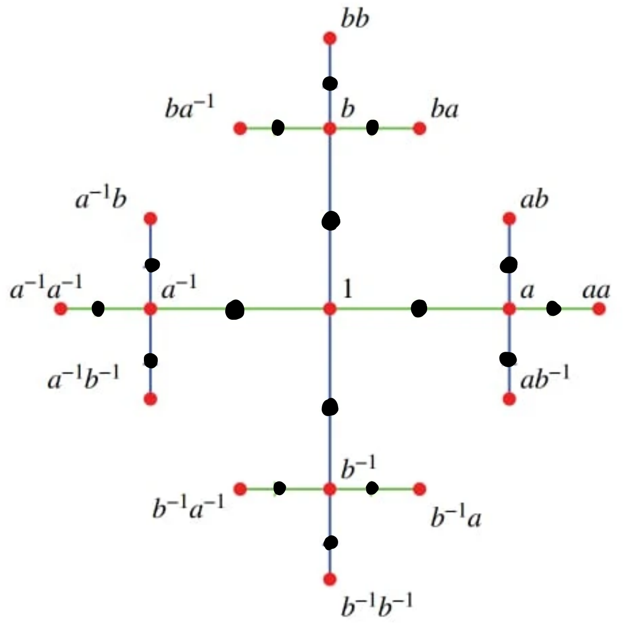
Theorem: If G acts freely on a tree then G is free. (upto isomorphism)

Proof: The proof is a 3-step proof.

Step 1 ("Tiling" the tree): let's start with;

Def$^{\underline{n}}$ of tiling of T: By a "tile," we mean a subtree T_0 of the barycentric subdivision T' of T.

Barycentric Subdivision: Graph obtained by subdiving each edge; i.e. we place a new vertex at the center of each edge of the original graph.



Tiling of T is a collection of tiles, with the following properties:

(i) No two tiles share an edge, so two tiles intersect atmost at one vertex of T'

(ii) $\bigcup_{g \in G} T_g = T'$. (i.e. union of all tiles $= T'$)

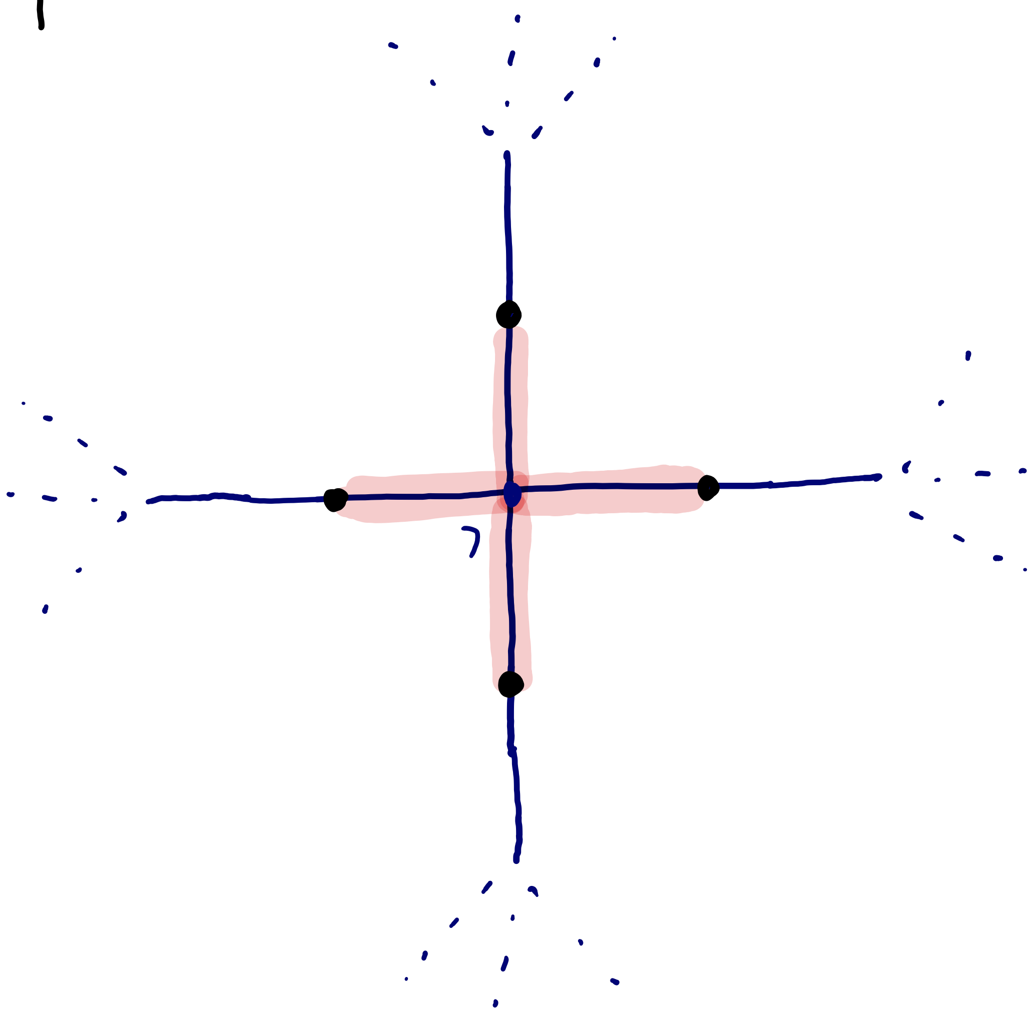
(iii) $\exists$ a tile "T_0" s.t. the set of tiles is equal to $\{gT_0 : g \in G\}$.

