Analytical methods for the study of reversible difference schemes for classical nonlinear oscillators

Mikhail Malykh

Peoples' Friendship University of Russia, Moscow, Russia
Joint Institute for Nuclear Research, Dubna, Russia

Feb. 25, 2025, ver. February 21, 2025

This work is supported by the Russian Science Foundation (grant no. 20-11-20257).

About

- The Department of Mathematical Modeling and Artificial Intelligence was created at RUDN University in Sept. 2023.
- The department was created by combining IT Department, created by I.L. Tolmachev, and the scientific school of E.P. Zhidkov (JINR, Dubna).

Why elliptic functions?

Why are finitely integrable dynamical systems integrable in elliptic functions?
Painlevé's answer: any dynamical system that defines a birational correspondence between initial and final values on algebraic integral varieties is integrable in classical transcendental functions, usually in elliptic ones.

Transformation given by the initial problem

Let us consider the initial problem

$$\frac{d\mathfrak{x}}{dt} = f(\mathfrak{x}), \quad \mathfrak{x}|_{t=0} = \mathfrak{x}_0 \tag{1}$$

on the segment $[0, \Delta t]$ of the real axis t. For brevity, we will use vector notation, meaning by $\mathfrak{x}$ the list $(x_1, \dots, x_n)$.
In standard approach, t is a variable, but $\mathfrak{x}_0$ is a constant vector. In Painlevé approach $\mathfrak{x}_0$ is a list of symbolic variables, but Δt is a given constant or a parameter.
From such viewpoint, the initial problem give us the correspondence of the affine space $\mathbb{A}^n$ between the initial value of the variable $\mathfrak{x}$ and the value of $\mathfrak{x}$ at $t = \Delta t$.
Using notation, which is standard for difference scheme theory, we will denote initial value as $\mathfrak{x}$ without index 0 and final value as $\hat{\mathfrak{x}}$.

The function $\hat{\mathfrak{x}}(\mathfrak{x})$

For some initial values $\mathfrak{x}$, the procedure for analytic continuation of the solution obtained in the Cauchy theorem along a segment does not encounter singular points other than poles, and in this case the final value of $\hat{\mathfrak{x}}$ is uniquely determined by the initial value of $\mathfrak{x}$.
However, if the path encounters a branch point, then the final value depends on the way it is passed.
Therefore, in general case, the final value $\hat{\mathfrak{x}}$ is a multivalued function of the initial value of $\mathfrak{x}$ and the transformation given by the initial problem is also multivalued.

What are classical transcendental functions?

Let the system of odes have several independent algebraic integrals of motion, that is,

$$h_1(\mathfrak{x}) = C_1, \dots, h_r(\mathfrak{x}) = C_r.$$

They define integral manifolds of dimension $n - r$, which is invariant with respect to the transformation given by the initial problem.

Definition (Painlevé)

If, for any choice of $\Delta t > 0$, the restriction of the transformation given by the Cauchy problem to the integral manifolds is a birational transformation of these manifolds, then we will say that the dynamic system of odes integrates in classical transcendental functions.

Jacobi oscillator

Jacobi oscillator, i.e., dynamical system

$$\dot{p} = qr,\ \dot{q} = -pr,\ \dot{r} = -k^2 pq, \tag{2}$$

has two quadratic integrals

$$p^2 + q^2 = c_1 \quad \text{and} \quad k^2 p^2 + r^2 = c_2 \tag{3}$$

which define an elliptic curve in the space pqr. On this curve, Jacobi system can be described by the quadrature

$$\int \frac{dp}{\sqrt{(c_1 - p^2)(c_2 - k^2 p^2)}} = t + c_3 \tag{4}$$

and thus is integrable in terms of elliptic Jacobi functions.

Why the restriction?

Using the addition theorem for Jacobi elliptic functions, we can express p, q, r rationally through their values at $t = 0$ and vice versa.

Theorem

The restriction of the correspondence defining by the Jacobi oscillator to the integral curve is birational.

The correspondence in space pqr can be described by a system of transcendental equations

$$f_i(p, q, r, p_0, q_0, r_0, c_1, c_2) = 0, \quad i = 1, \ldots, 6$$

After eliminating of c_1, c_2 we will not get a birational correspondence. Thus the amendment about the restriction to the integral manifolds is important.

