Lecture №6

Introduction to the Branch and Bound Method

1. The scheme of the method

The Branch and Bound Method (BBM) – is the universal method, which can be applied to any of the Optimization Problems.

Let we have the problem of minimization. Find $< F(q^*),\ q^* >$

$$F(q^*) = \min_{q \in Q} F(q);$$

$$q^* = \arg\min_{q \in Q} F(q).$$

The Branch and Bound Method is the method of directed search.

The set of the possible solutions is divided into subsets:

$Q_1, Q_2, ..., Q_n$.

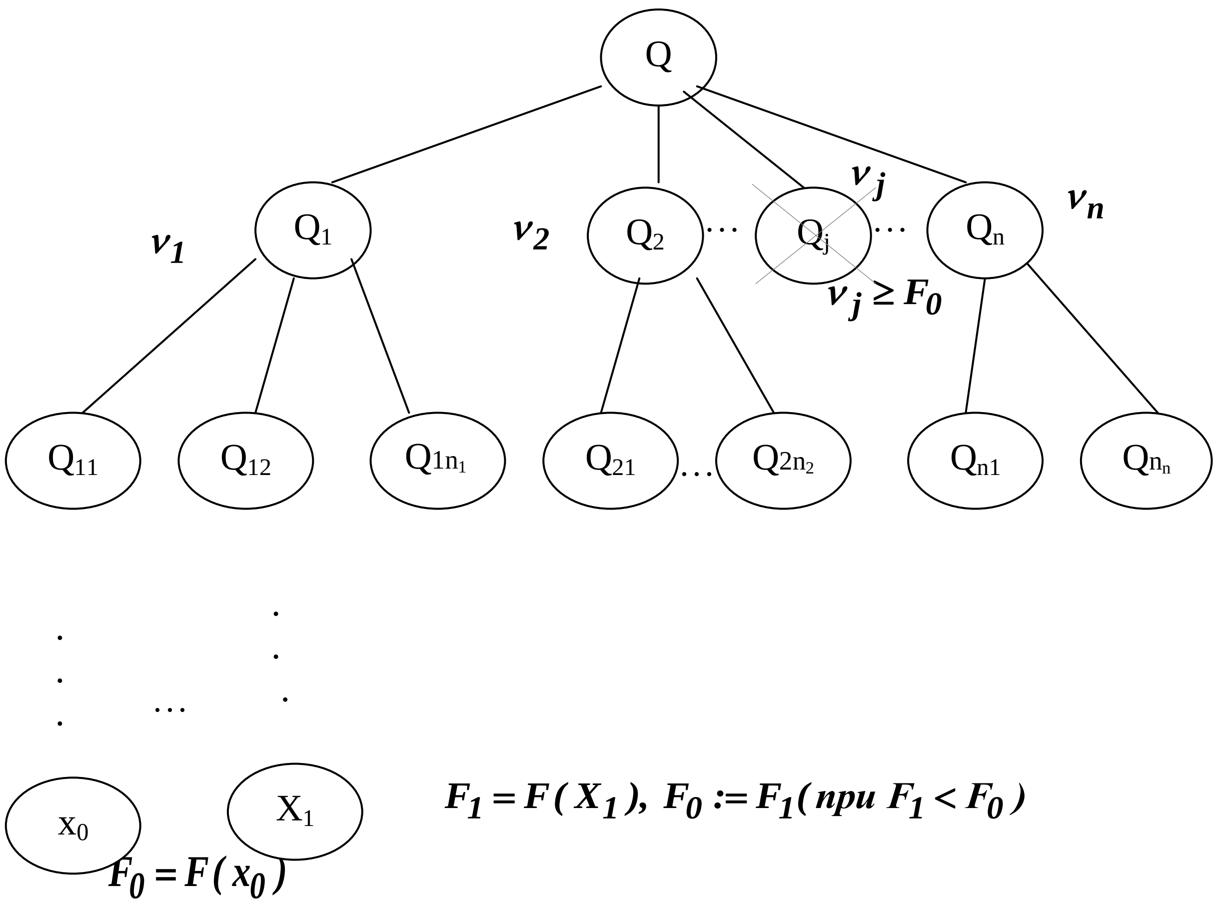

Figure 1 – Illustration to the Method

The Rules of Branching:

1. $Q_1 \cup ... \cup Q_n = Q$; $Q_i \neq \varnothing$.
2. $Q_i \cap Q_j = \varnothing$.

