Implementation of a new statistical comparison among signals

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Pattern Recognition Exam

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Contents

1 Introduction

In real-world scenarios, signals are inevitably affected by noise. While extensive research has been dedicated to noise reduction, this study focuses on assessing whether two signals, contaminated by different noise levels, can be considered equivalent based on their spectral characteristics. Specifically, the aim is to develop a statistical test that determines, with a given confidence level, whether two signals should be classified as identical or distinct.

In this context, signal equivalence is defined in terms of frequency content: two signals are considered the same if they exhibit identical frequency content, regardless of the superimposed noise. Ideally, signals differing only by noise should be identified as equivalent, whereas those with distinct spectral compositions should be classified as different.

To achieve this, the analysis is conducted on a controlled dataset of simulated signals, allowing a systematic evaluation of statistical methodologies. The objective is to establish a statistical test that, given any two signals, applies a rigorous procedure to determine their equivalence, providing a decision with an associated p-value to quantify confidence.

Among various noise types, white noise represents a fundamental and commonly encountered case. Consequently, it is the primary focus of this study, where the goal is to assess whether two signals, each affected by different levels of white noise (or slight variations thereof, see 2), correspond to the same underlying signal or distinct ones. This analysis serves as a foundation for extending the methodology to more complex noise models and real-world scenarios.

All the processing and analysis conducted in this study were implemented in Python, and the associated code is available in the following GitHub repository:
https://github.com/malvasochiara/statistical-signal-comparison

The repository contains two folders:

- $main\_classification\_pipeline$: This folder includes the functions for dataset generation and preparation for statistical analysis, as detailed in sections 2 and 3, as well as the statistical methodology outlined in 4.3.

- $supplementary\_material$: This folder contains additional approaches and analyses referenced throughout the report.

1.1 White Noise Properties

White Gaussian Noise (WGN) is one of the most commonly employed stochastic models in practical applications. In signal processing, the term "white noise" traditionally refers to a stochastic process consisting of independent random variables. The origin of this terminology lies in the spectral properties of these stochastic processes, which exhibit a flat spectrum, meaning that all frequencies have equal power.

A stochastic process $\mathbf{X}(t)$ is defined as WGN if $\mathbf{X}(\tau)$ follows a normal distribution for any random variable τ, and if the values $\mathbf{X}(t_1)$ and $\mathbf{X}(t_2)$ are statistically independent for $t_1 \neq t_2$. The first condition characterizes the "Gaussian" nature of the noise, while the second justifies the term "white."

An important property of WGN is that its Power Spectral Density (PSD) remains constant across all frequencies [1], i.e.,

$$PSD = \frac{N_0}{2}, \quad \forall f.$$

It is crucial to emphasize that WGN does not accurately describe any real physical phenomenon. By definition, spectrally white noise possesses a noise spectral density that is constant across all frequencies. However, in practical systems, noise is never truly white but rather "spectrally pink," meaning that the noise spectral density remains approximately constant up to a certain cutoff frequency and subsequently decreases to ensure a finite variance.

In most practical applications, the cutoff frequency is sufficiently high, and the noise spectral density below this threshold remains sufficiently constant to justify the use of the white noise model [2]. Consequently, WGN serves as a useful abstraction for generating processes that approximate real physical phenomena [3].

In numerical simulations, sequences of normally distributed random numbers are often employed as an approximation of a WGN process.

2 Dataset construction

The initial phase of the study involved the construction of a suitable dataset to address the problem at hand. The approach was to maintain a general framework, minimizing unnecessary complexities, particularly in the preliminary stages of the analysis. Guided by Fourier's theorem, which asserts that any periodic signal can be represented as a sum of sine and cosine functions, the dataset was constructed as a superposition of sinusoidal waves. This choice ensured a straightforward yet robust foundation for the subsequent analysis. Each signal $S(t)$ is represented as a sum of sinusoidal components, as given by the following equation:

$$S(t) = \sum_{n=1}^{N} \sin(2\pi f_n t) \tag{1}$$

where f_n denotes the n-th frequency in an array consisting of N distinct frequencies. It is important to note that, in the above expression, the amplitude of each sinusoidal component is set to unity. For the sake of simplicity, all signals in this analysis are constructed from sinusoidal waves of unitary amplitude.

The parameters that are allowed to vary, and which must be considered when interpreting the results, are as follows:

- **Sampling rate** [Hz]: This parameter determines both the range of allowable frequencies that can compose the signal, as it must satisfy Nyquist's theorem, and the number of discrete data points sampled in the signal.
- **Duration** [s]: The temporal extent of the signal. Together with the sampling rate, this defines the total number of sampled points.
- **Number of components** (N): The total number of distinct sinusoidal components included in the signal.
- **Frequencies**: The frequency of each sinusoidal component, which may vary from 1 Hz up to the Nyquist frequency.

In addition to the selection of signals, the simulation of noise is also a critical aspect of the study. Two types of noise are considered:

1. **White Gaussian Noise**: This type of noise is described in Section 1.1. In Python, white noise is simulated using the *numpy.random.randn* function, which generates a sample of normally distributed random numbers [4]. To provide a controllable parameter that effectively quantifies the amount of noise added to the signal, the Signal-to-Noise Ratio (SNR) is chosen. The SNR is defined as:

$$\text{SNR} = \frac{\text{signal power}}{\text{noise power}}.$$

   By providing the signal (to compute its power) and the desired SNR as input, the appropriate noise level can be computed by adjusting the noise power and scaling the Gaussian distribution obtained with *numpy.random.randn*.

