2.19 Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.

(a) **Claim:** If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

Proof: f_n converges uniformly such that for any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|f - f_n| < \epsilon$ for all $n > N$. Thus $|f| \leq |f - f_n| + |f_n| > \epsilon + |f_N|$. This implies that

$$\int |f| < \int (\epsilon + |f_n|) = \epsilon\mu(X) + \int |f_n| < \infty$$

Thus $f \in L^1(\mu)$. Furthermore we also have that $|f_n| \leq |f_n - f| + |f| < \epsilon + |f| \in L^1(\mu)$ and so we can apply the Dominated convergence theorem to get

$$\lim_{n\to\infty} \int f_n = \int f$$

(b) **Claim:** The previous is not necessarily true if $\mu(X) = \infty$.

Proof: We can take as an example the sequence of functions $f_n = n^{-1}\chi_{(0,n]}$. Here, $f_n \to f$ uniformly where $f = 0$. However

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int n^{-1}\chi_{(0,n]} = 1 \neq 0 = \int f$$

and so the results from part (a) do not hold.

2.25 Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

(a) **Claim:** $g \in L^1(m)$, and in particular $g < \infty$ a.e.

Proof: $f(x)$ is Rieman integrable and so is Lebesgue measurable. It is also strictly positive and so we can apply the Monotone Convergence Theorem (along with Thrm 2.28) to the increasing sequence $f_n = x^{-1/2}\chi_{[(1/n,1)]}$ to get

$$\int f = \lim_{n\to\infty} \int_{1/n}^1 x^{-1/2} = \lim_{n\to\infty} (2 - 2n^{-1/2}) = 2$$

Again, by the MCT, we see that

$$\int |g| = \int g = \lim_{n\to\infty} \sum_1^n 2^{-n} \int f(x - r_n) = \sum_1^\infty 2^{-n} \int f(x) = 2$$

and so $g \in L^1(\mu)$. Since we also have $g \in L^+$, theorem 2.20 tells us that $g < \infty$ a.e.

(b) **Claim:** g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

Proof: Let g' be the modifiction of g on some null set N such that $g' = g$ a.e. On any interval I there exists an interior point such that we can create the set $(r_n, k^{-1}) \cap I$ having positive measure. Let $E_k = ((r_n, k^{-1}) \cap I) \setminus N$. Then any sequence $\{x_k\}$ for which $x_k \in E_k$ converges to r_n. But $\lim\limits_{k\to\infty} f(x_k - r_n) = \infty$ and $f(x_k - r_n) \leq g(x_k)$ meaning that g and any related g' are unbounded on the (arbitrary) open interval I. This also implies that any g' is everywhere discontinuous since $\mathbb{Q}$ is dense in $\mathbb{R}$ and for any $x_0 \in \mathbb{R}$ we can never find a $\delta > 0$ such that $|x - x_0| < \delta \implies |g'(x) - g'(x_0)| < \epsilon$ for $\epsilon > 0$.

(c) **Claim:** $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.
Proof: $\mathbb{Q}$ is a null set and g is finite everywhere else so by propotision 2.23 $\int g < \infty$. The same can be said for g^2 so $\int g^2 < \infty$.
For any interval I, There exists a rational r_n in it's interior such that $[r_n, r_n + \delta] \subset I$ for some $\delta > 0$. Thus

$$\int_I g^2 \geq \int_{r_n}^{r_n+\delta} 2^{-2n} f(x - r_n)\,dx = 2^{-2n} \int_0^{\delta} x^{-1}\,dx = \infty$$

so $\int_I g^2$ is unbounded over any interval I.

2.26 **Claim:** If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t)\,dt$, then F is continous on $\mathbb{R}$.

Proof: For any x_0 we can create a sequence $f_n = f\chi_{[-\infty, x_n]}$ for which $f_n \to f$ as $x_n \to x_0$. We also have that $|f_n| \leq |f|$ where $|f| \in L^1(m)$, so we can apply the dominated convergence theorem and get

$$F(x_0) = \int f\chi_{[-\infty, x_0]} = \lim_{n\to\infty} \int f\chi_{[-\infty, x_n]} = \lim_{n\to\infty} F(x_n)$$

Thus $F(x)$ is continuous at any point $x_0 \in \mathbb{R}$ and so is continuous on $\mathbb{R}$.

