5.29 (a) $\mathcal{X}$ contains 0 and for $f, g \in \mathcal{X}$

$$\sum n|(f+\lambda g)(n)| \leq \sum n|f(n)| + n|\lambda g(n)| \leq \sum n|f(n)| + |\lambda| \sum |g(n)|$$

meaning that $\mathcal{X}$ is a supspace of $\mathcal{Y}$. It is proper since the function $f(n) = n^{-1}$ is contained within $\mathcal{X}$ but not $\mathcal{Y}$. To show that $\overline{\mathcal{X}} = \mathcal{Y}$ not that for any $f \in \mathcal{Y}$ we can pick N such that $\sum_{N+1}^{\infty} f(n) < \epsilon$. We can then define g_ϵ by

$$g_\epsilon(n) = \begin{cases} f(n) \text{ for } n \leq N \\ 0 \text{ for } n > N \end{cases}$$

which is contained in $\mathcal{X}$ as it is a finite sum. Furthermore,$\|f - g_\epsilon\| = \sum_{N+1}^{\infty} f(n) < \epsilon$ so $\mathcal{X}$ is dense in $\mathcal{Y}$. Since $\mathcal{X} \neq \overline{\mathcal{X}}$, we know that $\mathcal{X}$ is not complete.

(b) We want to show that T maps closed sets to closed sets. This is equivalent to ensuring that for every convergent sequence $f_n \to f$ such that $Tf_n \to g$ for some $g \in \mathcal{Y}$, we have $Tf_n \to Tf$. We can then pick integers N and M such that $\sum_{N+1}^{\infty} nf(n) < \epsilon$, $\sum_{N+1}^{\infty} g(n) < \epsilon$, $\|Tf_m - g\| < \epsilon$ and $\|f - f_m\| < \epsilon/N$. This gives us

$$\begin{aligned} \sum_{n=1}^{\infty} |Tf(n) - Tf_m(n)| &\leq \sum_{n=1}^{N} n|f(n) - f_m(n)| + \sum_{N+1}^{\infty} n|f(n)| \\ &+ \sum_{N+1}^{\infty} |Tf_m(n) - g(n)| + \sum_{N+1}^{\infty} |g(n)| \\ &< N\|f - f_m\| + \epsilon + \|Tf_m - g\| + \epsilon = 4\epsilon \end{aligned}$$

and so $Tf_n \to Tf$ meaning T is closed.
T is not bounded since we can define

$$f_k(n) = \begin{cases} n^{-2} \text{ for } n \leq k \\ 0 \text{ for } n > k \end{cases}$$

such that $f_k \in \mathcal{X}$ and $\|f_k\| < \sum_1^{\infty} n^{-2} < \infty$ but $\|Tf_k\|$ can be made arbitrarily large by taking k large enough.

(c) We know that T is not bounded and so not continuous, hence $S = T^{-1}$ cannot be open. It is also obviously surjective since T is defined on $\mathcal{X}$. Boundedness follows from

$$\|Sf\| = \sum_1^{\infty} |n^{-1} f(n)| \leq \sum_1^{\infty} f(n) = \|f\|.$$

5.37 Let $T^\dagger f = f \circ T$ and note that it is a linear map from $\mathcal{Y}^*$ to $\mathcal{X}^*$. Then for any $f \in \mathcal{Y}^*$ for which $\|f\| = 1$ we have

$$|\widehat{x}(T^\dagger f)| = |f(Tx)| < \|f\|\|T(x)\| = \|Tx\|$$

such that $\widehat{x} \circ T^\dagger \in L(\mathcal{Y}^*, K)$. We can subsequently form the set $\mathcal{A} = \{\widehat{x} \circ T^\dagger : \|x\| = 1\}$ and observe that

$$\sup_{A \in \mathcal{A}} \|Af\| = \sup_{\|x\|=1} \|\widehat{x} \circ T^\dagger f\| \leq \|Tx\| < \infty.$$

The uniform boundedness principle then tells us that $\sup_{\|x\|=1}\|\widehat{x}\circ T^\dagger\| < \infty$ for $\|f\|=1$.

By Hahn-Banach we can always find a function $f_0 \in \mathcal{Y}^*$ for which $\|f_0\|=1$ and $f_0(Tx) = \|Tx\|$. This means that for $\|x\|=1$,

$$\|Tx\| = f_0(Tx) = \widehat{x}\circ T^\dagger f_0 \leq \sup_{\|x\|=1}\|\widehat{x}\circ T^\dagger\|\|f_0\| < \infty$$

and so T is bounded.

5.38 Given $Tx = \lim T_n x$, we see that T is linear since

$$T(x\lambda y) = \lim T_n(x+\lambda y) = \lim T_n(x) + \lambda \lim T_n(y) = Tx + \lambda Ty.$$

To show boundedness note that $\sup_{n\in\mathbb{N}}\|T_n x\| < \infty$ for all $x \in \mathcal{X}$. The Uniform Boundedness Principle then tells us that $C = \sup_{n\in\mathbb{N}}\|T_n\| < \infty$. But then

$$\|Tx\| = \|\lim T_n x\| = \lim \|T_n x\| \leq \sup \|T_n x\| \leq \sup \|T_n\|\|x\| = C\|x\|$$

and so T is bounded.

5.45 Define a family of seminorms $p_{n,k} = \|f^{(k)}|_{[-n,n]}\|_u$ on $\mathcal{X} = C^\infty(\mathbb{R})$ and let $\mathcal{T}$ be the topology which they generate. For every $f \neq 0$ we have that $f(x) \neq 0$ for some $x \in \mathbb{R}$ such that $p_{n,k}(f) \neq 0$ for any $n \geq |x|$. Proposition 5.16 then tells us that $\mathcal{X}$ is Hausdorff.

To show completeness, take any Cauchy sequence $\langle f_n\rangle \subset \mathcal{X}$. Proposition 5.14b tells us that $f_n \to f$ uniformly on any set $[-n,n]$, where $f(x) = \lim f_n(x)$. Thus f is continuous on any compact set and so $f \in C(\mathbb{R})$. We can make the same argument for any $\langle f_n^{(k)}\rangle$ such that $f_n^{(k)} \to h_k$ where $h_k \in C(\mathbb{R})$. As all of these sequences are Cauchy, they are each eventually dominated by $|h_k|+1$ so we can apply the Dominated Convergence Theorem to get

$$h_k = \lim f_n^{(k)} = \lim \int_0^x f_n^{(k+1)} = \int_0^x \lim f_n^{(k+1)} = \int_0^x h_{k+1}$$

which tells us that $h_k' = h_{k+1}$ or equivalently $f^{(k)} = h_k$. Thus $\mathcal{X}$ is complete and a Fréchet space.

If $f_n \to f$, then prop 5.14b tells us that $f_n^{(k)} \to f^{(k)}$ uniformly on any set $[-n,n]$. Any compact set K can be encompasses by such a set and so $\langle f_n^{(k)}\rangle$ converges uniformly on all compact sets.

Likewise, if $f_n^{(k)} \to f^{(k)}$ uniformly on all compact sets, this includes all sets $[-n,n]$. This means $\langle f_n\rangle$ converges with respect to each pseudonorm defining the toplogy on $\mathcal{X}$ and so, by prop 5.14b, $f_n \to f$.

5.51 The norm topology if finer than the weak topology on $\mathcal{X}$, so any weakly open set is also open in the norm topology. But this means that weakly closed sets are also norm-closed. Conversely, we know that each $f \in \mathcal{X}^*$ is continuous in the weak topology. If we have a norm-closed subspace $\mathcal{M} \subset \mathcal{X}$ then for each $x \in \mathcal{X}\setminus\mathcal{M}$ we can find some $f_x \in \mathcal{X}^*$ for which $f_x(x) \neq 0$ and $f_x(\mathcal{M}) = 0$. By continuity we know that $\ker(f_x)$ is weakly-closed, so $\mathcal{M} = \cap_{x\in\mathcal{X}\setminus\mathcal{M}}\ker(f_x)$ is also weakly-closed.

5.53 (a) Given that $T_n \to T$ strongly, exercise 5.47 tells us that $M = \sup_n \|T_n\| < \infty$. Observing that

$$\|T_n x_n - Tx\| \leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leq \|T_n\| \|x_n - x\| + \|T_n x - Tx\|,$$

for any $\epsilon > 0$ we can pick N such that $\|x_n - x\| < \epsilon/M$ and $\|T_n x - Tx\| < \epsilon$ for $n > N$. This gives us an upper bound

$$\|T_n x_n - Tx\| \leq \|T_n\| \|x_n - x\| + \|T_n x - Tx\| \leq M \frac{\epsilon}{M} + \epsilon = 2\epsilon.$$

Since ϵ was arbitrary this tells us that $T_n x_n \to Tx$.

(b) We know that $S_n x \to Sx$ for $x \in \mathcal{X}$ since $\langle S_n \rangle$ converges strongly. But then the result from part (a) tells us that $T_n S_n x \to TSx$ since $\langle S_n x \rangle \subset \mathcal{X}$. Since this holds for all $x \in \mathcal{X}$ we have that $T_n S_n \to TS$ strongly.