Each set Q_i is estimated. Estimation ν_i-number, which (in minimization problems) is not greater than the value of the function to minimize $F(q)$ for points from Q_i:

$$\nu_i \leq F(x) \quad \forall x \in Q_i.$$

You have to keep branching the sets. When you get one element set with possible solution x_0, calculate $F_0 = F(x_0)$.

Compare each estimation ν_j with F_0:

1) $\nu_j < F_0$,

2) $\forall x \in Q_j \;\; F(x) \geq \nu_j > F_0$,

3) $\forall x \in Q_j \qquad F(x) \geq \nu_j = F_0$.

There are no better solutions in Q_j (less than F_0) in the second and third cases. We cut off such set.

The Cutting Off: If $\nu_j \geq F(x_0) = F_0$ then the set Q_j is not branched further – cutting off. (Other rules are possible).

Hanging vertices are further branched.

If $F_1 > F_0$, than the vertex is cut off, F_0 doesn't change.

If $F_1 < F_0$, than $F_0 := F_1$.

When we change F_0, we compare F_0 with the estimations ν_j and cut off some of the sets if it is necessary. We do the same verification and when we receive new vertexes until all the subsets will be branched or cut off. The solution of the problem is the last value of F_0.

Stop – when there are no unbranched vertexes. The least F_0 gives the solution of the problem.

It is obvious that the method gives the exact solution of any problem by the directed (not complete) search.

Because the sets with the greater or equal objective function value than the estimation are cut off, and the estimation of the set is greater or equal F_0 current minimum value of $F(x)$, it guarantees, that the optimum point q^* with the value $F(q^*)$ will not be cut off.

Disadvantage: dependence of the time and memory capacity on the ways of the estimation of the branching.

2. Examples of the Usage of the Method

Example 1. The net of the cities and roads, is set by the first and last destination points and distances between them. Find the shortest route from 1 to 7.

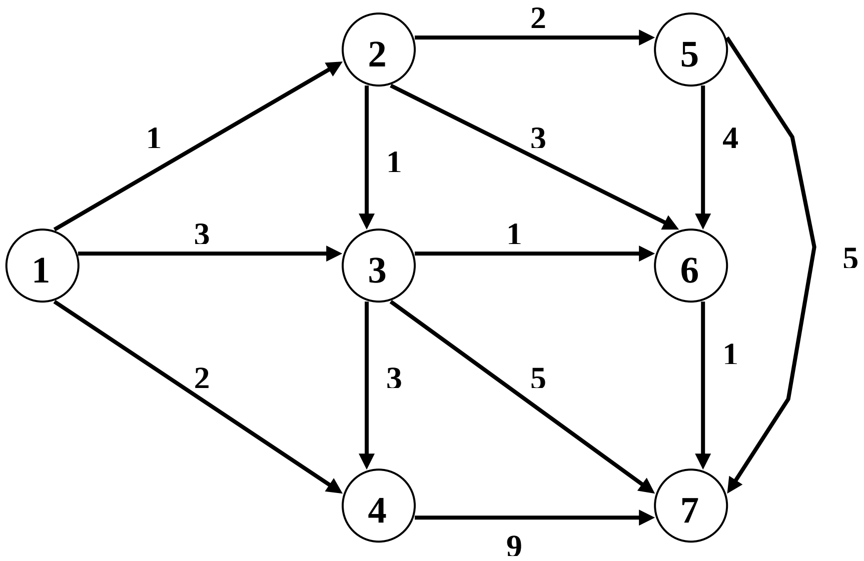

Figure 2 – Statement of the Problem in the example 1

In this problem:

$\boldsymbol{Q}$ − the length of all routes from the city 1 to the city 7.

$\boldsymbol{F(q)}$ − length of the route.

As the estimation of the set you can use the length of the passed route, that is shown last in the subset.

Each arc has the number of branching in the parenthesis: ($\boldsymbol{i}$).

The rule of the cutting off №1: $F_0 \le v_i$.

