2.6 **Claim:** The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions X can fail to be measurable.

Proof: Define

$$g = \sup_{\alpha \in \mathcal{A}} f_\alpha(x)$$

where $\mathcal{A}$ is an uncountable index set. Then we have that

$$g^{-1}((a,\infty]) = \bigcup_{\mathcal{A}} f_\alpha^{-1}((a,\infty])$$

but in general, $f_\alpha^{-1}((a,\infty]) \in \mathcal{M}$ and $\mathcal{M}$, being a σ-algebra, is not necessarily closed under uncountable unions. For instance, if $\mathcal{M} = \mathcal{B}_{\mathbb{R}}$ and $f_\alpha^{-1}((a,\infty])$ is a closed set for each α, then $g^{-1}((a,\infty])$ is an uncountable union of closed sets and so not in $\mathcal{B}_{\mathbb{R}}$. Thus, in general, g as defined above is not measurable.

2.7 **Claim:** Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = X$, and $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \emptyset$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \leq \alpha$ on E_α and $f(x) \geq \alpha$ on E_α^c for every α.

Proof: Let $f(x) = \inf\{q \in \mathbb{Q} : x \in E_q\}$. By definition $f^{-1}((q,\infty]) = E_q \in \mathcal{M}$ for any $q \in \mathbb{Q}$. By the result of exercise 2.4, this tells us that f is measurable.

Given some $x \in E_\alpha$ for any $\alpha \in \mathbb{R}$, it holds that $x \in E_q$ for every rational $q \geq \alpha$. But $f(x) \leq q$ for all such q and, as $\mathbb{Q}$ is dense in $\mathbb{R}$, $f(x) \leq \alpha$ must also hold. Likewise, for any $q' \leq \alpha$, $x \in E_a^c$ implies that $x \in E_{q'}^c$. Thus $f(x) > q'$ for all $q' \leq \alpha$ implying $f(x) \geq \alpha$ on E_α^c.

2.8 **Claim:** If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Proof: By Propositions 2.6 we know that if f is measurable then $-f$ is measurable, so assume that f is increasing.

Take a finite open interval (a,b). For any $x \in f^{-1}((a,b))$, $a < f(x) < b$ holds, meaning we can always find c,d such that $a < c < f(x) < d < b$. But then $f^{-1}(a) < f^{-1}(c) < x < f^{-1}(d) < f^{-1}(b)$ by the monotonicity of f. So then for any $x \in f^{-1}((a,b))$ we can always find some open interval $(f^{-1}(c), f^{-1}(d)) \subset f^{-1}((a,b))$ which contains x. Thus $f^{-1}((a,b))$ is an open set and an element of $\mathcal{B}_{\mathbb{R}}$. Since $\mathcal{B}_{\mathbb{R}}$ is generated by the set of all open intervals on $\mathbb{R}$, this means that f is measurable.

2.11 **Claim:** Suppose f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x,\cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f(\cdot,y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \leq x \leq a_{i+1}$ let

$$f_n(x,y) = \frac{f(a_{i+1},y)(x-a_i) - f(a_i,y)(x-a_{i+1})}{a_{i+1}-a_i}$$

Then f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Also, every function on $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Proof: $f(\cdot,y)$ is continuous function between metric spaces and so is Borel measurable. Thus $f_n(x,y)$ is Borel measurable on any $[a_i,a_{i+1}]$ as it is the sum and product of Borel measurable functions. In general, this means that for any $E \in \mathcal{B}_{\mathbb{R}}$, the intersection $f_n^{-1}(E) \cap [a_i,a_{i+1}]$ is

Borel measurable. But $f_n^{-1}(E) = \bigcup_{i\in\mathbb{Z}} \big(E \cap [a_i, a_i + 1]\big)$, which is just a countable union of Borel sets, and so $f_n^{-1}(E)$ is Borel measurable on its entire domain.

For any $\epsilon > 0$ we can find $\delta > 0$ (for a point $x \in \mathbb{R}$) such that $|f(x', y) - f(x, y)| < \epsilon$ whenever $|x' - x| < \delta$. For any delta, we can also find an $N \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ for all $n > N$. Thus at any point $x \in [a_i, a_{i+1}]$, with fixed ϵ, δ, and N such that $a_i, a_{i+1} \in (x - \delta, x + \delta)$ for all $n > N$, we have

$$\begin{aligned}
|f_n(x,y) - f(x,y)| &= \Big|\frac{f(a_{i+1},y)(x-a_i) - f(a_i,y)(x-a_{i+1}) - (a_{i+1}-a_i)f(x,y)}{a_{i+1}-a_i}\Big| \\
&= \Big|\frac{f(a_{i+1},y)(x-a_i) - f(a_i,y)(x-a_{i+1}) - (a_{i+1}-x+x-a_i)f(x,y)}{a_{i+1}-a_i}\Big| \\
&= \Big|\frac{\big(f(a_{i+1},y)-f(x,y)\big)(x-a_i) - \big(f(a_i,y)-f(x,y)\big)(x-a_{i+1})}{a_{i+1}-a_i}\Big| \\
&\le \Big|\frac{\big(f(a_{i+1},y)-f(x,y)\big)(x-a_i)}{a_{i+1}-a_i}\Big| - \Big|\frac{\big(f(a_i,y)+f(x,y)\big)(x-a_{i+1})}{a_{i+1}-a_i}\Big| \\
&\le \frac{\epsilon(x-a_i)}{a_{i+1}-a_i} + \frac{\epsilon(a_{i+1}-x)}{a_{i+1}-a_i} = \epsilon
\end{aligned}$$