(iv) At any vertex atmost 2 tiles meet.

Qn. Find a nice tiling of T w.r.t. G.

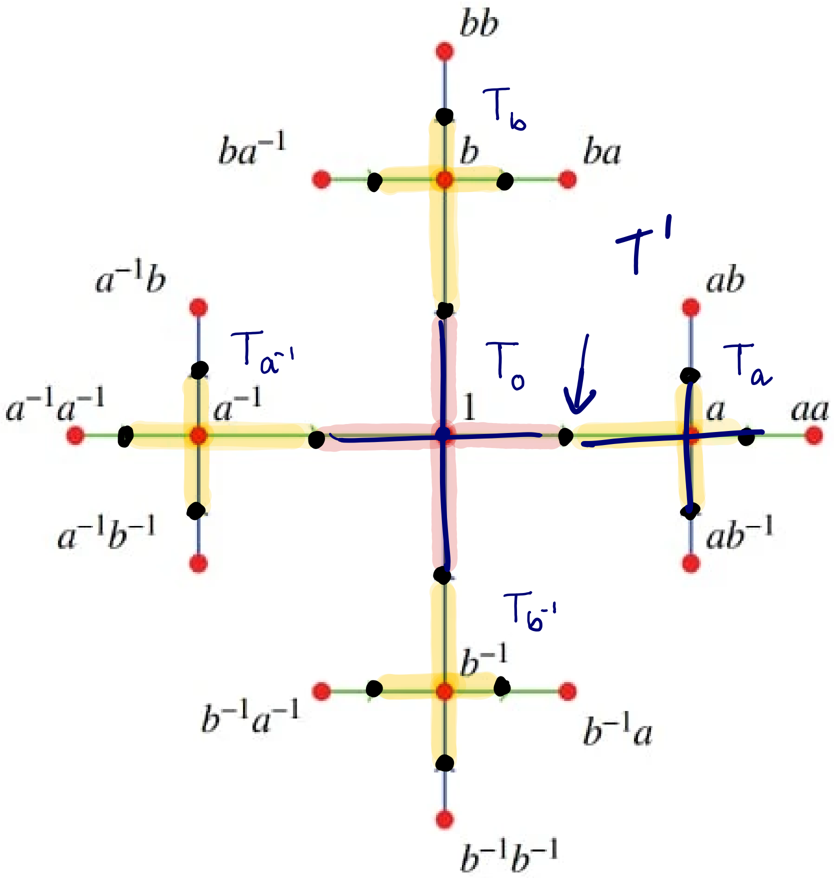
→ Take any vertex v of T, we define "star(v)" of T' to be union of edges of T' (half-edges of T) incident to v.

