Def^n:

① Free product :

Let, H and G be two groups

Then, $H * G$ = { group formed by concatenating alternating words from H and G }

free pdt

i.e. $\forall\ x \in H * G$

$$x = h_1 g_1 \cdots h_n g_n$$

or

$$h_1 g_1 \cdots h_n \quad \text{for some}$$

or

$$g_1 h_1 \cdots g_n h_n \qquad n \in \mathbb{N}$$

or

$$g_1 h_1 \cdots g_n$$

If we have presentations for H and G

i.e. $$H = \langle S_H \mid R_H \rangle$$

$$G = \langle S_G \mid R_G \rangle$$

Then,

$$H * G = \langle S_H \cup S_G \mid R_H \cup R_G \rangle$$

(II) Free action :

Let, $G \curvearrowright X$ Then, the action is called free iff $\forall x \in X$, $\mathrm{Stab}(x)$ is the trivial subgroup of G.

(III) Transitive action

Let, $G \curvearrowright X$ Then the action is called transitive iff $\forall x, y \in X$ $\exists g \in G$ s.t. $g \cdot x = y$.

(IV) Action on a tree without inversions :

Let, $G \curvearrowright T \rightarrow$ tree Structure (More generally any graph)

Let, $e \in E(T)$ Then, if

$g \cdot e \neq \overline{e}$ where $\overline{e}$ is just e inverted [i.e., if $e = uv$, $\overline{e} = vu$]

More formally, $\forall g \in G$ and $y \in E(T)$ Let, $y = u_y \cdot v_y$ [$u_y, v_y \in V(T)$]

$g \cdot y \neq \overline{y}$ means,

$g \cdot u_y = v_y$ and $g \cdot v_y = u_y$ is not happening.

Thm: Let, $\mathcal{G} \curvearrowright T_{(\text{tree})}$ ~~without~~ s.t. $orb(u) \neq orb(v)$ $\forall$ adjacent vertices u, v. It is also given that $\mathcal{G}$ acts freely and transitively on the edges. Then given any edge e and it's end point vertices say V_1 and V_2.

Let, $H_1 = Stab(V_1)$, $H_2 = Stab(V_2)$

Then, $\mathcal{G} \cong H_1 * H_2$.

Pf: Let, $e \in E(T)$, and ends of e be V_1 and V_2.

Let, $H_i = Stab(V_i)$ $[i = 1, 2]$

$\because$ T is a tree $\exists$ a unique edge path from e to ge. Say,

$(e_0, e_1, \cdots, e_n)$ where $e_0 = e$ and $e_n = g.e$

As, $\mathcal{G}$ acts transitively on the edges of T we have $e_i = g_i \cdot e$ for some $g_i \in \mathcal{G}$.

Base Case: $i = 0$

We have $1_g \cdot e = e$ and that,

$1_g \in H_1 * H_2$ (We are done here)

Sps, for $i = k$ our hypothesis that $g_k \in H_1 * H_2$ was true.

We have that e_k and e_{k+1} is connected by a vertex as they lie as adjacent edges on our path. $\Longrightarrow g_k^{-1} e_k$ and $g_k^{-1} e_{k+1}$ are also connected by a vertex. $\therefore$ e and $g_k^{-1} g_{k+1} e$ has a common vertex.

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Say, Wlog this vertex is V_1

So,

$$(g_K^{-1} g_{K+1}) \cdot e = e'$$

Notice,

$$(g_{K+1}^{-1} g_K) \cdot e' = e$$

$$\Rightarrow (g_{K+1}^{-1} g_K) \cdot V_1 \in \{V_1, V_2\}$$

$$\left[\text{Note: } (g_{K+1}^{-1} g_K) \cdot V_1 = V_2 \Rightarrow \text{orb}(V_1) = \text{orb}(V_2) \quad [\Rightarrow \Leftarrow] \right]$$

$$\therefore (g_{K+1}^{-1} g_K) \cdot V_1 = V_1$$

$$\Rightarrow g_{K+1}^{-1} g_K \in H_1$$

$$\Rightarrow g_{K+1} \in H_1 * H_2 \quad \text{(By induction)}$$

$\therefore$ We have proven $\forall g \in G$,

$$g \in H_1 * H_2$$

$$\Rightarrow G \leq H_1 * H_2$$

$\therefore$ We have that given any g it can be written as some element in $H_1 * H_2$.

