5.9 (a) Given some $f \in C([0,1])$, if $f \in C^{(k)}([0,1])$ then f is k times continuously differentiable on $(0,1)$ and one-sided derivatives exist at 0 and 1 for $f^{(j)}$ where $j \in \{0, \dots, k-1\}$. Conversely if f is k times continuously differentiable on $(0,1)$ and $\lim_{x\searrow 0} f^{(j)}(x)$ and $\lim_{x\nearrow 1} f^{(j)}(x)$ exist for $j \leq k$ then we can show that $f^{(k)} \in C^{(k)}([0,1])$ as follows.

Assume $f^{(j-1)}$ is continuous on $[0,1]$ and let $\alpha = \lim_{x\searrow 0} f^{(j)}(x)$, then for any $\epsilon > 0$ we have $\delta > 0$ such that $|f^{(j)}(x) - \alpha| \leq \epsilon$ for all $x \leq \delta$. The mean value theorem then tells us that

$$\mathrm{Re}(f^{(j)}(c)) = \mathrm{Re}(\frac{f^{(j-1)}(x) - f^{(j-1)}(0)}{x})$$

for some $c \in (0,x)$ implying that

$$|\mathrm{Re}(\frac{f^{(j-1)}(x) - f^{(j-1)}(0)}{x}) - \mathrm{Re}(\alpha)| = |\mathrm{Re}(f^{(j)}(c)) - \mathrm{Re}(\alpha)| \leq \epsilon.$$

In the same way, we can determine that

$$|\mathrm{Im}(\frac{f^{(j-1)}(x) - f^{(j-1)}(0)}{x}) - \mathrm{Im}(\alpha)| = |\mathrm{Im}(f^{(j)}(c)) - \mathrm{Im}(\alpha)| \leq \epsilon$$

where ϵ, and hence c, can be made to be arbitrarily close to 0. Thus $f^{(j)}(0) = \alpha$.

An analogous argument can be made to show that $f^{(j)}(1) = \lim_{x\nearrow 1} f^{(j)}(x)$ and that $f^{(j)} \in C([0,1])$ as a result. By induction we have that $f \in C^{(k)}([0,1])$.

(b) We see that $\|\cdot\|$ is a norm by first noting that $\|f\|$ is the sum of non-negative values and hence non-negative. Also $\|0\| = 0$ and if $f \neq 0$ then $\|f\| \geq \|f\|_u > 0$. Furthermore,

$$\|f+g\| = \sum_0^k \|f^{(j)} + g^{(j)}\|_u \leq \sum_0^k \|f^{(j)}\|_u + \|g^{(j)}\|_u = \|f\| + \|g\|$$

$$\|\lambda f\| = \sum_0^k \|\lambda f^{(j)}\|_u = \sum_0^k |\lambda| \|f^{(j)}\|_u = |\lambda| \sum_0^k \|f^{(j)}\|_u = |\lambda| \|f\|$$

Assume that we have a sequence $\langle f_n \rangle \subset C^1([0,1])$ such that $f_n \to f$ and $f_n' \to g$ uniformly. We know, since $C([0,1])$ is closed that f and g are continous. We also have that, for $n > N$, $\|f_n' - g\|_u \epsilon$ such that $|f_n'| \leq |g| + \epsilon$ for all $n > N$. Thus we can apply the dominated convergence theorem to see

$$f(x) - f(0) = \lim f_n(x) - f_n(0) = \lim \int_0^x f_n' = \int_0^x \lim f_n' = \int_0^x g.$$

Thus $f' = g$ and $f \in C^1([0,1])$.

A generalization of this can be used to perform an induction. Take some sequence $\langle f_n \rangle \subset C^k([0,1])$ which is Cauchy, and let's assume that $C^{(k-1)}([0,1])$ is a complete space wrt $\|\cdot\|$. Our sequence is then also Cauchy in $C^{(k-1)}([0,1])$ and converges to some $f \in C^{(k-1)}([0,1])$. Since $C([0,1])$ is complete, $\langle f_n^{(k)} \rangle$ also converges to some $g \in C([0,1])$. By our previous argument we know that $f^{(k)} = g$, hence $f \in C^k([0,1])$. Furthermore $f_n \to f$ in $C^k([0,1])$ since $\|f_n - f\| = \sum_0^{(k-1)} \|f_n^{(j)} - f^{(j)}\|_u + \|f_n^{(k)} - g^{(k)}\|_u$ which is the sum of two values which converge to 0 as $n \to \infty$.

