A.3) (a) Given a set $A \subseteq B$, we know that for each element $a \in A$, its the case that $a \in f^{-1}(f(a))$. Since $f^{-1}(f(A)) = \bigcup_{a \in A} f^{-1}(f(a))$, it holds that $A \subseteq f^{-1}(f(A))$.

(b) If f is injective, then for every $a \in A$ it holds that $f^{-1}(f(a)) = a$. Thus it must also hold that $f^{-1}(f(a)) \in A$ for arbitrary a. This implies that $f^{-1}(f(A)) \subseteq A$ and we have equality.

Likewise, if the equality $A = f^{-1}(f(A))$ holds for every $A \subseteq S$, then f must be injective. If not, then pick some non-empty set, A, containing two distinct elements, a and b, for which $f(a) = f(b)$. Then we can create a new set $A' = A \setminus \{a\}$ for which $f(A') = f(A)$. But this implies that $A' = f^{-1}(f(A')) = f^{-1}(f(A)) = A$ which gives us a contradiction $\Rightarrow\Leftarrow$.

B.2) It is a basic property of sets that countable unions of countable sets are countable. We can represents the set of dyadic rationals, D, as:

$$D = \bigcup_{n \in \mathbb{N}} \{\frac{k}{2^n} | k \in \mathbb{Z}\}$$

Since $\mathbb{N}$ is countable, if $\{\frac{k}{2^n}|k \in \mathbb{Z}\}$ is countable then D is as well.

Note that $\{\frac{k}{2^n}|k \in \mathbb{Z}\} \cong \mathbb{Z}$. Furthermore $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n|n \in \mathbb{N}\}$, meaning that $\mathbb{Z}$ (being a countable union of countable sets) is countable. This means that every $\{\frac{k}{2^n}|k \in \mathbb{Z}\}$ is countable and we're done.

C.1) We're given that $E \subset \mathbb{R}$ and $E \neq \emptyset$.

Let inf E = sup $E = \gamma$ and assume that E is not a singleton such that it contains at least two unique elements a and b. As $\mathbb{R}$ has a linear ordering, also assume $a < b$ without loss of generality. Since γ is the supremum of E we have that $a < b \leq \gamma$. But, γ is also the infimum such that $\gamma \leq a < b$. This leads to a contradiction as we now have that $a < b \leq a < b$. And so E must be a singleton.

Likewise if we assume that E is a singleton, then it's sole element, $e \in E$, is it's own least upper bound. Assume there is some other $\gamma \in \mathbb{R}$ such that sup $E = \gamma \neq e$. Then, since $e < \gamma$, we can always find $\delta \in \mathbb{R}$ such that $e < \delta < \gamma$, implying that γ is not actually the supremum. The element, e, is also it's own greatest lower bound by analogous reasoning about its greatest uppper bound. Thus inf E = sup $E = e$.

C.5) Just take the subsets $A = (0, 1)$ and $B = (1, 2)$. Their closures are $\bar{A} = [0, 1]$ and $\bar{B} = [1, 2]$ with interesection $\bar{A} \cap \bar{B} = [0, 1] \cap [1, 2] = \{1\} \neq \emptyset$.

C.6) Let S be the set of interior points of E as defined in the problem statement. If $x \in S$ then $\exists\ \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq E$. This ball is an open set and E^{o} is the union of all open sets contained in E. Thus $x \in B_\epsilon(x) \subseteq E^{\text{o}}$ and so $S \subseteq E^{\text{o}}$.

Likewise, E^{o} is an open set and so by definition, for any $y \in E^{\text{o}}$, $\exists\ \epsilon > 0$ s.t. $B_\epsilon(y) \subseteq E^{\text{o}}$. This immediately implies that every element of E^{o} is an interior point and so $E^{\text{o}} \subseteq S$. This, combined with the previous inclusion implies that $S = E^{\text{o}}$.

