Math 275A: Homework 1

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1.1.3

Claim: $\mathcal{R}^d$ is countably generated.

Proof: We know that the collection of open rectangles with rational vertices, $\mathcal{C}$, is countable as it can be represented by a finite produce of countable spaces

$$\mathcal{C} = \{(x_{11}, x_{12}, \dots, x_{1d}) \times \cdots \times (x_{d1}, x_{d2}, \dots, x_{dd}) : x_{11}, \dots, x_{1d}, x_{21}, \dots, x_{dd} \in \mathbb{Q}\}$$

It follows from this and the density of $\mathbb{Q}$ in $\mathbb{R}$ that every open set $U \subset \mathbb{R}^d$ is a countable union of such rectangles.

In class, we showed that given two collections $\mathcal{A}$ and $\mathcal{B}$ for which $\mathcal{A} \subset \mathcal{B}$, it follows that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. We also showed that, by definition, $\sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$ for any $\mathcal{A}$, as $\sigma(\mathcal{A})$ is the minimal σ-algebra containing $\mathcal{A}$.

Now let $\mathcal{U}$ represent the collection of open sets in $\mathbb{R}^d$. $\mathcal{U}$ contains $\mathcal{C}$ such that, $\sigma(\mathcal{C}) \subset \sigma(\mathcal{U}) = \mathcal{R}^d$. At the same time, $\sigma(\mathcal{C})$ contains every countable union of sets from $\mathcal{C}$ (i.e. the open sets), such that $\mathcal{U} \subset \sigma(\mathcal{C})$. But this implies that $\sigma(\mathcal{U}) \subset \sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$, establishing that $\sigma(\mathcal{C})$ and $\mathcal{R}^d$ are equivalent

1.2.4

Claim: Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(x)$ has a uniform distribution of $(0, 1)$, the is, if $y \in [0, 1], P(Y \leq y) = y$.

Proof: I take it for granted here that $F(X)$ is a measurable function in the sense that $F(X)^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{R}$ where $\mathcal{F}$ is the σ-algebra associated with our underlying probability space. Thus we know that there exists a distribution function $F'(y)$ that describes the measure induced on Y by $F(X)$.

Define an function F^{-1} as

$$F^{-1}(y) = \sup\{x : F(x) = y\}.$$

Note that if F^{-1} were only right-continuous, it may not be well-defined at some points y in $Y = [0, 1]$. As is, evey y is mapped to a unique real number by F^{-1}. Furthermore for any given y, it holds that $F(F^{-1}(y)) = y$.

The distribution F' can be written as

$$F'(y) = P(Y \leq y) = P\{w : F(X(w)) \leq y\} = P\{w : X(w) \leq F^{-1}(y)\}.$$

To verify the last equality note that if x is such that $F(x) \leq y$ then x must be less than the supremum of x such that $F(x) = y$. Likewise if $F(x) > y$, then x must be greater that $\sup\{x : F(x) = y\}$.

By the definition of the distribution function of X, it follows that

$$F'(y) = F(F^{-1}(y)) = y.$$

1.3.1

Claim: If $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) = \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

Proof: By the fact that $\mathcal{A}$ is a subcollection of sets in $\mathcal{S}$, we know that $X^{-1}(\mathcal{A}) \subset \sigma(X)$, from which it follows that $\sigma(X^{-1}(\mathcal{A})) \subset \sigma(X)$. To show the reverse inclusion, first note two equalities:

$$\{X \in \cup_i A_i\} = \cup_i \{X \in A_i\}$$
$$\{X \in A^c\} = \{X \in A\}^c.$$

The equalities imply that $\sigma(X^{-1}(\mathcal{A})) = \{\{X \in A\} : A \in \sigma(\mathcal{A})\}$. The prior inclusion $\sigma(X^{-1}(\mathcal{A})) \subset \sigma(X)$ then becomes an equality and so $X^{-1}(\mathcal{A})$ must generate $\sigma(X)$.

1.3.4

Claim: (i) Show that a continuous function from $\mathbb{R}^d \to \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$. (ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.

Proof: (i) If $f : \mathbb{R}^d \to \mathbb{R}$ is continuous then its inverse $f^{-1}(A)$ is open for any open set A. Let $\mathcal{A}$ represent the open sets in $\mathbb{R}$ which generate $\mathcal{R}$. Then all the sets in $f^{-1}(\mathcal{A})$ are open and members of $\mathcal{R}^d$. Applying Theorem 1.3.1, we can conclude that f must be measurable.

(ii) From (i) we can conclude that $\mathcal{R}^d$ is a sufficiently rich collection of sets so as to make any continuous function $f : \mathbb{R}^d \to \mathbb{R}$ measurable. To show that it is also the smallest such σ-field, note that any candidate σ-field $\mathcal{F}$ must contain $f^{-1}(\mathcal{A})$ for every continuous function f.

Take the set of d projection functions π_i that map each vector in $x \in \mathbb{R}^d$ onto its i-th coordinate x_i. Let $\mathcal{A}_i = \pi_i^{-1}(\mathcal{A})$. Every open set in $\mathbb{R}^d$ of the form $U_1 \times \cdots \times U_d$, for $U_i \in \mathbb{R}$, can be described as some intersection $\cap_{i=1}^d A_i$ of sets where $A_i \in \mathcal{A}_i$. The collection of such sets defines a basis for the standard topology on $\mathbb{R}^d$. Thus any σ-field that renders all continuous functions measurable must contain $\mathcal{R}^d$, as it will contain at least all the open sets in $\mathbb{R}^n$. So then $\mathcal{R}^d$ itself must be the smallest σ-algebra that reders such functions measureable.

Bonus

Claim: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $E \in \mathcal{F}$. Define:

$$\mu_E(A) := \mu(A \cap E) \quad \text{for} \quad A \in \mathcal{F}$$

Prove that μ_E is also a measure on $(\Omega, \mathcal{F})$

Proof: For any set $A \in \mathcal{F}$, $A \cap E$ is also a set in $\mathcal{F}$, it being a σ-algebra. This means that μ_E is well-defined for all sets $A \in \mathcal{F}$ and that it must satisfy the non-negativity and null empty set properties of a measure.

As for countable additivity, given a countable collection of disjoint sets $\{A_i\}_i$,

$$\mu_E(\cup_i A_i) = \mu(\cup_i A_i \cap E) = \cup_i \mu(A_i \cap E) = \cup_i \mu_E(A_i)$$

where in the second equality we exploit the fact that the sets $\{A_i \cap E\}_i$ must also be disjoint. Thus countable additivity holds as well.