1.18 We're given an algebra $\mathcal{A} \subset \mathcal{P}(X)$, the collections $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma\delta}$ induced by $\mathcal{A}$, and an outer-measure μ^* induced by the premeasure μ_0 on $\mathcal{A}$.

(a) **Claim:** For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
Proof: By definition

$$\mu^*(E) = \inf\left\{ \sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j \right\}$$

Since $\mu^*(E)$ is an infimum, it is always possible to find a sequence $\{A_j\}_1^\infty \subset \mathcal{A}$ with $E \subset \bigcup_1^\infty A_j$ and $\sum_1^\infty \mu_0(A_j) \leq \mu^*(E) + \epsilon$. Any such set $\bigcup_1^\infty A_j$ is contained in $\mathcal{A}_\sigma$ since it is a countable union of elements of $\mathcal{A}$. So let $A = \bigcup_1^\infty A_j$ such that the previous relations become

$$E \subset A \text{ and } \mu^*(A) \leq \mu^*(E) + \epsilon \text{ with } A \in \mathcal{A}_\sigma$$

(b) **Claim:** If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
Proof: ($\Rightarrow$) Given E, μ^*-measurable, we know that for any $\epsilon_i > 0$, we can find an infinite union $B_i \in \mathcal{A}_\sigma$ containing E such that $\mu^*(B_i) \leq \mu^*(E) + \epsilon_i$. Let $\{\epsilon_i\}_1^\infty$ be a monotonically decreasing sequence of real numbers that converges to 0. Then define $B = \bigcap_1^\infty B_i$ such that $E \subset B$ and $B \in \mathcal{A}_{\sigma\delta}$. Since by Caratheodory's Theorem we know that the μ^*-measurable sets form a σ-algebra (by definition containing $\mathcal{A}$), we also know that any element of $\mathcal{A}_\sigma$ or $\mathcal{A}_{\sigma\delta}$ is μ^*-measurable as well. Thus we can do the following

$$\mu^*(B \setminus E) = \mu^*(B_i \cap E^c) = \mu^*(B_i) - \mu^*(B_i \cap E) = \mu^*(B_i) - \mu^*(E) \leq \epsilon_i \quad \forall i$$

implying that $\mu^*(B \setminus E) = 0$.
($\Leftarrow$) Given some $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$, we make use of the fact that any element of $\mathcal{A}_{\sigma\delta}$ is μ^*-measurable to show that for any $F \subset X$

$$\begin{aligned}
\mu^*(F) &= \mu^*(F \cap B) + \mu^*(F \cap B^c) \\
&= \mu^*(F \cap B) + \mu^*(F \cap B^c) + \mu^*(F \cap (B \setminus E)) \\
&\geq \mu^*(F \cap B) + \mu^*(F \cap E^c) \\
&\geq \mu^*(F \cap E) + \mu^*(F \cap E^c)
\end{aligned}$$

It's imediatly clear from the subadditivity of outer measures that $\mu^*(F \cap E) + \mu^*(F \cap E^c) \geq \mu^*(F)$ so E must be μ^*-measurable.

(c) **Claim:** If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.
Proof: ($\Rightarrow$) μ_0 being σ-finite, there exists some countable union such that $X = \bigcup_1^\infty X_i$ and $\mu_0(X_i) < \infty$. Thus the measurable sets $E_i = E \cap X_i$ have finite measure and their countable union equals E. By part (a), for any $\epsilon_{ik} > 0$ there exists $A_{ik} \in A_\sigma$ such that $E_i \subset A_{ik}$ and $\mu^*(A_{ik}) \leq \mu^*(E_i) + \epsilon_{ik}$. E_i being measureable, it is also μ^*-measurable such that

$$\mu^*(A_{ik}) = \mu^*(A_{ik} \cap E_i) + \mu(A_{ik} \cap E_i^c) = \mu^*(E_i) + \mu(A_{ik} \setminus E_i)$$

This in turn implies that

$$\mu^*(A_{ik} \setminus E_i) = \mu^*(A_{ik}) - \mu^*(E_i) \leq \epsilon_{ik} \implies \mu^*(A_{ik} \setminus E_i) = 0$$