2.34 Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

(a) **Claim:** $\int f = \lim \int f_n$.
Proof: Here we can assume that f_n is real valued since $|f_n| > |\text{Re } f|$ and $|f_n| > |\text{Im } f|$. By theorem 2.30 we know that $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ that converges to a function that equal f almost everywhere. Thus by the DCT, $f \in L^1$ and

$$\int f = \lim \int f_{n_k} = \lim \int f_n$$

(b) **Claim:** $f_n \to f$ in L^1.
Proof: $|f - f_n|$ converges to 0 in measure and also $|f - f_n| \leq |f| + |f_n| < 2g$, so by part (a)

$$\lim \int |f - f_n| = \int 0 = 0$$

and $f_n \to f$ in L^1.

2.36 **Claim:** If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

Proof: By corollary 2.32 of Folland we know that there exists a subsequence $\{f_{n_k}\}$ that converges to f a.e. This tells us that f either equals 0 or 1, except on a set of measure zero. Since f is measurable, this tells us that $f^{-1}(1) = E \cup F$ for some measurable set E and null-set F such that $f = \chi_E$ a.e.

2.38 Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) **Claim:** $f_n + g_n \to f + g$ in measure.
Proof: We know that $\mu\{x : f_n(x) - f(x) > \epsilon\} \to 0$ and $\mu\{x : g_n(x) - g(x) > \epsilon\} \to 0$. Furthermore $|(f_n(x)+g_n(x)) - (f(x)+g(x))| < |f_n(x) - f(x)| + |g_n(x) - g(x)|$ such that

$$\begin{aligned}
&\{x : |(f_n(x) + g_n(x)) - (f(x) + g(x))| > \epsilon\} \\
&\subset \{x : |f_n(x) - f(x)| + |g_n(x) - g(x)| > \epsilon\} \\
&\subset \{x : |f_n(x) - f(x)| > \epsilon\} \cup \{x : |g_n(x) - g(x)| > \epsilon\}
\end{aligned}$$

But

$$\begin{aligned}
&\mu(\subset \{x : |f_n(x) - f(x)| > \epsilon\} \cup \{x : |g_n(x) - g(x)| > \epsilon\}) \\
&< \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) + \mu(\{x : |g_n(x) - g(x)| > \epsilon\}) \to 0
\end{aligned}$$

so it must hold that $f_n + g_n \to f + g$ in measure.

(b) **Claim:** $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.
Proof: Let $F_n = \{x : |f_n(x)| > \epsilon\}$ and $g_n = \{x : |g_n(x)| > \epsilon\}$. Both sets decrease as $n \to \infty$ and $\cap_1^\infty F_n = \emptyset$, so we can conclude that $\mu(F_n) \to 0$ and likewise for G_n. But for fixed $\epsilon > 0$

$$\begin{aligned}
&\{x : |f_n(x)g_n(x) - f(x)g(x)| > \epsilon\} \\
&\subset \{x : |f_n(x)(g_n(x) - g(x))| > \epsilon\} \cup \{x : |g_n(x)(f_n(x) - f(x))| > \epsilon\}
\end{aligned}$$

and

$$\begin{aligned}
\{x : |f_n(x)(g_n(x) - g(x))| > \epsilon\} &\subset \{x : |f_n(x)| > \epsilon\} \cup \{x : |g_n(x) - g(x)| > \epsilon\} \\
\{x : |g_n(x)(f_n(x) - f(x))| > \epsilon\} &\subset \{x : |g_n(x)| > \epsilon\} \cup \{x : |f_n(x) - f(x)| > \epsilon\}
\end{aligned}$$

Both unions on the right have measures that converge to 0 so it must hold that
$\mu(\{x : |f_n(x)g_n(x) - f(x)g(x)| > \epsilon\}) \to 0$. Thus $f_n g_n \to fg$.
If $\mu(X) = \infty$. We can't always assume that F_n or G_n approach 0. For example, take $f_n = g_n = x + n^{-1}$ which converge to $f = g = x$ in measure. If $\mu(X) = \infty$, then

$$\mu(\{x : |f_n(x)g_n(x) - f(x)g(x)| > \epsilon\}) = \mu(\{x : |2xn^{-1} + n^{-2}| > \epsilon\})$$

but $|2xn^{-1} + n^{-2}| > \epsilon$ for all x such that $|x| > (\epsilon n - n^{-1})/2$ and so $\mu(X) = \infty$ for any n and any ϵ. Thus $f_n g_n \not\to fg$.