The main theorem in differential case

Theorem (Painléve, 1897)

Let the algebraic integral manifolds of the dynamical system be lines of genus $p = 1$. The dynamic system integrates in classical transcendental functions iff

$$\frac{dx_1}{f_1}$$

is is an Abelian differential of the first kind on integral curves. The system on the integral manifolds can be described by the quadrature

$$\int \frac{dx_1}{f_1} = t + c$$

and x_i are meromorphic doubly-periodic functions of t.

Finite difference method

Within the framework of the finite difference method the system of differential equations is replaced with the system of algebraic equations

$$g_i(\mathfrak{x}, \hat{\mathfrak{x}}, \Delta t) = 0, \quad i = 1, \dots, n, \tag{5}$$

called as a difference scheme. In this case, $\mathfrak{x}$ is interpreted as the value of the solution at the time t, and $\hat{\mathfrak{x}}$ as the solution at the time $t + \Delta t$.

Example

- Euler scheme $\hat{x} - x = f(x)\Delta t$,
- midpoint scheme $\hat{x} - x = f\left(\frac{\hat{x}+x}{2}\right)\Delta t$,
- trapezoid scheme $\hat{x} - x = \frac{f(\hat{x})+f(x)}{2}\Delta t$, ...

Difference scheme as algebraic correspondence

By the definition, any difference scheme is a system of equations, which define a correspondence between the initial value $\mathfrak{x}$ and the final values $\hat{\mathfrak{x}}$.
This is the correspondence, which was indicated by Painlevé in differential case only in 1897. Painlevé's approach is transferred to the finite differences naturally.
By analog of def 1 we can investigate when dynamical system can be approximate by the difference scheme, defining the birational correspondence between initial and final values.
We expected the results in two cases to be similar, but it is true only in one dimensional case.

One dimensional case

Theorem

If ode define the birational correspondence between initial and final values (as point of projective right line $\mathbb{P}$), then this ode is Riccati equation

$$\frac{dx}{dt} = ax^2 + bx + c.$$

Theorem

If ode can be approximate by difference scheme, defining the birational correspondence between initial and final values, then this ode is Riccati equation

$$\frac{dx}{dt} = ax^2 + bx + c.$$

Ref.: Malykh, M.D. et al. // Journal of Mathematical Sciences, 2019, doi 10.1007/s10958-019-04380-0

Reversibility of difference schemes

Definition

By reversibility, we should understand the possibility to uniquely determine the final data $\hat{\mathfrak{x}}$ from the initial data $\mathfrak{x}$ and vice verse using the system

$$g_i(\mathfrak{x}, \hat{\mathfrak{x}}, \Delta t) = 0, \quad i = 1, \ldots, n,$$

for any fixed value of the step Δt.

Ref.: Malykh M.D. et al. // Journal of Mathematical Sciences, Vol. 261, No. 5, 2022. DOI 10.1007/s10958-022-05781-4

The dynamical system with a quadratic right-hand side

Theorem

For any dynamical system with a quadratic right-hand side, a t-symmetric and reversible difference scheme can be constructed:

$$\hat{x}_i - x_i = F_i(x, \hat{x})\Delta t, \quad i = 1, \dots, n, \tag{6}$$

where F_i is obtained from f_i by replacing monomials: x_j with $(\hat{x}_j + x_j)/2$, $x_j x_k$ with $(\hat{x}_j + x_j)(\hat{x}_k + x_k)/4$, and x_j^2 with $x_j \hat{x}_j$.

The reversible difference scheme is Cremona quadratic transformation of projective space $\mathbb{P}^n$.

There is no analogy between differential ans difference cases

The difference case is more interesting and richer than the differential case.

- In differential case we need in the amendment about the restriction to the integral manifolds, in difference case this amendment interferes with us.
- In differential case only a few dynamical systems with a quadratic right-hand side possess the Painlevé property and can be integrate in classical transcendental functions [Kowalewski, 1880s], in difference case any such a system can be approximate by reversible difference scheme

Approximate solutions by reversible scheme

Let a transition from layer to layer be described by the Cremona transformation C depending on Δt:

$$\hat{\mathfrak{x}} = C\mathfrak{x}.$$

Definition

By the approximate solution released from the point $\mathfrak{x}$, we mean the sequence

$$O(\mathfrak{x}) = \{C^m\mathfrak{x},\, m \in \mathbb{Z}\},$$

i.e., the orbit of Cremona transformation C.