If we set each $\epsilon_{ik} = 1/(k2^i)$ and let $A_k = \bigcup_{i=1}^{\infty} A_{ik}$ then $E \subset A_k$, $A_k \in \mathcal{A}_\sigma$, and we have

$$\mu^*(A_k \setminus E) = \mu^*\Big(\bigcup_{i=1}^{\infty} A_{ik} \cap E^c\Big) \leq \sum_{i=1}^{\infty} \mu^*(A_{ik} \setminus E) \leq \sum_{i=1}^{\infty} \mu^*(A_{ik} \setminus (E_i) = \frac{1}{k}$$

Then we can set $A = \bigcap_{k=1}^{\infty} A_k$ such that $E \subset A$ and $A \in \mathcal{A}_{\sigma\delta}$. Since

$$\mu^*(A \setminus E) \leq \mu^*(A_k \setminus E) \leq \frac{1}{k}$$

it must hold that $\mu^*(A - E) = 0$ and we're done.

($\Leftarrow$) This follows from the previous proof in part (b). There, we never actually assumed that $\mu^*(E) \leq \infty$.

1.22.a We're given $(X, \mathcal{M}, \mu)$ with outer measure $\mu*$ (induced by μ), $\mathcal{M}^*$ (the σ-algebra of μ^*-measurable sets), and $\bar{\mu} = \mu^*|\mathcal{M}^*$.

Claim: If μ is σ-finite, then $\bar{\mu}$ is the completion of μ.

Proof: From Caratheodory's Theorem we know that $\bar{\mu}$ is a complete measure on $\mathcal{M}^*$. Furthermore, Theorem 1.9 in Folland tells us that $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$ (where $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$) is a σ-algebra for which there is a unique extension of μ to a complete measure of $\overline{\mathcal{M}}$. Thus if we can show that $\mathcal{M}^* = \overline{\mathcal{M}}$, we're done.

Given a set $S \in \overline{\mathcal{M}}$ we have that $S = E \cup F$ where $E \in \mathcal{M}$ and $F \in \mathcal{N}$. Furthermore, $F \subset B$ for some $B \in \mathcal{M}$ for which $\mu(B) = 0$. This means that $S \subset (E \cup B) \in \mathcal{M}$ and that $\mu^*(E \cup B \setminus S) = \mu^*((E \cup B) \setminus (E \cup F)) \leq \mu^*(B) = 0$. Noting that $\mathcal{M} = \mathcal{M}_{\sigma\delta}$, we can apply parts (b,c) of exercise 1.18 to determine that S is μ^*-measurable. Since μ is σ-finite, this analysis holds for any $S \in \overline{\mathcal{M}}$.

Likewise, if we're given a μ^*-measurable set S, then we know by exercise 1.18 that there exists a set $B \in \mathcal{M}_{\sigma\delta}$ such that $S \subset B$ and $\mu^*(B \setminus S) = 0$. $B \in \mathcal{M}^*$ so $(B \setminus S) \in \mathcal{M}^*$ as well. This and another application of the result in ex. 1.18 tell us that there exists some $C \in \mathcal{M}_{\sigma\delta}$ such that $(B \setminus S) \subset C$ and $\mu^*(C \setminus (B \setminus S)) = 0$.

$(S \cap C) \subset (C \setminus (B \setminus S))$ implies that $\mu^*(S \cap C) = 0$ by subadditivity. What's more, we can write S as the union $S = (B \setminus C) \cup (S \cap C)$. Once again, there exists some $D \in \mathcal{M}_{\sigma\delta}$ containing $S \cap C$ such that $\mu^*(D \setminus (S \cap C)) = 0$. Thus $S = E \cup F$ where $E = (B \setminus C) \in \mathcal{M}$ and $F = (S \cap C) \subset D \in \mathcal{N}$, implying that $S \in \overline{\mathcal{M}}$.

1.24 Given finite measure μ on $(X, \mathcal{M})$ with outer measure μ^*. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ but $E \notin \mathcal{M}$.

(a) **Claim:** If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

Proof: $A, B \in \mathcal{M}$ so it holds that

$$\mu(A \cup B) = \mu(A \cap B^c) + \mu(B) = \mu(A) + \mu(A^c \cap B)$$

Since $E \subset (A \cup B^c)$, we also have

$$\mu(X) = \mu^*(X) = \mu^*(E) < \mu^*(A \cup B^c) = \mu((A^c \cap B)^c) = \mu(X) - \mu(A^c \cap B)$$

implying that $\mu(A^c \cap B) = 0$. In the same way, we can show that $\mu(A \cap B^c) = 0$. Thus $\mu(A) = \mu(B)$.

