Math 275B: Homework 1

Marcus Lucas

B.21.18

Claim: Suppose $\mathcal{F} \subset \mathcal{G}$ are two σ-fields and X and Y are bounded $\mathcal{G}$ measurable random variables. Prove that

$$\mathbb{E}\left[X\mathbb{E}\left[Y|\mathcal{F}\right]\right] = \mathbb{E}\left[Y\mathbb{E}\left[X|\mathcal{F}\right]\right].$$

Proof: The functions X, Y, $\mathbb{E}\left[X|\mathcal{F}\right]$, and $\mathbb{E}\left[Y|\mathcal{F}\right]$ are all $\mathcal{G}$ measurable and bounded meaning that their products are $\mathcal{G}$ measurable and bounded (hence integrable w.r.t. $\mathcal{G}$) as well. Thus

$$\mathbb{E}\left[X\mathbb{E}\left[Y|\mathcal{F}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X\mathbb{E}\left[Y|\mathcal{F}\right]|\mathcal{F}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{F}\right]\mathbb{E}\left[Y|\mathcal{F}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Y\mathbb{E}\left[X|\mathcal{F}\right]|\mathcal{F}\right]\right] = \mathbb{E}\left[Y\mathbb{E}\left[X|\mathcal{F}\right]\right]$$

where the first and last equalities follow from the definition of conditional expectation (applied to the entire probability space Ω) while the second and third equalities follow from Proposition 21.20 (Bass), taking into account that $\mathbb{E}\left[X|\mathcal{F}\right]$ and $\mathbb{E}\left[Y|\mathcal{F}\right]$ are both $\mathcal{F}$ measurable.

B.21.19

Claim: Let $\mathcal{F} \subset \mathcal{G}$ be two σ-fields and let X be a bounded $\mathcal{G}$ measurable random variable. Prove that if

$$\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\mathbb{E}\left[Y|\mathcal{F}\right]\right]$$

for all bounded $\mathcal{G}$ measurable random variables Y, then X is $\mathcal{F}$ measurable.

Proof: From the previous problem we can infer that

$$\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\mathbb{E}\left[Y|\mathcal{F}\right]\right] = \mathbb{E}\left[Y\mathbb{E}\left[X|\mathcal{F}\right]\right]$$

for any bounded $Y \in \mathcal{G}$. Then for any set $A \in \mathcal{G}$,

$$\mathbb{E}\left[1_A X\right] - \mathbb{E}\left[1_A\mathbb{E}\left[X|\mathcal{F}\right]\right] = \mathbb{E}\left[1_A(X - \mathbb{E}\left[X|\mathcal{F}\right])\right] = 0$$

Thus if $A = \{X > \mathbb{E}\left[X|\mathcal{F}\right]\}$ it follows that A must be a set of measure zero. Else it would contain some set A' over which $\mathbb{E}\left[1_A(X - \mathbb{E}\left[X|\mathcal{F}\right])\right] > 0$. A similar argument can be made for $A = \{X < \mathbb{E}\left[X|\mathcal{F}\right]\}$ such that $X = \mathbb{E}\left[X|\mathcal{F}\right]$ a.e. except possibly on a $\mathcal{G}$ measurable set of measure zero. In particular the equality must hold on every positve measure set in $\mathcal{F}$.

Since $\mathbb{E}\left[X|\mathcal{F}\right]$ is by definition $\mathcal{F}$ measureable, it follows that X is almost $\mathcal{F}$ measureable as well. Let $N_{\mathcal{G}} \in \mathcal{G}$ be the measure zero set where $\mathbb{E}\left[X|\mathcal{F}\right]$ and X differ. It follows that there exists a measure zero set $N_{\mathcal{F}} \in \mathcal{F}$ containing the points where the functions differ, as they are equal on every positive measure set in $\mathcal{F}$. Thus

$$X^{-1}((c,\infty]) = (X^{-1}((c,\infty]) \cap N_{\mathcal{F}}^c) \cup (X^{-1}((c,\infty]) \cap N_{\mathcal{F}}) = (\mathbb{E}\left[X|\mathcal{F}\right]^{-1}((c,\infty]) \cap N_{\mathcal{F}}^c) \cup (X^{-1}((c,\infty]) \cap N_{\mathcal{F}})$$

The first set intersection on the right-hand side obsiously lies in $\mathcal{F}$. Thus there must be an $\mathcal{F}$ measure zero set over which we can modify X so as to render the second intersection (and thus X) $\mathcal{F}$ measurable. In lieu of a modification, X may only be $L^1(\mathcal{F})$ equivalent to $\mathbb{E}\left[X|\mathcal{F}\right]$.

