2.20 If $f_n, g_n, f, g, \in L^1$, $f_n \to f$, and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Proof: Proposition 2.12 tells us that given a measure μ, for any $\overline{\mu}$-measurable function f', we can find a μ-measureable function f such that $f' = f$ $\overline{\mu}$-a.e. Functions which are μ-measurable are also $\overline{\mu}$-measurable. Thus by proposition 2.11, we have that f and g are $\overline{\mu}$-measurable and so are μ-measurable after a possible redefinition on a null set.

By taking real and imaginary parts it suffices to assume that all f_n, g, f, g are real-valued, in which case $g_n + f_n \geq 0$ a.e. and $g_n - f_n \geq 0$ a.e. We also have that $g_n + f_n \to g + f$. Thus by corollary 2.19 of Fatou's Lemma

$$\int g + \int f = \int \liminf \int (g_n + f_n) \leq \liminf \int (g_n + f_n) = \int g + \liminf \int f_n$$
$$\int g - \int f = \int \liminf \int (g_n - f_n) \leq \liminf \int (g_n - f_n) = \int g - \limsup \int f_n$$

$g \in L^+$ therefore $\liminf \int f_n \geq \int f \geq \limsup \int f_n$, implying that $\int f = \lim \int f_n$.

2.21 Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof: Assume $\int |f_n - f| \to 0$. Then by proposition 2.22 and the fact that $f_n - f \in L^1$, we have that

$$\left| \int f_n \right| - \left| \int f \right| \leq \left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f|.$$

Since we can bound the right hand side of the inequality with arbitrary $\epsilon > 0$ by taking n sufficiently large, it follows that $\int |f_n| \to \int |f|$

Likewise, let $g_n = |f_n| + |f|$, $g = 2|f|$, $h_n = |f_n - f|$, and $h = 0$ such that $g_n \to g$ a.e. and $h_n \to h$ a.e. Then $g_n, h_n, g, h \in L^1$, and

$$|h_n| \to |f_n - f| \leq |f_n| + |f| = g_n \in L^1.$$

If, in addition, $\int |f_n| \to \int |f|$ such that $\int g_n \to \int g$, we meet all the criteria of the generalized Dominated Convergence Theorem. Thus we can infer that

$$\int h_n = \int h \implies \int |f_n - f| \to 0.$$

2.32 Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Proof: The integrand of $\rho(f, g)$ is continuous and so is a measurable function. It is also non-negative and real-valued, so $\rho(f, g) \geq \int 0 = 0$ for any f, g. Also by prop 2.23b, $\rho(f, g) = 0$ iff it's integrand equals zero a.e. This only occurs when $f = g$ a.e. That $|f - g| = |g - f|$ implies symmetry of $\rho(f, g)$. Subbadditivity follows from that fact that $|f - h| < |f - g| + |g - h|$ such that

$$|f - h| + |f - h|\big(|f - g| + |g - h|\big) < |f - g| + |g - h| + |f - h|\big(|f - g| + |g - h|\big)$$
$$\implies |f - h|\big(1 + |f - g| + |g - h|\big) < \big(1 + |f - h|\big)\big(|f - g| + |g - h|\big)$$
$$\implies \frac{|f - h|}{1 + |f - g|} < \frac{|f - g| + |g - h|}{1 + |f - g| + |g - h|} < \frac{|f - g|}{1 + |f - g|} + \frac{|g - h|}{1 + |g - h|}$$

And since the resulting composite functions are all real-valued and measurable, we have by the basic properties of the Lebesgue integral that

$$\rho(f,h) = \int \frac{|f-h|}{1+|f-h|} < \int \frac{|f-g|}{1+|f-g|} + \int \frac{|g-h|}{1+|g-h|} = \rho(f,g) + \rho(g,h)$$

So ρ is a metric.

Regarding the convergence of $\{f_n\}$, assume that $f_n \to f$ with respect to ρ and let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$. Then

$$\rho(f_n, f) = \int \frac{|f_n - f|}{1+|f_n - f|} \geq \int_{E_{n,\epsilon}} \frac{|f_n - f|}{1+|f_n - f|} \geq \frac{\epsilon}{1+\epsilon}\mu(E_{n,\epsilon})$$

implying that $\mu(E_{n,\epsilon}) \leq \frac{1+\epsilon}{\epsilon}\rho(f_n, f) \to 0$ such that $\mu(E_{n,\epsilon}) \to 0$. Thus $f_n \to f$ in measure.

