Ping - Pong lemma:

Free groups:

A free group F generated by a set X is defined as:

For any group G and any map $\varphi : X \to G$ there is a unique group homomorphism $\bar{\varphi} : F \to G$ extending φ.

A group is free if it contains a free generating set.

Ping Pong lemma:

Let G be a group, generated by elements a, b. Suppose there is a G-action on a set X such that there are non-empty subsets $A, B \subset X$ with B not included in A and such that for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$a^n . B \subset A \qquad b^n . A \subset B$$

Then G is a free group of rank 2 freely generated by a and b.

Proof: $\alpha \neq \beta$

We need to find $F(\{\alpha, \beta\}) \cong G$

such that $\{\alpha, \beta\} \to \{a, b\}$.

Also by properties of free groups there is a group homomorphism

$$\varphi : F(\{\alpha, \beta\}) \to G$$

mapping α to a and β to b.

Since G is generated by $\{a, b\}$ φ is surjective

Assume φ is not injective.

$\therefore$ there exists a reduced word $w \in F \setminus \{e\}$ such that $\varphi(w) = 1_G$

Case 1: Word is of the form $\alpha^{n_1} \beta^{m_1} \cdots \alpha^{n_k}$

$$B = 1_G B = \varphi(w).B$$
$$= a^{n_1} b^{m_1} \cdots a^{n_k} . B$$
$$\subset a^{n_1} b^{m_1} \cdots b^{n_{k-1}} . A.$$

$\vdots$

$\subset a^{n_1} . B.$

$\subset A.$

a contradiction

Case 2: w is of the form $\beta^{n_0} \alpha^{n_1} \dots \beta^{n_k}$

But $\alpha w \alpha^{-1}$ is of form case 1.

$$
\begin{aligned}
\text{So } 1_G &= \varphi(\alpha) . 1_G . \varphi(\alpha^{-1}) \\
&= \varphi(\alpha) . \varphi(w) . \varphi(\alpha^{-1}) \\
&= \varphi(\alpha w \alpha^{-1})
\end{aligned}
$$

a contradiction.

Case 3: $w = \alpha^n w' \beta^m$

$$\alpha^r w \alpha^{-r} = \alpha^{r+n} w' \beta^m \alpha^r$$

$$1_G = \varphi(\alpha^r w \alpha^{-r})$$

a contradiction.

Case 4: $w = \beta^m w' \alpha^n$

$$\varphi(w^{-1}) = 1_G$$

contradiction.

for free groups:

Let G be a group G_1 and G_2 be two subgroups of G

$|G_1| \geq 3 \quad |G_2| \geq 2 \quad G = \langle G_1 \cup G_2 \rangle$

Also G has action on X where $X_1, X_2 \subset X$

$X_2 \not\subset X_1$

$\forall_{g \in G_1 \setminus \{e\}}\ g.X_2 \subset X_1 \qquad \forall_{g \in G_2 \setminus \{e\}}\ g.X_1 \subset X_2,$

$G \cong G_1 * G_2$

Proof: $f_1 : G_1 \hookrightarrow G$

Then there exists $\hat{f} : G_1 * G_2 \to G$,

$\hat{f} : G_1 \hookrightarrow G$ (so $\mathrm{Im}(\hat{f}) = G$)

If $\ker(\hat{f}) \neq 1_G$ then there is a contradiction by Ping-Pong lemma.

<u>Example:</u> $a, b \in SL(2, \mathbb{Z})$

$$a := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad b := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Through Ping-Pong lemma:

$$A := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| > |y| \right\}$$

$$B := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| < |y| \right\},$$

$A, B \subset \mathbb{R}^2$.

for all $n \in \mathbb{Z} \setminus \{0\}$.

$$a^n . \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ny \\ y \end{pmatrix}$$

$$\begin{aligned} |x + 2ny| &\geq |2ny| - |x| \\ &\geq |2y| - |x| \\ &> |2y| - |y| \\ &= |y| \end{aligned}$$

$a^n . B \subset A$.

$b^n . A \subset B$.