5.15 We're given that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces with $T \in L(\mathcal{X}, \mathcal{Y})$ and $\mathcal{N}(T) = \{x \in \mathcal{X} : Tx = 0\}$.

(a) We see that $\mathcal{N}(T)$ is a subspace of $\mathcal{X}$ since it includes the zero vector and for any two $x, y \in \mathcal{N}(T)$, $T(x+y) = Tx + Ty = 0$ and so $\mathcal{N}(T)$ is closed under vector addition.
Assume we have a convergent sequence $\langle x_n \rangle \subset \mathcal{X}$ such that $x_n \to x$. Then

$$\|Tx\ Vert = \|T(x - x_n)\| \leq C\|x - x_n\|$$

where the last expression can be made arbitrarily small by picking n large enough. Thus $Tx = 0$ and $x \in \mathcal{N}(T)$, meaning that $\mathcal{N}(T)$ is closed.

(b) Since for any $x_{\mathcal{N}} \in \mathcal{N}(T)$ we have that $T(x + x_{\mathcal{N}}) = Tx + Tx_{\mathcal{N}} = Tx$, it follows that some S exists such that $T = S \circ \pi$. We just define S as $S(x\mathcal{N}(T)) = T(x)$, where x is a representative element of $\mathcal{X}/\mathcal{N}(T)$. This is well defined since for $x\mathcal{N}(T) = y\mathcal{N}(T)$ we have $(x - y) \in \mathcal{N}(T)$ such that $T(x) = T(x) - T(x - y) = T(y)$. To show that S is unique, note that π is surjective and that if we had another map $R \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = R \circ \pi$, then there would have to be some coset in $K \in \mathcal{X}/\mathcal{N}(T)$ such that $S(K) \neq R(K)$ such that $R \circ \pi \neq S \circ \pi = T$. So S must be unique.
Finally, exercise 12 tells us that for any $\epsilon > 0$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{N}(T)\| \geq 1 - \epsilon$. Thus

$$\frac{\|S(x + \mathcal{N}(T))\|}{\|x + \mathcal{N}(T)\|} \leq \frac{\|T(x + x_{\mathcal{N}}(T))\|}{1 - \epsilon} \leq \frac{\|T\|\|x\|}{1 - \epsilon} = \frac{\|T\|}{1 - \epsilon}$$

which implies $\|S\| \leq \|T\|$. But we also have that $\|T\| = \|S \circ \pi\| \leq \|S\|\|\pi\| = \|S\|$ and so $\|S\| = \|T\|$.

5.20 If $\mathcal{M}$ has finite dimension, let $\{x_1, \ldots, x_n\}$ be a basis for it. For any $i \in \{1, \ldots, n\}$ we can define a continuous linear functional f_i on $\mathcal{M}$ by $f_i(x) = \lambda_i x$ such that $|f_i(x)| \leq |\lambda_i| x_i$ for all $x \in \mathcal{M}$. $|\lambda_i| x_i$ is a semi-norm so we can apply the Hahn-Banach Theroem to generate bounded linear functions F_i on $\mathcal{X}$ for which $F_i|\mathcal{M} = f_i$.

The kernel of any linear functional is a closed set so let $\mathcal{N} = \cap_1^n \ker(F_i)$ which is a closed. Then $\mathcal{M} \cap \mathcal{N} = \{0\}$ since $\mathcal{M} \cap (\cap_1^n \ker(F_i)) = \cap_1^n \ker(f_i) = \{0\}$. To show that $\mathcal{M} + \mathcal{N} = \mathcal{X}$ note that for any $x \in \mathcal{X}$ we have $\sum F_i(x) x_i \in \mathcal{M}$. But $F_j(x - \sum F_i(x) x_i) = F_j(x) - F_j(x)(F_j(x_j)) = 0$ such that $x - \sum F_i(x) x_i \in \mathcal{N}$ and $x = m + n$ for some $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

5.21 We're given $\alpha : \mathcal{X}^* \times \mathcal{Y}^* \to (\mathcal{X} \times \mathcal{Y})^*$ defined by $\alpha(f, g)(x, y) = f(x) + g(y)$. Let $\alpha^{-1} : (\mathcal{X} \times \mathcal{Y})^* \to \mathcal{X}^* \times \mathcal{Y}^*$ be defined by $\alpha^{-1}(h)(x, y) = (h(x, 0), h(0, y))$. Then

$$\begin{aligned} \alpha^{-1}(h_1 + h_2) &= (h_{1x} + h_{2x}, h_{1y} + h_{2y})) = \alpha^{-1}(h_1) + \alpha^{-1}(h_2) \\ \alpha^{-1}(\lambda h) &= (\lambda h_x, \lambda h_y) = \lambda(\alpha^{-1}(h)) \end{aligned}$$

and also

$$\|\alpha^{-1}\| = \sup_{\|h\|=1} \|(h_x, h_y)\| = \sup_{\|h\|=1} \|h_x\| + \|h_y\| \leq 2\|h\|$$

so α^{-1} is linear and bounded.