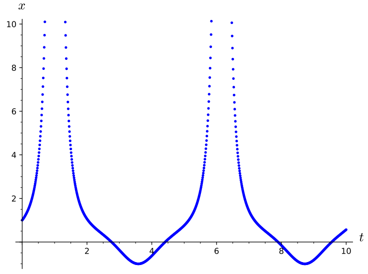
In classical works, the points $\mathfrak{x}$ and $\hat{\mathfrak{x}}$ treat as the points of two different planes. We have treat they as the points of one plane.

Points at infinity

If at some value k the denominator of the transformation becomes zero, then the point $\mathfrak{x}_{k+1}$ will be infinitely remote. Thus we consider $\mathfrak{x}$ as a point in the projective space $\mathbb{P}_n$.

Example ($\wp$-oscillator)

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{d}{dt}y = 6\,x^2 - 1, \\ x(0) = 1, \quad y(0) = 2 \end{cases}$$



The exact solution is periodic and has pols of 2nd degree.

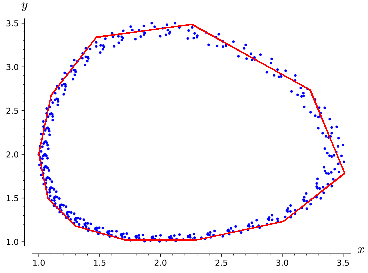
The approximate solutions describe correct the behavior at infinity.

Invariant curves

In two dimensional case, the points of the approximate solution lie on some curve even at big step Δt.

Example (Volterra-Lotka system)

$$\begin{cases} \dfrac{dx}{dt} = -x(y-2), \\ \dfrac{dy}{dt} = (x-2)y, \\ x(0) = 1, \quad y(0) = 2 \end{cases}$$



Two solutions were found at $\Delta t = 0.30083$

blue by the Runge-Kutta scheme and

red by the reversible scheme

Generalizations

Ways to construct a reversible scheme for a system with a non-square right-hand side:

- Appelroth quadratization: utilizing Appelroth's technique (1902), we reduce any system of ordinary differential equations with a polynomial right-hand side to a quadratic form, enabling the application of Kahan's method.
- Abrashin doubling:

$$\frac{dx}{dt} = f(x) \quad \to \quad \hat{x} - x = f(y)\Delta t,\ \hat{y} - y = f(\hat{x})\Delta t,$$

Refs:

1. Malykh, M. et al. // Mathematics 2024, 12, 2725.
2. Sytova S. N. // Differ. Equ. — 1996. — Vol. 32, no. 7. — P. 995–998.

An unconventional integrator of W. Kahan

Firstly, indicated method to construct reversible schemes was presented by William "Velvel" Kahan in 1993 at conference in Ontario.

> *I have used these unconventional methods for 24 years without quite understanding why they work so well as they do, when they work. That is why I pray that some reader of these notes will some day explain the methods' behavior to me better than I can, and perhaps improve them.*

In 1994 Sanz-Serna applied the method to Volterra-Latka system and explain the successes of the method to the inheritance of the symplectic structure

$$\frac{dx \wedge dy}{xy}.$$

Ref.: J.M. Sanz-Serna // Applied Numerical Mathematics 16 (1994) 245-250.

Kahan's method for Hamiltonian systems

1. Geometric properties of Kahan's method restricted to quadratic vector fields.
2. For the systems with cubic hamiltonian, Kahan's method conserved the modified Hamiltonian

$$H + \frac{\Delta t}{3} \nabla H^T \left(E - \frac{\Delta t}{2} \frac{\partial \mathfrak{f}}{\partial \mathfrak{x}} \right)^{-1} \mathfrak{f}$$

3. For the systems with cubic Hamiltonian, Kahan's method preserves the measure

$$\frac{dx_1 \wedge dx_2 \cdots \wedge dx_n}{\det \left(E - \frac{\Delta t}{2} \frac{\partial \mathfrak{f}}{\partial \mathfrak{x}} \right)}$$

Ref.: E. Celledoni et al // J. Phys. A: Math. Theor. 46 (2013) 025201

Systems with cubic Hamiltonian

Kahan's scheme perfectly imitates a Hamiltonian system with a cubic Hamiltonian H, for example, a elliptic $\wp$-oscillator.