(b) **Claim:** Given $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$ and the function ν defined by $\nu(A \cap E) = \mu(A)$, $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Proof: $\mathcal{M}_E$ is clearly a σ-algebra. $A^c \in \mathcal{M}$ and $((A \cap E)^c) \cap E = (A^c \cup E^c) \cap E = A^c \cap E$. Also, $\bigcup_i (A_i \cap E) = (\bigcup_i A_i) \cap E$ is an element of $\mathcal{M}_E$.

As for the function ν, $\nu(\emptyset) = 0$. To show countable additivity, note that for any countable disjoint sequence $\{E_i\}_1^\infty$ for which $E_i \in \mathcal{M}_E$, there must exist corresponding sets $A_i \in \mathcal{M}$ such that $E_i = A_i \cap E$. Thus the sequence $\{A_i\}_1^\infty$ may not be disjoint itself, however we can create a new one, $\{A_i'\}_1^\infty$ by letting $A_i' = A_i \setminus \left(\bigcup_{i=1}^\infty \bigcup_{j=i+1}^\infty (A_i \cap A_j)\right)$. This new sequence is disjoint and for every A_i', it still holds that $E_i = A_i' \cap E$. Hence

$$\nu\big(\bigcup_1^\infty E_i\big) = \nu\Big(\bigcup_1^\infty (A_i' \cap E)\Big) = \nu\Big(\big(\bigcup_1^\infty A_i'\big) \cap E\Big) = \mu\Big(\bigcup_1^\infty A_i'\Big) = \sum_1^\infty \mu(A_i') = \sum_1^\infty \nu(E_i)$$

and we have countable additivity for ν. Thus ν is a measure on $\mathcal{M}_E$.

1.30 We're given $E \in \mathcal{L}$ and $m(E) > 0$ and want to show that for any $\alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Proof: Assume that there exists some $\alpha < 1$ such that $m(E \cap I) \leq \alpha m(I)$ for all open interavals I.

By Theorem 1.18 in Folland, we know that E contains some compact set K which must be bounded. This set must also have finite measure, since any compact set must be bounded and so can be contained within a bounded interval of finite measure. This is all to say that we can assume $m(E) < \infty$ without loss of generality.

Also by Theorem 1.18, we can always find an open set U containing E such that $m(U) \leq m(E) + \epsilon$ for any $\epsilon > 0$. Any open set, can be represented by a countable union of disjoint open intervals, so $U = \bigcup_i I_i$. For each such interval

$$m(I_i) = m(I_i \setminus E) + m(I_i \cap E) \leq m(I_i \setminus E) + \alpha m(I_i)$$

implying that for the entire open set U

$$(1-\alpha)m(U) = \sum_{i=1}^\infty (1-\alpha)m(I_i) \leq \sum_{i=1}^\infty m(I_i \setminus E) = m\Big(\bigcup_i (I_i \setminus E)\Big) = m(U \setminus E) < \epsilon$$

If we let $\epsilon = (1-\alpha)\mu(E)$, this implies that $m(U) < m(E)$. But subadditivity of m means that $U \supset E$ implies $mu(U) \geq m(E)$, so we have a contradiction. Thus it must be the case that for any $\alpha < 1$, there exists an open intervals I such that $m(E \cap I) > \alpha m(I)$.

2.3 Given that $\{f_n\}$ is a sequence of measurable functions on X, prove that $\{x : \lim\, f_n(x) \text{ exists}\}$ is a measurable set.

Proof: Assume $f_n : X \to \overline{\mathbb{R}}$. From Proposition 2.7 in Folland, we know that when $f = \lim\, f_n(x)$ exists, $f = g_3 = g_4$ where $g_3(x) = \limsup_{n\to\infty} = f_n(x)$ and $g_4(x) = \liminf_{n\to\infty} = f_n(x)$. We also know that g_3 and g_4 are measurable functions.

Claim: Fixing $a \in \overline{\mathbb{R}}$, the function h such that $h(x) = a$ when $g_3(x) = g_4(x) = \pm\infty$ and $h = g_3 - g_4$ otherwise, is also measurable.