2. **Colored Noise**: This is defined as white noise that is modified in the frequency domain such that it varies linearly with frequency. The white noise is generated as described above, and then the Fourier Transform is computed using *numpy.fft.fft*. In the frequency domain, the noise is modified according to the following relation:

$$\text{colored\_noise\_spectrum} = \text{white\_noise\_spectrum} + \text{slope} \cdot \text{frequencies},$$

   where the slope is user-defined and the frequencies are those computed using the numpy function *numpy.fft.fftfreq*. Finally, the inverse Fourier Transform is applied, and the resulting colored noise is added to the signal.

Some considerations must be taken into account. White noise is added directly in the time domain, whereas colored noise, due to its definition, involves performing manipulations in the frequency domain. Since the signal is finite, edge effects and spectral leakage may occur. Consequently, the application of colored noise might introduce additional artifacts into the signal.

Furthermore, it is important to note that the colored noise also takes an input SNR value, which is used to compute the signal power. However, in this case, since the noise is modified in the frequency domain, the power of the resulting noise in the time domain is generally higher than the computed value.

The following figures (1 and 2) present an example of the generated signal, both in its clean form and with the addition of noise, in both the time and frequency domains.

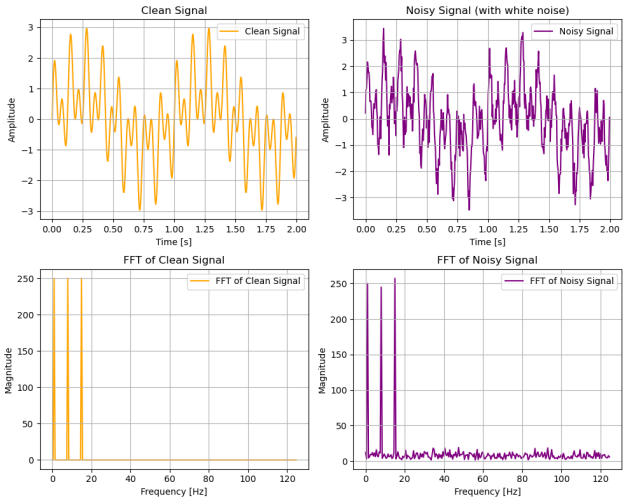


Figure 1: Example of a simulated signal affected by white noise. The left panels display the clean simulated signal, while the right panels illustrate the effect of introducing white noise.

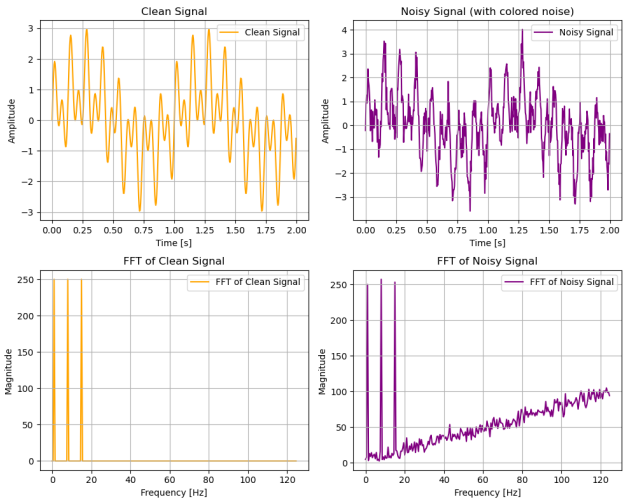


Figure 2: Example of a simulated signal affected by colored noise. The left panels display the clean simulated signal, whereas the right panels illustrate the impact of colored noise. As observed, colored noise exhibits a distinct behavior in the frequency domain, whereas its effect in the time domain is less straightforward to interpret.

In addition, the same function was implemented to generate a sum of cosinusoidal signals, following the same logic and parameters described above. In the subsequent sections, unless stated otherwise, the signal used for analyzing each approach consists of a sum of sinusoidal waves. The use of cosinusoidal signals was intended to validate the generality of the approach and assess whether its effectiveness was limited to sinusoidal signals.

Finally, a third function was implemented to generate signals composed of both sine and cosine waves. The input parameters remain the same as previously described; however, for each frequency component, it is randomly determined whether the contribution is in the form of a sine or a cosine wave. This function is employed to evaluate the generalizability of the proposed approach.

3 Data preparation

After establishing the methodology for constructing the dataset to address the problem at hand, the next step is to identify the properties of the signals that should be exploited for the classification task described previously.

Since the definition of "same signal" is based on the frequency content, it is logical to approach the problem directly in the frequency domain. The objective is to identify features that are common among signals originating from the same process but differ for signals originating from distinct processes. Given that the noise introduced into the signals is white noise, its distinctive properties in both the time and frequency domains can be exploited.

To perform the analysis in the frequency domain, the Fourier transform is employed as the primary tool. The central idea behind this choice is to utilize the linearity of the Fourier transform. Specifically, if two signals are identical but affected by additive noise, their difference in the frequency domain should correspond to the Fourier transform of the noise. Since the noise is assumed to be white noise, the difference between the signals should ideally exhibit the characteristics of white noise itself, both in the time and

frequency domains.

Mathematically, let two signals be defined as $\mathbf{X}(t) = \mathbf{x}(t) + \mathbf{n}_1(t)$ and $\mathbf{Y}(t) = \mathbf{x}(t) + \mathbf{n}_2(t)$ where $\mathbf{x}(t)$ represents the common underlying signal and $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are the noise components affecting each signal. The difference between the signals in the frequency domain can then be expressed as:

$$\Delta\hat{X}(f) = \mathcal{F}[\mathbf{X}(t) - \mathbf{Y}(t)] = \mathcal{F}[\mathbf{n}_1(t) - \mathbf{n}_2(t)] = \hat{N}(f)$$

where $\mathcal{F}$ denotes the Fourier transform and $\hat{N}(f)$ represents the Fourier transform of the difference between the noise components. This difference should ideally retain the characteristics of white noise, as both $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are assumed to be white noise processes.