- According to 1st Celledoni's theorem, the symplectic structure is inherit, i.e.

$$d\hat{x} \wedge d\hat{y} = (1 + O(\Delta t))dx \wedge dy.$$

- According to 2nd Celledoni's theorem, the energy integral is inherit, thus the approximate solution itself is a sequence of points $\mathfrak{x}_n = (x_n, y_n)$ of an elliptic curve $f(x, y, \Delta t) = c$.

Ref.: Suris et al. // Proc. R. Soc. A. 2019. 475: 20180761

Method of Hirota and Kimura

- In 2000, reversible scheme was written by Hirota and Kimura for odes, describing the motion of the top. For 2 classical cases, modified integrals was written. The expressions for two integral for Euler-Poinsot case are the same which we presented at PCA'2022.
- In 2010, Suris et al. indicated that the scheme of Hirota and Kimura define Cremona transformation between the layers.
- In 2019, Suris et al. described the method of Hirota and Kimura for finding the integrals. It is, of course, the variation around Lagutinski method (1912).

Refs.: 1.) Hirota and Kimura // Journal of the Physical Society of Japan Vol. 69, No. 3, March, 2000, pp. 627-630; No. 10, October, 2000, pp. 3193-3199; 2.) Suris et al. // Math. Nachr. 283, No. 11, 1654 – 1663 (2010); 3.) Suris et al. // Experimental Mathematics, 26:3, 324-341 (2019).

Elliptic oscillators

Dynamical system, which is integrable in elliptic functions, we will call as elliptic oscillators.
Elliptic oscillators are notable not for the fact that reversible schemes exist for them, but for something else. What is it?

Example

Jacobi oscillator

$$\dot p = qr,\ \dot q = -pr,\ \dot r = -k^2 pq,$$

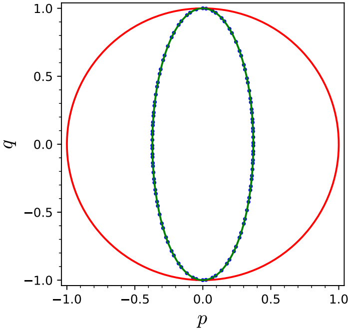
can approximate by reversible difference scheme

$$\hat p - p = \frac{\Delta t}{2}(\hat q r + q \hat r),\ \hat q - q = -\frac{\Delta t}{2}(\hat p r + p \hat r),\ \hat r - r = -\frac{k^2 \Delta t}{2}(\hat p q + p \hat q).$$

This is Cremona quadratic transformation in the space pqr.

Jacobi oscillator, approximate solution

The solution projection on the plane pq, points marked approximate solution, red line is exact integral curve, green is approximate integral curve. In all our experiments $k = \frac{1}{2}$.



The projections of the exact solution points lie on the circle, and the approximate – on the ellipse.

Carambola of Kowalewski

For Kowalewski top, when

$$A = B = 2,\ C = 1,$$

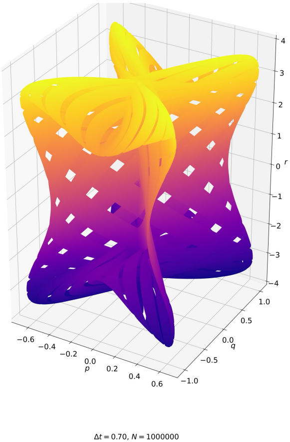
$$x_0 = 1,\ y_0 = z_0 = 0,$$

solution points do not line up, but fill an area everywhere dense. In our experiments $M = 1$, $g = 10$, and

$$(p, q, r) = (0, 1, 2),$$

$$(\gamma, \gamma', \gamma'') = (1, 0, 0).$$

The step $\Delta t = 1/10$.



Picture by V. Kadrov, 2025.

Results of experiments

Our experiments suggest that:

- the points of solutions line up in case of oscillators and fill multidimensional areas in other cases
- in case of oscillators, the approximate integral lines don't coincide with exact integral curve
- in case of elliptic oscillators, the approximate integral lines look like algebraic curves of small order.

