Homework-4

1. Let M be param. hypersurface in $\mathbb{R}^3$, $p \in M$. Prove that for $v, w \in T_pM$, $L_p(v) \times L_p(w) = K(p)\, v \times w$, K the Gaussian curvature.

2. Suppose the principal curvatures of a (connected Ω) param. hypersurface $: \Omega \to \mathbb{R}^3$ vanish. Show that the hypersurface is part of a plane.

3. Recall, the mean curvature of param hypersurface $M \subseteq \mathbb{R}^n$ is $H(p) = tr(L_p)/(n-1)$, L_p the Weingarten map (same notⁿ as (1)). For M as above, $M \subseteq \mathbb{R}^3$, Show that $H^2 \geq K$. What are the points where $H = K$ holds?

4. Determine the Weingarten map for the param sphere S^n of radius r, Write $H(p)$ for $p \in S^n \subseteq \mathbb{R}^{n+1}$

5. Let M, N be manifolds of dim m & n resp. Prove (i.e. give an atlas) that $M \times N$ is a manifold of dim $(m+n)$.

6. Let $\mathcal{F}$ denote the set of all <u>flags</u> of subspaces $V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V$, $\dim V_i = i$. Give (?) a manifold structure to $\mathcal{F}$.

Homework - 3

1. Let S be the cylinder $x_1^2 + x_2^2 = r^2$, $r > 0 \subseteq \mathbb{R}^3$ (in its parametric form). Show that $\alpha : I \to S$ is a geodesic of $S \iff \alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$ for suitable $a, b, c, d \in \mathbb{R}$.

2. A param. curve α on $S \equiv S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$ (in its param. form) is a geodesic $\iff$
$$\alpha(t) = (\cos at)e_1 + (\sin at)e_2$$
for an orthonormal pair $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$.

3. Let S be a param hypersurface $\subseteq \mathbb{R}^n$, $p \in S$, $v \in T_p S$ and $\alpha : I \to S$ a maximal geodesic in S with initial velocity v. Show that the maximal geodesic β in S with initial velocity cv, $c \in \mathbb{R}$, is given by $\beta(t) = \alpha(ct)$.

*4. Let S be an n-plane $a_1 x_1 + \cdots + a_{n+1} x_{n+1} = b$ in $\mathbb{R}^{n+1}$, $P, Q \in S$ and let $v = (P, v) \in T_P S$. Show that if α is <u>any</u> param. curve in S from P to Q, then $P_\alpha(v) = (Q, v)$. Conclude from this that in an n-plane, parallel transport is path independent.

*5. Let $\alpha : [0, \pi] \to S^2$ be the half great circle in S^2 running from the north pole $P = (0,0,1)$ to the south pole $Q = (0,0,-1)$, defined by $\alpha(t) = (\sin t, 0, \cos t)$. Show that, for $v = (P, (v_1, v_2, 0)) \in T_P S^2$, $P_\alpha(v) = (Q, (-v_1, v_2, 0))$.

(Hint: Check this first for $v = (P, (1,0,0))$ & for $v = (P, (0,1,0))$; then use linearity of P_α).

*6. Let S be a param n-hypersurface $\subseteq \mathbb{R}^{n+1}$; $P \in S$; let $G_p = \{\phi : T_pS \to T_pS,\ \phi$ bijective linear$\}$
$= GL(T_pS)$.

Let $H_p := \{T \in G_p : T = P_\alpha$ for some piecewise smooth $\alpha : [a,b] \to S$ with $\alpha(a) = \alpha(b) = P\}$.

Show that H_p is a subgroup of G_p by showing

(i) for each pair of piecewise smooth α, β in $S : P \to P$, $\exists$ a piecewise smooth curve $\gamma : P \to P$ such that $P_\gamma = P_\beta \circ P_\alpha$,

(ii) for each α in $S : P \to P$, $\exists$ β in S $P \to P$ such that $P_\beta = P_\alpha^{-1}$.