Thus, by calculating the difference in the frequency domain between the two signals, it becomes possible to isolate the noise components. These components can then be analyzed to determine whether the signals are the same (in terms of the underlying signal $\mathbf{x}(t)$) or if they originate from different processes.

Another important step in the signal processing pipeline is the detrending of the signal. This step is essential not only in the context of the present simulation to mitigate the effects of colored noise, as described above, but also to account for any potential frequency drift that may be present in the data. The detrending process ensures that only the relevant signal characteristics are retained for comparison, thereby improving the reliability of the subsequent analysis.

The pipeline followed for processing the signals is as follows:

1. Compute the complex Fourier transform of the signal. To reduce redundancy, only the positive frequency components of the transformed signal are retained for further processing, as the negative frequencies carry the same information due to the symmetry of the Fourier transform. This simplifies the analysis without losing essential information.

2. Apply a linear detrending procedure to both the real and imaginary components of the signal. This step removes any linear trends or drifts in the signal, which could obscure the underlying frequency content.

3. Compute the difference between each possible pair of signals. In this step, it is crucial to track whether the two signals that originated the difference are from the same underlying process or from different processes.

4. Normalize the difference by dividing it by its standard deviation.

Due to the linearity of the Fourier transform, the resulting output is a complex-valued signal that corresponds to:

- The Fourier transform of white noise, if the original signals are identical.
- The Fourier transform of a generic signal, if the original signals differ.

A complementary procedure was also tested, where the difference between the signals was first computed in the time domain, and then the Fourier transform was applied. Although this approach is valid, it was discarded due to significantly longer computational times. This is because the number of FFTs required is much higher, as it involves computing the FFT for every pairwise combination of signals. The results obtained were identical to the primary method, but the computational inefficiency led to its exclusion. The Python code for this alternative approach is provided in the Supplementary Material folder in the GitHub repository.

4 Analysis

Several lines of analysis were explored with the objective of identifying a statistical test capable of correctly detecting true positives with a probability of 95%. The central idea underlying the following analyses is to leverage specific properties of white noise. In particular, if two signals are identical, their difference should exhibit characteristics typical of white noise.

In order to ensure a valid comparison, the parameters used to generate the dataset for

the statistical analysis were kept fixed. Specifically, the sampling rate was set to 500 Hz, and the signal duration was 2 seconds. A total of 20 distinct signals were generated, each with 20 noisy versions. The number of components in each signal was randomly chosen between 1 and 40 to introduce variability, and the signal-to-noise ratio was distributed between 10 and 100 dB.

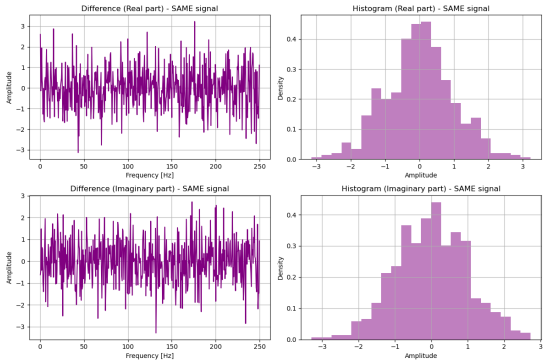
The differences between signals were computed pairwise. Since the input to the statistical test is always the difference between two signals, the sample size is sufficiently large, even with a relatively small number of signals (400 in total, considering all noisy versions of each distinct signal), as it accounts for all possible combinations of signal pairs.

The parameters defined above were utilized to compute the p-values and other statistical quantities. Consequently, the analysis of signals generated with these parameters resulted in the p-value distributions presented in the following section (with the exception of those in section 4.3.1, where specific details are provided). Conversely, for the purpose of illustrating the amplitude distribution and presenting example plots of the differences, certain parameters were adjusted to enhance the clarity of the visual representations.

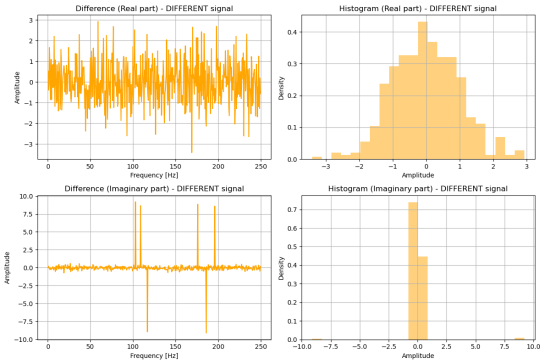
4.1 Normality of Real and Imaginary part's amplitude

White noise exhibits characteristic spectral properties; in particular, the real and imaginary parts of its Fourier transform both follow a Gaussian distribution [5]. This property can be effectively exploited, as the differences between two identical signals and two distinct signals exhibit notable discrepancies. As illustrated in Figure 6a, when two signals are identical, their difference does not present any distinct peaks and instead resembles a random process. Conversely, as shown in Figure 6b, when two signals have different frequency content, their difference reveals pronounced peaks corresponding to their constituent frequencies, along with minor variations attributable to noise.

As evidenced by the plots in Figure 3, the imaginary part of the Fourier transform appears to be more effective in distinguishing between identical and distinct signals, as the real part exhibits fewer differences between these cases. This behavior is a consequence of the specific nature of the chosen signals. As discussed in Section 2, the simulated signals



(a) Same signal.



(b) Different signal.