To investigate approximate integral varieties found in computer experiments we generalize Lagutinski theory of algebraic integrals.
Ref.: Malykh M.D. et al. // Записки научных семинаров ПОМИ. Т. 517, 2022. С. 17–35.

Approximate integral varieties

Definition

The set of all hypersurfaces of the form

$$a_0 g_0(\mathfrak{x}) + \cdots + a_m g_m(\mathfrak{x}) = 0,$$

where $a_0, \ldots, a_m$ are parameters and $g_0, \ldots g_m$ are polynomials, is called a linear system of dimension m.

Theorem

If for any approximate solution $\mathfrak{x}_0, \mathfrak{x}_1, \ldots$

$$\det(g_i(\mathfrak{x}_j)) = 0, \quad (i, j = 0, 1, \ldots m)$$

then any approximate solution lies on a hypersurface of the linear system.

Jacobi oscillator, integral curve

The points of a approximate solution lie on a curve V, defined by the system of eqs

$$p^2 + q^2 = c_1 \left(1 + \frac{k^2 \Delta t^2}{4} q^2\right), \quad k^2 p^2 + r^2 = c_2 \left(1 - \frac{\Delta t^2}{4} r^2\right).$$

This is elliptic curve. It passes into an exact integral curve

$$p^2 + q^2 = c_1, \quad k^2 p^2 + r^2 = c_2.$$

at $\Delta t \to 0$, but they do not coincide.

Non algebraic integral curve

Controversy, if any approximate solution lies on a hypersurface of the linear system, than

$$\det(g_i(\mathfrak{x}_j)) = 0, \quad (i, j = 0, 1, \dots m)$$

At $\Delta t \to 0$, this determinant passes into Lagutinski determinant. Thus Lagutinski determinant with respect to the linear system is equal to zero and, via Lagutinski's theorem, the dynamical system has a rational integral.

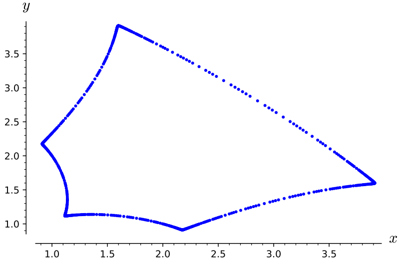
Example

There are no rational integrals for the system Volterra-Lotka. Points of a solution of Volterra-Lotka system line up, however the approximate integral curve is transcendental.

Volterra-Lotka system

At the fig. we can see points of solution of Cauchy problem

$$\begin{cases} \dfrac{dx}{dt} = -x(y-2), \\ \dfrac{dy}{dt} = (x-2)y, \\ x(0) = 1, \quad y(0) = 2 \end{cases}$$



at a big enough step Δt.
They lie on a curve with cusps. We use a big interval $0 < t < 500$ ans 740 points.

Invariant sets of Crenona transformation

Approximate integral manifold is an invariant set of Crenona transformation.
In general case, Crenona transformation hasn't simple invariant sets.
In contrary of that, in case of elliptic oscillators Crenona transformation has invariant sets of the smallest dimension, that are curves

$$h_1(\mathfrak{x}, \Delta t) = c_1, \ldots, h_{n-1}(\mathfrak{x}, \Delta t) = c_{n-1}.$$

Furthermore, these curves are algebraic and have the genus 1 (elliptic curve). It's very special case!
We are used to thinking that quadrature is obtained due to the separation of variables. However in this special case we can describe difference scheme as a quadrature also.

Quadrature for difference scheme

Theorem

Let the dynamical system $\dot{\mathfrak{x}} = \mathfrak{f}(\mathfrak{x})$ *can be approximate be revertible difference scheme*

$$\hat{\mathfrak{x}} = \mathfrak{R}(\mathfrak{x}, \Delta t),$$

and let Cremona transformation has an invariant curve V *of genus* 1*. Then difference scheme can be written in form of quadrature*

$$\int\limits_{\mathfrak{x}}^{\hat{\mathfrak{x}}} v(\mathfrak{x}, \Delta t) dx_1 = \Delta t,$$

where $v dx_1$ *is elliptic differential of the 1st kind on the curve* V.

