1. (a) Theorem 3.1 assumes $\mathcal{F}$ is paritially ordered by set inclusion. Also $\bigcup \mathcal{C} \in \mathcal{F}$ for any chain in $\mathcal{F}$, so each chain has an upper bound. These are the prerequisits for Zorn's lemma and the conclusion, $\mathcal{F}$ has a maximal element, follows.

   (b) $\mathcal{Y}$ is nonempty since each X_α is nonempty and we can always pick the pair $(\{\alpha\}, \langle x_\alpha \rangle)$ for some $\alpha \in A$.

   We know that $\preceq$ is a partial ordering by checking the definition. If $(B, \langle x_\alpha \rangle_{\alpha \in B}) \preceq (C, \langle y_\alpha \rangle_{\alpha \in C})$ and $(C, \langle y_\alpha \rangle_{\alpha \in C}) \preceq (D, \langle z_\alpha \rangle_{\alpha \in D})$, it follows that $B \subset D$ and $x_\alpha = z_\alpha \forall\ \alpha \in B$. Thus $(B, \langle x_\alpha \rangle_{\alpha \in B}) \preceq (D, \langle z_\alpha \rangle_{\alpha \in D})$. Obviously this relation is reflexive. And if $(B, \langle x_\alpha \rangle_{\alpha \in B}) \preceq (C, \langle x_\alpha \rangle_{\alpha \in C})$ and $(C, \langle y_\alpha \rangle_{\alpha \in C}) \preceq (B, \langle x_\alpha \rangle_{\alpha \in B})$ then $B = C$ and $\langle x_\alpha \rangle = \langle y_\alpha \rangle$ and equality must hold.

3. (a) Take the set of all finite subsets of $\mathbb{N}$. It is down-closed and finite chains are closed. But it is not chain-closed since $\mathbb{N}$ is not included in this set. Likewise there is not maximal element of the set. For any finite subset, we can always add another integer no in that set.

   (b) $\mathcal{C}_\sigma$ contains $\emptyset$ and $\mathbb{N}$. Furthermore each successive set $\{\sigma(1) \cdots \sigma(n)\}$ is formed by adding a single unique element to the previsous set. Thus we can't insert an additional set anywhere into $\mathcal{C}_\sigma$ such that it will still be a chain. This means that any chain which includes $\mathcal{C}_\sigma$ must be $\mathcal{C}\sigma$ itself, making it a maximal chain in $\mathcal{P}(\mathbb{N})$.

   (c) It is possible to create a bijection between $\mathbb{N}$ and $\mathbb{Q} \cap (0,1)$ and so between $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Q} \cap (0,1))$. We can construct a chain by taking set of rational numbers from $\mathbb{Q} \cap (0,1)$ which converge to r for each $r \in \mathbb{R}$. For any two r_a, r_b not equal to eachother, there is a uncountably infinite number of reals between the two. Thus there is an uncountable number of sets in the chain between any two sets A and B (definted by r_a and r_b). This then means that the corresponding chain in $\mathcal{P}(\mathbb{N})$ has the same property.

4.8 Given that X has the cofinite topology, pick any $x \in X$. Every neighborhood U of x contains all but a finite number of points. Let J be the largest index of any member of the sequence $\{x_j\}$ not included in U (there are only a finite number of points to check). Then $x_j \in U$ for all $j > J$, meaning $x_j \to x$.

4.13 We're given the U is open and A is dense in X, meaning $X = A \cup Acc(A)$. By proposition 4.1 we know that $\overline{U} = U \cup Acc(U)$. Also it's clear that $\overline{U \cap A} \subseteq \overline{U}$, so we just need to show that $\overline{U \cap A} \subseteq \overline{U}$ or that

$$U \cup Acc(U) \subseteq (U \cap A) \cup Acc(U \cap A)$$

A point $x \in U$ is either in $U \cap A$ or is an accumulation point of A. If the latter is the case, then we also have $x \in Acc(U \cap A)$. Likewise, any $x \in Acc(U)$ is also an accumulation point for $U \cap A$. Thus the above inclusion holds and $\overline{U} = \overline{U \cap A}$.

4.15 We already know that g is continuous on A. This means that g^{-1} maps any closed set V (let's say in $\mathbb{C}$) to an closed set in the relative topology of A. Since A itself is closed, this means that $g^{-1}(V))$ must also be closed in the topolgoy of X. We also know that, since A is closed, $A = \partial A \cup A^o$ and thus $A^c \cup \partial A$ is a closed set.

Let h represent the extended function g. Then $h^{-1}(0)$ is the union of some closed set and $A^c \cup \partial A$ is also closed. In general, h^{-1} maps any closed set $V \subset \mathbb{C}$ to a closed subset of X. If $V \ni 0$ then the mapping is the same as with g. Otherwise $h^{-1}(V) = g^{-1}(V) \cup (A^c \cup \partial A)$. As such, h is continuous.