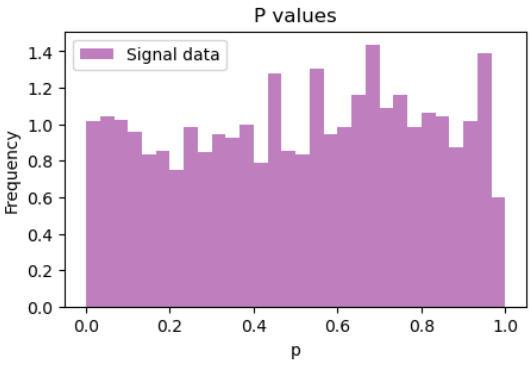
Figure 3: Difference in the Fourier transform of two signals. In both figures, the upper panel represents the real part, while the lower panel corresponds to the imaginary part. The plots illustrate the amplitude of the difference as a function of frequency, whereas the histograms depict the amplitude distribution.

are composed entirely of sinusoidal waves. Since the Fourier transform of a pure sine wave is purely imaginary, the imaginary part of the Fourier transform primarily represents sinusoidal components. To verify this observation, the same analysis was repeated using purely cosinusoidal waves, yielding the opposite result: in this case, the real part effectively captured the differences in frequency content between pairs of identical and distinct signals, as expected.

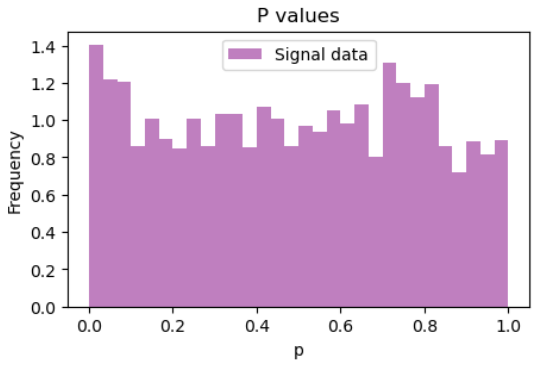
After an initial visual inspection, a statistical analysis was performed on both the real and imaginary components of the difference. To assess the normality of the data, the Shapiro-Wilk test was employed, utilizing the *SciPy* implementation of the function *scipy.stats.shapiro*. Figure 4 displays the distribution of the p values both for the imaginary and the real part.

The obtained results are consistent across both parts of the signal. In particular, the Shapiro-Wilk test applied to the real part yields a false negative rate of 5.2%, while for the imaginary part, the rate is 6.2%.

Upon examining the p-value distributions for the different signals, which were ex-



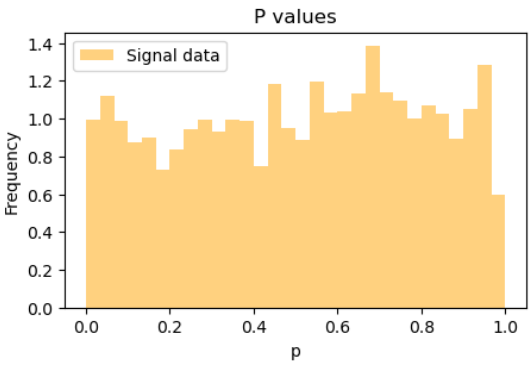
(a) P-values obtained from the Shapiro test on the real part of the difference.



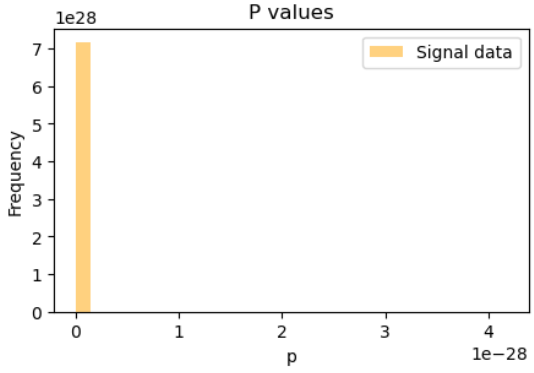
(b) P-values obtained from the Shapiro test on the imaginary part of the difference.

Figure 4: Distribution of p-values derived from performing the Shapiro-Wilk test on the real (left) and imaginary (right) components of the difference of same signals.

pected to be mostly below the threshold of 0.05, it was observed that, for the real part, the distribution is instead approximately uniform, with p-values spread over the entire range, as shown in Figure 5. This indicates that approximately 95% of the differences in the real part are classified as originating from pairs of identical signals. In contrast, the distribution for the imaginary part aligns with expectations, as anticipated. These



(a) P-values obtained from the Shapiro test on the real part of the difference.



(b) P-values obtained from the Shapiro test on the imaginary part of the difference.

Figure 5: Distribution of p-values derived from performing the Shapiro-Wilk test on the real (left) and imaginary (right) components of the difference of different signals.

results are consistent with the observations shown in Figure 3: since the signal is entirely

composed of sinusoidal waves, the relevant frequency information is predominantly contained in the imaginary part of the spectrum [6]. The test applied to the imaginary part effectively classifies two given signals, achieving approximately 94% true positive rate. However, this outcome is highly dependent on the type of dataset used. For instance, when testing with cosinusoidal waves, the results were conversely opposite, with the real part of the difference correctly capturing the distinction between two signals.

Therefore, a universal recommendation for signal classification cannot be drawn from these results. As a logical next step, the approach was to explore a property of white noise that involves both the real and imaginary parts of the spectrum, thus eliminating the need to select between the two components.