Now we will show that each element in $H_1 * H_2$ can be made into a path from e to ge.

given, $g = h_1 k_1 \cdots$ [We will show $g \in \mathcal{G}$]

$h_i \in H_1$, $k_i \in H_2$

The first edge in our path is e, ofcourse

The second " is $h_1 e$ and share V_1 with e.
[$\because h_1 V_1 = V_1$ as $h_1 \in H_1$]

The 3rd " is $h_1 k_1 e$

and share $h_1 V_2$ as [$h_1 k_1 V_2 = h_1 V_2$]

Continuing this way inductively we have a path from e to ge.

$\Rightarrow h_1 k_1 \cdots \in \mathcal{G}$

$\Rightarrow \forall x \in H_1 * H_2$, $x \in \mathcal{G}$

$\Rightarrow H_1 * H_2 \leq \mathcal{G}$

$\Rightarrow \mathcal{G} \cong H_1 * H_2$

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Define,

$$PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) / \{I_2, -I_2\}$$

Consider the induced action on $PSL_2(\mathbb{Z})$ on the farey tree by the action of $SL_2(\mathbb{Z})$ on the farey tree. $SL_2(\mathbb{Z})$ acts on the farey tree without inversions therefore so does the induced action on $PSL_2(\mathbb{Z})$. $SL_2(\mathbb{Z})$ also acts transitively on the edges as:

(I) Consider the vertex corresponding to the edge in the ~~caley gra~~ farey graph connecting $\pm(1,0)$ and $\pm(0,1)$ call this vertex V_0 and the vertex corresponding to the triangle $\{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ call it w_0.

Let, e_0 be the edge connecting V_0 and W_0

(II) Say that e is an edge with ends V and W. And say that V corresponds to $\{\pm(a,b), \pm(c,d)\}$

$$\Rightarrow A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ or } A' = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}$$

has det $= 1$.

Wlog sps, $\det(A) = 1$

Then, $A V_0 = V$

Claim: $\exists$ matrix B s.t.

$$B V_0 = V_0 \text{ and } B W_0 = A^{-1} w_0$$

If this was true

$\Rightarrow (AB) e_0 = e$ [Which would prove that our action is transitive]

Proof of Claim:

W_0 is the vertex corresponding to $\{\pm(0,1), \pm(1,0), \pm(1,1)\}$

Let, W_1 be the vertex corresponding to $\{\pm(0,1), \pm(1,0), \pm(-1,1)\}$

$\Rightarrow A^{-1} w = W_0$ or W_1 ($\because A^{-1}w$ must be a vertex of farey tree that is connected to $A^{-1}V = V_0$)

If, $A^{-1}w = W_0$ then take $B = I_2$ and we are done.

But if $A^{-1}w = W_1$ then we have,

$W_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_0 \quad \therefore B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ works

as, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Stab}(V_0)$ [Check!]

$\therefore$ We have that $SL_2(\mathbb{Z})$ acts transitively on the farey tree.

$\therefore$ So does the induced action by $PSL_2(\mathbb{Z})$

Note: $SL_2(\mathbb{Z})$ does not act freely on the edges of farey tree as both I and $-I$ fix the tree

Claim: $PSL_2(\mathbb{Z})$ acts freely on the farey tree.

Proof: Let, $A \in SL_2(\mathbb{Z})$ and, e be an edge of the farey tree.

