Math 275A: Homework 6

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2.1.10

Claim: If X and Y are integer-valued r.v.'s then $P(X+Y=n)=\sum_m P(X=m)P(Y=n-m)$.

Proof: This follows almost by inspection. For a given n, $P(X+Y=n)$ is the probability that X equals some integer m and Y equals some other integer $n-m$ such that they sum to n. The total probability that $X+Y=n$ is just $\sum_m P(X=m\cap Y=n-m)$ where $m\in\mathbb{Z}$. Applying independence of X and Y completes the proof.

2.1.11

Claim: Use 2.1.10 to show that if $X=\text{Poisson}(\lambda)$ and $Y=\text{Poisson}(\mu)$ are independent then $X+Y=\text{Poisson}(\lambda+\mu)$

Proof: Take advantage of independence and just do the computation.

$$\begin{aligned}
P(X+Y=n) &= \sum_m P(X=m)P(Y=n-m)\\
&= \sum_m e^{-\lambda}\lambda^m/m!\cdot e^{-\mu}\mu^{(n-m)}/(n-m)!\\
&= e^{-\lambda-\mu}\sum_m \frac{\lambda^m\mu^{(n-m)}}{m!(n-m)!}\\
&= e^{-\lambda-\mu}\sum_m \frac{\lambda^m\mu^{(n-m)}n!}{m!(n-m)!n!}\\
&= e^{-\lambda-\mu}\sum_m \frac{(\lambda+\mu)^n}{n!}
\end{aligned}$$

3.1.3

Claim: Suppose $P(X_i=1)=P(X_i=-1)=1/2$. Show that if $a\in(0,1)$

$$\frac{1}{2n}\log P(S_{2n}\geq 2na)\to-\gamma(a)$$

where $\gamma(a)=\frac{1}{2}\{(1+a)\log(1+a)+(1-a)\log(1-a)\}$.

Proof: We can make use of the definition of $P(S_n=2n)$ and Sterling's approximation to arrive at

$$P(S_n=k)=(1+\frac{k}{n})^{-n-k}(1-\frac{k}{n})^{-n+k}(\pi n)^{1/2}(1+\frac{k}{n})^{-1/2}(1-\frac{k}{n})^{-1/2}$$

Let assume that $k/n = a$. Then for $j \in \mathbb{N}$ we can rewrite the last equation as

$$P(S_n = k + j) = (1 + a + \frac{j}{n})^{-n(1+a)-j}(1 - a - \frac{j}{n})^{-n(1-a)+j}(\pi n)^{1/2}(1 + (a + \frac{j}{n})^2)^{-1/2}$$

Observe that

$$\begin{aligned}
P(S_n \geq k) &= \sum_j P(S_n = k + j) \\
&= \sum_j (1 + a + \frac{j}{n})^{-n(1+a)-j}(1 - a - \frac{j}{n})^{-n(1-a)+j}(\pi n)^{1/2}(1 + (a + \frac{j}{n})^2)^{-1/2} \\
&\leq \sum_j (1 + a)^{-n(1+a)-j}(1 - a)^{-n(1-a)+j}(\pi n)^{1/2}(1 + a^2)^{-1/2} \\
&= (1 + a)^{-n(1+a)}(1 - a)^{-n(1-a)}(\pi n)^{1/2}(1 + a^2)^{-1/2} \sum_j (1 + a)^{-j}(1 - a)^j \\
&= CP(S_n = k)
\end{aligned}$$

such that $\frac{1}{2n} \log P(S_{2n} > 2k)$ and $\frac{1}{2n} \log P(S_{2n} > 2k)$ will converge to the same value. Finally note that

$$\begin{aligned}
\frac{1}{2n} \log P(S_{2n} = 2k) &= \frac{1}{2n}(-2n)(1 + a)\log(1 + a) + \frac{1}{2n}(-2n)(1 - a)\log(1 - a) + \frac{1}{2n}\log\Big((\pi 2n)^{1/2}(1 + a)\Big) \\
&= -(1 + a)\log(1 + a) - (1 - a)\log(1 - a) = -\gamma(a)
\end{aligned}$$

1 Extra 1

Claim: Let $X_1, X_2, \ldots$ be random variables with absolutely continuous distributions μ_n, μ and density functions f_n, f.

(a)

$$\sup_{A \in \mathcal{B}(\mathbb{R})} |\mu_n A - \mu A| = \frac{1}{2} \int_{\mathbb{R}} |f_n(x) - f(x)| \, dx$$

(b) If $f_n(x) \to f(x)$ pointwise then $\|\mu_n - \mu\| \to 0$.

Proof: (a)

2 Extra 2

Claim: Give $\mathbb{Z}$ the sigma algebra of all subsets.

(a) Let X and Y be $\mathbb{Z}$-valued r.v.'s with distributions μ and ν. Prove that $P(X \neq Y) \geq \|\mu - \nu\|$.

(b) Let μ and ν be probability distributions on $\mathbb{Z}$. Prove that there are $\mathbb{Z}$-valued r.v.'s X' and Y' on some probability space that have distributions μ and ν and satisfy $P(X' \neq Y') \geq \|\mu - \nu\|$.

Proof: