1. (a) The set of balls defined by ρ and ρ' form bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ which they generate. Given an open set $U \in \mathcal{T}$, for every point $x \in U$ there exists an open ball $\mathcal{B}_\rho(x)$ containing x which lies completely within U. But we can also generate another ball $\mathcal{B}'_{\rho'}(x)$ of possibly smaller radius which fits within $\mathcal{B}_\rho(x)$ and still contains x. Thus $\mathcal{T} \subseteq \mathcal{T}'$.

   The same argument can be made to show that $\mathcal{T}' \subseteq \mathcal{T}$ such that the topologies generated by ρ and ρ' are equivalent.

   (b) $\rho(\mathbf{x}, \mathbf{y})$ is clearly non-negative since each ρ_n is a metric. Also $\rho(\mathbf{x}, \mathbf{y}) = 0$ implies that $\rho_n(x_n, y_n) = 0$ for all $n \in \mathbb{N}$ such that $\mathbf{x} = \mathbf{y}$. Symmetry holds since

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{\rho_n(x_n, y_n)/n\} = \max\{\rho_n(y_n, x_n)/n\} = \rho(\mathbf{y}, \mathbf{x})$$

   where we exploit the symmetry of each metric ρ_n. Finally we can demonstrate the triangle inequality as follows:

$$\begin{aligned}
\rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}) &= \max\{\rho_n(x_n, y_n)/n\} + \max\{\rho_n(y_n, z_n)/n\} \\
&\geq \max\{\rho_n(x_n, y_n)/n + \rho_n(y_n, z_n)/n\} \\
&\geq \max\{\rho_n(x_n, z_n)/n\} \\
&= \rho(\mathbf{x}, \mathbf{z})
\end{aligned}$$

   Take any ball $\mathcal{B}_\rho(\mathbf{x})$ of finite radius r and let N equal the smallest integer greater than or equal to r^{-1}. For all integers $n \geq N$, we will have that $\pi_n(\mathcal{B}_\rho(\mathbf{x})) = X_n$. Thus any such ball can be represented as a product of open sets $\prod_{n\in\mathbb{N}} U_n$ where $U_n \neq X_n$ for finitely many n. But this corresponds to the product topology generated by the ρ_ns.

   (c) More generaly. If we have a product topology on a couuntably infinite product of metrizable spaces, i.e. $X = \prod_{n\in\mathbb{N}} X_n$, we can generate that same topology in two steps. First, replace each metric ρ_n on X_n by $\rho'(x, y) = \min\{\rho(x, y), 1\}$. As shown in part (a), each ρ'_n generates the same topology on X_n as the orinial metric ρ_n. Then apply the result from part(b) to create $\rho(\mathbf{x}, \mathbf{y}) = \max\{\rho'_n(x_n, y_n)/n : n \in \mathbb{N}\}$. As we've just shown, this metric will generate the desired product topology, meaning that it is metrizable.

4.20 If each X_α is first countable then every point $x_\alpha \in X_\alpha$ possesses a countable neighborhood. Given some $x \in X = \prod_{\alpha\in A} X_\alpha$, any open set U containing x has a finite number of components $U_\alpha \neq X_\alpha$. We can categorize all such open sets contain x by the finite sequence $\langle \alpha_j \rangle$ of indices of such U_{α_j}.

Now the number of finite subsets of $\mathbb{N}$ (and thus any countable set A) is countable. Furthermore for any finite sequence $\langle \alpha_j \rangle$, we can form a countable neighborhood base around x by taking the set $\{\prod_{\alpha_j \in \langle \alpha_j \rangle} \pi^{-1}_{\alpha_j} V_{\alpha_j}\}$ where V_{α_j} is any member of the countable neighborhood base for $\pi_{\alpha_j}(x)$. This set is a finite product of countable sets and so is countable. The sum of all these sets over all the finite sequences of A then is also countable and forms a neighborhood basis for x in the product topology.

The same line of reasoning can be used to the second countable case. If each X_α has a countable basis B_α, the collection of all sets of the form $\prod_{\alpha_j \in \langle \alpha_j \rangle} U_{\alpha_j}$ where $U_{\alpha_j} \in B_{\alpha_j}$ forms a countable basis for X.