Jaconi oscillator, quadrature

Due to the th. 16 the quadrature

$$\int_{p}^{\hat{p}} \frac{dp}{\sqrt{(c_1 - p^2)(c_2 - k^2 p^2)}}$$

on approximate integral curve

$$p^2 + q^2 = c_1 \left(1 + \frac{k^2 \Delta t^2}{4} q^2\right), \quad k^2 p^2 + r^2 = c_2 \left(1 - \frac{\Delta t^2}{4} r^2\right).$$

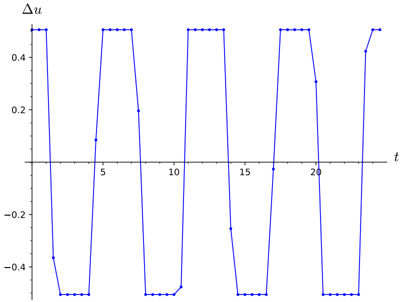
is a function of $\Delta t, c_1, c_2$ only.
We calculate this function in Sage, it's a transcendental function of Δt.

Jaconi oscillator, quadrature

At the fig. we can see the value

$$\int_{p_m}^{p_{m+1}} \frac{dp}{\sqrt{(c_1 - p^2)(c_2 - k^2 p^2)}}$$

on the approximate solution form previous example.



It is not a constant because we must select the root branch correctly.

The periodicity of approximate solution

The periodicity of approximate solution is consequence of quadrature form for difference scheme.
One step is described as

$$\int\limits_{\mathfrak{x}_{m-1}}^{\mathfrak{x}_m} v(\mathfrak{x},\Delta t)dx_1 = \Delta t,$$

m steps – as

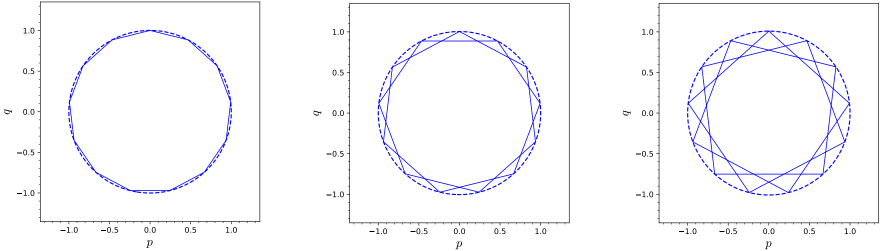
$$\int\limits_{\mathfrak{x}_0}^{\mathfrak{x}_m} v(\mathfrak{x},\Delta t)dx_1 = m\Delta t.$$

Sequence $\mathfrak{x}_0, \mathfrak{x}_1, \dots$ has the period M, iff

$$M\Delta t = \oint v(\mathfrak{x},\Delta t)dx_1 = N\omega(\Delta t),$$

where ω is real period of elliptic integral.

Example: Jacobi oscillator



Periodic solutions with period $M = 13$ and with $N = 1, 2$ and 3.

Jaconi oscillator, the periodicity

We demonstrated such a solution at PCA'2021. Our old algorithm was purely analytical and lacked resources already at $N \simeq 10$. Using the quadrature, we proved the existence such a solution at any M and found that

$$\beta\Delta t = \sqrt{c_1}\,\mathrm{sn}\left(\frac{4N}{M}K\left(\sqrt{\frac{c_1}{c_2}}k\right), \sqrt{\frac{c_1}{c_2}}k\right),$$

where

$$\beta = \frac{4\sqrt{c_1 c_2\left(4 - c_1 k^2 \Delta t^2\right)\left(4 + c_2 \Delta t^2\right)}}{16 + 8c_2\Delta t^2 - c_1 c_2 \Delta t^4 k^2}$$

These exact formulas work only for small enough steps Δt.

The meromorphic representation of approximate solution

The quadrature

$$\int_{\mathfrak{x}_0}^{\mathfrak{x}_m} v(\mathfrak{x}, \Delta t) dx_1 = m\Delta t$$

showed also that the approximate solution can be represented as a set of values of a meromorphic doubly periodic function:

$$\mathfrak{x}_m = \wp(m\Delta t).$$

Conclusion

The discrete and continuous theories of elliptic oscillators are described by the same formulas: the quadrature describes the transition from initial to final data, the motion is periodic, it is described by meromorphic functions, and so on.
The whole difference lies in the fact that in the discrete theory the birational transformation describing the transition from the old position of the system to the new one is continued to the Cremona transformation

The End

Additional materials: `https://malykhmd.neocities.org`