So for any ϵ we can find N such that $|f_n(x, y) - f(x, y)| < \epsilon$ and thus f_n converges pointwise to f. This along with Proposition 2.7 implies that f is Borel measurable.

Finally, we know that functions on $\mathbb{R}$ that are continuous are Borel measurable. Now assume that every function on $\mathbb{R}^k$ which is continuous in each variable separately is also Borel measurable. If we're given a function f on $\mathbb{R}^{k+1} = \mathbb{R} \times \mathbb{R}^k$ that is continuous in each variable separately, we know that $f(\cdot, y) : \mathbb{R} \to \mathbb{R}$ is a continuous function and hence Borel measurable. Also, $f(x, \cdot) : \mathbb{R}^k \to \mathbb{R}$ is continuous in each variable separately and so, by our assumption, it is Borel measurable. As we've just proved, this means that f is Borel measurable on $\mathbb{R}^{K=+1}$. So, in general, functions on any finite dimensional space $\mathbb{R}^n$ which are continous in each variable separately are also Borel measurable.

2.14 **Claim:** If $f \in L^+$, let $\lambda(E) = \int_E f\,d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g\,d\lambda = \int fg\,d\mu$.

Proof: Theorem 2.10 tells us that we can construct a sequence $\{\phi_n\}$ of increasing simple functions which converge pointwise to f. For any disjoint sequence $\{E_j\} \subset \mathcal{M}$ such that $\bigcup_j E_j = E$ we have

$$\int_E f = \lim_{n\to\infty} \int_E \phi_n$$

by the monotone convergence theorem. By thereom 2.13(d) we already know that restrictions of integrals of simple functions to sets in $\mathcal{M}$ define measures on $\mathcal{M}$. Thus

$$\lim_{n\to\infty} \int_E \phi_n = \lim_{n\to\infty} \sum_j \int_{E_j} \phi_n = \sum_j \lim_{n\to\infty} \int_{E_j} \phi_n$$

Again by the monotone convergence theorem, the simple functions ϕ_n are still increasing and approaching f when restricted to E_j, so

$$\sum_j \lim_{n\to\infty} \int_{E_j} \phi_n = \sum_j \int_{E_j} f$$

and so

$$\lambda(E) = \int_E f = \sum_j \int_{E_j} f = \sum_j \lambda(E_j)$$

establishing that λ is a measure on $\mathcal{M}$. $\square$

Now suppose we have a simple function $g \in L^+$ defined over a collection of measures $\{E_j\}_1^N$. Then

$$\int g\,\lambda = \sum_j a_j \lambda(E_j) = \sum_j a_j \int_{E_j} f\,d\mu = \sum_j \int a_j f \chi_{E_j}\,d\mu = \int f \sum_j a_j \chi_{E_j}\,d\mu = \int fg\,d\mu$$

More generally, if g is any function in L^+, prop 2.10 tells us that we can create a sequence of increasing simple functions $\{\psi_n\}$ that approach g pointwise. Therefore $\{f\psi_n\}$ approaches fg pointwise and we can apply the monotone convergence theorem twice to get

$$\int g\,d\lambda = \lim_{n\to\infty} \int \psi_n\,\lambda = \lim_{n\to\infty} \int f\psi_n\,\mu = \int fg\,d\mu \quad \square$$

2.16 **Claim:** If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Proof: We know by proposition 2.10 that there exists a series of increasing simple functions $\{\phi_n\}$ whice approach f pointwise and that (by the MCT)

$$\int f = \lim_{n\to\infty} \int \phi_n$$

Thus for any $\epsilon > 0$, there exists a simple function ϕ such that $0 \leq \phi \leq f$ and

$$\int \phi > (\int f) - \epsilon$$

But each $\phi = \sum_j a_j \chi_{E_j}$ for some finite collection of sets $\{E_j\}_1^N \subset \mathcal{M}$. This being the standard representation of ϕ, only one a_j, let's assume a_1 can equal zero. Furthermore, $\int \phi < \infty$ implies that $E = E_2 \cup \cdots \cup E_N$ must have finite measure since

$$\int \phi = \sum_{j=2}^N a_j \mu(E_j) \geq \ \min\{a_j\}_2^N \mu(E)$$

So E is a set such that $E \in \mathcal{M}$, $\mu(E) < \infty$, and

$$\int_E f \geq \int_E \phi = \int \phi \geq (\int f) - \epsilon$$