Math 275B: Homework 5

Marcus Lucas

D.5.5.6

Claim. Use theorems 5.5.7 and 5.5.9 to show that for a simple random walk, the expected number of visits to k between successive visits to 0 is 1.

Proof. We know by theorem 5.5.4 that simple random walks in one dimension are recurrent. Also, since $\rho_{ij} > \sum_{k=0}^{j-i-1} p(i+k, i+k+1) = \frac{1}{2^{(j-i)}}$ for any $i, j \in \mathbb{Z}$ it follows that p is irreducible as well.

Simple random walks are doubly stochastic random processes so $\nu(k) \equiv 1$ is a stationary distribution of p. Theorem D.5.5.9 tells us $\nu(k)$ is unique up to a constant multiple. But $\mu_0(k)$ is also a stationary distribution of p by theorem D.5.5.7, so $\mu_0(k)$ is equivalent to $v(k)$ up to a constant multiple. Since $\mu_0(0) = 1$, it follows that $\mu_0(k) \equiv 1$. This proves the claim, as $\mu_0(k) = E_0\left(\sum_{n=0}^{T_0-1} 1_{\{X_n=k\}}\right)$ is defined to be the expected number of visits to k starting from 0 up until 0 is reached again. □

B.6.4

Claim: On $\Omega = [0,1)$, define $T : \Omega \to \Omega$ by $Tx = (2x)[1]$. Use $\mathcal{F} = \mathcal{B}([0,1))$, $P = dx$. Define:

$$X(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x < 1. \end{cases}$$

Show that the sequence $X_n(x) = X(T^{n-1}x)$ consists of independent zeros and ones with probability $\frac{1}{2}$ each.

Proof. For any n,

$$P(X_n = 0) = P(XT^{n-1} = 0) = P(x \in A) \quad \text{for} \quad A = \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right).$$

Clearly $P(A) = \frac{1}{2}$, implying that $P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$.

For any sequence $w_{n+1}, \ldots, w_{n+k}$ of zeros and ones,

$$P(X_{n+1} = w_{n+1}, X_{n+2} = w_{n+2}, \ldots, X_{n+k} = w_{n+k}) = P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

where

$$A_i = \bigcup_{k=1}^{2^{n+i-1}-1} \left[\frac{2k}{2^{n+i}}, \frac{2k+1}{2^{n+i}}\right).$$

The sets A_i are defined such that the intersection above has measure $P(A_1 \cap A_2 \cap \cdots \cap A_k) = \frac{1}{2^k}$. From this we can conclude that

$$P(X_{n+1} = w_{n+1}, \ldots, X_{n+k} = w_{n+k}) = \frac{1}{2^k} = P(X_{n+1} = w_{n+1}) \cdots P(X_{n+k} = w_{n+k}),$$

and so every sequence w is independent. □