Example:



We build up a tile T_0 containg v, one star at a time. We start with the star 's' of v in T'. Then for each $g \in G$, we add the star $g \cdot s$ (induced action) to tile T_g. Choose any stars that is incident to T_0 and doesnot belong to any other tile, add that to T_0 & its g-translates to T_g's. and inductively we obtain the desired tiling.

Clearly, (i) is satisfied. Any two tiles meet atmost at on vertex in T'



(ii) is also satisfied.

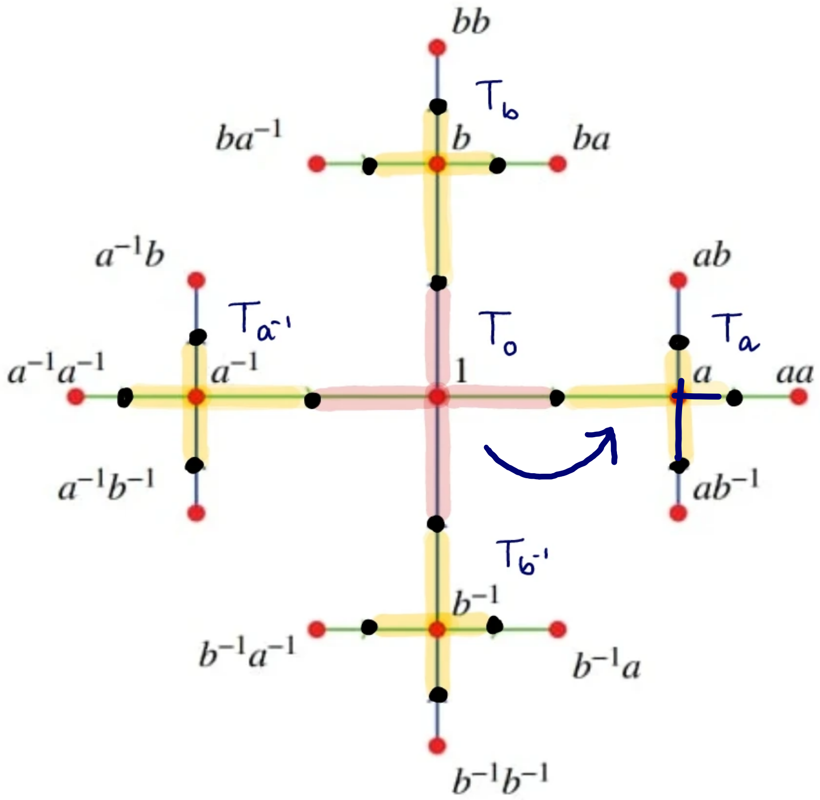
(iii) Here $T_0 = T_e$ does the job.

(iv) at any vertex of T', atmost 2 tiles meet.

Claim: For any $g, h \in G$, we have:

$$gT_h = T_{gh}.$$

Pf:



Let's see: aT_0 which is clearly T_a.

