4.56) (a) For two values $t, s \in [0, \infty]$, if $s < t$ then

$$st + s < st + t \implies s(t+1) < t(s+1) \implies \frac{s}{s+1} < \frac{t}{t+1} \text{ or } \Phi(s) < \Phi(t)$$

Also

$$\Phi(t+s) = \frac{t+s}{t+s+1} = \frac{t}{t+s+1} + \frac{s}{t+s+1} < \frac{t}{t+1} + \frac{s}{s+1} = \Phi(t) + \Phi(s)$$

(b) Φ is monotic and so it has an inverse. This means that $\rho(x, y) \leq r$ implies $\Phi \circ \rho(x, y) \leq \Phi(r)$ and also $\Phi \circ \rho(x, y) < r$ implies $\rho(x, y) < \Phi^{-1}(r)$. Thus set U open in (Y, ρ) is also open under the topology generated by $\Phi \circ \rho$ since any open ball with respect to ρ must include an open ball with respect to Φ and vice versa. This implies that the topologies generated by either metric are the same.

(c) Let $\gamma(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Then $(C(X), \gamma)$ is a metric space with the topology of uniform convergence. The previous result tells us that $\Phi \circ \gamma$ is also a metric on $C(X)$ with the same topology.

(d) We already know that each summand of ρ is a metric. Thus ρ is a metric since a positive sum of metrics is still a metric.

For any open ball $B_r(f)$, each function $g \in B_r(f)$ must be contained within some ball $B_s(g)$ such that $B_s(g) \subset B_r(f)$. If we let $B'_{k,N}(g) = \{h : \sup_{x \in \overline{U_n}} |h(x) - g(x)| < k^{-1}\}$ such that $\Phi(k^{-1} < \frac{s}{2})$ and $\sum_{n=N}^{\infty} 2^{-n} < \frac{s}{2}$ then it folows that

$$B'_{k,N}(g) \subset \{h : \sum_{n=1}^{N-1} 2^{-n} \Phi(\sup_{x \in \overline{U_N}} |h(x) - g(x)|) < \frac{s}{2}\} \subset B_s(g).$$

So $B_r(f)$ is open in the topology of uniform convergence of compact sets.

Likewise, for any open ball $B'_{k,N}(f)$ in the topology of uniform convergence. For any $g \in B'_{k,N}(f)$ there exists an s such that $\{h : \sup_{x \in \overline{U_n}} |h(x) - g(x)| < s\} \subset B'_{k,N}$. But this implies that $B'_{k,N}(f)$ also contains $B_{2^{-N}\Phi(s)}$ since

$$\begin{aligned}
B_{2^{-N}\Phi(s)} &= \{h : \sum_{n=1}^{N-1} 2^{-n} \Phi(\sup_{x \in \overline{U_n}} |h(x) - g(x)|) < 2^{-n}\Phi(s)\} \\
&\subset \{h : 2^{-N} \Phi(\sup_{x \in \overline{U_N}} |h(x) - g(x)|) < 2^{-N}\Phi(s)\} \\
&= \{h : \sup_{x \in \overline{U_N}} |h(x) - g(x)| < s\} \subset B'_{k,N}
\end{aligned}$$

So any $B'_{k,N}$ is open in the metric topology and thus the two topologies are equivalent.

Lastly, a sequence of functions converges in the metric iff they converge in the topology of uniform convergence of compact sets (since it is equivalent to the metric topology). Since X is LCH an thus every point $x \in X$ has a compact neighborhood. This implies that such a convergent sequence will converge uniformly on those compact neighborhoods, i.e. it will converge locally uniformly.

4.61) **Restatement:** Let X be a compact Hausdorff space and Y be a metric space. If $\mathcal{F}$ is an equicontinuous, pointwise totally bounded subset of functions mapping X to Y, then $\mathcal{F}$ is totally bounded in the uniform metric and $\mathcal{F}$ is compact in the space of all functions from X to Y.

The proof is almost the same as that for theorem 4.43. As in the original proof, once we have a finite set $x_1, \dots, x_n \in X$ for which $\cup_1^n U_x = X$, we need to find a finite set $\{z_n\} \subset Y$ that is $\frac{1}{4}\epsilon$-dense in $\{f(x_j) : f \in \mathcal{F}, 1 \leq j \leq n\}$. Here, we need pointwise total bounded-ness to be able to claim there is a finite covering of each set $\{f(x_j) : f \in \mathcal{F}\}$ with balls of radius $\frac{1}{4}\epsilon$, the centers of which we can use to form our finite set $\{z_n\}$.

From here, the proof proceeds the same as for theorem 4.43.

4.63) Any functions K and f have compact images since their domains are compact. Compact sets are bounded in $\mathbb{C}$ and so their images are also bounded. In particular, $\|f\|_u$ is finite. Also by continuity of K we can pick a δ such that $|K(x_1, y) - K(x_2, y)| < \epsilon/\|f\|_u$ whenever $|x_1 - x_2| < \delta$. This implies the following:

$$\begin{aligned}
|Tf(x_1) - Tf(x_2)| &= \int_0^1 K(x_1, y) f(y)\, dy - \int_0^1 K(x_2, y) f(y)\, dy \\
&\leq \int_0^1 |K(x_1, y) - K(x_2, y)| \|f\|_u\, dy \\
&< \int_0^1 \frac{\epsilon}{\|f\|_u} \|f\|_u\, dy = \epsilon
\end{aligned}$$

and so $Tf \in C([0, 1])$.

