18.10) **Claim:** Let f_n be a sequence of continuous functions on $\mathbb{R}$ that converge at every point. Prove there exists an interval and a number M such that $\sup_n|f_n|$ is bounded by M on that interval.

Proof: We're given that the sequence $\langle f_n\rangle$ converges at every point. For any $m \in \mathbb{N}$ we can form the set $A_m = \{x : \sup_n|f_n| > m\}$ which is just the union of open sets $\bigcup_n |f_n|^{-1}((m,\infty))$ which is itself open. If there is no open interval in X such that $\sup_n|f_n| \leq m$, this means there is no open set in X satisfying this condition, implying that A_m is dense in X. If this holds for every $m \in \mathbb{N}$, then the Baire Category Theorem tells us that $\bigcap_{m\in\mathbb{N}} A_m = \{x : \sup_n|f_n| = \infty\}$ is dense in X, contradicting our assumption on the convergence of $\langle f_n\rangle$.

4.76) Given that X is normal and second countable let $\mathcal{B}$ represent a countable basis. For every pair $(U,V) \in \mathcal{B}\times\mathcal{B}$ satisfying $\overline{U} \subset V$, let $f_{U,V} \in C(X,I)$ be some function defined using Urysohn's Lemma for which $f_{U,V}(\overline{U}) = 0$ and $f_{U,V}(\overline{V}^c) = 1$. Let $\mathcal{F} \subset C(X,I)$ be a set of functions $f_{U,V}$ defined in this manner, which will be countable since $\mathcal{B}\times\mathcal{B}$ is countable.

Now for any closed set A and point $x \in A^c$ we know there exists some basis element $V \subset A^c$ containing x. Since X is normal we can find open sets B_1, B_2 containing V^c and x respectively. But since B_2 is open, it must also contain a basis element U containing x. Thus $x \in U \subset \overline{U} \subset B_2 \subset V$ and there is a continuous function $f_{U,V} \in \mathcal{F}$ which separates x and A.

5.3) Define $T : \mathcal{X} \to \mathcal{Y}$ by $Tx = \lim T_n$. Clearly T is a linear map since for $a, b \in K$ we have

$$T(ax+by) = \lim T_n(ax+by) = a(\lim T_nx) + b(\lim T_ny) = a(Tx) + b(Ty)$$

It is also bounded since

$$\|Tx\| = \|\lim T_nx\| = \lim \|T_nx\| \leq \lim \|T_n\|\|x\|$$

where $\lim \|T_n\| < \infty$ since all $\lim \|T_nx\| < \infty$. Lastly we have that

$$\lim \|T_n - T\| = \lim_{n\to\infty} \sup\{\|T_nx - Tx\| : \|x\| = 1\} = \sup\{\|\lim T_nx - Tx\| : \|x\| = 1\} = 0$$

meaning that $\|T_n - T\| \to 0$ and thus $L(\mathcal{X},\mathcal{Y})$ is complete.

5.8) To verify that $\|\mu\| = |\mu|(X)$ is a norm note that any is non-negative and only equals 0 on sets of measure zero. Since any complex measure can be expressed as $\mu = \int f\,d\nu$ we have that $\|\alpha\mu\| = \int |\alpha f|\,d\nu = |\alpha|\int |f|\,d\nu = |\alpha|\|\mu\|$. Also, proposition 3.14 in Folland tells us that for any two measures,

$$\|\mu_1+\mu_2\| = |\mu_1+\mu_2|(X) \leq |\mu_1|(X) + |\mu_2|(X).$$

So $\|\cdot\|$ is a norm.