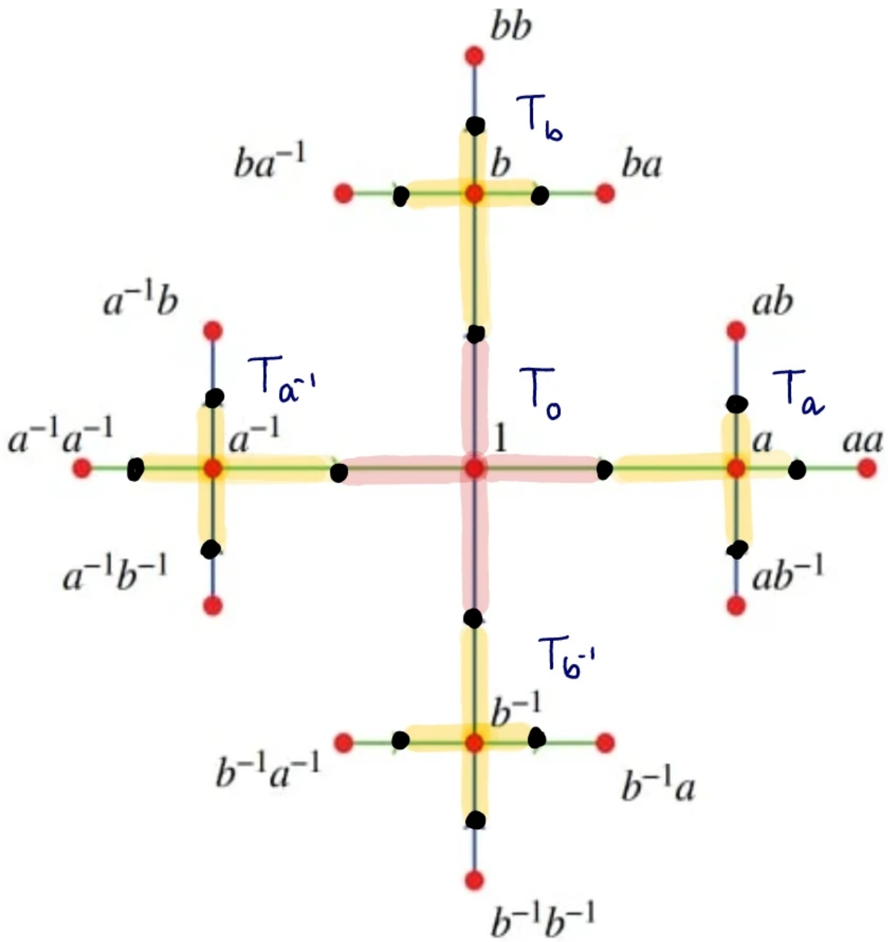
try to convince yousellf that this holds, & prove the claim, as an exc.

Step 2 (Finding the gen. set) : $T_g \cap T_0 \neq \phi$

Define $S = \{ g \in G : (gT_0) \cap T_0 \neq \phi \}$

$\Leftrightarrow$ they meet at some vertex at T'

$\Leftrightarrow$ " " " exactly one " " "



Here : $S = \{a, b, a^{-1}, b^{-1}\}$.

<u>Claim</u>: S is symmetric.

Qn. Why do we need symmetric?

cuz we need to show for any $g \in G$, $\exists$ s_i's s.t. $g = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}$, $\varepsilon_i \in \{1, -1\}$.

<u>Pf idea</u>: Say $s \in S$, so, it means.

$$sT_0 \cap T_0 = \{w\}, \quad w \in T'.$$

applying s^{-1}

$$T_0 \cap (s^{-1} T_0) = \{s^{-1}(w)\}$$

$$\Leftrightarrow s^{-1} \in S. \text{ (by def}^{\underline{n}}\text{)}$$

justify that it is a valid step!

<u>Claim</u>: S is a generating set.

<u>Pf</u>: Let $g \in G$ be arbitary

Look at gv, $\exists!$ path from v to gv.

since T is tree.

(T is tree $\Leftrightarrow$ for any vertex $\exists!$ path)

Keep a track of tiles in this paths b/w v & gv.

Say:

$$T_{g_n}, T_{g_{n-1}}, \ldots T_{g_1}, \underline{\underline{T_{g_0}}}$$

Here $g_n = g$, $g_0 = e$.

<u>Claim</u>: $g_i^{-1} g_{i+1} \in S$.

<u>Pf</u>: If a path travels thru, $T_{g_{i+1}}$ & T_{g_i} without any tiles in b/w, then $\exists\ \omega \in \underline{T}'$ s.t.