And, $A \cdot e = e$

$\Rightarrow A \in Stab(v)$, $A \in Stab(w)$ [v, w are ends of e] [$\because SL_2(\mathbb{Z})$ acts without inversions]

$\therefore A \in Stab(v) \cap Stab(w)$

Now, $Stab(Mv) = M \, Stab(v) \, M^{-1}$

$$\left[\because Stab(Mv) = \{ g \in G \mid g.(M.v) = M.v \} \right.$$

$$= \{ g \in G \mid (M^{-1} g M) v = v \}$$

$$\left. = M \, Stab(v) \, M^{-1} \right]$$

$\therefore$ We just have to find $Stab(v_0)$ to find $Stab(v)$ $\forall$ v and $Stab(w_0)$ to find $Stab(w)$ $\forall$ w

Doing the calculations we get,

$$\mathrm{Stab}(V) \cong \mathbb{Z}/4\mathbb{Z}$$

$$\mathrm{Stab}(W) \cong \mathbb{Z}/6\mathbb{Z}$$

and

$$\mathrm{Stab}(V) \cap \mathrm{Stab}(W) = \{I, -I\}$$

$\Rightarrow PSL_2(\mathbb{Z})$ acts freely on the farey tree [$\because I = -I$ in $PSL_2(\mathbb{Z})$

$(PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{I, -I\})$

$\therefore$ Now we can apply the theorem on $PSL_2(\mathbb{Z})$.

Images of $\mathrm{Stab}(V_0)$ and $\mathrm{Stab}(W_0)$ in $PSL_2(\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ respectively.

$$\Rightarrow PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

[By the prev thm]

$$\Rightarrow PSL_2(\mathbb{Z}) \cong \langle a, b \mid a^2 = b^3 = e \rangle$$

Free product with amalgamation :

Let, G, H and K be groups.

Fix injective homomorphisms,

$i_G : K \longrightarrow G$ and $i_H : K \rightarrow H$

Let, $N = \langle\langle \{ i_G(k)\, i_H(k)^{-1} \mid k \in K \} \rangle\rangle$

The smallest normal subgroup generated by this given set.

We have $N \trianglelefteq H * G$

Define,

$$G *_K H = G * H / N$$

(Free pdt with amalgamation)

[The pdt depends ~~...~~ on i_G and i_H but to avoid clutter in notation we just write $G *_K H$]

Note : $G = \langle S_G \mid R_G \rangle$

means, $S_G \rightarrow$ gen set, $R_G \rightarrow$ relation set

If, $x \in R_G$ Then we assign $x = e$

eg: $S_G = \{a\}$, $R_G = a^n$

$\Rightarrow G = \langle S_G \mid R_G \rangle = \langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$

Note:

$$G *_K H = \langle S_G \cup S_H \mid R_G \cup R_H \cup \{ i_G(k)\, i_H(k)^{-1} \mid k \in K \} \rangle$$

~~Not~~

Observe that we can replace

$\{ i_G(k)\, i_H(k)^{-1} \mid k \in K \}$ with,

$\{ i_G(k)\, i_H(k)^{-1} \mid k \in S_K \}$ where

$S_K \longrightarrow$ gen. set for K.

Example: Choose positive integers m and n and say that $\exists\, d \in \mathbb{N}$ s.t. $d \mid n$, $d \mid m$.

Then, $i_H : \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$

$$1 \longmapsto \frac{m}{d}$$

and, $i_G : \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$

$$1 \longmapsto \frac{n}{d}$$

Then we get,

$$\mathbb{Z}/m\mathbb{Z} *_{\mathbb{Z}/d\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \langle a, b \mid a^m = b^n = 1 \text{ ~~and~~ } a^{\frac{m}{d}} = b^{\frac{n}{d}} \rangle$$

W.r.t. this homomorphisms.

Thm: Sps that a group G acts without inversions on a tree T in such a way that G acts transitively on edges of T. Choose one edge of T and say that the stabilizer of this edge is K and the stabilizers of it's vertices are H_1 and H_2 $(\mathrm{Orb}(v) \neq \mathrm{Orb}(w)$ $\forall\, v, w$ adjacent vertices$)$

Then,

$$G \cong H_1 *_K H_2$$

where the maps $K \to H_i$ are the inclusions of the edge stabilizer into the two vertex stabilizers.

~~i.e.? $\varphi: K \to H_1$ s.t. $\varphi(x) = x$~~

~~We can do this~~

This theorem is proved similarly as the previous one.

~~Usind~~

Using this theorem one can infer

$$SL_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$

$[\because \mathrm{stab}(e_0) = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{stab}(v_0) = \mathbb{Z}/4\mathbb{Z}$

$\mathrm{stab}(w_0) = \mathbb{Z}/6\mathbb{Z}\,]$