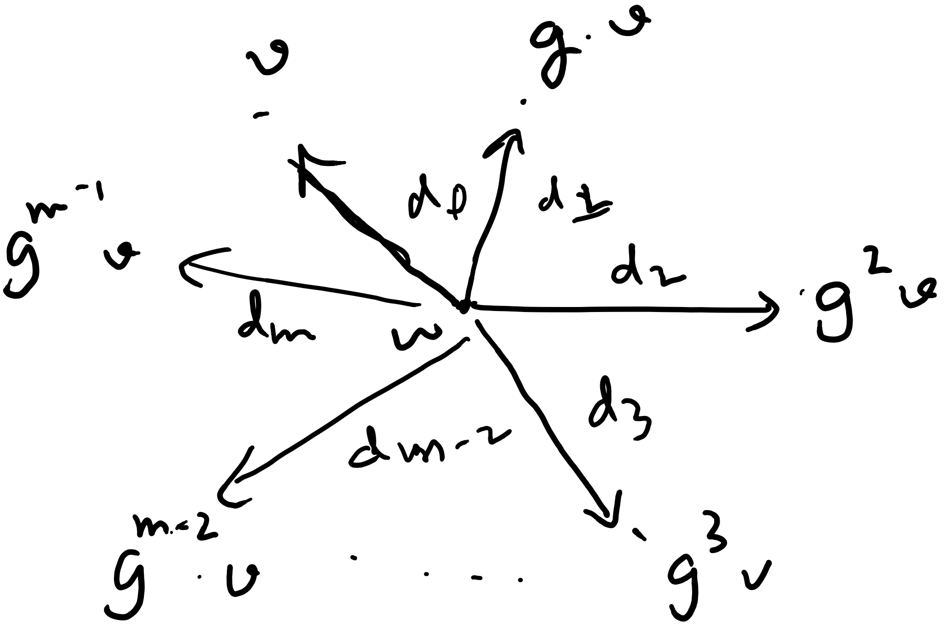


$$\left(\sum d_i\right)_{min} =$$

$$\mathrm{orb}(v) = \{v, g\cdot v, \dots, g^{m-1} v\}.$$

$$\Big[d(w, v) + d(w, g\cdot v) + \dots + d(w, g^{m-1}\cdot v)\Big]_{\min}$$

$$\Big[\sum_{i=0}^{m-1} d(\underline{\underline{w}}, g^{i}\cdot v)\Big]_{\min}$$



$$\Phi : G \times X \to X$$
$$g \cdot v \to \tilde{v}$$

$g \in G(\mathrm{fix})$ $\quad$ $\mathrm{orb}(v) = \{v, g\cdot v, \ldots g^{m-1}\cdot v\}$ $\to \mathbb{R}^n$

$v, g\cdot v, \ldots \subseteq X$

$g\cdot \underline{\mathrm{orb}(v)} = \{g\cdot v, g^2 \cdot v, \ldots g^m \cdot v\}$

$g^m \cdot v = e\cdot v = e$

2

$\mathrm{orb}(v) = \{v, g\cdot v, \ldots g^{m-1}\cdot v\}$

$\underline{g\cdot \mathrm{orb}(v)} = \{g\cdot v, g^2\cdot v, \ldots g^m \cdot v\}$