$$T_{g_{i+1}} \cap T_{g_i} = \{\omega\},$$

applying g_i^{-1}

$$(g_i^{-1} T_{g_{i+1}}) \cap (g_i^{-1} T_{g_i}) \neq \phi$$

$$\Rightarrow T_{g_i^{-1} g_{i+1}} \cap T_0 \neq \phi$$

$$\Rightarrow (g_i^{-1} g_{i+1}) T_0 \cap T_0 \neq \phi$$

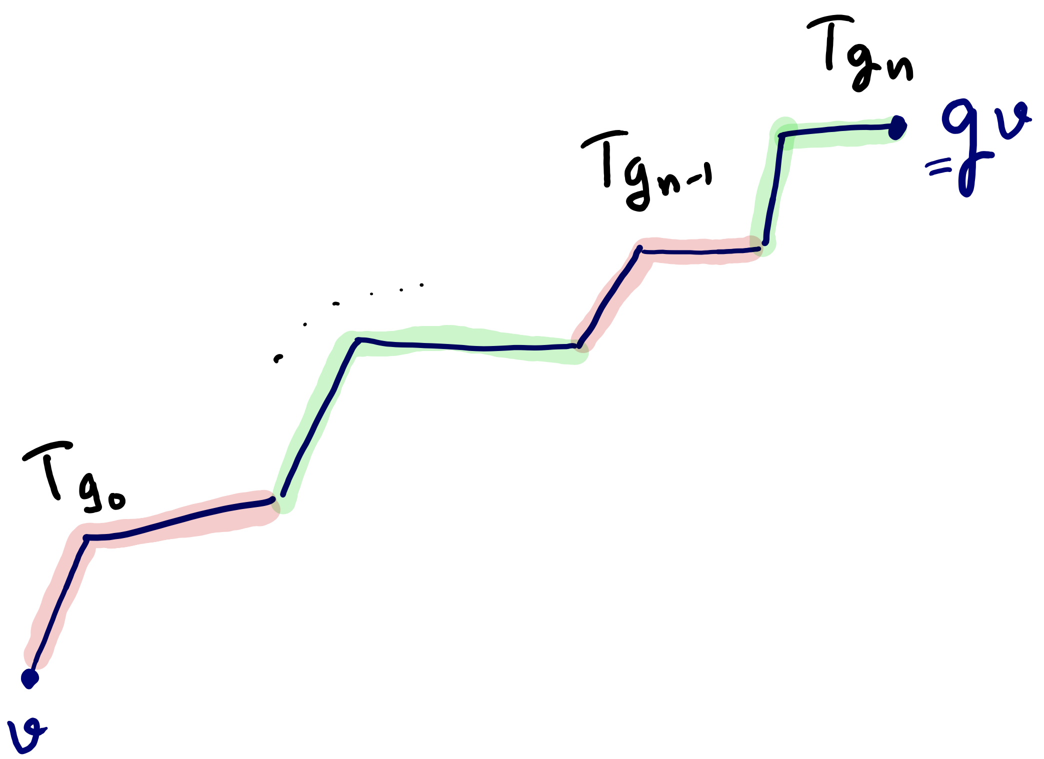
$$\Rightarrow \quad g_i^{-1} g_{i+1} \in S.$$

Now, we kept a track of tiles in this paths b/w v & gv.

Say:

$$T_{g_n}, T_{g_{n-1}}, \ldots T_{g_1}, T_{g_0}$$

Here $g_n = g$, $g_0 = e$.



We can write:

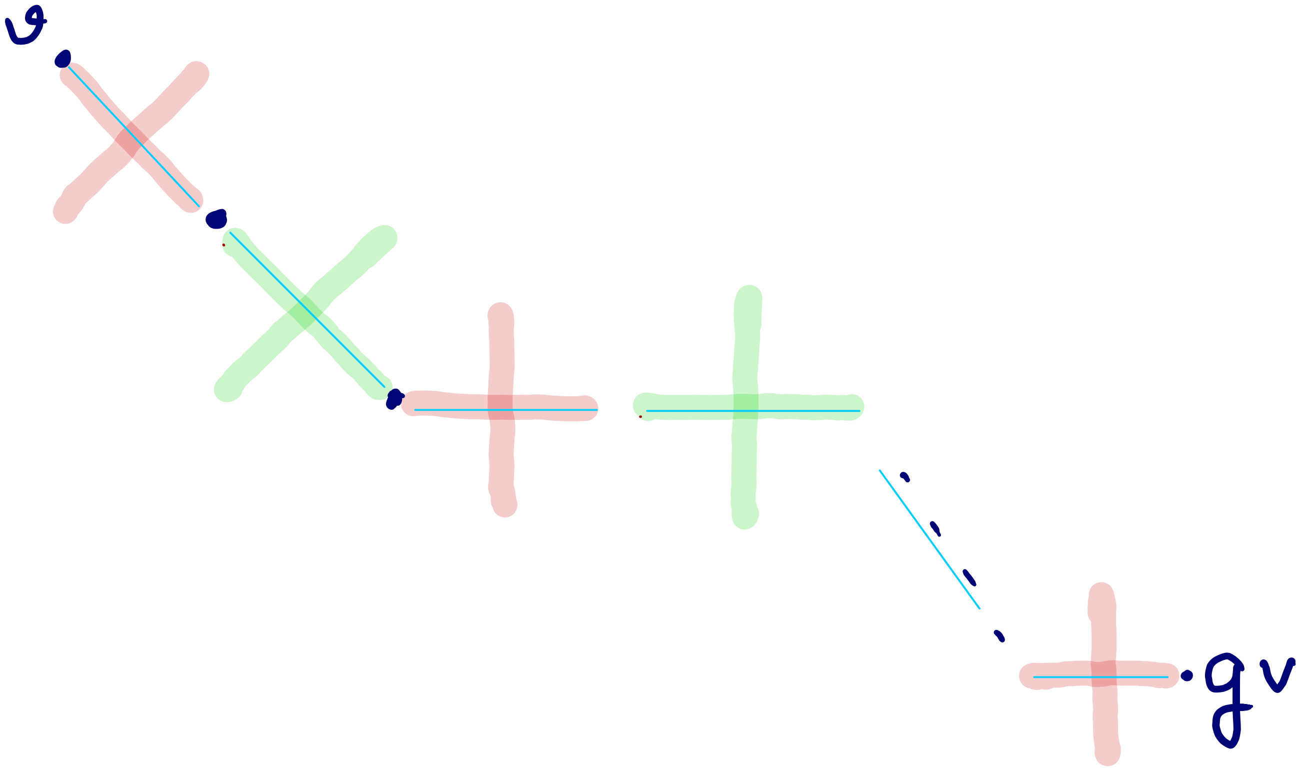
$$g := g_n$$
$$= g_0^{-1} g_n \quad (\because g_0 = e)$$
$$= \underbrace{g_0^{-1} g_1}_{\in S} \underbrace{g_1^{-1} g_2}_{\in S} \underbrace{g_2^{-1} \cdots}_{\in S} \underbrace{\cdots g_{n-1}}_{\in S} \underbrace{g_{n-1}^{-1} g_n}_{\in S}$$

$$g = s_1 \cdots s_k$$

Step 3 (The gen. set is free):

It is enough to show that $\exists!$ way to write g as freely reduced product of elements of S.

Observation: At any vertex of T' atmost 2 tiles meet. (Step (iv))

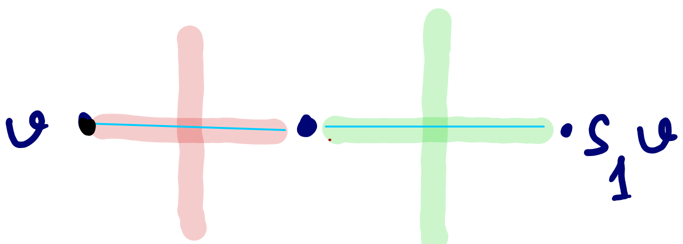


So, what we are saying is that path is unique (due to T being tree) but tiles are unique too, once path is choosen; if not then $\exists$ w $\in T'$ s.t. at that vertex more than 2 tiles meet $\rightarrow \Leftarrow$.

So, how do we find the path associated with $s_1 \ldots s_k$?

we can just start back tracking.

say we know, $\exists!$ path b/w v & $s_1 v$, So, by defn of S the tiles: T_0 & T_{s_1} $(= s_1 T_0)$ meet at a single vertex.



$$\underbrace{\qquad\qquad\qquad}_{T_0 \cup s_1 T_0}$$

So, $T_0 \cup s_1 T_0$ is a tree. $\exists!$ path contained in $T_0 \cup s_1 T_0$.

Now to get to s_1s_2v we again repeat the same process.

We get a unique path from s_1v to s_1s_2v,

$$v \;\text{———}\; s_1v \;\text{———}\; s_1s_2v \;\cdots\cdots\; \underbrace{s_1s_2\cdots s_k}_{g} v$$

Homework qstn:

Q1. Identify the step where we used that the action of G on T is free?

Q2. Find rank of G in a general situation.