Given an absolutely convergent series $\langle\mu_n\rangle \subset \mathcal{M}(X)$ for which $\sum\|\mu_n\| < \infty$ it follows that any series $\sum\mu_n(E)$ converges absolutely on any measurable set E since it is bounded above by $\sum|\mu_n|(X)$. Let $\mu = \sum\mu_n$. Then μ is non-negative and $\mu(\emptyset) = 0$. To show that countable additivity holds, assume we have a countable disjoint sequence of measurable sets $\langle E_k\rangle \subset X$ for which $\mu(E) = \bigcup E_n$. Tonelli's theorem tells us that $\mu(E)$ is bounded since

$$\sum_k |\mu(E_k)| = \sum_k\sum_n |\mu_n(E_k)| = \sum_n\sum_k |\mu_n(E_k)| \leq \sum_n\sum_k |\mu_n|(E_k) = \sum_n |\mu_n(E)| \leq \sum_n \|\mu\|$$

Then we can apply Fubini's theorem to show that

$$\sum_k \mu(E_k) = \sum_{n,k} \mu_n(E_k) = \sum_n \mu_n(E) = \mu(E).$$

So $\mu \in \mathcal{M}(X)$.

I'm not sure about this last part. Need to show that the series $\sum \mu_n$ converges to μ?

5.27) Let $\langle r_j \rangle_1^\infty$ be an enumeration of the rational numbers and given some $n \in \mathbb{N}$ let $I_j^{(n)}$ be the interval centered at r_j of length $n^{-1}2^{-j}$. Then the set $U_n = \cup_1^\infty I_j^{(n)}$ is an open and dense set in $\mathbb{R}$ of measure less than n^{-1}. We can see that U_n^c is nowhere dense since if it contained an open set, it would also have to contain an open interval and thus some rational number. If we let $U = \cap_1^\infty U_n$, then U has measure zero and it's compliment is $\cup_1^\infty U_n^c$, a countable unions of nowhere dense sets. This set is by definition meager.

5.42) (a) First we need to find the closure of E_n. Assume we have a sequence $\langle f_i \rangle \subset E_n$ such that $f_i \to f$. For each f_i there exists a point $x_i \in [0,1]$ such that $|f_i(x) - f_i(x_i)| \leq n|x - x_i|$ for all $x \in [0,1]$. The sequence formed from these points exists in a compact set and so has a convergence subsequence. Let x_0 be the limit of this subsequence. Then we have that for any $\epsilon > 0$ there always exists an index m for which $\|f - f_m\|_u < \epsilon$ and x_0 satisfies $|f_m(x) - f_m(x_0)| < n|x - x_0|$ for all $x \in [0,1]$. This means that

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \epsilon + n|x - x_0| + \epsilon \end{aligned}$$

and since ϵ was arbitrary, we end up with the desired inequality $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$. So E_n itself is actually closed.

To show that it has empty interior, assume otherwise. Then there is some $g \in E_n^o$ and $\epsilon > 0$ such that $\mathcal{B}_\epsilon(g) \subset E_n^o$. We know that g can be uniformly approximated by piecewise linear functionals having slopes of magnitude greater than $2n$ everywhere. This means that there is a function $h \in C([0,1])$ such that $\|h - g\|_u < \epsilon$ with slope greater than $2n$ or less than $-2n$. But then for any point $x_0 \in [0,1]$ we can always find a point $x_1 \in [0,1]$ such that $|h(x_1) - h(x_0)| > 2n|x - x_0|$. So $h \neq E_n$ and there is no open set contained within E_n.

(b) If a function $f \in C([0,1])$ is anywhere differentiable then it contains at least one point $x_0 \in [0,1]$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} < \epsilon \quad \text{for all} \quad x \in \mathcal{B}_\delta(x_0).$$

Since any such f has a compact domain. We know that it's image is compact and hence bounded. This means $\alpha = \|f\|_u$ is finite giving us another bound

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} < \frac{\alpha}{\delta} \quad \text{for all} \quad x \in [0,1] \setminus \mathcal{B}_\delta(x_0).$$

So any continuous function f that is somewhere differentiable is a member of any set E_n for which $n > \max\ \{\epsilon, \frac{\alpha}{\delta}\}$ given a fixed ϵ and δ. This means that the set of all functions that are somewhere differentiable, let's call it D, is a subset of $\cup E_n$. But then D is meager as it is the subset of a meager set. And so, the set of nowhere differentiable functions must be residual in $C([0,1])$.