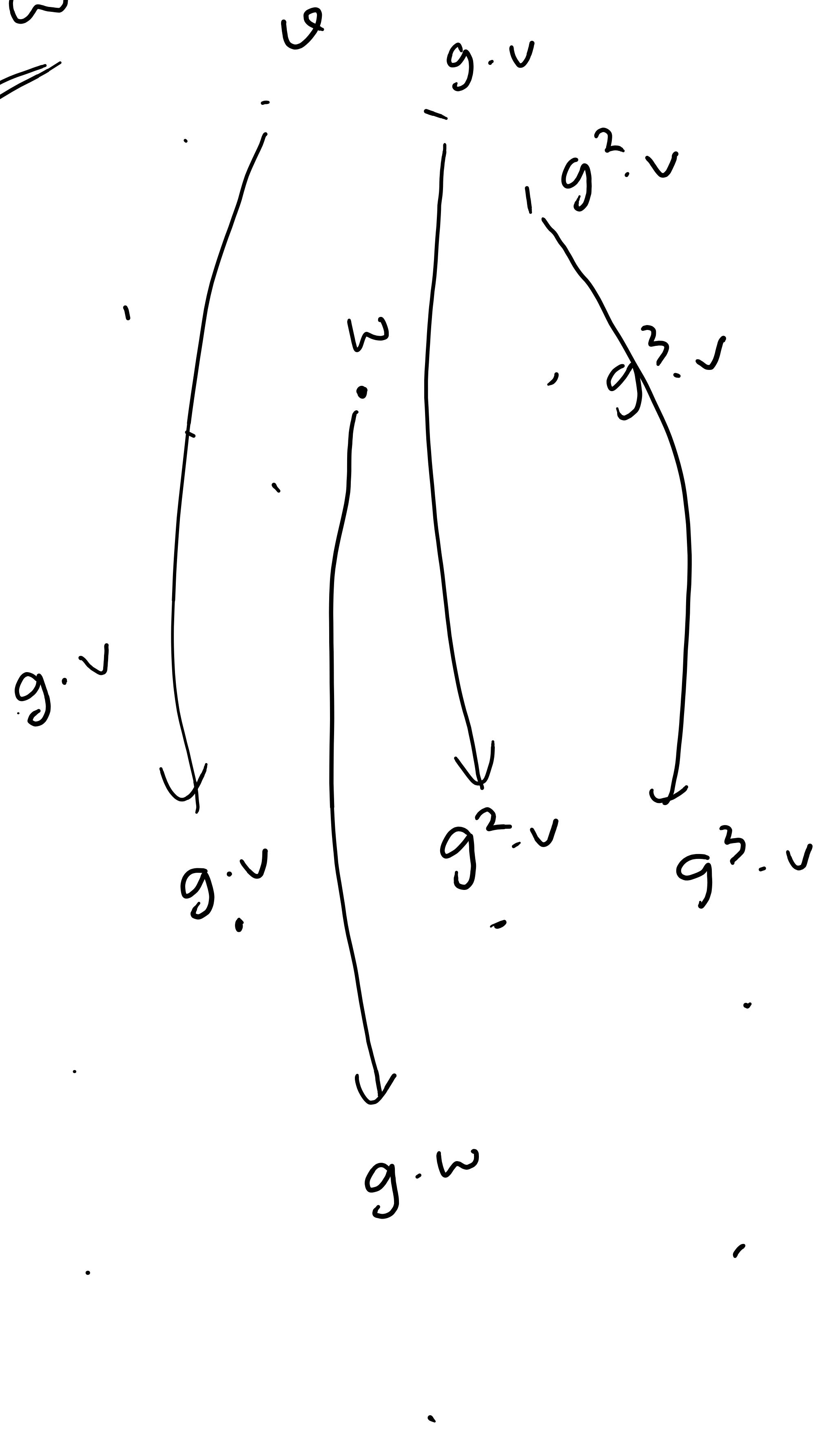
$g^m = e$

$e\cdot v = v$

$= \underline{\mathrm{orb}(v)}$ $\leftarrow$ ω

$g \cdot \mathrm{orb}(v) = \{ g \cdot v, \ldots\ldots, g^m \cdot v \}$

$g \cdot w = w$

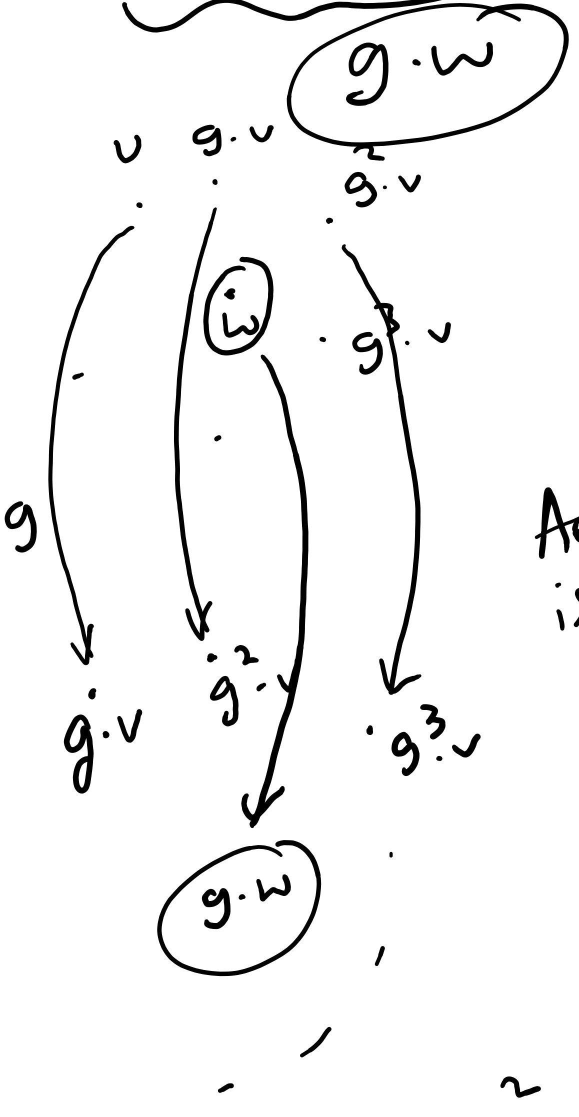


$$\underbrace{g \cdot \mathrm{orb}(v)} = \underbrace{\mathrm{orb}(u)}$$

ω

"for some x"

$$g \cdot x = x$$
$$\Rightarrow g = e$$



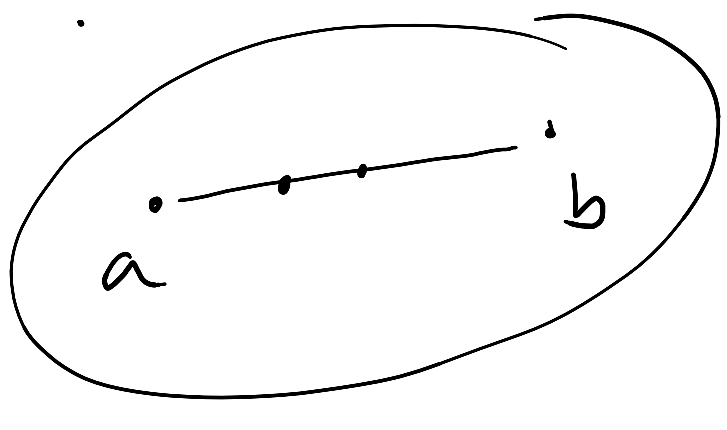
Action is free $g \cdot w = w$

$g \in G$

$$\Rightarrow g = e$$

$G \leftarrow$ torsion free

$\mathbb{R}^2$



via isom

free

$G \curvearrowright \mathbb{R}^n$

$\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$

$g \cdot v = w'$

G is tr

$G \cong \mathbb{Z}^r = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \mathbb{Z}}_{r}$

$G \curvearrowright X$

<u>Thm</u>: G is free $\iff$ G acts freely on a tree.

$\Downarrow$

Subgrps of free grps are free.

$\boxed{\mathbb{F}_2 \curvearrowright \Gamma \text{ (free)}}$

Γ

$H \leq \mathbb{F}_2$

$\mathbb{F}_2 \curvearrowright \Gamma$

$H \curvearrowright \underline{\Gamma}$ tree

$H \in$ free.