Furthermore

$$\alpha\alpha^{-1}(h) = \alpha(h_x, h_y) = h_x + h_y = h$$
$$\alpha^{-1}\alpha(f,g) = \alpha^{-1}(f+g) = (f + g(x,0), f + g(0,y)) = (f,g)$$

such that α^{-1} is a right and left inverse of α, implying that α is bijective. Thus α is an isomorphism. To show that it is isometric, note that

$$\begin{aligned}
\|\alpha(f,g)\| = \|f+g\| &= \sup_{\|(x,y)\|=1} |f(x)+g(y)| = \sup_{\max(\|x\|,\|y\|)=1} |f(x)+g(y)| \\
&= \sup_{\|x\|=1} |f(x)| + \sup_{\|y\|=1} |g(y)| = \|f\| + \|g\| \\
&= \|(f,g)\|
\end{aligned}$$

5.22 (a) Given $T \in L(\mathcal{X}, \mathcal{Y})$ and $T^\dagger : \mathcal{X}^* \to \mathcal{Y}^*$ defined by $T^\dagger f = f \circ T$, we know that $T^\dagger$ is linear since it is the composition of linear functions. Also

$$\|T^\dagger\| = \sup_{\|f\|=1} \|T^\dagger f\| \leq \sup_{\|f\|=1} \|f\|\|T\| = \|T\|$$

So $T^\dagger \in L(\mathcal{X}^*, \mathcal{Y}^*)$.
For any $\epsilon < \|T\|$ we can find an $x \in \mathcal{X}$ such that $\|T\| - \epsilon < \|Tx\|$. We also have, by Prop 5.8, that there exists some $f \in \mathcal{Y}^*$ such that $\|f\| = 1$ and $f(Tx) = \|Tx\|$. Thus

$$\|T\| - \epsilon < \|Tx\| = f(Tx) < \|T^\dagger\|$$

ans so $\|T\| = \|T^\dagger\|$.

(b) Given that $\hat{x} : f \mapsto f(x)$ and $\widehat{Tx} : f \mapsto f(Tx)$ we have

$$T^{\dagger\dagger}(\hat{x})(f) = \hat{x} \circ T^\dagger f = (T^\dagger f)(x) = f(Tx) = \widehat{Tx}(f)$$

Thus $T^{\dagger\dagger}|\widehat{\mathcal{X}} = T$. Since $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images, this gives us our result.

(c) s

5.25 Let $\langle f_n \rangle$ be a countable dense subset in $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Let $\mathcal{M}$ be the closure of the subspace generated by $\langle x_n \rangle$. If there exists $x \in \mathcal{X} \setminus \mathcal{M}$ then by Hahn-Banach we can construct a function $f \in \mathcal{X}^*$ for which $f(x) \neq 0$, $f(\{\mathcal{M}\}) = \{0\}$, and $\|f\| = 1$. But then since $\langle f_n \rangle$ is dense, there exists some f_n for which $\|f - f_n\| < 1/3$ such that

$$\|f\| \leq \|f - f_n\| + \|f_n\| < \frac{1}{3} + 2|f_n(x_n)| = \frac{1}{3} + 2|f_n(x_n) - f(x_n)| < \frac{1}{3} + \frac{2}{3} = 1$$

Here we have a contradiction and so it must hold that $\mathcal{X} = \mathcal{M}$. To find a countable dense subset in $\mathcal{X}$ note that any point $x \in \mathcal{X}$ can be approximated by some $\sum_{i=1}^n a_i x_i$ such that $\sum a_i x_i \in B_\epsilon(x)$. If B is a countable dense subset of K then for each a_i we can find an element b_{β_i} such that $|b_{\beta_i} - a_i| < \epsilon/n$. But then we have that

$$\|x - \sum b_{\beta_i} x_i\| \leq \|x - \sum a_i x_i\| + \|\sum a_i x_i - \sum b_{\beta_i} x_i\| = 2\epsilon$$

This means that the set $\{\sum_{i=1}^n b_i x_i : b_i \in B, n \in \mathbb{N}\}$ is dense in $\mathcal{X}$. It is countable since the set of all such linear combinations maps into $B^{\mathbb{N}}$.