Proof: $h^{-1}(a) = \{-\infty, \infty\} \in \mathcal{M}$. Since $\{a\} \in \mathcal{B}_{\overline{R}}$, we can decompose any set $E \in \mathcal{B}_{\overline{R}}$ into $E = (E \setminus \{a\}) \cup \{a\}$, both of which are Borel sets. Thus

$$h^{-1}(E) = h^{-1}((E \setminus \{a\}) \cup \{a\}) = h^{-1}((E \setminus \{a\})) \cup h^{-1}(\{a\})$$

which is a union of measurable sets. Thus h is measurable. □

Let $a > 0$. Then the set $F = h^{-1}((0, \infty])^c$ is measurable. The set $I = (g_3^{-1}(\infty) \cap g_4^{-1}(\infty)) \cup (g_3^{-1}(-\infty) \cap g_4^{-1}(-\infty))$ is also measurable and so

$$\{x : \lim f_n(x) \text{ exists}\} = F \cup I$$

is measurable as well.

If we instead have $f : X \to \mathbb{C}$, then corollary 2.5 tells us that f is measurable iff Re f and Im f are measurable. Thus the sequences $\{\text{Re } f_n\}$ and $\{\text{Im } f_n\}$ are composed of measurable functions, implying that the sets on which their limlits are defined are also measurable. Since

$$\{x : \lim f_n(x) \text{ exists}\} = \{x : \lim \text{Re } f_n(x) \text{ exists}\} \cap \{x : \lim \text{Im } f_n(x) \text{ exists}\}$$

this means that $\{x : \lim f_n(x) \text{ exists}\}$ is measurable as well.

2.4 If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Proof: Any set $(a, \infty]$ can be approximated by a sequence $\{a_n\} \subset (\mathbb{Q} \cap (a, \infty])$ such that $f^{-1}((a, \infty]) = \bigcup f^{-1}((a_n, \infty])$ and so is a member of $\mathcal{M}$. Such intervals are enough to generate $\mathcal{B}_{\overline{\mathbb{R}}}$ so f is measurable.

2.9 We're given that $f : [0, 1] \to [0, 1]$ is the Cantor function and $g(x) = f(x) + x$.

(a) **Claim:** g is a bijection from [0,1] to [0,2], and $h = g^{-1}$ is continuous.

Proof: We know that f is continuous and non-decreasing and that x (the identity function) is continuous and increasing. Thus g is continuous and increasing. As $g(0) = 0$ and $g(1) = 2$, g must be a bijection and $h = g^{-1}$ exists.

Given that $h^{-1} = g$, for any open interval (a, b), $h^{-1}(a, b) = g(a, b) = (g(a), g(b))$. Since any open set is just a countable union of open intervals, h is continuous.

(b) **Claim:** If C is the Cantor set, $m(g(C)) = 1$.

Proof: By definition, $C = [0, 1] \setminus \bigcup_n E_n$ where $\{E_n\}$ is a countable disjoint sequence of open intervals. f is constant valued on each interval E_n such that $g|E_n$ becomes a translation. Theorem 1.21 in Folland then tells us that $m(g(E_n)) = m(E_n)$ for all $n \in \mathbb{N}$. Thus

$$m(g([0, 1] \setminus C)) = m(g(\bigcup_n E_n)) = m(\bigcup_n E_n) = m([0, 1] \setminus C) = 1$$

Since g is strictly increasing, $g(C)$ and $g([0, 1] \setminus C)$ are disjoint. So

$$m(g(C)) + m(g([0, 1] \setminus C)) = m([0, 2]) = 2$$

implying $m(g(C)) = 1$.

(c) **Claim:** Given that $g(C)$ contains a Lebesgue nonmeasurable set A and $B = g^{-1}(A)$, B is Lebesgue measurable but not Borel.

Proof: $A \subset g(C)$ so $B \subset C$. C is a null set and m is a complete measure so B is Lebesgue measurable. However, if B were Borel-measurable, then $g = (g^{-1})^{-1}$ is a continuous function and so Borel-measurable. This would imply that A were Borel-measurable which it is not. Thus B can't be Borel.

(d) **Claim:** There exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Proof: Define $F(x) = 1$ if $x \in B$ and $F(x) = 0$ otherwise. Also let $G(x) = g^{-1}$. F is Lebesgue measurable since it's inverses are either the $\emptyset$, B, or X. We already have shown that G is continuous. Therefore

$$(F \circ G)^{-1}((0, \infty)) = G^{-1}(B) = A$$

and so $F \circ G$ is no Lebesgue measurable.