Math 275A: Homework 5

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Extra

Claim: Let Ω be a set. For subsets $A_1, A_2, \dots$ and A in Ω, let us say that $A_n \to A$ **pointwise** if $\mathbf{1}_{A_n} \to \mathbf{1}_A$ pointwise.

(a) Assume that $A_1 \subset A_2 \subset \cdots$ and A is their union. Show that $A_n \to A$ pointwise.

(b) Prove carefully that A_n converges pointwise to some limit set if and only if $\{A_n \text{ i.o.}\} = \{A_n \text{ eventually}\}$.

(c) Let $\mathcal{F}$ be a sigma-algebra and let $\mu : \mathcal{F} \to [0, \infty)$ be finitely additive. Prove that μ is a measure if and only if

$$\mu A_n \to \mu A \quad \text{whenever} \quad A_n \to A \text{ pointwise}$$

where all sets are in $\mathcal{F}$.

Proof:

(a) Assuming the $A_n \uparrow A$, it follows that for every $\omega \in A$, there exists an N such that $\omega \in A_n$ for $n \geq N$. Thus $\mathbf{1}_{A_n}(\omega) = \mathbf{1}_A(\omega)$ for all such n, implying that $\mathbf{1}_{A_n} \to \mathbf{1}_A$ pointwise.

(b) ($\implies$) By definition $\{A_n \text{ eventually}\} \subset \{A_n \text{ i.o.}\}$. To estabilish the reverse inclusion note that if $A_n \to A$ pointwise then any $\omega \in \{A_n \text{ i.o.}\}$ must also be a member of $\{A_n \text{ eventually}\}$. Otherwise, $\mathbf{1}_{A_n}(\omega) \not\to \mathbf{1}_A(\omega)$ at that point, violating our assumption on $\langle A_n \rangle$.

($\impliedby$) Let $A = \{A_n \text{ i.o.}\} = \{A_n \text{ eventually}\}$. For any $\omega \in A$, we know there exists finite N such that ω eventually lies in A_n for $n \geq N$. Thus $\mathbf{1}_{A_n}(\omega) \to \mathbf{1}_A(\omega)$. For any $\omega' \in A^c$, ω' must lie within only finitely many A_n's (since it can't hit an A_n inifinitely often) implying that $\mathbf{1}_{A_n}(\omega') \to 0$. Thus $\mathbf{1}_{A_n} \to \mathbf{1}_A$ pointwise.

(c) ($\implies$) Assume μ is a measure and that $A_n \to A$ pointwise. We know from part (b) that $A = \limsup A_n = \liminf A_n$. Let $B_n = \cup_{i=n}^{\infty} A_i$ such that $B_1 \supset B_2 \supset \cdots \supset A$ and $\mu B_n \downarrow \mu A$ (continuity from above). Let $C_n = \cap_{i=n}^{\infty} A_i$ such that $C_1 \subset C_2 \subset \cdots \subset A$ and $\mu C_n \uparrow \mu A$ (continuity from below). Because $C_n \subset A_n \subset B_n$ for each n such that $\mu C_n \leq \mu A_n \leq \mu B_n$, it must hold that $\mu A_n \to \mu A$.

($\impliedby$) Note that finite additivity implies that $\mu(\emptyset) = 0$, since $\emptyset$ is disjoint from itself and so $\mu\emptyset = \mu\emptyset + \mu\emptyset$. Assume that $\mu A_n \to \mu A$ whenever $A_n \to A$ pointwise. Let A_n be a telescoping sequence of sets for which $A_n \uparrow A$. Then $A_n \to A$ pointwise as established in part (a) and $\mu A_n \to \mu A$ by our assumption. This establishes continuity of the measure μ from below which can be shown to be equivalent to countable additivity (assuming finite additivity).

2.3.8

Claim: Let A_n be a sequence of indepenedent events with $P(A_n) < 1$ for all n. Show that $P(\cup A_n) = 1$ implies $\sum P(A_n) = \infty$ and hence $P(A_n \text{ i.o.}) = 1$.

Proof: The independence of the sequence $\langle A_n \rangle$ along with $P(\cup A_n) = 1$ implies that $P(\cap A_n^c) = \prod P(A_n^c) = \prod(1 - P(A_n)) = 0$. Taking the log of that infinite product yields $\sum \log(1 - P(A_n)) = -\infty$.

Since $P(A_n) < 1$, each term $\log(1 - P(A_n))$ is finite and negative. What's more, $\frac{-\log(1-P(A_n))}{P(A_n)} \to 1$ for $P(A_n) \to 0$. So we can apply the limit comparison test to determine that $\sum \log(1 - P(A_n))$ diverges if and only if $\sum P(A_n)$ does as well (which they both do). The last part of the claim follows from BCII.

Alternate Proof?: After establishing that $\prod(1-P(A_n)) = 0$ it seems that we could also note that $P(A_n) < 1$ implies $1-P(A_n) > 0$ for all n. Hence $\prod_{i=1}^{m}(1-P(A_n)) = P(\cap_{i=1}^{m} A_i^c) > 0$ for finite m, such that $P(\cap_{i=m}^{\infty} A_i^c)$ must equal zero. Now $\{A_n \text{ i.o.}\}^c = \{A_n^c \text{ eventually}\} = \lim P(\cap_n^{\infty} A_n^c) = 0$, so that $\{A_n \text{ i.o.}\}$ must equal 1. The first Borel-Cantelli lemma then implies that $\sum P(A_n) = \infty$.