4.2 Rayleigh distribution of magnitude

The Rayleigh distribution [7] is defined as the square root of the sum of the squares of two independent Gaussian components, i.e.,

$$y = \sqrt{x_1^2 + x_2^2}$$

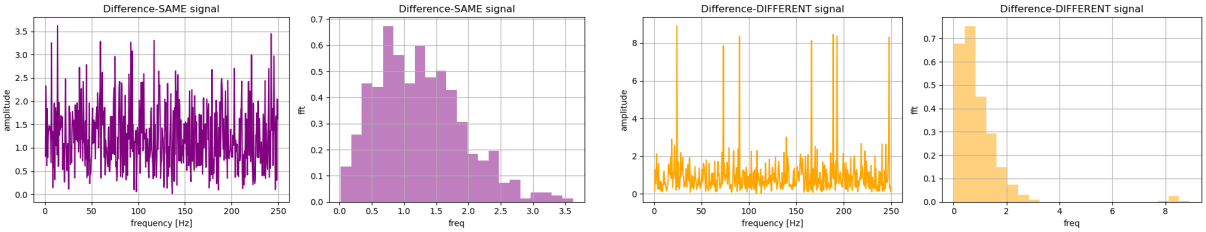
where x_1 and x_2 are independently Gaussian-distributed variables. The probability density function is given by:

$$f(x, \sigma) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$$

with σ being the scale parameter of the distribution.

Given that the objective was to identify a property of white noise that involves both the real and imaginary components of the spectrum, and knowing that these components are gaussianly distributed and independent, it follows that the magnitude of the FFT of white noise should follow a Rayleigh distribution.

The magnitude of the difference in the FFT was computed, with differences calculated pairwise as described previously, and labeled as either 'same signal' or 'different signal.' Figure 6 illustrates the amplitude and its distribution for a representative example signal. As described in the previous section, the difference between two identical signals exhibits a behavior similar to that of noise, while the difference between two distinct signals preserves the frequency content of both original signals, as evidenced by the presence of peaks.



(a) Same signal. (b) Different signal.

Figure 6: Magnitude of the difference in the Fourier transform of two signals. The plots illustrate the magnitude as a function of frequency, whereas the histograms depict the magnitude distribution.

The statistical analysis of the distribution was conducted using the Kolmogorov-Smirnov test, implemented via the *SciPy* function *scipy.stats.kstest*. Specifically, the amplitude distribution was tested against a Rayleigh distribution, with the parameter σ estimated from the test dataset. The Kolmogorov-Smirnov test assesses whether a given set of observations follows a fully specified continuous distribution [8]. Figure 7 illustrates the resulting distribution of p-values.

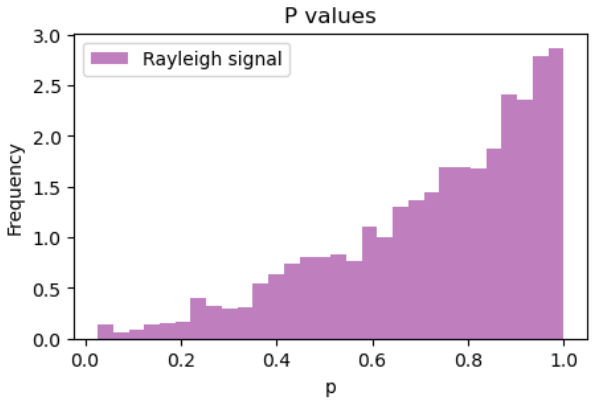


Figure 7: Distribution of p-values obtained from testing the amplitude distribution of the magnitude of the difference between the FFT of signal pairs against a Rayleigh distribution.

As evident from the plot, the procedure resulted in an overly conservative test, as indicated by the p-value histogram exhibiting a strong bias toward 1. This suggests that the empirical distribution is systematically closer to the theoretical distribution than

would be expected under a truly uniform null hypothesis.

This effect may arise from the fact that the parameters of the true Rayleigh distribution of white noise are not assumed to be known a priori but are instead estimated from the data. When certain distribution parameters are inferred from the sample, the Kolmogorov-Smirnov test tends to be conservative, meaning that the probability of a Type I error is lower than the nominal significance level reported in standard tables of the Kolmogorov-Smirnov statistic [8].

4.2.1 Focus on the most extreme value

Up to this point, the analysis has focused on the entire amplitude distribution. Despite being grounded in a solid theoretical framework, the obtained results were either not generalizable (as discussed in 4.1) or excessively conservative (see 4.2).

An examination of the plots in Figure 6 reveals that the primary difference between the two cases—same signals versus different signals—lies in the peak values, which characterize the frequency content and appear as clear outliers in the histogram. The qualitative interpretation is that, for same signals, the highest values in the distribution arise from noise fluctuations and should therefore conform to a Rayleigh distribution. Conversely, for different signals, the highest amplitude values are associated with the signal itself and cannot be attributed to noise, implying that they should not follow a Rayleigh distribution.

The first approach involved testing whether the most extreme values belong to a Rayleigh distribution, specifically the one estimated from the data. In this context, the most extreme value was defined as the maximum, as the magnitude is always positive and the distinguishing characteristic between "same" and "different" signals is the presence of prominent peaks. To perform this test, the survival function was employed. This function provides the probability that a random variable exceeds a given value. A small p-value from the survival function suggests that the observed maximum is unlikely under the Rayleigh distribution, indicating it may be an extreme deviation. This approach is particularly useful for extreme value analysis, as it directly evaluates the likelihood of observing such large values within the assumed distribution. The plot in Figure 8 illustrates

the resulting distribution of p-values.

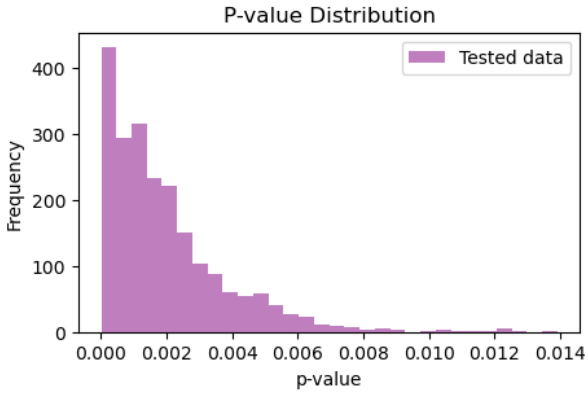


Figure 8: Distribution of p-values obtained by applying the survival function to the maximum value of the magnitude of the difference between the FFTs of pairs of identical signals.

