Math 275B: Homework 3

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D.4.6.2

Claim: f is said to be **Lipschitz continous** if $|f(t) - f(s)| \leq K|t - s|$ for $0 \leq s, < 1$. Show that $X_n = (f((k+1)2^{-n}) - f(k2^{-n}))/2^{-n}$ on $I_{k,n}$ defines a martingale, $X_n \to X_\infty$ a.s. and in L^1, and

$$f(b) - f(a) = \int_a^b X_\infty(\omega)\, d\omega.$$

Proof: For any n, $E|X_n| = \sum_{k=0}^{(2^n-1)} |(f((k+1)2^{-n}) - f(k2^{-n}))| \leq \sum_{k=0}^{(2^n-1)} K|2^{-n}| = K$, so $X_n \in L^1$. X_n is clearly adapted to $\mathcal{F}_n = \{I_{k,i} : 0 \leq i \leq n, 0 \leq k < 2^n\}$ since it is a simple function constructed on sets in $\mathcal{F}_n$. Lastly, for any set $I_{k,n}$,

$$\begin{aligned} E[X_{n+1}; I_{k,n}] &= \left(f(\frac{2k+2}{2^{(n+1)}}) - f(\frac{2k+1}{2^{(n+1)}}) + f(\frac{2k+1}{2^{(n+1)}}) - f(\frac{2k}{2^{(n+1)}}) \right) /2^{-n} \\ &= \left(f(\frac{2k+2}{2^{(n+1)}}) - f(\frac{2k}{2^{(n+1)}}) \right) /2^{-n} = \left(f(\frac{k+1}{2^{-n}}) - f(\frac{k}{2^{-n}}) \right) /2^{-n} = E[X_n; I_{k,n}], \end{aligned}$$

indicating the $E[X_{n+1}; I_{k,n}] = X_n$. Thus X_n is a martingale.

By definition $|X_n| \leq K$ for any n. Thus, $E(|X_n|; |X_n| > K) = 0$ such that X_n is uniformly integrable. Almost sure and L^1 convergence follow from theorem D.4.6.7. The theorem also tell us that there exists $X_\infty \in L^1$ such that $X_n = E(X|\mathcal{F}_n)$. So for any $a, b \in [0, 1)$ for which $a \leq b$, let A_n denote the largest interval $I_{k,n} \in \mathcal{F}_n$ contained within $[a, b)$. Then $A_n \to [a, b)$ and $X_\infty \mathbf{1}_{A_n} \to X_\infty \mathbf{1}_{[a,b)}$. An application of the dominated convergence theorem proves the last part of the claim.

D.4.6.7

Claim: Show that if $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y$ in L^1 then $E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ in L^1.

Proof: Jensen's inequality tells us that $|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \leq E(|Y_n - Y||\mathcal{F}_n)$. That $Y_n \to Y$ in L^1, implies $E|Y_n - Y| \to 0$. This implies $E(|Y_n - Y||\mathcal{F}_n) \to 0$ in L^1 since $E(|Y_n - Y|; A) \leq E(|Y_n - Y|)$ for $A \in \mathcal{F}_n$. Theorem D.4.6.8 tells us that $|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0$ in L^1 as well. Applying the triangle inequality, we can conclude that

$$|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \leq |E(Y_n|\mathcal{F}_n) - E(Y_n|\mathcal{F}_n)| + |E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0 \text{ in } L^1.$$

D.4.8.7

Claim: Let S_n be a symmetric simple random walk starting at 0, and let $T = \inf\{n : S_n \notin (-a, a)\}$ where a is an integer. Find constants b and c so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale, and use this to compute ET^2.

Proof: $Y_n \in L^1$ because it is a finite sum of L^1 random variables. It is adapted to $\mathcal{F}_n$ because each of the terms in its definition are themselves adapted. To verify $E(Y_{n+1}|\mathcal{F}_n) = Y_n$ we exploit the indepedence of