D.4.1.3

Claim: Imitate the proof in the remark after Theorem 1.5.2 to prove the Cauchy-Schwarz inequality.

$$E(XY|\mathcal{G})^2 \leq E(X^2|\mathcal{G})E(Y^2|\mathcal{G})$$

Proof: For any set $A \in \mathcal{G}$ and real value θ we can write

$$\int_A (X+\theta Y)^2 \; dP = \int_A X^2 \; dP + 2\theta \int_A XY \; dP + \theta^2 \int_A Y^2 \; dP.$$

As is mentioned in the refenced remark, the fact that the quadratic expression on the right-hand side is positive for all θ implies that it must have either one or zero real roots. In particular, its descriminant must be less than or equal to zero such that

$$\left(\int_A XY \; dP \right)^2 \leq \int_A X^2 \; dP \int_A Y^2 \; dP$$

always holds. But this relation implies the claim up to equivalence on $\mathcal{G}$. To see this let

$$B = \left\{ E(XY|\mathcal{G})^2 > E(X^2|\mathcal{G})E(Y^2|\mathcal{G}) \geq \epsilon > 0 \right\}.$$

The integral inequality that we derived implies that B must have measure zero for all ϵ. Thus the claim can only be violated on a set of measure zero.

D.4.1.6

Claim: Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 \leq \infty$, then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2).$$

Proof: The proof of Theorem 4.1.15 (Durrett) demonstrates that $E(X|\mathcal{F})$ and $E(X|\mathcal{G})$ can be viewed as orthogonal projections of $X \in L^2$ onto $L^2(\mathcal{F})$ and $L^2(\mathcal{G})$. It shows that $E((X - E(X|\mathcal{F}))Z) = 0$ for any $Z \in L^2(\mathcal{F})$. Because $\mathcal{G} \subset \mathcal{F}$, $L^2(\mathcal{G})$ is a subspace of $L^2(\mathcal{F})$ and we can write

$$\begin{aligned}
E(\{X - E(X|\mathcal{G})\}^2) &= E(\{X - E(X|\mathcal{F}) + E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) \\
&= E(\{X - E(X|\mathcal{F})\}^2 + 2E(\{X - E(X|\mathcal{F})\}\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}) \\
&\quad + \{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) \\
&= E(\{X - E(X|\mathcal{F})\}^2 + 2E(\{X - E(X|\mathcal{F})\} E(X|\mathcal{F})) \\
&\quad - 2E(\{X - E(X|\mathcal{F})\} E(X|\mathcal{G})) + \{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) \\
&= E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2)
\end{aligned}$$

where the cancellations of the middle terms of the second-to-last equation follow from $(X - E(X|\mathcal{F}))$ being orthogonal to both $E(X|\mathcal{F})$ and $E(X|\mathcal{G})$.

D.4.1.7

Claim: An important special case of the previous result occurs when $\mathcal{G} = \{\emptyset, \Omega\}$. Let $\text{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\text{var}(X) = E(\text{var}(X|\mathcal{F})) + \text{var}(E(X|\mathcal{F})).$$

Proof: When $\mathcal{G}$ is the trivial sub-algebra we know that $E(X|\mathcal{G}) = EX$. Thus the statement proved in the previous problem becomes

$$\text{var}(E) = E(\{X - EX\}^2) = E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - EX\}^2)$$

Expanding the right hand expression above gives us

$$\begin{aligned}
\mathrm{var}(E) &= E(X^2) - 2E(XE(X|\mathcal{F})) + E(E(X|\mathcal{F})^2) + E(E(X|\mathcal{F})^2) - 2EX \cdot E(E(X|\mathcal{F})) + E((EX)^2) \\
&= E(E(X^2|\mathcal{F})) - 2E(E(X|\mathcal{F})E(X|\mathcal{F})) + E(E(X|\mathcal{F})^2) + E(E(X|\mathcal{F})^2) - 2(EX)^2 + (EX)^2 \\
&= E(E(X^2|\mathcal{F})) - 2E(E(X|\mathcal{F})^2) + E(E(X|\mathcal{F})^2) + E(E(X|\mathcal{F})^2) - (EX)^2 \\
&= E(E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2) + E(E(X|\mathcal{F})^2) - E(E(X|\mathcal{F}))^2 \\
&= E(\mathrm{var}(X|\mathcal{F})) + \mathrm{var}(E(X|\mathcal{F}))
\end{aligned}$$

where I make use of Theorem 4.1.14 (Durrett) and the fact that $E(X|\mathcal{F}) \in \mathcal{F}$ to determine

$$E(XE(X|\mathcal{F})) = E(E(XE(X|\mathcal{F})|\mathcal{F})) = E(E(X|\mathcal{F})E(X|\mathcal{F})).$$

I also exploit the equivalence $EX = E(E(X|\mathcal{F}))$ several times.