Math 275A: Homework 2

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1 Bonus

Claim: Let m be a measure on the Borel set $\mathcal{R}$, with $A \in \mathcal{R}$ such that $mA < \infty$. Let $\epsilon > 0$. Show that there is a finite collection of bounded intervals $I_1, I_2, \dots, I_k$ such that

$$m(A\Delta(I_1 \cup \cdots \cup I_k)) < \epsilon.$$

Proof: Lebesque measure m is, by definition, the restriction of outer measure m^* to the m^*-measurable sets, themselves equivalent to $\mathcal{R}$. The explicit definition of m^* is

$$m^*(A) = \inf\{\sum_i (b_i - a_i) : A \subset \cup_i (a_i, b_i], -\infty < a_i \leq b_i < \infty\}.$$

This implies that for any ϵ and set A of finite measure, there exists a countable set of half-open intervals $\{I_i\}_i$ covering A such that

$$m(A) + \frac{\epsilon}{2} \geq \sum_i m(I_i).$$

Since $\sum_i m(I_i)$ converges, we can select an integer k such that

$$\sum_{i=1}^{k} m(I_i) + \frac{\epsilon}{2} \geq \sum_i m(I_i)$$

Note, that sets of half-open intervals I_i can be assumed to be disjoint, following from the fact that any two such overlapping intervals I_i and I_j can be rewritten as a pair of disjoint intervals, so that we can build collection of disjoint intervals from any generic collection. Thus

$$\sum_i m(I_i) = m(\cup_i I_i)$$

This equality lets us rewrite the previous inequalities as

$$m(A) + \frac{\epsilon}{2} > m(\cup_i I) \quad \text{and} \quad m(\cup_{i=1}^k I_i) + \frac{\epsilon}{2} \geq m(\cup_i I_i).$$

That A and $\cup_{i=1}^k I_i$ are both subsets of $\cup_i I_i$, lets us use additivity to rewrite them again as

$$m(\cup_i I_i \setminus A) \leq \frac{\epsilon}{2} \quad \text{and} \quad m(\cup_i I_i \setminus \cup_{i=1}^k I_i) \leq \frac{\epsilon}{2},$$

thus implying that $m(A\Delta \cup_{i=1}^k I_i) \leq \epsilon$.

1.2.1

Claim: Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Proof: We just need to show that $Z^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{R}$. But given any such B,

$$Z^{-1}(B) = (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c).$$

Each of the intersections are in $\mathcal{F}$ since $A \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra. Thus their union $Z^{-1}(B)$ is also in $\mathcal{F}$ for any $B \in \mathcal{R}$ and $Z(\omega)$ is a random variables.

1.3.3

Claim: Show that if f is continuous and $X_n \to X$ a.s. then $f(X_n) \to f(X)$ a.s.

Proof: Because f is continuous it is also measurable. Thus the functions $f(X_n)$ and $f(X)$ must all be random variables. One can use this fact to argue that the set

$$\Omega_o' := \{w : \lim_{n\to\infty} f(X_n) \text{ exists}\} = \{w : \limsup f(X_n) - \liminf f(X_n) = 0\}$$

is measurable, as the expressions $\liminf X_n$ and $\limsup X_n$ are themselves random variables. Lastly, for any given $\omega \in \Omega_o\{w : \lim_{n\to\infty} X_n \text{ exists}\}$ it follows from continuity of f that $f(X_n(\omega)) \to f(X(\omega))$. This means that Ω_o' contains Ω_o which itself has measure equal to one. Thus $P(\Omega_o') = 1$ by monotonicity of measures and $f(X_n)$ converges a.s.

1.4.1

Claim: Show that if $f \geq 0$ and $\int f \, d\mu = 0$ then $f = 0$ a.e.

Proof: It follows from our the assumptions that $f < 1/n$ a.e. Assume otherwise, such that there exists a $E \in \mathcal{F}$ for which $f(E) \geq 1/n$ and $\mu E > 0$. Then we can define a simple function $\phi = \frac{1}{n}\mathbf{1}_E$, less than f and with integral equal to $\int \phi \, d\mu = \frac{1}{n}\mu E > 0$. But we know from the properties we've derived for the integral that $f > \phi$ implies $\int f > \int \phi = \mu E > 0$ Thus f must be less than any constant function $\frac{1}{n}$ a.e.. Taking that limit as $n \to \infty$ implies then that $f = 0$ a.e.

1.4.3

Claim: Let g be an integrable function on $\mathbb{R}$ and $\epsilon > 0$. (i) Use the definition of the integral to conclude there is a simple function $\phi = \sum_k b_k \mathbf{1}_{A_k}$ with $\int |g - \phi| \, dx < \epsilon$. (ii) Approximate the A_k by finite unions of intervals to get a step function

$$q = \sum_{j=1}^{k} c_j \mathbf{1}_{(a_{j-1}, a_j)}$$

with $a_0 < a_1 < \cdots < a_k$, so that $\int |\phi - q| < \epsilon$.

Proof: (i) First note that $g = g^+ - g^-$ for two non-negative functions. Thus, we'll assume the $g \geq 0$ without loss of generality.

We know by step 3 of the definition of the integral that for a given $\epsilon > 0$ there exists a bounded function h, zero everywhere but a set E of finite measure, such that $0 \leq h \leq g$ and $\int h\ dx + \frac{\epsilon}{2} > \int g\ dx$. This along with the other properties of the integral implies that $\frac{\epsilon}{2} > \int g - h\ dx \geq 0$.

Likewise, step 2 of the definition of the integral implies the existence of simple functions $\psi > h > \phi \geq 0$ defined only on E such that $\int \phi\ d\mu + \frac{\epsilon}{2} > \int h\ d\mu$ and $\int h\ d\mu + \frac{\epsilon}{2} > \int \psi\ dx$. These inequalities indicate that more generaly, there always exists some function φ on E for which $\int |h - \varphi|\ dx < \frac{\epsilon}{2}$.

Combined with the contraint derived for $\int g - h\ dx$, this last inequality shows that there always exists a simple function ϕ, defined on a set E of finite measure, such that $\int |g - \phi|\ dx < \epsilon$.

(ii) We know that ϕ is a simple function, defined over a finite collection of sets A_i such that $\phi = \sum_{i=1}^{n} b_i \mathbf{1}_{A_i}$. As shown in the bonus problem, we can cover each A_i with a finite collection of m_i bounded intervals $I_{A_i}^j$. Let $b = \max_i\{b_i\}$. Then for a given ϵ we can select an interval set so that for $I_i = I_{A_i}^1 \cup \cdots \cup I_{A_i}^{m_i}$ we have $\mu(A_i \Delta I_i) < \frac{\epsilon}{nb}$.

We can define a similiar set I_i satisfying the same bound for each set A_i. Define a set of simple functions $q_i = b_i \mathbf{1}_{I_i}$ and then define $q = \max_i q_i$. The function q has the form suggested in the promp (up to a finite, measure-zero set of points). We can see this by noting that $\cup_i I_i$ is an open set and the endponts of intervals comprising the sets I_i can be ordered and labeled as $a_0 < a_1 < \cdots < a_k$. On each interval (a_i, a_{i-1}), q just takes the maximum value between the two functions q_i active on it.

The way we've defined q, the difference $\phi - q$ is zero is zero on each set A_i excepting small sets where the intervals I_i overlap. On these sets, q equals at most b. Thus the integral $\int |\phi - q|$ can be bounded by $b * \sum_{i=1}^{n} m(A_i \Delta I_i) \leq bn \frac{\epsilon}{nb} = \epsilon$. This completes the proof.