As shown in the plot, the null hypothesis is consistently rejected, indicating that either the assumption regarding the distribution the data should follow is incorrect, or that the chosen test is not suitable for the analysis. Given that the underlying theory appears sound—such as the fact that, for identical signals, both the real and imaginary components follow a Gaussian distribution, as confirmed by the Shapiro-Wilk test performed in section 4.1—it is more likely that the issue lies with the application of the test rather than the theoretical framework itself.

Since the theory behind the previous method seemed well supported, an alternative approach was explored, still relying on the idea that the maximum value of a set of identical signals should follow a Rayleigh distribution. Here, the Gumbel distribution is utilized to model the distribution of the maximum (or minimum) value from a sample of various distributions, including a Rayleigh [9]. The cumulative distribution function (CDF) of the Gumbel distribution is:

$$F(x) = exp(-exp(\alpha(x - u)))$$

where $\alpha = \frac{1.283}{\sigma(x)}$ and $u = \mu(x) - 0.45\sigma(x)$, with $\mu(x)$ representing the mean and $\sigma(x)$ the

variance of the data.

As in the previous approach, the survival function was calculated using the cumulative distribution function (CDF) of a Gumbel distribution. The parameters u and α were estimated from the data, and a statistical test was conducted to assess whether the most extreme values followed the expected distribution. The resulting distribution of the p-values is shown in Figure 9.

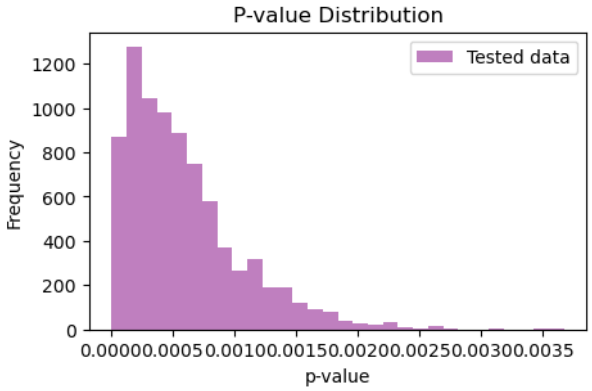


Figure 9: Distribution of p-values obtained by applying the survival function to the maximum value of the magnitude of the difference between the FFTs of pairs of identical signals, in order to test their compatibility with a Gumbel distribution.

In this case, the results consistently do not align with expectations, as the null hypothesis is always rejected due to the small p-values. These inconsistent results may stem from an incorrect estimation of the Gumbel distribution parameters. Since the Gumbel distribution models the extreme value of a distribution, the focus should be on the tail(s) of the data. To accurately estimate the parameters of the extreme value distribution, it is essential to avoid using the entire dataset. Instead, one should concentrate exclusively on the most extreme values, requiring a sampling strategy that isolates the tail of the distribution. Including the entire dataset is inappropriate, as the central values can lead to a biased estimation of the parameters, which could explain why the null hypothesis is consistently rejected.

The underlying concept of this method was that, while it is acknowledged that the peak value could occasionally represent an outlier for identical signals, these instances

could ideally be confined to a 5% error rate for misclassified identical signals. However, the outcomes suggest that relying on a single value rather than the entire distribution may still be inappropriate.

4.3 Normality of concatenated Real and Imaginary part

Among all the approaches tested, the one outlined in section 4.2 appeared to be the most promising. However, it became evident that this method was too specific for the current dataset. The optimal test was focused on assessing the normality of the imaginary part of the difference, whereas for a cosinusoidal dataset, the appropriate test should involve the real part of the spectrum. Attempting to combine both parts using the magnitude resulted in an overly conservative test. Nonetheless, an approach that incorporates both the real and imaginary components should, in principle, offer greater generalizability.

Since both the real and imaginary components follow a Gaussian distribution (and are normalized), their combined distribution should also approximate a Gaussian distribution. Therefore, the procedure consists of concatenating the two *numpy* arrays containing the real and imaginary parts, and then applying a Shapiro-Wilk test, as described in section 4.1, to the concatenated data. Figure 10 illustrates the distribution of p-values obtained from this procedure. The proposed approach exhibits a true positive rate of 95.04% and

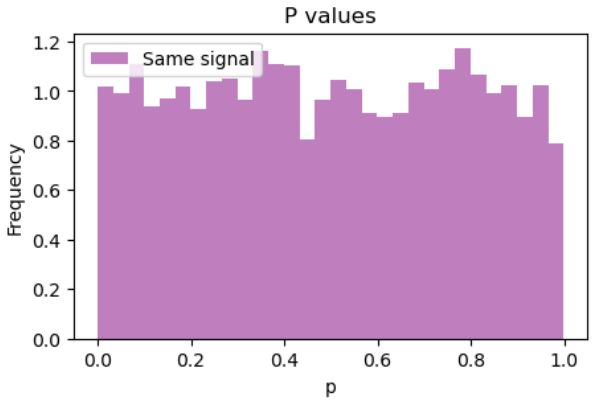


Figure 10: Distribution of p-values obtained by testing the normality of the concatenated amplitude distributions of the real and imaginary parts of the difference between the FFTs of pairs of identical signals.

produces a flat distribution, yielding a reasonable outcome. To further substantiate the results, Figure 11 presents the distribution of p-values for different signals, providing additional validation for this method:

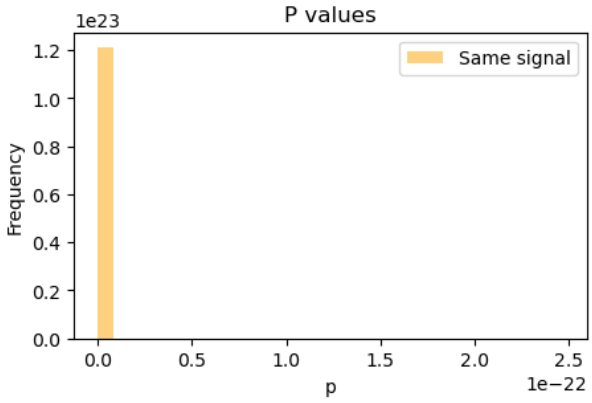


Figure 11: Distribution of p-values obtained by testing the normality of the concatenated amplitude distributions of the real and imaginary parts of the difference between the FFTs of pairs of different signals.

