Math 275A: Homework 3

Marcus Lucas

1.5.4

Claim: If f is integrable and E_m are disjoint sets with union E then

$$\sum_{m=0}^{\infty} \int_{E_m} f \ d\mu = \int_E f \ d\mu$$

So if $f \geq 0$, then $\nu(E) = \int_E f \ d\mu$ defines a measure.

Proof: The easiest way to show this is to apply the dominated convergence theorem. Define $f_n = f\mathbf{1}_{A_n}$ for $A_n = \cup_{m=0}^{n} E_m$. Since $A_n \uparrow E$ it follows that $f_n \to f$ on E a.e. By construction, each f_n is dominated by $|f|$ in that $|f_n| \leq |f|$. Given that f, and therefore $|f|$, is integrable, we can apply the DCT to our sequence. Thus $\int_E f_n \ d\mu \to \int_E f \ d\mu$.

To prove the claim, note that each A_n over which each f_n is defined is a union of n disjoint sets so that

$$\int_E f_n \ d\mu = \lim_n \int_{A_n} f \ d\mu = \lim_n \sum_{m=0}^{n} \int_{E_m} f \ d\mu = \sum_{m=0}^{\infty} \int_{E_m} f \ d\mu$$

The second equality comes from the linearity property of integration and the fact that we can write f_n as $\sum_{m=0}^{n} f\mathbf{1}_{E_m}$ for disjoint sets E_m. If, in addition, $f \geq 0$ such that $\nu(E) = \int_E f \ d\mu \geq 0$ for every set E, then $\nu(E)$ satisfies the definition of a measure.

1.5.8

Claim: Show that if f is integrable on $[a,b]$, $g(x) = \int_{[a,x]} f(y) \ dy$ is continuous on (a,b).

Proof: Given that $g : \mathbb{R} \to \mathbb{R}$ we just need to show continuity in terms of limits of sequences. Take any sequence $x_n \to x$ where $x_n, x \in (a,b)$. Then define $f_n = f\mathbf{1}_{[a,x_n]}$ such that the functions f_n converge to $f\mathbf{1}_{[a,x]}$ a.e. Since f is integrable over $[a,b]$, the absolute value of each f_n is upper-bounded by an integrable function $|f|$ on $[a,b]$. Thus we can apply the Dominated Convergence Theorem to conclude that

$$g(x_n) = \int_{[a,x_n]} f(y) \ dy = \int_{[a,b]} f_n(y) \ dy \to \int_{[a,b]} f(y)\mathbf{1}_{[a,x]} \ dy = \int_{[a,x]} f(y) \ dy = g(x)$$

We did not make any assumptions about the sequence x_n other than convergence to x, so continuity follows.

1.6.5

Claim: Show that: (i) if $\epsilon > 0$, $\inf\{P(|X| > \epsilon) : EX = 0, \ \text{var}(X) = 1\} = 0$; (ii) if $y \geq 1$, $\sigma^2 \in (0,\infty)$, $\inf\{P(|X| > y) : EX = 1, \ \text{var}\ (X) = \sigma^2\} = 0$

Proof: Given that my own understanding of set theory is lacking, I'll make the assumption that the infimums in the claim are taken over all possible random variables X defined on $([0,1], \mathcal{R}, \mu)$ where μ is taken to be the Lebesgue measure.

(i) For a given $\epsilon > 0$, define a sequence of random variables $X_n = \epsilon\sqrt{n}\mathbf{1}_{E_n} - \epsilon\sqrt{n}\mathbf{1}_{E_{-n}}$ where $E_n = (0, \frac{1}{2n\epsilon^2})$ and $E_{-n} = (-\frac{1}{2n\epsilon^2}, 0)$. Then

$$EX_n = \sqrt{n}\epsilon\mu(E_n) - \sqrt{n}\epsilon\mu(E_{-n}) = 0$$

and

$$\text{var}(X_n) = n\epsilon^2\mu(E_n) + n\epsilon^2\mu(E_{-n}) = \frac{2n\epsilon^2}{2n\epsilon^2} = 1$$

while

$$P(|X_n| > \epsilon) = \mu(E_n) + \mu(E_{-n}) = \frac{1}{n\epsilon^2} \quad \text{for} \quad n > 1$$

This probablity goes to zero as $n \to \infty$, proving the claim.

(ii) Given $y \geq 1$ and $\sigma^2 \in (0, \infty)$, define a sequence of random variables $X_n = \sqrt{n}y\mathbf{1}_{E_n} - \sqrt{n}y\mathbf{1}_{E_{-n}} + \mathbf{1}_{[0,1]}$ where $E_n = (0, \frac{\sigma^2}{2ny^2})$ and $E_{-n} = (-\frac{\sigma^2}{2ny^2}, 0)$. Then

$$EX_n = \sqrt{n}y\mu(E_n) - \sqrt{n}y\mu(E_{-n}) + \mu([0,1]) = 1$$

and

$$\text{var}(X_n) = E(X_n - 1)^2 = ny^2\mu(E_n) + ny^2\mu(E_{-n}) = \frac{2ny^2\sigma^2}{2ny^2} = \sigma^2$$

Due to the way we've defined X_n and because $y \geq 1$, $|X_n(\omega)|$ only has the potential to be greater than y when ω falls within E_n or E_{-n}. Thus

$$P(|X_n| > y) \leq \mu(E_n) + \mu(E_{-n}) = \frac{\sigma^2}{ny^2} \quad \text{for} \quad n > 1$$

which goes to zero as $n \to \infty$, proving the claim.

1.6.8

Claim: Suppose that the probability measure μ has $\mu(A) = \int_A f(x)\, dx$ for all $A \in \mathcal{R}$. Use the proof technique of Theorem 1.6.9 to show that for any g with $g \geq 0$ or $\int |g(x)|\, \mu(dx) < \infty$ we have

$$\int g(x)\ \mu(dx) = \int g(x)f(x)\ dx$$

Proof: Let's go through the cases as in 1.6.9.

Case 1 (Indicator Functions): For some $A \in \mathcal{R}$, let $g = \mathbf{1}_A$. Then

$$\int g(x)\ \mu(dx) = \int \mathbf{1}_A\ \mu(dx) = P(X \in A) = \mu(A) = \int_A f(x)\ dx = \int \mathbf{1}_A(x)f(x)dx = \int g(x)f(x)dx$$

Case 2 (Simple Functions): Let $g(x) = \sum_{n=1}^m c_m\mathbf{1}_{A_m}$ be some simple function where $c_m \in \mathbb{R}$ and $A_m \in \mathcal{R}$. Then linearity of integration lets us write

$$\begin{aligned}\int g(x)\ \mu(dx) &= \int \sum_{n=1}^m c_m\mathbf{1}_{A_m}(x)\ \mu(dx) = \sum_{n=1}^m c_m \int \mathbf{1}_{A_m}(x)\ \mu(dx) \\ &= \sum_{n=1}^m c_m \int \mathbf{1}_{A_m}(x)f(x)\ dx = \int \sum_{n=1}^m c_m\mathbf{1}_{A_m}(x)f(x)\ dx = \int g(x)f(x)\ dx\end{aligned}$$

where Case 1 was applied to get the third equality.

Case 3 (Nonnegative Functions): If $g \geq 0$ define

$$g_n(x) = ([2^n g(x)/2^n]) \wedge n$$

where $[x]$ is the largest integer less than or equal to x and $a \wedge b = \min\{a, b\}$. As defined g_n are simple nonnegative functions such that $g_n \uparrow g$ and $g_n f \uparrow gf$ Applying Case 2 and the Monotone Convergence Theorem (twice) yields

$$\int g(x)\ \mu(dx) = \lim_n \int g_n(x)\ \mu(dx) = \lim_n \int g_n(x) f(x)\ dx = \int g(x) f(x)\ dx$$

Case 4 (Integrable Functions): For the general case can write $g(x) = g^+(x) - g^-(x)$. That $\int |g(x)|\ \mu(dx) < \infty$ guarantes $\int g^+(x)\ \mu(dx)$ and $\int g^-(x)\ \mu(dx)$ are finite. Thus

$$\int g(x)\ \mu(dx) = \int g^+(x)\ \mu(dx) - \int g^-(x)\ \mu(dx) = \int g^+(x) f(x)\ dx - \int g^-(x) f(x)\ dx = \int g(x) f(x)\ dx$$

thereby completing the proof.

1.6.13

Claim: If $EX_1^- < \infty$ and $X_n \uparrow X$, then $EX_n \uparrow EX$.

Proof: Each X_n can be decomposed as $X_n = X_n^+ - X_n -$. $X_n \uparrow X$ implies that $X_n^+ \uparrow X^+$ so the Monotone Convergence Theorem tells us that $EX_n^+ \uparrow EX^+$.

Similarly, $X_n^- \downarrow X^-$ where each X_n^- is a nonnegative function. This implies that $|X_n| \leq X_1$ for which we know that EX_1 is finite. Thus we can apply the Dominated Convergence Theorem to determine $EX_n^- \to EX^-$.

Combing results, we have that

$$EX_n = EX_n^+ - EX_n^- \to EX^+ - EX^- = EX$$

To get monotonic convergence from bellow, observe that since $X_n^- \downarrow X^-$ it follows that EX_n^- decreases with each n as $n \to \infty$. This combined with the monotonicity of EX_n^+ implies $EX_n \uparrow EX$.