ξ_{n+1} from $\mathcal{F}_n$ along with the fact that even moments of ξ_n equal 1 while odd momments equal 0:

$$\begin{aligned} E(Y_{n+1}|\mathcal{F}_n) &= E\big((S_n+\xi_{n+1})^4 - 6(n+1)(S_n+\xi_{n+1})^2 + b(n+1)^2 + c(n+1)|\mathcal{F}_n\big) \\ &= E\big(S_n^4 + S_n^3\xi_{n+1} + S_n^2\xi_{n+1}^2 + S_n\xi_{n+1}^3 + \xi_{n+1}^4 \\ &\quad - 6(n+1)(S_n^2 + S_n\xi_{n+1} + \xi_{n+1}^2) + b(n+1)^2 + c(n+1)|\mathcal{F}_n\big) \\ &= S_n^4 + S_n^2 + 1 - 6(n+1)(S_n^2+1) + b(n^2+2n+1) + c(n+1) \\ &= Y_n + 1 - 6(n+1) + b(2n+1) + c. \end{aligned}$$

From the last expression it is clear that if we let $b = 3$ and $c = 2$, then Y_n becomes a martingale.

Applying the stopping theorem for the bounded stopping time $T \wedge n$ gives us

$$E(Y_0) = E(Y_{T\wedge n}) = E(S_{T\wedge n}^4 - 6(T\wedge n)S_{T\wedge n}^2 + 3(T\wedge n)^2 + 2(T\wedge n)) = 0.$$

We know from the first part of the proof of Theorem D.4.8.7 that $E(T) < \infty$. The MCT then tells us that $E(T\wedge n)^i \uparrow E(T)^i$ for positive integers i. Likewise the bounded convergence theorem (noting $|S_n| \leq a$) tells us that $S_{T\wedge n}^i \to S_T^i$. Thus our expectation equation becomes $E(S_T^4 - 6TS_T^2 + 3T^2 + 2T) = 0$.

From D.4.8.7 (and symmetry) we know that $P(S_N = a) = P(S_N = -a) = 1/2$ such that $E(S_T^4) = a^4$ and $E(TS_T^2) = \sum_{i=0}^{\infty} ia^2 P(T=i) = E(T)a^2$. We also know tha from D.4.8.7 that $E(T) = a^2$ so our expectation equation reduces to $a^4 - 6a^2a^2 + 3E(T^2) + 2a^2 = 0$ such that

$$E(T^2) = \frac{5a^4 - 2a^2}{3}$$

D.5.1.1

Claim: Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\in \{1, 2, \ldots, N\}$ and taking each value with probability $1/N$. Show that $X_n = |\,\{\xi_1, \ldots, \xi_n\}\,|$ is a Markov chain and compute its transition probability.

Proof: X_n equals the number of unique elements, ξ_i, that have been chosen by step n, not the specific elements. For any X_n, X_{n+1} can only ever equal X_n or $X_n + 1$ and the likelihood of either outcome depends strictly on the value of X_n. If $X_n = i$, the chance of picking a new element is just $\frac{N-i}{N}$. Thus $P(X_{n+1} = j|X_n = i) = \frac{N-i}{N}$ and $P(X_{n+1} = i|X_n = i) = \frac{i}{N}$. All other transition probabilities are zero.

D.5.1.2

Claim: Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\in \{1, -1\}$, taking each value with probability $1/2$. Let $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$ and $X_n = \max\{S_m : 0 \leq m \leq n\}$. Show that X_n is not a Markov chain.

Proof: In general it's unclear from the value of X_n if $X_n = S_n$. If we also know $X_{n-1} = i-1$, then $X_n = i$ ensures $X_n = S_n$ such that

$$P(X_{n+1} = i+1|X_n = i, X_{n-1} = i-1) = P(X_{n+1} = i|X_n = i, X_{n-1} = i-1) = 1/2.$$

If we're only given that X_n, there is a nonzero chance that $X_n > S_n$, i.e. $P(X_n > S_n|X_n = i) > 0$ for $i \neq n$.. In that case, since ξ_{n+1} is independent of the previous outcomes,

$$\begin{aligned} P(X_{n+1} = i|X_n = i) &= P(\xi_{n+1} = -1 \cap X_n = S_n|X_n = i) + P(X_n > S_n|X_n = i) \\ &= P(\xi_{n+1} = -1)P(X_n = S_n|X_n = i) + P(X_n > S_n|X_n = i) \\ &> 1/2\,(P(X_n = S_n|X_n = i) + P(X_n > S_n|X_n = i)) = 1/2, \end{aligned} \tag{1}$$

implying $P(X_{n+1} = i|X_n = i, X_n = i-1) \neq P(X_{n+1} = i|X_n = i)$. This contradiction shows that X_n cannot be a Markov chain.