4.3.1 Approach Validation

To further evaluate the validity and generalizability of the proposed approach, various signal scenarios were analyzed. All figures presented in this section depict the distributions of p-values obtained using the method described in Section 4.3. Specifically, the procedure involves computing the difference between the FFTs of pairs of identical signals, concatenating their real and imaginary components, and subsequently testing the resulting array for normality. This section focuses on different input signals, while the analysis methodology remains unchanged.

The following type of signals were generated:

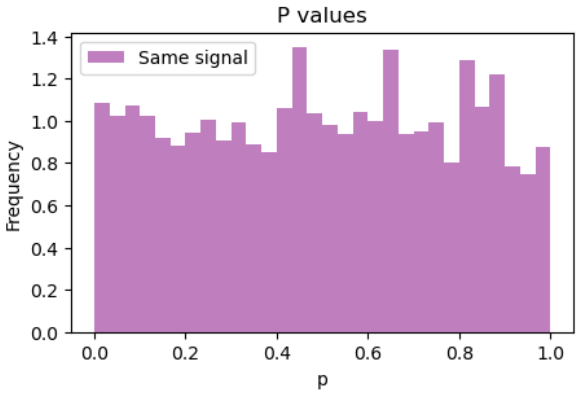
- Sinusoidal signals, as described in Section 2, perturbed by colored noise with a spectral slope of 0.6. The corresponding results are presented in Figure 12a, yielding a true positive rate of 94.8%.

- Cosinusoidal signals affected by white noise. The dataset was constructed following the methodology outlined in Section 2, maintaining the same parameters used for

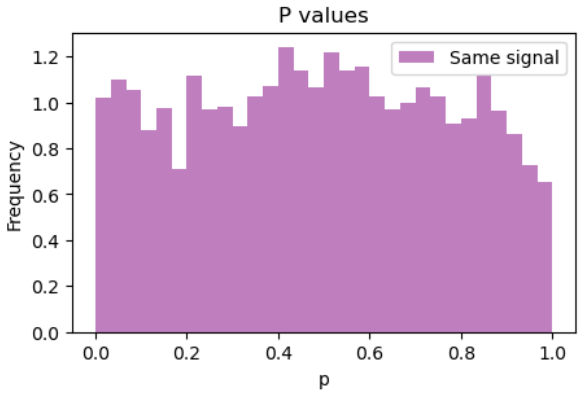
sinusoidal signals. The results, shown in Figure 12b, indicate a true positive rate of 95%.

- A composite signal consisting of both sinusoidal and cosinusoidal components, affected by white noise. The results, displayed in Figure 13a, show a true positive rate of 94.2%.

- A signal composed of both sinusoidal and cosinusoidal components, perturbed by colored noise. The results are reported in Figure 13b, with a corresponding true positive rate of 94.4%



(a) Original signals are composed by a sum of sinusoidal waves affected by colored noise.

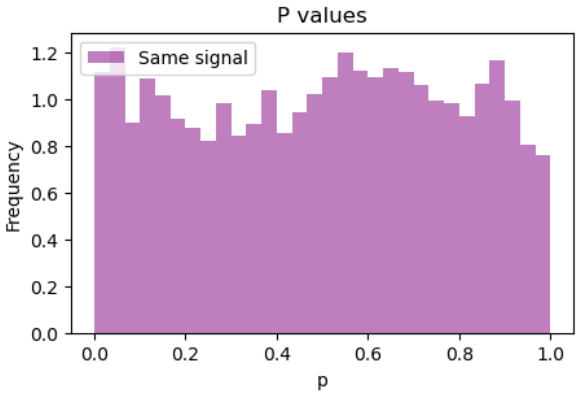


(b) Original signals are composed by a sum of cosinusoidal waves affected by white noise.

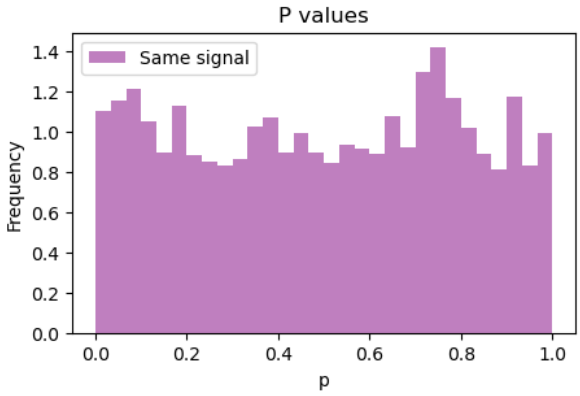
Figure 12: Resulting p value distribution applying the approach in section 4.3 to various signals type.

Finally, the sum of sinusoidal signals was analyzed under different parameter configurations. All parameters, except for the one under investigation, were maintained at the values specified in Section 4.

- The same parameters were used, but with a lower signal-to-noise ratio (SNR), ranging from 1 to 10, corresponding to a reduction by one order of magnitude. The distribution of p-values is shown in Figure 14a, with a resulting true positive rate of 95.3%.