To show that $\mathcal{F} = \{Tf : \|f\|_u \leq 1\}$ is equicontinuous note that for any point $x_1 \in [0, 1]$, we can find a δ such that $|K(x_1, y) - K(x_2, y)| < \epsilon$ for all x_2 such that $|x_1 - x_2| < \delta$. Since, we have $\|f\|_u \leq 1$ this also means that $|K(x_1, y) - K(x_2, y)| < \epsilon/\|f\|_u$ for any $f \in \mathcal{F}$. Thus by the proof of continuity of a specific Tf we have that $\mathcal{F}$ is equicontinuous at a point x_1. The choice of x_1 was arbitrary so $\mathcal{F}$ is actually just equicontinuous.

Furthermore, since $|Tf(x)| \leq \int_0^1 |K(x, y)||f(y)|\, dy \leq \int_0^1 |K(x, y)|\, dy$ we know that $\mathcal{F}$ is pointwise bounded. Thus by the Arzela-Ascoli Theorem, we know that it is precompact.

4.64) Let $\mathcal{F} = \{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ where

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}.$$

For any finite $\epsilon > 0$ let $\delta = \epsilon^{1/\alpha}$. Then

$$|f(x) - f(y)| \leq \rho(x, y)^\alpha \leq \delta^\alpha = \epsilon$$

implying that $\mathcal{F}$ is equicontinuous. It is also pointwise bounded by definition and so $\mathcal{F}$ is precompact by the Arzela-Azcoli Theorem.

For any sequence $\langle f_n \rangle \subset \mathcal{F}$ convergent to some $f \in C(X)$ and for each point $x \in X$, each sequence $\langle f_n(x) \rangle$ is a convergent sequence of points in a closed set $[-1, 1]$ such that $f(x) \in [-1, 1]$. But this means that $\|f\|_u \leq 1$. Also, since $N_\alpha(f_n) \leq 1$ we have that $|f_n(x) - f_n(y)| \leq \rho(x, y)^\alpha$. But $\langle f_n(x) - f_n(y) \rangle \to f(x) - f(y)$ and so it holds that $|f(x) - f(y)| \leq \rho(x, y)^\alpha$. Thus $N_\alpha(f) \leq 1$ and $f \in \mathcal{F}$. This means that $\overline{\mathcal{F}} = \mathcal{F}$ and so $\mathcal{F}$ is compact.

4.68) Letting $\mathcal{F} = \{f(x,y) = g(x)h(y) : g \in C(X), h \in C(Y)\}$, note that the set of finite sums of elements of $\mathcal{F}$ form an algebra. For instance, given any two elements f, f' in this algebra, it holds that

$$ff' = \Big(\sum_i g_i h_i\Big)\Big(\sum_j g_j' h_j'\Big) = \sum_{i,j} (g_i g_j')(h_i h_j')$$

which is still an element of $\mathcal{F}$. The algebra $\mathcal{A}$ generated by $\mathcal{F}$ contains this algebra and so they must be equivalent. Also for any $f \in \mathcal{A}$ we have that $f^* = \sum g_i^* h_i^* \in \mathcal{A}$ such that the algebra is closed under complex conjugation. This also implies that $\overline{\mathcal{A}}$ is closed under conjugation since it is a continuous function which we can apply to any convergent sequence in $\mathcal{A}$.

Assume we have two unique points $(x_1, y_1), (x_2, y_2) \in X \times Y$ and that $x_1 \neq x_2$ without loss of generality. Then, since X is compact Hausdorff and hence normal, Urysohn's Theorem tells us that there exists a function $g \in C(X)$ that separates x_1 and x_2. This in turn means that $f = g \cdot 1 \in \mathcal{A}$ separates (x_1, y_1) and (x_2, y_2). Thus $\overline{\mathcal{A}}$ separates points in $X \times Y$. Since $\overline{\mathcal{A}}$ also contains every constant function and since $X \times Y$ is compact, the Complex Stone-Weierstrass Theorem tells us that $\overline{\mathcal{A}} = X \times Y$, i.e. $\mathcal{A}$ is dense in $X \times Y$.

4.69) Assume we have two points $x, y \in X$ and that $x \neq y$. Then there exists some coordinate map such that $\pi_\alpha(x) \neq \pi_\alpha(y)$. Thus $\mathcal{A}$ and hence it's closure separates X. We know that $\overline{\mathcal{A}}$ is closed under complex conjugation because the algebras of projections and the constant function are closed under conjugation and because the conjugation function is continuous. We know that X is compact by Tyconoff's Theorem, and that the entire set $\mathcal{A}$ vanishes nowhere because it contains constant functions. Thus we can apply the Complex Stone-Weierstrass Theorem to show that $\mathcal{A}$ is dense in X.