Thus $\langle a, b \rangle$ is freely generated by $\{a, b\}$.

$$\langle a, b \rangle \cong \langle a \rangle * \langle b \rangle \cong \mathbb{Z} * \mathbb{Z} \cong F_2$$

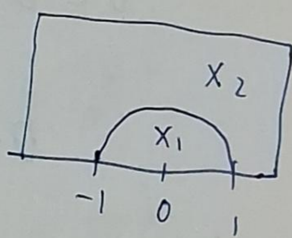
Free group in 2 generations.

Example: $\left\langle \overline{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}, \overline{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\rangle \cong \mathbb{Z} * 2\mathbb{Z}$

Herge $\bar{g}$ means $g\,\mathbb{Z}(SL_2(\mathbb{R})) \in PSL_2(\mathbb{R})$

Using actions on Schottky groups

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$



$$\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \cdot z = z + 2n$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot z = \frac{-1}{z}$$

Thus $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ sends $z \in X_1$ to X_2 and $z \in X_2$ to X_1

$$\left(\overline{\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle} \setminus I\right) . X_1 \subseteq X_2$$

$$\left(\overline{\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle} \setminus I\right) . X_2 \subseteq X_1$$

$$\overline{\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle} \cong \overline{\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle} * \overline{\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle}$$

$$\cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

Suppose $\lambda > 1$ and let $a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$

$b \in SL_2(\mathbb{R})$ such that

$$b . \{0, \infty\} \cap \{0, \infty\} = \phi .$$

then $SL_2(\mathbb{R}) \curvearrowright \mathbb{R} \cup \{\infty\}$,

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} . r = \frac{xr + y}{zr + t} .$$

$a.0 = 0$ $\quad a.\infty = \infty$

a contracts $X \setminus \{0\}$ towards ∞

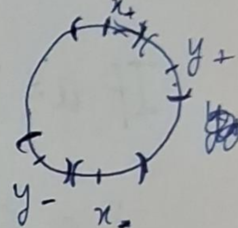
$\text{Fix}(bab^{-1}) = b.\,\text{Fix}(a) = \{b.0, b.\infty\}$.

$\text{Fix}(bab^{-1}) \cap \text{Fix}(a) = \phi$.

a^{-1} contracts everything except ∞ to 0.

bab^{-1} contracts everything except $b.0$ to $b.\infty$.

$ba^{-1}b^{-1}$ contracts everything except $b.\infty$ to $b.0$.



$a_1, a_2 \in \text{Homeo}(S^1)$

a_1 has fixed points x^- and x^+.

$a_1^n.(S^1 \setminus N_{x^-}) \subseteq N_{x^+} \quad \forall n \in \mathbb{Z}^+$.

$a_1^{-n}.(S^{-1} \setminus N_{x^+}) \subseteq N_{x^-} \quad \forall n \in \mathbb{Z}^+$

a_2 has fixed points y^- and y^+.

$a_2^n.(S^1 \setminus N_{y^-}) \subseteq N_{y^+} \quad \forall n \in \mathbb{Z}^+$.

$a_2^{-n}.(S^{-1} \setminus N_{y^+}) \subseteq N_{y^{-1}} \quad \forall n \in \mathbb{Z}^+$

$X_1 = N_{x^+} \cup N_{x^-}$

$X_2 = N_{y^+} \cup N_{y^-}$

$(\langle a_1 \rangle \setminus I) . X_2 \subseteq X_1$

$(\langle a_2 \rangle \setminus I) . X_1 \subseteq X_2$.

By Ping-Pong lemma:

$\langle a_1, a_2 \rangle \cong \langle a_1 \rangle * \langle a_2 \rangle \cong \mathbb{Z} * \mathbb{Z}$.

Thus.

If $a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ where $\lambda > 1$ and ~~$b \in SL_2(\mathbb{R})$~~

$b \in SL_2(\mathbb{R}) \setminus \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cup \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \cup \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right)$

Then for large n, $\langle a^n, ba^nb^{-1} \rangle \cong F_2$.

Thus $\langle a, b \rangle$ has a non-commutative free subgroup.

A finitely generated subgroup Γ of $GL_n(F)$ (where F is a field) is virtually solvable or contains a non-commutative free subgroup.