2.3.11

Claim: Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that (i) $X_n \to 0$ in probability if and only if $p_n \to 0$ and (ii) $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

Proof: (i) If $X_n \to 0$ in probability then for any $\epsilon > 0$, $P(|X_n| > \epsilon) \to 0$. Since each X_n is an indicator function, $P(|X_n| > \epsilon) = P(X_n = 1) = p_n$ for $\epsilon < 1$. Thus p_n goes to zero in the limit. This same equation tells us that if $p_n \to 0$, then $P(X_n = 1) = P(|X_n| > \epsilon) \to 0$.

(ii) If $X_n \to 0$ a.s. then $P(X_n = 1 \text{ i.o.})$ must go to zero, since for every $\omega \in \Omega$ excluding some null set, $X_n(\omega) \to 0$ and cannot equal 1 beyond some finite integer N. The events $X_n = 1$ are independent, so by the second Borel-Cantelli lemma, $\sum P(X_n = 1) = \sum p_n$ must be finite. If on the other hand we are given that $\sum p_n < \infty$, it follows form the first Borel-Cantelli lemma that $P(X_n = 1 \text{ i.o.}) = 0$. Thus for every ω excluding some null set, $X_n(\omega) = 1$ for only finitely many $n \in \mathbb{N}$. So $X_n(\omega) \to 0$ almost surely.

2.3.12

Claim: Let $X_1, X_2, \ldots$ be a sequence of r.v.'s defined on $(\Omega, \mathcal{F}, P)$ where Ω is a countable set and $\mathcal{F}$ consists of all subsets of Ω. Show that $X_n \to X$ in probability implies $X_n \to X$ a.s.

Proof: Assume X_n does not converge almost surely to X, then there exists a subset $A \subset \Omega$ with non-zero measure on which $|X_n - X| \not\to 0$. Instead there exists some constant $\alpha > 0$ for which $|X_n - X| > \alpha$ infinitely often. Since $\mu A > 0$ and Ω is discrete, the finite additivity of μ tells us that there is at least one element $\omega \in A$ for which $\mu\omega > 0$. But if such an ω exists, then X_n cannot coverge to X in probability either as $P(|X_n - X| > \alpha) > \mu\omega$ for an infinite subsequence of $\langle X_n \rangle$.

2.3.14

Claim: Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ a.s. if and only if $\sum_n P(X_n > A) < \infty$ for some A.

Proof: ($\Longleftarrow$) Assume there exists some A for which $\sum_n P(X_n > A) < \infty$. Then BCI implies that $P(X_n > A \text{ i.o.}) = 0$ and so $P(X_n < A \text{ eventually}) = 1$. For any $\omega \in \{X_n < A \text{ eventually}\}$, there is a finite integer N beyond which $X_n(\omega) < A$. That combined with the fact that X_n is bounded a.s. implies that $\sup_n X_n < \infty$ a.s.

($\implies$) Now assume that $\sum_n P(X_n > A) = \infty$ for every A. Then the indepence of the random variables X_n along with BCII tell us that $P(X_n > A \text{ i.o.}) = 1$ for any A. For $\omega \in \{X_n > A \text{ i.o.}\}$, $\sup X_n(\omega) > A$ almost surely and since A is arbitrary it follows that $\sup X_n(\omega) = \infty$ almost surely. Thus if $\sup X_n < \infty$ a.s., it must hold that $\sum_n P(X_n > A) < \infty$

2.4.2

Claim: Let $X_0 = (1, 0)$ and define $X_n \in \mathbb{R}^2$ inductively by declaring that X_{n+1} is chosen at random from the ball of radius $|X_n|$ centered at the origin, i.e. $X_{n+1}/|X_n|$ is uniformly distributed on the ball of radius 1 and independent of $X_1, \ldots, X_n$. Prove that $n^{-1} \log |X_n| \to c$ a.s. and compute c.

Proof: Place a uniform distribution on the unit ball in $\mathbb{R}^2$ and let Y_i be a sample taken from that distribution. Then the probability that $|Y_i| < r$ is just the normalized area. Specifically $P(|Y_i| < r) = r^2$ for $r \in [0, 1]$. Because each X_n can be represented as a sample from the unit ball scaled by the previous sample's magnitude $|X_{n-1}|$, we can write $|X_n| = \prod_{i=1}^n |Y_i|$ where the Y_i's are i.i.d. random variables as described previously. Noting then that $\log(|X_n|) = \log(\prod_{i=1}^n |Y_i|) = \sum_{i=1}^n \log(|Y_i|)$, we can apply the SLLN to determine that $n^{-1} \log(|X_n|) = n^{-1} \sum_{i=1}^n \log |Y_i| \to \mu$ where $\mu = E|Y_i| = \int_0^1 r(2r)\ dr = 2/3$