Math 275A: Homework 4

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1.7.4

Claim: Let μ be a finite masure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$. Show that

$$\int (F(x+c) - F(x))\ dx = c\mu(\mathbb{R})$$

Proof: Define the product measure $\lambda \times \mu$ on $\mathbb{R}^2$ where λ is the standard Lebesgue measure on $\mathbb{R}$. Also define a subset of $\mathbb{R}^2$,

$$E = \{(x, y) : x < y \leq x + c\} = \{(x, y) : y - c < x \leq y\},$$

over which we will integrate. From Fubini's theorem, we can determine that

$$\int_{\mathbb{R}^2} \mathbf{1}_E\ d(\lambda \times \mu) = \int_{\mathbb{R}} \int_{x<y\leq x+c} d\mu dx = \int_{\mathbb{R}} F(x+c) - F(x)\ dx.$$

Likewise, if we flip the order of integration (and use the alternative expression for E above), we get

$$\int_{\mathbb{R}^2} \mathbf{1}_E\ d(\lambda \times \mu) = \int_{\mathbb{R}} \int_{y-c<x\leq y} dx d\mu = \int_{\mathbb{R}} c\ d\mu = c\mu(\mathbb{R}),$$

proving the claim.

2.1.5

Claim: (i) If X and Y are independent with distributions μ and ν then

$$P(X+Y=0) = \sum_y \mu(\{-y\})\nu(\{y\}).$$

(ii) If X has a continuous distribution $P(X = Y) = 0$.

Proof: Since X and Y are independent, we know the vector (X, Y) has distribution $\mu \times \nu$. Thus can define the $P(X + Y = 0)$ as an integral over the indicator function for $x + y = 0$, that is

$$P(X+Y=0) = \int \mathbf{1}_{(x+y=0)} d(\mu \times \nu).$$

This integrand is nonzero only on a line, passing through the origin of R^2. Thus we can rewrite the expression for the probability as

$$P(X+Y=0) = \int \mathbf{1}_{(x+y=0)}\ d\mu d\nu = \int \mu(\{-y\}) d\nu = \sum_y \mu(\{y\})\nu(\{-y\}),$$

proving the first claim. Note here that the sum over y makes sense in that each distribution has only countably many discontinuities such that μ and ν only evaluate to zero at a finite number of points.

To prove the second claim, just note that if μ (r.e. ν) has a continuous distribution function $F(x)$, then $\mu(x) = F(x) - F(x^-) = 0$ by definition. Thus

$$P(X+Y=0) = \sum_y \mu(\{y\})\nu(\{-y\}) = \sum_y 0 \cdot 0 = 0.$$

2.1.15

Claim: Let Ω be the unit interval $(0,1)$ equipped with the Borel sets $\mathcal{F}$ and Lebesque measure P. Let $Y_n(\omega) = 1$ if $[2^n\omega]$ is odd and 0 if $[2^n\omega]$ is even. Show that $Y_1, Y_2, \dots$ are independent with $P(Y_k = 0) = P(Y_k = 1) = 1/2$.

Proof: For a given k, each successive h-interval $[\frac{i-1}{2^k}, \frac{i}{2^k})$ for $i \in 1, 2, \dots, 2^k$ maps alternately to either 0 or 1, and has an equivalent measure under P. Note that all intervals specified as $[0, \frac{1}{2^k})$ are actually open intervals, but this doesn't affect their measure under P so I will not differentiate these intervals in the proof.

Given a specific y_1, $\{\omega : Y_1(\omega) = y_1\} = [\frac{i}{2}, \frac{i+1}{2})$ for some $i \in \{1,2\}$. If we also specify y_2, then $\{\omega : Y_1(\omega) = y_1, Y_2(\omega) = y_2\} = \{\omega : Y_1(\omega) = y_1\} \cap \{\omega : Y_2(\omega) = y_2\} = [\frac{i}{2}, \frac{i+1}{2})$ for some $i \in \{0,1,2,3\}$. Proceeding by induction, it follows that for any given sequence of y_k's of length n,

$$\{\omega : Y_1(\omega) = y_1, \dots, Y_n(\omega) = y_n\} \cap_{i=k}^n \{\omega : Y_k(\omega) = y_k\} = [\frac{i}{2^n}, \frac{i+1}{2^n})$$

for some $i \in \{0, 2^n - 1\}$.

