Math 275B: Homework 4

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D.5.2.4

Claim. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that:

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m)p^{n-m}(y,y)$$

Proof. Let $Y_m : \Omega_0 \to \mathbb{R}$ define a family of random varibles for which $Y_m(w) = 1$ if $w_{(n-m)} = y$ and $Y_m(w) = 0$ otherwise. Then, using θ_n to specify the left-shift of a sequence w by n steps, we can use the strong Markov property to determine:

$$\begin{aligned} p^n(x,y) &= E_x(Y_N \circ \theta_N; N \leq n) = E_x(E_x(Y_N \circ \theta_N | \mathcal{F}_N); N \leq n) = E_x(E_{X_N} Y_N; N \leq n) \\ &= E_x(p^{n-N}(y,y); N \leq n) = \sum_{m=1}^{n} p^{n-m}(y,y)E_x(; N = m) = \sum_{m=1}^{n} p^{n-m}(y,y)P_x(T_y = m). \end{aligned}$$

□

D.5.3.3

Claim. Show that in the Ehrenfest chain, all states are recurrent.

Proof. The set $S = \{1, 2, \ldots, r\}$ is obviously closed since probabilities $\rho_{ij} > 0$ are only defined for $i, j \in S$. Likewise S is irreducible since for any $i, j \in S$ where $i < j$:

$$\rho_{ij} \geq p^{j-i}(i,j) = \prod_{k=0}^{j-i-1} p(i+k, i+k+1) = \prod_{k=0}^{j-i-1} \frac{r-k}{r} > 0.$$

Theorem 5.3.3 in Durrett tells us that because S is finite, closed, and irreducible, all states in S are recurrent, proving the claim. □

D.5.5.2

Claim. Let $w_{xy} = P_x(T_y < T_x)$. Show that $\mu_x(y) = w_{xy}/w_{yx}$.

Proof. Here I'll instead show that $\mu_x(y)w_{yx} = w_{xy}$. Starting with the definition of $\mu_x(y)$ in theorem D.5.5.7:

$$\begin{aligned} \mu_x(y)w_{yx} &= \left(\sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)\right) P_y(T_x < T_y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)P_y(T_x < T_y) \\ &= \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n, X_m \neq y \text{ for } n < m < T_x). \end{aligned}$$

The last summand just describes the probablity that $X_n = y$ is the last occurance of y (starting from y) before $X_m = x$ for some $m > n$. But this description describes all the possible sequences $w \in \Omega_0$ for which $X_0 = x$ and $T_y < T_x$ are satisfied. Furthermore, each set $\{X_0 = x, X_n = y, X_m \neq y \text{ for } n < m < T_x\}$ is disjoint from the others for unique n, such that the sum of their probabilities gives $P_x(T_y < T_x) = w_{xy}$. $\square$

D.5.6.2

Claim. Show that if S is finite and p is irreducible and aperiodic, then there is an m that $p^m(x, y) > 0$ for all $x, y \in S$.

Proof. S is irreducible so for any $x, y \in S$, $\rho_{xy} > 0$. Theorem D.5.6.4 then tells us that $d_x = d_y$ for all $x, y \in S$. Since S is also aperiodic, actually $d_x = 1$ for all $x \in S$.

Lemma then D.5.6.5 tells us that for each $x \in S$ there exists a positive integer m_x such that for $m \geq m_x$, $p^m(x, x) > 0$. Since S is finite, let $m_0 = \max_{x \in S} m_x$ such that $p^m(x, x) > 0$ for any $x \in S$ and $m > m_0$.

Lastly, since $\rho(x, y) > 0$ for each pair $x, y \in S$, there exists some positive integers n_{xy} such that $p^{n_{xy}}(x, y) > 0$. Note that for such integers, it must hold that $p^{n_{xy}+m}(x, y) > p^{n_{xy}}(x, y)p^m(y, y) > 0$ for any $m \geq m_0$ as well. Thus let $n_0 = m_0 + \max_{x,y \in S} n_{xy}$. Then the claim holds for any $x, y \in S$ and $m > n_0$. $\square$