Math 275B: Homework 2

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Claim: Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be an increasing family of σ-fields and let $\mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$. If N is a stopping time, define

$$\mathcal{F}_N = \{A \in \mathcal{F}_\infty : A \cap (N \leq n) \in \mathcal{F}_n \ \forall \ n\}.$$

1. Prove that $\mathcal{F}_N$ is a σ-field.
2. If M is another stopping time with $M \leq N$ a.s., and we define $\mathcal{F}_M$ analogously, prove that $\mathcal{F}_M \subset \mathcal{F}_N$.
3. If X_n is a martingale with respect to $\{\mathcal{F}_n\}$ and N is a stopping time bounded by the real number K, prove that $E[X_K|\mathcal{F}_N] = X_N$.

Proof:

1. For $A \in \mathcal{F}_N$, if $A \cap (N \leq n) \in \mathcal{F}_n$ then its compliment $A^c \cup (N \leq n)^c \in \mathcal{F}_n$. $\mathcal{F}_n$ is also closed under intersections such that $A^c \cup (N \leq n)^c \cap (N \leq n) = A^c \cap (N \leq n) \in \mathcal{F}_n$ for all n, so $A^c \in \mathcal{F}_N$.
2. $M \leq N$ implies that $(N \leq n) \subset (M \leq n)$. Thus if $A \in \mathcal{F}_M$ such that $A \cap (M \leq n) \in \mathcal{F}_n$ for all n, it follows that $A \cap (M \leq n) \cap (N \leq n) = A \cap (N \leq n) \in \mathcal{F}_n$. Hence $A \in \mathcal{F}_N$ for any $A \in \mathcal{F}_M$ implying $\mathcal{F}_M \subset \mathcal{F}_N$.
3. For any $A \in \mathcal{F}_N$,

   $$E[X_N; A] = \sum_{k=0}^{K} E[X_N; A \cap (N = k)] = \sum_{k=0}^{K} E[X_k; A \cap (N = k)].$$

   As in Durrett 21.24, we can can make the observation that since $A \in \mathcal{F}_N$, $A \cap (N = k) = A \cap (N \leq k) - A \cap (N \leq k-1)$ is $\mathcal{F}_j$ measurable for $j \geq k$. Thus

   $$E[X_k; A \cap (N = k)] = E[X_{k+1}; A \cap (N = k)] = \cdots = E[X_K; A \cap (N = k)]$$

   such that

   $$E[X_N; A] = \sum_{k=0}^{K} E[X_K; A \cap (N = k)] = E[X_K; A].$$

   This proves the claim.

D.4.2.1

Claim: Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.

Proof: By definition X_n is measurable w.r.t. $\mathcal{G}_n$. Because $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$, $X_i \in \mathcal{G}_n$ for $i \in \{1, \ldots, n\}$ such that $\sigma(X_i) \subset \mathcal{G}_n$. $\mathcal{F}_n$ is the smallest σ-algebra containing each $\sigma(X_i)$ so $\mathcal{F}_n \subset \mathcal{G}_n$ as well. Clearly $X_n \in \mathcal{F}_n$, so we can verify that it's a martingale w.r.t. $\mathcal{F}_n$ by noting that

$$E[X_n|\mathcal{F}_{n-1}] = E[E[X_n|\mathcal{G}_{n-1}]|\mathcal{F}_{n-1}] = E[X_{n-1}|\mathcal{F}_{n-1}] = X_{n-1}$$

D.4.2.6

Claim: Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. By example 4.2.3 that $X_n = \prod_{m \le n} Y_m$ defines a martingale. (i) Use Theorem 4.2.12 and an argument by contradiction to show $X_n \to 0$ a.s. (ii) Use the strong law of large numbers to conclude $(1/n) \log X_n \to c < 0$.

Proof: (i) $Y_1, Y_2, \ldots$ are nonnegative, thus $X_1, X_2, \ldots$ are nonnegative. Also X_n, being a martingale, is a supermartingale. Thus theorem 4.2.12 tells us that $X_n \to X$ a.s. for some X.

Assume $X \neq 0$ on some positive measure set A. Since $P(Y_m = 1) < 1$ it follows that $P(|Y_m - 1| > 1/k) > \epsilon$ for some integer k and $\epsilon > 0$. The second Borel-Cantelli lemma tells us that $P(|Y_m - 1| > 1/k \text{ i.o.}) = 1$. This means that $P(|X_n - X| > X/k \text{ i.o.}) = 1$ such that X_n does not converge to X on the set A where $X \neq 0$.

(ii) We can rewrite the random sequence as $(1/n) \log X_n = (1/n) \sum_{m \le n} \log Y_m = S_n/n$. The random variables $\log Y_m$ are i.i.d. so the SLLN tells us that $S_n/n \to E \log Y_m$. Note that $\log(x)$ is a concave function and in particular, $\log(x) \le x - 1$ for $x \neq 1$. Thus $E \log Y_m < E(Y_m - 1) = 0$ where the strict inequality comes from the fact that $Y_m \neq 1$ on a set of positive measure.

D.4.4.9

Claim: Let X_n and Y_n be martingales with $EX_n^2 < \infty$ and $EY_n^2 < \infty$.

$$EX_nY_n - EX_0Y_0 = \sum_{m=1}^{n} E(X_m - X_{m-1})(Y_m - Y_{m-1})$$

Proof: We know from theorem 4.4.7 of Durrett that $E(X_m - X_{m-1})Y_{m-1} = 0$ which lets us immediately reduce the right-hand side above to $\sum_{m=1}^{n} E(X_m - X_{m-1})Y_m$. Noting that

$$E(X_{m-1}Y_m) = E(E[X_{m-1}Y_m|\mathcal{F}_{m-1}]) = E(X_{m-1}E[Y_m|\mathcal{F}_{m-1}]) = E(X_{m-1}Y_{m-1}),$$

the right-hand side further reduces to

$$\sum_{m=1}^{n} EX_mY_m - EX_{m-1}Y_{m-1} = EX_nY_n - EX_0Y_0$$

proving the claim.

RP.3.8

Claim: Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with mean zero and finite variance σ^2. If T is a stopping time with finite mean, show that

$$\text{var}(\sum_{i=1}^{T} X_i) = \sigma^2 E(T)$$

Proof: Let $Y_i = X_i^2$ such that $E(Y_i) = E(X_i^2) = \sigma^2$. Then $Y_1, Y_2, \ldots$ are clearly i.i.d. with mean $\mu = \sigma^2$. This along with T being finite allows us to apply Wald's equation (Cor. 3.15) from R&P,

$$E(\sum_{i=1}^{T} Y_i) = \mu E(T).$$

Furthermore, $\text{var}(\sum_{i=1}^{T} X_i) = \sum_{i=1}^{T} \text{var}(X_i) = \sum_{i=0}^{T} X_i^2 = \sum_{i=0}^{T} Y_i$ for zero mean i.i.d. random varianbles. Substituting this and $\mu = \sigma^2$ into the equation above proves the claim.