Thus the probability of any length-n sequence occuring is $P([\frac{i}{2^n}, \frac{i+1}{2^n})) = \frac{1}{2^n}$. This combined with the fact that each Y_k has and equal probability of being 0 or 1 lets us write

$$P(Y_1 = y_1, Y_2 = y_2, \dots, Y_n = y_n) = \frac{1}{2^n} = \prod_{i=1}^n P(Y_i = y_i) \quad \text{for all} \quad n \in \mathbb{N},$$

confirming that $Y_1, Y_2, \dots$ are independent.

2.2.3

Claim: (i) Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$. Let $U_1, U_2, ...$ be independent and uniformly distributed on $[0,1]$, and let

$$I_n = n^{-1}(f(U_1) + \cdots + f(U_n))$$

Show that $I_n \to I := \int_0^1 f\, dx$ in probability. (ii) Suppose $\int_0^1 |f(x)|^2\, dx < \infty$. Use Chebyshev's inequality to estimate $P(|I_n - I| > a/n^{1/2})$.

Proof: Each function $f(U_i)$ is a random variable since each U_i is a random variable and f is measurable. They also inherit the i.i.d. property from U_i. That $f(x)$ is integrable implies $Ef(U_i) = \int_0^1 f\, dU_i(x) = \int_0^1 f\, dx = I$ is finite. Thus we apply the L^1 weak law of large numbers (2.2.14) to determine that $I_n \to I$ in probability.

Letting $\phi(x) = x^2$, we can apply Chebyshev's inequality to the random variables $|I_n - I|$ to determine that

$$\frac{a^2}{n} P(|I_n - I| > a/n^{1/2}) \leq E|I_n - I|^2$$

I being equal to the expectation of each $f(U_i)$ means

$$E(I_n - I)^2 = E(n^{-1}(\sum_{i=1}^{n} f(U_i) - nI))^2 = n^{-2}\text{var}(\sum_{i=1}^{n} f(U_i)) = n^{-1}\text{var}(f(U_i)).$$

Combining this result with the previous inequality implies

$$P(|I_n - I| > a/n^{1/2}) \leq a^{-2}\left(\int_0^1 f(x)^2\, dx - \left(\int_0^1 f(x)\, dx\right)^2\right) < \infty.$$

Where the boundedness of $\text{var}(f(U_i))$ follows from $f \in L^2$.

2.2.4

Claim: Let $X_1, X_2, \ldots$ be i.i.d with $P(X_i = (-1)^k k) = C/k^2 \log k$ for $k \geq 2$ where C is chosen to make the sum of the probabilites = 1. Show that $E|X_i| = \infty$, but there is a finite constant μ so that $S_n/n \to \mu$ in probability.

Proof: To see that $E|X_i| = \infty$ observe that

$$E|X_i| = \sum_{k=2}^{\infty} k \cdot C/k^2 \log k = \sum_{k=2}^{\infty} C/k \log k = C\sum_{k=2}^{\infty} 1/k \log k.$$

The sum multiplying C in the last expression is a divergent series of nonnegative terms so $E|X_i| = \infty$.

To show convergence in probability, we apply Durrett's weak law of large numbers (2.2.12). First note that

$$nP(|X_n| > n) = nC\sum_{k=n+1}^{\infty} 1/k^2 \log k \leq \frac{nC}{\log n}\sum_{k=n+1}^{n} \infty \frac{1}{k^2} \leq \frac{nC}{\log n}\int_n^{\infty} \frac{x^2}{}dx = \frac{C}{\log n}.$$

Thus $nP(|X_n| > n) \to 0$ as $n \to \infty$, allowing us to apply the WLLN. Furthermore,

$$\mu_n = E(X_1 \mathbf{1}_{(|X_1| \leq n)}) = \sum_{k=2}^{n} \frac{(-1)^k C}{k \log k} \to \sum_{k=2}^{\infty} \frac{(-1)^k C}{k \log k} = \mu$$

Where the fact that μ is an alternating series implies convergence. So by the WLLN $S_n/n \to \mu_n \to \mu$ as $n \to \infty$ proving the claim.