D.3) We have from the problem statement that every subsequence $\{x_{n_m}\}$ of $\{x_n\}$ has a futher subsequence $\{x_{n_{m_k}}\}$ which converges to x. Now assume that there is a subsequence $\{x_{n_m}\}$ which does not converge to x. We know that this subsequence contains at least one subsubsequence which converges to x. Call S the set of all such subsubseqeuces in $\{x_{n_m}\}$. This set can be partially ordered by inclusion. Also note that any union of subsubsequences that converge to x also converges to x and so must be contained in S.

Every linearly ordered subset of S has an upper bound of X. Thus, by Zorn's Lemma, S has a maximal element, let's call it the subsubsequence $\{x_{n_{m'_k}}\}$. As this is the maximal subsubsequence of $\{x_{n_m}\}$, it must be the case that $\{x_{n'_m}\} = \{x_{n_m}\} - \{x_{n_{m'_k}}\}$ does not contain a convergent subsubsequence. This contradicts or initial assumption. Thus it must hold that every subsequence $\{x_{n_m}\}$ converges to x. But if this is the case, then it must also hold that $\{x_n\} \to x$.

E.4) (a) Given that f is continuous and $f(0) > 0$, we can find a positive real, γ, such that $f(0) > \gamma > 0$ and use it to find an open set $U = [f(0) - \gamma, f(0) + \gamma]$ containing $f(0)$. By the continuity of f, we have that $V = f^{-1}(U)$ is also open. It's also the case that $0 \in V$, since $f^{-1}(0) \subseteq V$.

V being open, we can find an open interval $(-a, a) \subset V$ containing 0. This interval is completely contained in V so that $f((-a, a)) \subset U$. Now, we constructed U to be a set of positive real numbers. Thus $f(x) > 0$ for all $x \in (-a, a)$.

(b) Given $f(x) \geq 0$ for all rational x, assume there is some real (and necessarily irrational) x', such that $f(x') < 0$. Then by an analogous argument to that in part a), there must existe a nonempty interval $(x' - \gamma, x' + \gamma)$ where $f(x) < 0$ for all x in said interval. But we know that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that if such an interval exists it must contain a rational number. Thus we have a contradiction.

Now, if we're only given that $f(x) > 0$ for all $x \in \mathbb{Q}$, it is possible to construct a function such that $f(x) = 0$ for some $x \in \mathbb{R}$ that's not rational. For example, take $f(x) = (x - \sqrt{2})^2$. In this case, $f(x) > 0 \; \forall \; x \in \{x | x \neq \sqrt{2}, \; x \in \mathbb{R}\}$ and $f(\sqrt{2}) = 0$.

E.5) By the definition of continutity, $f^{-1}(U)$ is open for any open set $U \subseteq X'$. This in turn implies that if U is closed, it's inverse is also closed.

Let's assume that f is continuous. Pick any $x \in \overline{E}$, then by proposition 0.22 in Folland, there exists a sequence $\{x_n\}$ contained in E that converges to x. As such, the sequence $\{f(x_n)\}$ is contained in $f(E)$ and we know that $f(x) \in X'$. We can show that $\{f(x_n)\}$ converges to $f(x)$ by noting that for any open ball $B_\epsilon(f(x))$ in X', there exists an open region $U = f^{-1}(B_\epsilon(x))$ in X which itself contains a ball $B_\delta(x) \subseteq U$ containing x. Given that $\{x_n\}$ converges to x, we know that $\exists \; N$ such that $x_n \in B_\delta(x) \; \forall \; n > N$. This then implies that $f(x_n) \in B_\epsilon(f(x)) \; \forall \; n > N$. Since our initial selection of $\epsilon > 0$ was arbitrary, we see that $\{f(x_n)\}$ must converge to $f(x)$. This in turn means that $f(x) \in \overline{f(E)}$ and thus $f(\overline{E}) \subseteq \overline{f(E)}$.

Now assume that $f(\overline{E}) \subseteq \overline{f(E)}$ for all $E \subseteq X$. Let's also assume that f is not continuous, i.e. there exists some closed set $V \subseteq X'$ such that $U = f^{-1}(V)$ is not also closed. Then $U \subset \overline{U}$ and $f(\overline{U}) \supseteq f(U) = \overline{f(U)}$ since $f(U) = V$ is closed. This contradicts our first assumption, thus f must be continuous.