The rule of the cutting off №2: if we came into the city using different routes, we cut off the set with the greater or equal estimation.

Solution (see. fig. 3) $F(q^*) = F_0 = 4$, $q^* = (1,2,3,6,7)$.

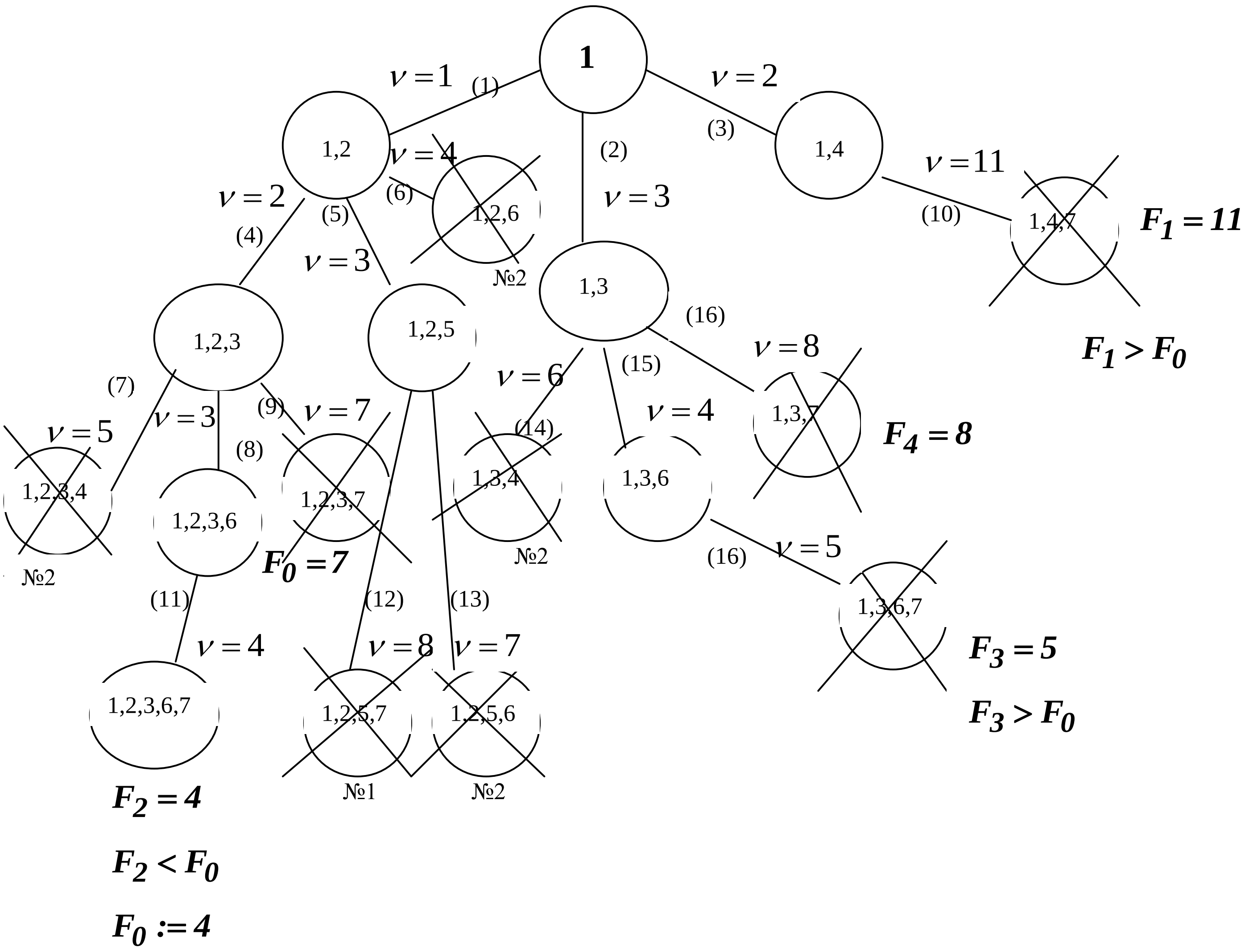

Figure 3 – Solution of the example 1

Example 2. Find $F = 2x_1 + 5x_2 + 3x_3 \to min$ on the set D of the permutations from numbers $\{1,4,5\}$ using branch and bound method.

There is the number of the step of branching on the edge of the tree in the round brackets on the fig. 4 (from (1) to (13)). We should denote the estimation $v_{(i)}$ for the set $D_{(i)}$ that we get after the step of branching i.

The estimation $v_{(i)}$ is known as the part of the objective function value on the already defined variables for the set (1,5), where $x_1 = 1$, $x_2 = 5$, we have $v_{(5)} = 2 \cdot 1 + 5 \cdot 5 = 27$ (see. fig. 4).

Let's remind, that the vertex with the estimation v (in the problem of minimization) is cut off, when $F_0 \le v_i$.

$\boldsymbol{v_{(1)} = 2 \cdot 1 = 2}$; $\boldsymbol{v_{(2)} = 2 \cdot 4 = 8}$; $\boldsymbol{v_{(3)} = 2 \cdot 5 = 10}$; $\boldsymbol{v_{(4)} = 2 \cdot 1 + 5 \cdot 4 = 22}$;

$\boldsymbol{v_{(5)} = 2 \cdot 1 + 5 \cdot 5 = 27}$; $\boldsymbol{v_{(6)} = F(x^0) = 2 \cdot 1 + 5 \cdot 4 + 5 \cdot 3 = 37 = F_0}$;

$\boldsymbol{v_{(7)} = F(x^1) = 1 \cdot 2 + 5 \cdot 5 + 3 \cdot 4 = F_1 = 39 > F_0}$; $\boldsymbol{v_{(8)} = 2 \cdot 4 + 5 \cdot 1 = 13 < F_0}$;

$\boldsymbol{v_{(9)} = F_2 = F(x^2) = 2 \cdot 4 + 5 \cdot 1 + 5 \cdot 3 = 28 < F_0}$; $\boldsymbol{F_0 := F_2 = 28}$;

$\boldsymbol{v_{(10)} = v_{(2)} + 5 \cdot 5 = 8 + 25 = 33 > F_0}$, the vertex is cut off.

$\boldsymbol{v_{(11)} = v_{(3)} + 5 = 15}$; $\boldsymbol{v_{(12)} = v_{(11)} + 3 \cdot 4 = 27 = F(x^3) = F_3 < F_0}$;

$\boldsymbol{F_0 := F_3 = 27;}$ $\boldsymbol{v_{(13)} = v_{(3)} + 5 \cdot 4 = 30 > F_0 = 27}$. Solving has been finished (see. fig. 4). Solution: $\boldsymbol{F_{min} = 27,\ x^* = (5,1,4)}$.

