Math 275A: Homework 7

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3.2.1

Define a collection of sets A_{n,m_n} for $n \in \mathbb{N}^+$ and $m_n \in \{1, \dots, 2^n\}$ where $A_{n,m_n} = (0,1) \setminus [\frac{m_n - 1}{2^n}, \frac{m_n}{2^n}]$. Define the collection of distributions μ_{n,m_n} to be uniform on each respective A_{n,m_n} such that their corresponding distribution functions F_{n,m_n} are have a constant positive slope, excepting the interval $(\frac{m_n - 1}{2^n}, \frac{m_n}{2^n})$ where the slope is zero. Let $F_1 = F_{1,1}, F_2 = F_{1,2}, F_3 = F_{2,1}, \dots$. Then the sequence F_k converges pointwise to F, the distribution function for the uniform distribution μ. And so $\mu_k \implies \mu$.

At the same time, for any $x \in (0,1)$ there is an infinite subcollection of density functions $f_k(x)$ which evaluate to zero at x. Thus $f_k(x)$ does not converge to $f(x) = 1$ anywhere in $(0,1)$.

3.2.2.iii

Claim: Let $X_1, X_2, \dots$ be independent with distribution F, and let $M_n = \max_{m \le n} X_m$. Then $P(M_n \le x) = F(x)^n$. Prove that if $F(x) = 1 - e^{-x}$ for $x \ge 0$ then for all $y \in (-\infty, \infty)$

$$P(M_n - \log n \le y) \to \exp(-e^{-}y)$$

Proof: First note that

$$\begin{aligned} P(M_n - \log n \le y) &= P(M_n \le \log n + y) \\ &= F(\log n + y)^n \\ &= (1 - e^{-\log n - y})^n \\ &= (1 - n^{-1} e^{-y})^n \end{aligned}$$

We can apply Lemma 3.1.1 from Durrett to the last equation (where $c_n = -n^{-1}e^{-y} \to 0$, $a_n = n \to \infty$, and $c_n a_n = -e^{-y}$) to determine that $P(M_n - \log n \le y) \to \exp(-e^{-y})$.

3.2.4

Claim: Let $g \ge 0$ be continuous. If $X_n \implies X_\infty$ then

$$\liminf_{n \to \infty} Eg(X_n) \ge Eg(X_\infty)$$

Proof: We know from Fatou's lemma that, in general

$$\liminf_{n \to \infty} Eg(X_n) \ge E(\liminf_{n \to \infty} g(X_n))$$

3.2.10

3.2.13

Extra 1