4.22 Each sequence $\langle f_n(x_0)\rangle$ is cauchy given the upper bound on the distance between successive points $f_n(x_0)$ and $f_m(x_0)$, where $m \geq n \geq N$ converges to zero as $N \to \infty$. Since $(\mathcal{Y}, \rho)$ is complete, each sequence $\langle f_n(x_0)\rangle$ then has a limit point $f(x_0)$ whithin the space. Let $f(x)$ be the map definied by these limit points. Since $\sup_{x\in X}(\rho(f_n(x), f_m(x))) \to 0$, it follows that $\sup_{x\in X}(\rho(f_n(x), f(x))) \to 0$.

If there is another function g such that $\sup_{x\in X}(\rho(f_n(x), g(x))) \to 0$ then we have that $\rho(f_N(x), f(x)) < \epsilon$ and $\rho(f_N(x), g(x)) < \epsilon$ from some $N \in \mathbb{N}$. But then $\rho(f(x), g(x)) < 2\epsilon$ and we can make ϵ arbitrarily small. Thus $f = g$ and the limiting function is unique.

Take some neighborhood V of $f(x_0)$ for any point $x_0 \in X$. Then there exists a ball $\mathcal{B}_{3\epsilon}(f(x_0)) \subseteq V$ containing x_0, and an N such that $\sup_{x\in X}(\rho(f_N(x), f(x))) < \epsilon$. Since f_N is continuous, $U = f_N^{-1}(\mathcal{B}_{3\epsilon}(x_0))$ is open in X. Furthermore if $x \in U$ then

$$\rho(f(x_0), f(x)) \leq \rho(f(x_0), f_N(x_0)) + \rho(f_N(x_0), f_N(x)) + \rho(f_N(x), f(x)) = 3\epsilon$$

meaning that $f^{-1}(U)$ must be a neighborhood of x_0. Since x_0 was arbitrary, this means f is continuous.

4.24 If X is normal then the conclusions of Urysohn and Tietze follow immediately. Now assume the conculsion of Urysohn. For any disjoint closed subsets A and B, there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B. Since f is continuous, $f^{-1}(-\infty, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, \infty)$ are disjoint open sets containing A and B respectively.

Furthermore, if we assume the conclusion of Tietze's Theorem, then for disjoint closed A and B we can form $X' = A \cup B$ on which A and B are in fact open under the restricted topology. Thus the characteristic function χ_B is continuous on X', implying the existence of some $f \in C(X, [0,1])$ which satisfies the conclusion of Urysohn's lemma. This in turn implies that X is normal.

4.38 If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, then it must be Hausdorff, since we can always use two open sets definted in both topologies to separate any points $x, y \in X$. Now assume $\mathcal{T}'$ is compact and let $V \in \mathcal{T}' \setminus \mathcal{T}$. V^c is closed and thus compact. The identity map from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$ is continuous and so V^c must also be compact relative to $\mathcal{T}$. But since $\mathcal{T}$ is Hausdorff, this means that V^c is closed in $\mathcal{T}$ and that $V \in \mathcal{T}$, violating our assumption. Thus $\mathcal{T}'$ cannot be compact.

If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then we can demonstrate compactness by observing the identity map from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$. It is continuous and so $\mathcal{T}'$ must be compact. If $\mathcal{T}'$, then we can apply the same line of reasoning as before to show that some $V \in \mathcal{T} \setminus \mathcal{T}'$ must actually be containted in $\mathcal{T}$. Thus $\mathcal{T}'$ cannot be Hausdorff.

4.43 Given the sequence $\langle a_n\rangle$, let $\langle a_{n_k}\rangle$ be any subsequence. We can define $x = \sum_{k\in\mathbb{N}}(1 + (-1)^k)2^{-(n_k+1)}$, such that $\langle a_{n_k}\rangle$ becomes $(0, 1, 0, 1, 0, \dots)$. This sequence does not converge, meaning $\langle a_{n_k}\rangle$ is not pointwise convergent and $\langle a_n\rangle$ has no pointwise convergent subsequence.