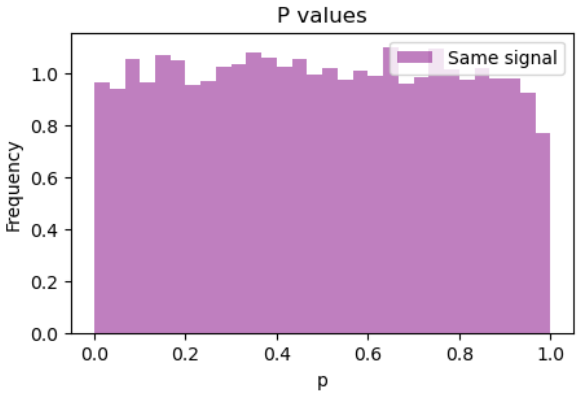
(a) Original signals are composed by a sum of sinusoidal and cosinusoidal waves affected by white noise.



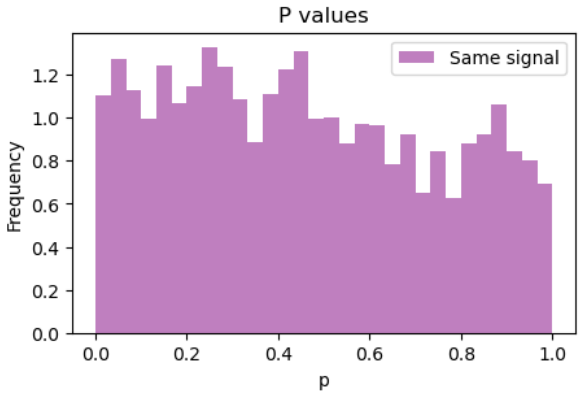
(b) Original signals are composed by a sum of sinusoidal and cosinusoidal waves affected by colored noise.

Figure 13: Resulting p value distribution applying the approach in section 4.3 to various signals type.

- An increased number of components, now ranging from 1 to 100. The corresponding o values are presented in Figure 14b, with an achieved true positive rate of 94.2%



(a) Original signals are composed by a sum of sinusoidal waves affected by white noise with low SNR.



(b) Original signals are composed by a sum of up to 100 sinusoidal waves affected by white noise.

Figure 14: Resulting p value distribution applying the approach in section 4.3 to various signals type.

5 Conclusion

The objective of this study was to identify a statistical test capable of determining whether two signals are identical with an ideal true positive rate of 95%. In this context, the term "identical" refers to signals that share the same frequency content but may differ due to the presence of noise.

Leveraging the linearity of the Fourier Transform, the difference between pairs of signals was analyzed under the well-supported assumption that, if the signals are identical, their difference corresponds to the noise component. Various approaches were explored to exploit the statistical properties of white noise in a manner that facilitates the classification of signal pairs.

The method proposed as a result of this study consists of the following steps:

1. Compute the (complex) Fourier Transform of the signals to be classified.
2. Apply a linear detrending procedure to both the real and imaginary components of the transformed signals.
3. Compute the normalized difference between the two spectra.
4. Concatenate the real and imaginary components of the resulting difference.
5. Perform a Shapiro-Wilk test to assess normality. If the p-value exceeds the significance threshold of 0.05, the two signals can be considered identical with an approximate sensitivity of 95%.

The proposed procedure demonstrated robustness across various conditions, including low signal-to-noise ratios, high-frequency content, and different signal types. However, a key limitation of the approach is that it requires prior knowledge of the noise affecting the signals, as it relies on statistical properties specific to white noise, which may not generalize to other noise types.

Future work could focus on extending this methodology to different noise models, potentially transforming them into an equivalent white noise representation. Additionally, the approach could be validated on a broader range of signals, including real experimental data, to further assess its applicability in practical scenarios.

References

[1] Friedrich K. Jondral. "White Gaussian Noise – Models for Engineers". In: *Frequenz* 71.5-6 (June 2017). Received March 28, 2017; published online June 8, 2017, pp. 231–236. DOI: 10.1515/freq-2017-0064.

[2] Leiden Institute of Physics. *Noise and Signal Processing*. Accessed: 18-Feb-2025. 2003. URL: https://home.physics.leidenuniv.nl/~exter/SVR/noise.pdf.

[3] University of Pennsylvania. *Week 11: Gaussian processes - White Gaussian noise*. Accessed: 18-Feb-2025. n.d. URL: https://www.seas.upenn.edu/~ese3030/homework/week_11/week_11_white_gaussian_noise.pdf.

[4] NumPy Developers. *numpy.random.randn — NumPy v1.24 Manual*. Accessed: 2025-02-18. 2025. URL: https://numpy.org/doc/stable/reference/random/generated/numpy.random.randn.html.

[5] University of Illinois. *Lecture 3: Noise*. 2020. URL: https://courses.grainger.illinois.edu/ece417/fa2020/slides/lec03.pdf.

[6] Princeton University. *Lecture 7: Signals and Systems*. 2011. URL: https://www.princeton.edu/~cuff/ele301/files/lecture7_4.pdf.

[7] Petr Beckmann. "Rayleigh Distribution and Its Generalizations". In: *Journal of Research of the National Bureau of Standards* 68D.9 (1964), pp. 927–932. URL: https://nvlpubs.nist.gov/nistpubs/jres/68D/jresv68Dn9p927_A1b.pdf.

[8] H. W. Lilliefors. "On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown". In: *Journal of the American Statistical Association* 62.318 (1967), pp. 399–402.

[9] National Institute of Standards and Technology. *1.3.6.6.16. Extreme Value Type I Distribution*. 2012. URL: https://www.itl.nist.gov/div898/handbook/eda/section3/eda366g.htm.