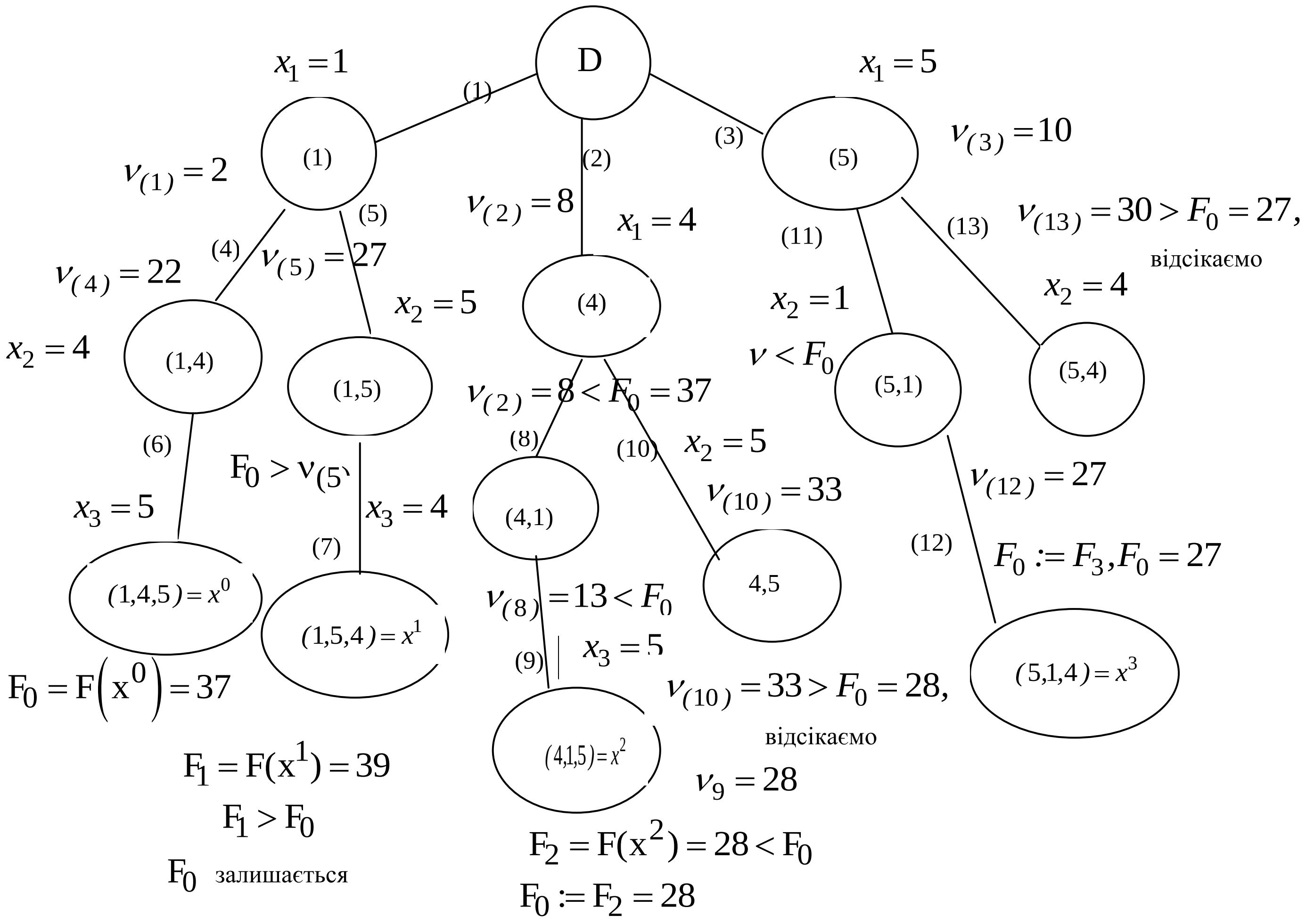

Figure 4 – The scheme of the solving the problem from the example 2 using branch and bound method

3. Branch and Bound Method for the Optimization Problems on Ordered Sets

Let it be the function (functional) $F(x)$, stated on the set X ($x \in X$), where the linear order is determined; $F(x) \in X$. Let $D \subset X$ – possible set.

Definition 1. Minimum x_{min} on a linear ordered set X is called and element, which foregoes all other elements: $x_{min} \prec x \quad \forall x \in X$.

The problem of optimization can be formulated in such a way: find

$$\min_{x \in D} F(x). \tag{1}$$

Definition 2. The estimation of the subset $D_i \subset D$ in the problem 1 is called the element $\nu \in X$, if $\nu \le F(x) \quad \forall x \in D_i$.

Let's denote S – as some list (array), n_{rec} – variable, that has the sense of the number examined by the method of possible solution. The algorithm of BBM for (1) is presented in the following steps.

0. $S = \varnothing$; $n_{rec} = 0$. Statement of the possible domain D ($D \ne \varnothing$), and objective function F on D.

1. The set D is divided into the subsets $D_1, ..., D_n$ with properties: $D_i \ne \varnothing$; $D_i \bigcap D_j = \varnothing \quad \forall i, j \in J_n = \{1, 2, ..., n\}$, $D = D_1 \bigcup ... \bigcup D_n$. The sets $D_1, ..., D_n$ we consider non-branched and non-cut off. Let's call this set «bud», and the properties of such sets – «the properties of the buds».

We should give each set, that doesn't belong to S and is a bud, the estimation $\nu_i(D_i) = \nu_i \in X$ – illegible number with the property $\nu_i \prec F(x) \quad \forall x \in D_i$, where the sign $\prec$ – sign of the linear order on the set X. Let's write them down in the list S of the bud with estimations. Let's denote the number of the buds $|S|$ via n.