(H_p is called the holonomy group of S at p)

[We'll discuss starred problems after the next class]

Homework-2

1. Find the Gaussian curvature of

(i) $\sigma(\theta,\varphi) = a(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ (sphere)

(ii) $\sigma(t,\theta) = (\cos\theta, \sin\theta, t)$ (right circular cylinder)

(iii) $\sigma(t,\theta) = (t\cos\theta, t\sin\theta, \theta)$ (helicoid)

2. Let M be a parameterized surface and X a vector field along M (i) give an example to show that, even if X is tangential, for $v \in T_pM$, $\partial_v X$ may fail to be a tangent vector.

(ii) For X tangential & $v \in T_pM$, set

$$D_v X := \partial_v X - (\langle \partial_v X, N(P)\rangle) N(P),$$ where

N is the unit normal field of M.

Prove that $D_v(X+Y) = D_v(X) + D_v(Y)$,

$D_v(f.X) = (\nabla_v f)X(P) + f(P)D_v X$,

$\partial_v(\langle X, Y\rangle) = \langle D_v X, Y(P)\rangle + \langle X(P), D_v(Y)\rangle$.

$\forall$ smooth tangential vector fields X, Y & smooth f. We call $D_v X$ the covariant derivative of X with respect to v, $D_v X \in T_pM$.

3. Let M be a param. hypersurface in $\mathbb{R}^n$, N its unit normal, X, Y tangential vector fields on M. For $p \in M$, prove that $\rightsquigarrow$ (PTO)

$$\left\langle \partial_{X(p)} Y, N(p) \right\rangle = \left\langle \partial_{Y(p)} X, N(p) \right\rangle . \quad (\forall p)$$

<u>Define</u> the vector field $[X,Y]$ along M by $[X,Y](p) = \partial_{X(p)} Y - \partial_{Y(p)} X$. Prove that, for X, Y tangential to M, $[X,Y]$ is tangential.

4. Let X be a smooth vector field on $\mathbb{R}^n$, $\sigma: \Omega \to \mathbb{R}^n$ a param. hypersurface, $v \in T(\Omega)$. Prove that $\partial_v (X \circ \sigma) = \partial_{D\sigma(v)} X$ where $D\sigma$ is the derivative of σ,
$(D\sigma)(p): T_p \Omega \to T_{\sigma(p)} \mathbb{R}^n$, $p \in \Omega$.

5. For X, Y, Z vector fields along a hypersurface M, prove the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

The "bracket" $[X,Y]$ of two vector fields is called the Lie bracket (pronounce as Lee bracket!) of X & Y.

6. For tangential vector fields X, Y on M, define $\partial_X Y$ by $(\partial_X Y)(p) := \partial_{X(p)} Y \in T_p M$. Compute $\partial_{[X,Y]} - [\partial_X, \partial_Y]$ for $\mathbb{R}^n$.

Homework-1

1. Describe the Weingarten map of a plane curve (assume regularity).

2(i) Let $f(x,y)$ be a smooth function $: \mathbb{R}^2 \to \mathbb{R}$ and assume that at every point of the "level curve"

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$$, at least one of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ ~~are~~ is non zero. Let $p = (x_0, y_0) \in \mathcal{C}$.

Then $\exists$ a regular parameterized curve $\gamma(t)$, defined on an open interval around 0, such that γ passes through p at $t=0$ and $\gamma(t) \in \mathcal{C}$ $\forall t$.

(ii) Let γ be a regular parameterized plane curve & $\gamma(t_0) = (x_0, y_0)$. Then $\exists$ a smooth map $f(x,y)$, defined for x, y in some open intervals containing x_0 & y_0 respectively, and the condition on $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ as above is satisfied, such that $\gamma(t) \in \mathcal{C} = \{(x,y) \mid f(x,y) = 0\}$ $\forall t$ in some open interval containing t_0.

3. Let P, Q be points in $\mathbb{R}^n$. Prove that the line segment joining P and Q has the shortest length among all parameterized curves joining P to Q.