Likewise assume that $f_n \to f$ in measure. Also note that $\frac{|f_n - f|}{1+|f_n - f|} < 1$ for any n. Then for any $E_{n,\epsilon}$ we have that

$$\rho(f_n, f) = \int \frac{|f_n - f|}{1+|f_n - f|} \leq \int \chi_{E_{n,\epsilon}} + \frac{\epsilon}{1+\epsilon}\chi_{E^c_{n,\epsilon}} = \mu(E_{n,\epsilon}) + \frac{\epsilon}{1+\epsilon}\mu(E^c_{n,\epsilon}) \qquad (\star)$$

But $\mu(E_{n,\epsilon}) \to 0$ as n increases and $\mu(E^c_{n,\epsilon}) \leq \mu(X) < \infty$ such that we can make the right-hand-side of $(\star)$ arbitrarily small by letting $n \to \infty$ and $\epsilon \to 0$. Thus $\rho(f_n, f) \to 0$ or $f_n \to f$ with respect to ρ.

2.40 In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Proof: Given a sequence $\{f_n\}$ such that $f_N \to f$ a.e., assume that there exists $g \in L^1$ such that $|f_n| \leq g$ for all n. Then it also holds that $|f_n - f| \leq 2g$ and by the dominated convergence theorem, we have that $0 = \int |f - f| = \lim \int |f_n - f|$. But if this is the case then for any $\epsilon > 0$, there exists n such that $\int |f_n - f| < \epsilon$.

Let $E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m - f| > k^{-1}\}$, such that $E_n(k)$ is decreases in size as $n \to \infty$ and $\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$. Then $k^{-1}\mu(E_n(k)) \leq \int |f_n - f| < \epsilon$ and, since ϵ is arbitrary, this means that $\mu(E_n(k)) \to 0$ as $n \to \infty$. The reset of the proof follow as in Folland.

2.41 If μ is σ-finite and $f_n \to f$ a.e., there exists measurable $E_1, E_2, \cdots \subset X$ such that $\mu((\bigcup_1^{\infty} E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Proof: As μ is σ-finite, there exists some collection $\{A_i\}$ such that $\mu(A_i) < \infty$ for all i and $X = \bigcup_i A_i$. Then by Egoroff's theorem, for any $\epsilon > 0$ we can find a collection $\{F_i\}$ such that $F_i \subset A_i$, $\mu(F_i) < \epsilon$, and $f_n \to f$ uniformly on each F_i^c. Letting $E_i = F_i^c$, we just need to show that $\mu((\bigcup_1^{\infty} E_i)^c) = 0$. But

$$\mu((\bigcup_1^{\infty} E_i)^c) = \mu(\bigcap_1^{\infty} E_i^c) = \mu(\bigcap_1^{\infty} F_i) < \mu(F_i) < \epsilon$$

and we can make ϵ as small as we want such that $\mu((\bigcup_1^{\infty} E_i)^c) = 0$ must hold.

2.42 Let μ be a counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Proof: If $f_n \to f$ in measure then $f_n \to f$ everywhere. Otherwise, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \geq \epsilon\chi_{\{i\}} = \epsilon$ for some $\epsilon > 0$ and $i \in \mathbb{N}$, invalidating our assumption. The counting measure is also σ-finite since we can represent $\mathbb{N}$ as the union of sets of a single integer $\{i\}$, each set having measure equal to 1 and the compliment of their union being the empty set. As such, we can apply the result of problem 2.41 to determine that $f_n \to f$ uniformly.

Now if $f_n \to f$ uniformly then for any $k \in \mathbb{N}$ there exists n_k such that $|f_n(x) - f(x)| < k^{-1}$ for all x and $n > n_k$. For any $\epsilon > 0$ there exists k such that $\epsilon > k^{-1}$ and so $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| > k^{-1}\}) = 0$ for any $n > n_k$. This means that $f_n \to f$ in measure.