2. Verification: $S = \varnothing$? If «yes» – move on to the step 16. If «no» – to the step 3.

3. We choose random bud $D_i \in S$.

4. We check: if the number of elements $|D_i|$ in the set D_i is equal unity: $|D_i| = 1$? If «yes» – to the step 6. If «no» – to the step 5.

5. We have $|D_i| \ne 1$ (it means $|D_i| > 1$), divide (branch) D_i as D, moving to the step 1.

6. One element bud we call «leaf», i.e. $D_i = \{x_{n_{rec}}\}$, $x_{n_{rec}} \in D$. The lesf D_i we remove from з S. Let's calculate $F_{n_{rec}} = F(x_{n_{rec}})$.

7. We verify: $n_{rec} > 0$? If «no» (i.e. $n_{rec} = 0$), then we move on to the step 8. Otherwise (i.e. $n_{rec} > 0$) – to the step 14.

8. We appropriate the point $x_{rec} := x_0$; $n_{rec} = 1$ to the point that gives the record value of the objective function.

9. We set $i = 1$ (organize the beginning of the cycle bud sorting).

10. Verification: $\nu_i \prec F_0$? If «yes» – move on to the step 12, if «no» – to the step 11.

11. Bud D_i we remove from the list S. (We should remark that in this case n does not change, it changes only in the step 1). It means that the bud D_i is cut off.

12. i is increased by 1. I.e. $i := i + 1$.

13. Verification: $i > n$? If «yes» – move on to the step 2. If «no» – move on to the step 10.

14. Verification: $F_{n_{rec}} \succ F_0$? If «yes» – move on to the step 2. If «no» – move on to the step 15.

15. We give to the record of the objective function F_0 the value $F_{n_{rec}}$, i.e. $F_0 := F_{n_{rec}}$, then $x_{rec} := x_{n_{rec}}$; $n_{rec} := n_{rec} + 1$. Move on to the step 9.

16. Output of the result: minimum value F_0 of the objective function and point x_{rec}, that gives it. Stop.

Remark. This algorithm is the algorithm of Branch and Bound Method for the problem on the set X, if the linear order is defined $\prec$ in the set X, and in the set of the objective function values, which belongs to X. That is why, when X - real numbers, then there are such concrete definitions in the algorithm: in the step 14 it is $\geq$; $<$ in the step 10).

The way of the branching of the possible set influences appreciably on the effectiveness of the BBM (step 1 (fragmentation D into buds D_i); step 3 – choice of D_i) and estimation of D_i (determination of the estimation in the step 1). Because the problem is considered in the general case, there is no methods that would work effectively. The ways of branching, cutting off, and estimation are predetermined by the specification of the type of the problems that are under consideration. The cutting of is made according to the classical for BBM condition as you can see from the algorithm : if $\nu_i(D_i) \prec F_0$ is not true, then D_i – is cut off.

Let's describe how the estimation of the subset can be calculated $D_{\tau_1 \tau_2 \ldots \tau_r}$ – buds of the r-th level.

When the given are the real numbers, the property of the estimation $\xi(D)$ of the subset D that makes the BBM works, is: if $D_i \subset D$, then $\xi(D_i) \geq \xi(D)$.

The following statement results from this property and the definition of the estimation.

Theorem 1. If $D_{i_1} \supset D_{i_2} \supset \ldots \supset D_{i_n} = \{x\}$, i.e. $|D_{i_n}| = 1$, and the functional ξ, given on the sets $D_{i_1}, \ldots, D_{i_n}$ is that $\xi(D_{i_n}) = \xi(x) = F(x)$, $\xi(D_{i_j}) \prec \xi(D_{i_{j+1}})$, $\forall j \in J_{n-1}$, than the value of the functional $\xi(D)$ could be the estimation of the possible subset in BBM.

Proof. It follows that $\xi(D_{i_j}) \prec F(x)$ $\forall x \in D_{i_j}$ $\forall j \in J_n$ because $\xi(D_{i_j}) \prec \xi(D_{i_{j+1}})$ $\forall j \in J_{n-1}$ and $\xi(D_{i_n}) = \xi(x) = F(x)$, which required to be proved.

Literature

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