Rinking

GGT

$f_\Gamma(x) \leq k f_G(kx+k)+k$

- **Thm** :- $\Gamma \to G$ is a q.i embedding $\Rightarrow f_\Gamma \precsim f_G$

$\left(\Gamma \rightleftarrows G \text{ are q.i groups, } f_\Gamma \approx f_G \right)$

- **Pf** :- Say h is the map. (λ, ε - q.i embedding)

$$\frac{1}{\lambda} d(1,g) - \varepsilon \leq d(h(1), h(g)) \leq \lambda d(1,g) + \varepsilon$$

So, $d(1,g) \leq x \Rightarrow d(h(1), h(g)) \leq \lambda x + \varepsilon$

- **Claim 1** :- $|B(1,r)| = |B(g,r)|$ for any group element g

- **Pf** :- $d(1,a) = x \Leftrightarrow d(g, ag) = x$ $\blacksquare$

- **Claim 2** :- Suppose $h: A \to B$ is q.i embed Then, for any point $p \in B$ if $X = h^{-1}(p)$, then $\text{diam}(X) \leq \lambda \varepsilon$

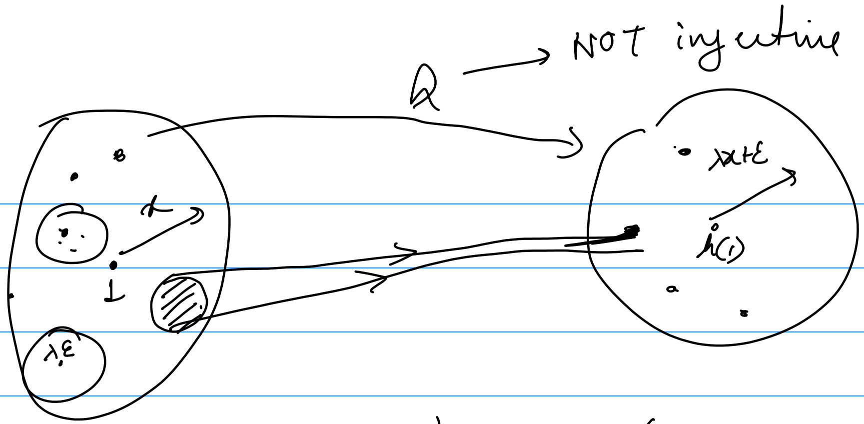
- **Pf** :- If $a, a' \in X$, then $\frac{1}{\lambda} d(a,a') - \varepsilon \leq 0$

$$\Rightarrow d(a,a') \leq \lambda\varepsilon$$

$$\Rightarrow \text{diam}(X) \leq \lambda\varepsilon \quad \blacksquare$$

$$d(1,g) \leq x \Rightarrow d(h(1), h(g)) \leq \lambda x + \varepsilon$$

So, $g \in B_\Gamma(1,x) \Rightarrow h(g) \in B_G(h(1), \lambda x + \varepsilon)$



$$|B(1,x)| \leq B(h(1), \lambda x+\varepsilon) \times \times$$

$$\frac{|B_\Gamma(1,x)|}{|B_\Gamma(1,\lambda\varepsilon)|} \leqslant |B_G(h(1), \lambda x+\varepsilon)|$$

So, now $f_\Gamma(x) = |B_\Gamma(1,x)| \leq |B_\Gamma(1,\lambda\varepsilon)| \, |B_G(h(1), \lambda x+\varepsilon)|$

Take $k > \lambda, \varepsilon, |B_\Gamma(1,\lambda\varepsilon)|$

$$\Rightarrow |B_\Gamma(1,x)| \leq k|B(h(1), kx+k)| + k$$

$$= k|B(1, kx+k)| + k$$

$$f_G(x)$$

So, $f_\Gamma \preceq f_G$ ∎

- Last day we proved – 1) $x^n \approx x^m \Longleftrightarrow n=m$

  2) $x^n + a_{n-1}x^{n-1} + \dots + a_0 \approx x^n$

- $\mathbb{Z}^n$ q.i to $\mathbb{Z}^m \Leftrightarrow n=m$ (Exercise: Find growth function)

  $\mathbb{R}^n$ q.i $\mathbb{R}^m \Leftrightarrow n=m$

$x^2 + \cdots$ $\quad x \cdots$

$\mathbb{Z}^2 \xrightarrow{q.i} \mathbb{Z}$?   Ans :- No

$x \preceq x^2$, $\quad x^2 \npreceq x \quad \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}^2$

$$\begin{tikzcd} & (G, X) \arrow[r, "f"] & (G, Y) \\ (H, X) \arrow[ur, "g"] & & \end{tikzcd}$$

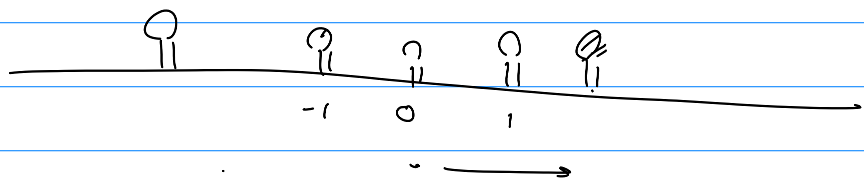
Presentation Topics

1. Ping-Pong lemma : + Application.

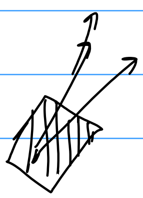
2. Farey Tree of a group :   Application :- presentation of $SL(2, \mathbb{Z})$

3. Lamplighter group :

$$L_2 = \langle a, t \mid a^2 = 1,\ t^k a t^{-k} t^j a t^{-j} = t^j a t^{-j} t^k a t^{-k} \rangle$$



L_n



4. Ends of groups :- (Prereq : Compactness, path connected, some ideas of topology)